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iCub Robot Modelling and Control of Its Biped Locomotion

Tony Dodd, David Owens, Fan Zhang, M Saiful Huq

Department of Automatic Control and Systems Engineering
The University of Sheffield
Mappin Street
S1 3JD UK

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1 Introduction

This report concludes the work have been done from January 2009 to January 2010.

The objective of the Sheffield research was to work in collaboration with the Italian Institute of Technology to investigate the control architecture(s) that may enable improved control of gait in a humanoid iCub robot and to identify issues that arise in this design process due to robot dynamics, gait specification and control structure.

The research had two key results,

1. The first one was the creation of an iCub-like robot model based on information from a Salford University thesis. The model was based on Mechanical Engineering methods using Kane's equations and Lagrange's equations and supported by data from IIT. Linearized equations were used for control design development. The impact model of iCub was also created.

2. The second one was that of considering the development of a systematic control design process based on real-time joint data and prescribed actuators. Key conclusions indicate that a systematic design process based on an inner multi-loop PD control scheme (feeding back angular and angular velocity data) can be achieved using concepts of multivariable second order systems followed by an outer loop design of a proportional plus repetitive controller that has an internal model reflecting the period nature of gait. Tracking accuracy can be excellent for a prescribed gait although relaxation of this requirement to excellent tracking of some phase shifted gait pattern is possibly more desirable. Potentially this may be achievable through adaptive changes to the gait reference signal.

The rest of the report is organised as followed: section 2 lists all the iCub parameters that used for the model creation and then section 3 gives the final symbolic mathematical result of iCub dynamical model; after that, section 4 and section 5 describes two different ways for the creation of iCub model, Kane's equation and Lagrange's equation respectively; the impact model of iCub robot is given in section 6; the controller structure in section 7 is based on the Salford University thesis; thereafter, a novel second order design process is fully developed in section 8, and finally, at the end of the reports are the Appendix and Bibliography.

M. Saiful Huq wrote sections 2, 3, 4 and 6. F. Zhang wrote all the other sections.

2 Model Parameters

m_i	mass of the i^{th} link, for $(i = 1, ..., 6)$
l_j	position vector of the j^{th} joint relative to $(j-1)^{th}$ joint in the $(j-1)^{th}$ frame of reference, for $j = 2, 3, 4$
l_j	position vector of the $(j+1)^{th}$ joint relative to j^{th} joint in the j^{th} frame of reference, $j = 5$
r_i	position vector of the center of mass (CoM) of the i^{th} link relative to i^{th} joint in the i^{th} frame of reference
I_i	inertia tensor matrix of the i^{th} link
T_i	applied torque vector at the i^{th} joint
α_i	absolute angle of the i^{th} link
θ_i	relative angle of the i^{th} link

3 Planer dynamic equations for iCub

Symbolic mathematical model for the iCub is obtained for the sagittal plane using Matlab symbolic math toolbox. Figure 1 shows the joints and torques sign conventions in the sagittal plane while Figure 2 shows the sagittal plane model parameters. The dynamic model assuming an open chain structure with the robot in its single support phase is derived using both Kane's and Lagrange's equation [3].

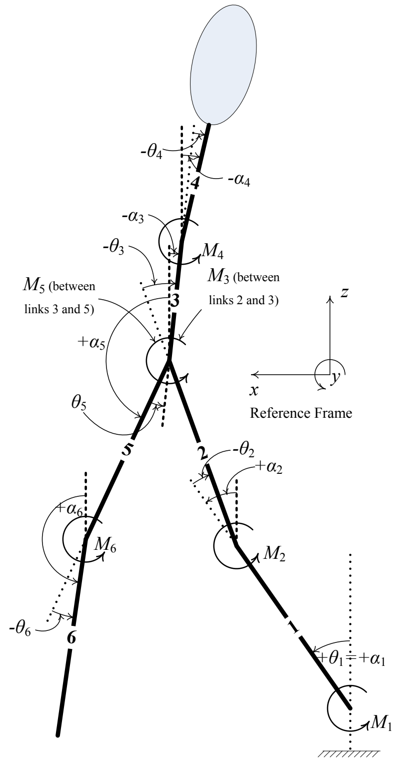


Figure 1: Sign conventions for the angles (absolute and relative) and torques used in the model. α_i and θ_i are the absolute and relative angle of the i'th joint respectively. The absolute angle α_i is related to the relative angle θ_i as $\alpha_i = \sum_{k=1}^{i} \theta_k$ for $i = 1, \ldots, 4$; $\alpha_5 = \alpha_3 + \pi - \theta_5$ and $\alpha_6 = \alpha_5 - \theta_6$

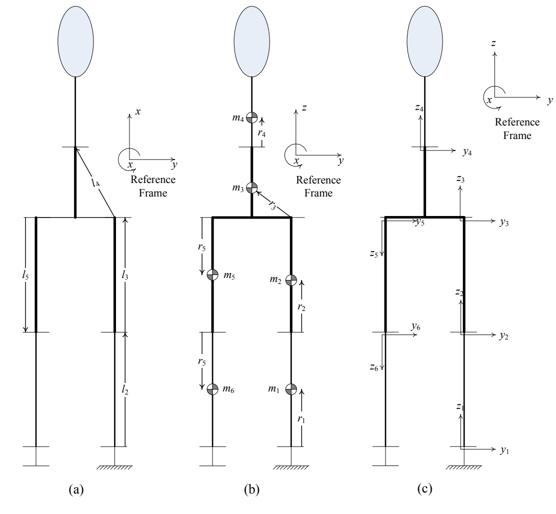


Figure 2: Sagittal plane parameters (a) distance between joints (b) location of the CoMs (c) inertial and local frames

$$\mathbf{A}(q)\ddot{\mathbf{q}} + \mathbf{B}(q,\dot{q})\dot{\mathbf{q}} + \mathbf{G}(\dot{q}) + \mathbf{C}(q) = \mathbf{D} \tag{1}$$

where

$\boldsymbol{q}$	the $n \times 1$ vector of generalized coordinates describing the pose of the system
$\dot{\boldsymbol{q}}$	the $n \times 1$ vector of joint velocities
$\ddot{\boldsymbol{q}}$	the $n \times 1$ vector of joint accelerations
$\boldsymbol{M}$	the $n \times n$ symmetric joint-space inertia matrix
$\boldsymbol{N}$	the $n \times n$ matrix describing Coriolis and centripetal effect
$\boldsymbol{G}$	$n \times n$ matrix describing viscous and coulomb frictions and not generally part of the rigid body dynamics
$\boldsymbol{C}$	$n \times n$ matrix of gravity loading
$\boldsymbol{D}$	the vector of generalized forces associate with the generalized coordinates q

$$\begin{aligned}\mathbf{A}(1,1) = &- m_4 l_{23} l_{33}\cos(\theta_2) - m_4 l_{23} l_{43}\cos(\theta_3+\theta_2) - m_4 l_{23} r_{43}\cos(\theta_4+\theta_3+\theta_2)\\ &- m_3 l_{23} l_{33}\cos(\theta_2) - m_2 l_{23} r_{23}\cos(\theta_2) - m_5 l_{23} l_{33}\cos(\theta_2)\\ &- m_6 l_{23} l_{53}\cos(\theta_3+\theta_2+\theta_5) - m_6 l_{23}^2 - m_3 l_{23} r_{33}\cos(\theta_3+\theta_2)\\ &- m_6 l_{23} r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6) - m_5 l_{23} r_{53}\cos(\theta_3+\theta_2+\theta_5)\\ &- m_6 l_{23} l_{33}\cos(\theta_2) - I_{12} - m_4 l_{23}^2 - m_2 l_{23}^2 - m_3 l_{23}^2 - m_1 r_{13}^2 - m_5 l_{23}^2\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbf{A}(2,1) = &- m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3) - I_{22}\\ &- m_4 l_{23} l_{33}\cos(\theta_2) - m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6) - m_5 l_{33} r_{53}\cos(\theta_3+\theta_5)\\ &- m_6 l_{33} l_{53}\cos(\theta_3+\theta_5) - m_2 r_{23}^2 - m_4 l_{33}^2 - m_3 l_{23} l_{33}\cos(\theta_2)\\ &- m_2 l_{23} r_{23}\cos(\theta_2) - m_5 l_{23} l_{33}\cos(\theta_2) - m_6 l_{23} l_{33}\cos(\theta_2) - m_6 l_{33}^2 - m_3 l_{33}^2\\ &- m_5 l_{33}^2\end{aligned} \tag{3}$$

$$\begin{aligned}\mathbf{A}(3,1) = &- m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3) - m_3 r_{33}^2 - 2m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6)\\ &- 2m_5 l_{33} r_{53}\cos(\theta_3+\theta_5) - 2m_6 l_{33} l_{53}\cos(\theta_3+\theta_5) - m_4 l_{43}^2 - 2m_5 r_{53}^2\\ &- 2m_6 l_{53}^2 - 2m_6 r_{63}^2 - 4m_6 r_{63} l_{53}\cos(\theta_6) - I_{32} - m_4 l_{43} r_{43}\cos(\theta_4)\\ &- m_4 l_{23} l_{43}\cos(\theta_3+\theta_2) - 2m_6 l_{23} l_{53}\cos(\theta_3+\theta_2+\theta_5) - m_3 l_{23} r_{33}\cos(\theta_3+\theta_2)\\ &- 2m_6 l_{23} r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6) - 2m_5 l_{23} r_{53}\cos(\theta_3+\theta_2+\theta_5)\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbf{A}(4,1) = &- I_{42} - m_4 l_{23} r_{43}\cos(\theta_4+\theta_3+\theta_2) - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) - m_4 r_{43}^2\\ &- m_4 l_{43} r_{43}\cos(\theta_4)\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{A}(5,1) =&m_5 l_{33} r_{53}\cos(\theta_3+\theta_5) + m_6 l_{33} l_{53}\cos(\theta_3+\theta_5) - I_{52} + m_5 r_{53}^2 + m_6 l_{53}^2\\ &+ m_6 r_{63} l_{53}\cos(\theta_6) + m_6 l_{23} l_{53}\cos(\theta_3+\theta_2+\theta_5) + m_5 l_{23} r_{53}\cos(\theta_3+\theta_2+\theta_5)\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{A}(6,1) =& - I_{62} + m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6) + m_6 r_{63}^2 + m_6 r_{63} l_{53}\cos(\theta_6)\\ &+ m_6 l_{23} r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6)\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{A}(1,2) =& - l_{23}(m_6 l_{53}\cos(\theta_3+\theta_2+\theta_5) + m_4 l_{43}\cos(\theta_3+\theta_2) + m_4 r_{43}\cos(\theta_4+\theta_3+\theta_2)\\ &+ m_5 r_{53}\cos(\theta_3+\theta_2+\theta_5) + m_6 r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6)\\ &+ m_3 r_{33}\cos(\theta_3+\theta_2) + m_4 l_{33}\cos(\theta_2) + m_2 r_{23}\cos(\theta_2) + m_5 l_{33}\cos(\theta_2)\\ &+ m_3 l_{33}\cos(\theta_2) + m_6 l_{33}\cos(\theta_2))\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{A}(2,2) =& - m_2 r_{23}^2 - m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3)\\ &- m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6) - m_6 l_{33} l_{53}\cos(\theta_3+\theta_5) - m_5 l_{33} r_{53}\cos(\theta_3+\theta_5)\\ &- I_{22} - m_4 l_{33}^2 - m_3 l_{33}^2 - m_6 l_{33}^2 - m_5 l_{33}^2\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbf{A}(3,2) =& - m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3) - 2m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6)\\ &- 2m_6 l_{33} l_{53}\cos(\theta_3+\theta_5) - 2m_6 r_{63}^2 - 2m_5 r_{53}^2 - 2m_6 l_{53}^2 - m_3 r_{33}^2\\ &- m_4 l_{43} r_{43}\cos(\theta_4) - 2m_5 l_{33} r_{53}\cos(\theta_3+\theta_5) - m_4 l_{43}^2 - 4m_6 r_{63} l_{53}\cos(\theta_6)\\ &- I_{32}\end{aligned} \tag{10}$$

$$\mathbf{A}(4,2) = -I_{42} - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) - m_4 r_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) \tag{11}$$

$$\begin{aligned}\mathbf{A}(5,2) =&m_6 l_{33} l_{53}\cos(\theta_3+\theta_5) + m_5 r_{53}^2 + m_6 l_{53}^2 + m_5 l_{33} r_{53}\cos(\theta_3+\theta_5) - I_{52}\\ &+ m_6 r_{63} l_{53}\cos(\theta_6)\end{aligned} \tag{12}$$

$$\mathbf{A}(6,2) = -I_{62} + m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6) + m_6 r_{63}^2 + m_6 r_{63} l_{53}\cos(\theta_6) \tag{13}$$

$$\begin{aligned}\mathbf{A}(1,3) =& - l_{23}(m_3 r_{33}\cos(\theta_3+\theta_2) + m_6 l_{53}\cos(\theta_3+\theta_2+\theta_5)\\ &+ m_6 r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6) + m_4 l_{43}\cos(\theta_3+\theta_2)\\ &+ m_4 r_{43}\cos(\theta_4+\theta_3+\theta_2) + m_5 r_{53}\cos(\theta_3+\theta_2+\theta_5))\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{A}(2,3) =& - l_{33}(m_4 r_{43}\cos(\theta_4+\theta_3) + m_4 l_{43}\cos(\theta_3) + m_6 r_{63}\cos(\theta_3+\theta_5+\theta_6) \\ &+ m_6 l_{53}\cos(\theta_3+\theta_5) + m_5 r_{53}\cos(\theta_3+\theta_5) + m_3 r_{33}\cos(\theta_3))\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbf{A}(3,3) =& - 2m_6 r_{63}^2 - 2m_6 l_{53}^2 - m_3 r_{33}^2 - m_4 l_{43}^2 - 2m_5 r_{53}^2 - m_4 l_{43} r_{43}\cos(\theta_4) \\ &- 4m_6 r_{63} l_{53}\cos(\theta_6) - I_{32}\end{aligned} \tag{16}$$

$$\mathbf{A}(4,3) = -I_{42} - m_4 r_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) \tag{17}$$

$$\mathbf{A}(5,3) = -I_{52} + m_6 l_{53}^2 + m_5 r_{53}^2 + m_6 r_{63} l_{53}\cos(\theta_6) \tag{18}$$

$$\mathbf{A}(6,3) = m_6 r_{63}^2 + m_6 r_{63} l_{53}\cos(\theta_6) - I_{62} \tag{19}$$

$$\mathbf{A}(1,4) = -m_4 l_{23} r_{43}\cos(\theta_4+\theta_3+\theta_2) \tag{20}$$

$$\mathbf{A}(2,4) = -m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) \tag{21}$$

$$\mathbf{A}(3,4) = -m_4 l_{43} r_{43}\cos(\theta_4) \tag{22}$$

$$\mathbf{A}(4,4) = -I_{42} - m_4 r_{43}^2 \tag{23}$$

$$\mathbf{A}(5,4) = 0 \tag{24}$$

$$\mathbf{A}(6,4) = 0 \tag{25}$$

$$\begin{aligned}\mathbf{A}(1,5) = &- l_{23}(m_6 l_{53}\cos(\theta_3+\theta_2+\theta_5) + m_5 r_{53}\cos(\theta_3+\theta_2+\theta_5)\\ &+ m_6 r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6))\end{aligned} \tag{26}$$

$$\mathbf{A}(2,5) = -l_{33}(m_6 r_{63}\cos(\theta_3+\theta_5+\theta_6) + m_6 l_{53}\cos(\theta_3+\theta_5) + m_5 r_{53}\cos(\theta_3+\theta_5)) \tag{27}$$

$$\mathbf{A}(3,5) = -2m_5 r_{53}^2 - 2m_6 l_{53}^2 - 2m_6 r_{63}^2 - 4m_6 r_{63} l_{53}\cos(\theta_6) \tag{28}$$

$$\mathbf{A}(4,5) = 0 \tag{29}$$

$$\mathbf{A}(5,5) = m_5 r_{53}^2 + m_6 l_{53}^2 + m_6 r_{63} l_{53}\cos(\theta_6) + I_{52} \tag{30}$$

$$\mathbf{A}(6,5) = m_6 r_{63}^2 + m_6 r_{63} l_{53}\cos(\theta_6) + I_{62} \tag{31}$$

$$\mathbf{A}(1,6) = -m_6 l_{23} r_{63}\cos(\theta_3+\theta_2+\theta_5+\theta_6) \tag{32}$$

$$\mathbf{A}(2,6) = -m_6 l_{33} r_{63}\cos(\theta_3+\theta_5+\theta_6) \tag{33}$$

$$\mathbf{A}(3,6) = -2m_6 r_{63}(l_{53}\cos(\theta_6) + r_{63}) \tag{34}$$

$$\mathbf{A}(4,6) = 0 \tag{35}$$

$$\mathbf{A}(5,6) = m_6 r_{63} l_{53}\cos(\theta_6) \tag{36}$$

$$\mathbf{A}(6,6) = m_6 r_{63}^2 + I_{62} \tag{37}$$

$$\begin{aligned}
\mathbf{B}(1,1) =& l_{23}(m_5 l_{33}\sin(\theta_2)\dot{\theta}_1 + m_3 r_{33}\sin(\theta_3+\theta_2)\dot{\theta}_1 + m_3 l_{33}\sin(\theta_2)\dot{\theta}_1 \\
&+ m_6 l_{33}\sin(\theta_2)\dot{\theta}_1 + m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_1 \\
&+ m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_1 + m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_1 \\
&+ m_4 r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_1 + m_4 l_{43}\sin(\theta_3+\theta_2)\dot{\theta}_1 + m_4 l_{33}\sin(\theta_2)\dot{\theta}_1 \\
&+ 2m_4 l_{33}\sin(\theta_2)\dot{\theta}_2 + 2m_2 r_{23}\sin(\theta_2)\dot{\theta}_2 + 2m_5 l_{33}\sin(\theta_2)\dot{\theta}_2 \\
&+ 2m_3 r_{33}\sin(\theta_3+\theta_2)\dot{\theta}_3 + 2m_3 r_{33}\sin(\theta_3+\theta_2)\dot{\theta}_2 + 2m_3 l_{33}\sin(\theta_2)\dot{\theta}_2 \\
&+ 2m_6 l_{33}\sin(\theta_2)\dot{\theta}_2 + 2m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_5 \\
&+ 2m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_3 + 2m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_2 \\
&+ 2m_4 r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_4 + 2m_4 r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_3 \\
&+ 2m_4 r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_2 + 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_5 \\
&+ 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_6 + 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_3 \\
&+ 2m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_3 + 2m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_2 \\
&+ 2m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_5 + 2m_4 l_{43}\sin(\theta_3+\theta_2)\dot{\theta}_3 \\
&+ 2m_4 l_{43}\sin(\theta_3+\theta_2)\dot{\theta}_2 + m_2 r_{23}\sin(\theta_2)\dot{\theta}_1 \\
&+ 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_2)
\end{aligned} \tag{38}$$

$$\begin{aligned}
\mathbf{B}(2,1) =& - l_{23}m_3 l_{33}\sin(\theta_2)\dot{\theta}_1 - l_{23}m_6 l_{33}\sin(\theta_2)\dot{\theta}_1 - l_{23}m_4 l_{33}\sin(\theta_2)\dot{\theta}_1 \\
&+ 2m_6 l_{33} r_{63}\dot{\theta}_6\sin(\theta_3+\theta_5+\theta_6) + 2m_4 l_{33}\dot{\theta}_4 r_{43}\sin(\theta_4+\theta_3) \\
&+ 2m_5 l_{33} r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_5 + 2m_5 l_{33} r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_3 \\
&+ 2m_5 l_{33} r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_2 + m_5 l_{33} r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_1 + 2m_4 l_{33} l_{43}\sin(\theta_3)\dot{\theta}_3 \\
&+ 2m_4 l_{33} l_{43}\sin(\theta_3)\dot{\theta}_2 + m_4 l_{33} l_{43}\sin(\theta_3)\dot{\theta}_1 + 2m_6 l_{33} l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_5 \\
&+ 2m_6 l_{33} l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_3 + 2m_6 l_{33} l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_2 \\
&+ m_6 l_{33} l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_1 + 2m_6 l_{33} r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_5 \\
&+ 2m_6 l_{33} r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_2 + m_6 l_{33} r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_1 \\
&+ 2m_6 l_{33} r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_3 + 2m_3 l_{33} r_{33}\sin(\theta_3)\dot{\theta}_3 + 2m_3 l_{33} r_{33}\sin(\theta_3)\dot{\theta}_2 \\
&+ m_3 l_{33} r_{33}\sin(\theta_3)\dot{\theta}_1 + 2m_4 l_{33} r_{43}\sin(\theta_4+\theta_3)\dot{\theta}_3 + 2m_4 l_{33} r_{43}\sin(\theta_4+\theta_3)\dot{\theta}_2 \\
&+ m_4 l_{33} r_{43}\sin(\theta_4+\theta_3)\dot{\theta}_1 - l_{23}m_2 r_{23}\sin(\theta_2)\dot{\theta}_1 - l_{23}m_5 l_{33}\sin(\theta_2)\dot{\theta}_1
\end{aligned} \tag{39}$$

$$
\begin{aligned}
\mathbf{B}(3,1) = & -l_{23}m_3r_{33}\sin(\theta_3+\theta_2)\dot{\theta}_1 - l_{23}m_4l_{43}\sin(\theta_3+\theta_2)\dot{\theta}_1 \\
& -2l_{23}m_5r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_1 - 2l_{23}m_6l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_1 \\
& -2l_{23}m_6r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_1 - 4m_5l_{33}r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_2 \\
& -2m_5l_{33}r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_1 - 2m_4l_{33}l_{43}\sin(\theta_3)\dot{\theta}_2 - m_4l_{33}l_{43}\sin(\theta_3)\dot{\theta}_1 \\
& -4m_6l_{33}l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_2 - 2m_6l_{33}l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_1 \\
& -4m_6l_{33}r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_2 - 2m_6l_{33}r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_1 \\
& +4m_6r_{63}\dot{\theta}_6l_{53}\sin(\theta_6) + 2m_4l_{43}\dot{\theta}_4r_{43}\sin(\theta_4) + 2m_4r_{43}\dot{\theta}_3l_{43}\sin(\theta_4) \\
& +2m_4l_{43}r_{43}\sin(\theta_4)\dot{\theta}_2 + m_4l_{43}r_{43}\sin(\theta_4)\dot{\theta}_1 \\
& -2m_3l_{33}r_{33}\sin(\theta_3)\dot{\theta}_2 - m_3l_{33}r_{33}\sin(\theta_3)\dot{\theta}_1
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
\mathbf{B}(4,1) = & -m_4r_{43}(\dot{\theta}_1l_{23}\sin(\theta_4+\theta_3+\theta_2) + 2l_{43}\sin(\theta_4)\dot{\theta}_3 \\
& +2l_{43}\sin(\theta_4)\dot{\theta}_2 + l_{43}\sin(\theta_4)\dot{\theta}_1 + 2l_{33}\sin(\theta_4+\theta_3)\dot{\theta}_2 + l_{33}\sin(\theta_4+\theta_3)\dot{\theta}_1)
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\mathbf{B}(5,1) = & l_{23}m_5r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_1 + l_{23}m_6l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_1 \\
& +2m_5l_{33}r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_2 + m_5l_{33}r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_1 \\
& +2m_6l_{33}l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_2 + m_6l_{33}l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_1 \\
& -2m_6r_{63}l_{53}\sin(\theta_6)\dot{\theta}_3 - 2m_6r_{63}l_{53}\sin(\theta_6)\dot{\theta}_2 - 2m_6r_{63}l_{53}\dot{\theta}_5\sin(\theta_6) \\
& -m_6r_{63}l_{53}\sin(\theta_6)\dot{\theta}_1 - 2m_6r_{63}\dot{\theta}_6l_{53}\sin(\theta_6)
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\mathbf{B}(6,1) = & m_6r_{63}(2l_{53}\sin(\theta_6)\dot{\theta}_2 + \dot{\theta}_1l_{23}\sin(\theta_3+\theta_2+\theta_5+\theta_6) + 2l_{53}\sin(\theta_6)\dot{\theta}_5 \\
& +l_{53}\sin(\theta_6)\dot{\theta}_1 + 2l_{33}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_2 + l_{33}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_1 \\
& +2l_{53}\sin(\theta_6)\dot{\theta}_3)
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\mathbf{B}(1,2) = & l_{23}(2m_4r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_4 + 2m_4r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_3 \\
& +m_3r_{33}\sin(\theta_3+\theta_2)\dot{\theta}_2 + 2m_3r_{33}\sin(\theta_3+\theta_2)\dot{\theta}_3 + m_5r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_2 \\
& +m_6r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_2 + m_6l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_2 \\
& +2m_5r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_5 + 2m_5r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_3 \\
& +2m_6r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_5 + 2m_6r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_6 \\
& +2m_6r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_3 + 2m_6l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_3 \\
& +2m_6l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_5 + m_4l_{43}\sin(\theta_3+\theta_2)\dot{\theta}_2 + 2m_4l_{43}\sin(\theta_3+\theta_2)\dot{\theta}_3 \\
& +m_4r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot{\theta}_2 + m_2r_{23}\sin(\theta_2)\dot{\theta}_2 + m_5l_{33}\sin(\theta_2)\dot{\theta}_2 \\
& +m_4l_{33}\sin(\theta_2)\dot{\theta}_2 + m_3l_{33}\sin(\theta_2)\dot{\theta}_2 + m_6l_{33}\sin(\theta_2)\dot{\theta}_2)
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\mathbf{B}(2,2) =& l_{33}(m_4 l_{43}\sin(\theta_3)\dot\theta_2 + 2m_4 l_{43}\sin(\theta_3)\dot\theta_3 + 2m_6 l_{53}\dot\theta_3\sin(\theta_3+\theta_5)\\
&+ 2m_6 l_{53}\sin(\theta_3+\theta_5)\dot\theta_5 + m_3 r_{33}\sin(\theta_3)\dot\theta_2 + m_4 r_{43}\sin(\theta_4+\theta_3)\dot\theta_2\\
&+ 2m_3 r_{33}\sin(\theta_3)\dot\theta_3 + 2m_4 r_{43}\sin(\theta_4+\theta_3)\dot\theta_4 + 2m_4 r_{43}\sin(\theta_4+\theta_3)\dot\theta_3\\
&+ m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot\theta_2 + m_5 r_{53}\dot\theta_2\sin(\theta_3+\theta_5)\\
&+ 2m_6 r_{63}\dot\theta_3\sin(\theta_3+\theta_5+\theta_6) + 2m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot\theta_5\\
&+ 2m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot\theta_6 + 2m_5 r_{53}\dot\theta_3\sin(\theta_3+\theta_5)\\
&+ 2m_5 r_{53}\sin(\theta_3+\theta_5)\dot\theta_5 + m_6\dot\theta_2 l_{53}\sin(\theta_3+\theta_5))
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
\mathbf{B}(3,2) =& - m_3 l_{33} r_{33}\sin(\theta_3)\dot\theta_2 - m_4 l_{33} l_{43}\sin(\theta_3)\dot\theta_2 - 2m_6 l_{33} r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot\theta_2\\
&+ 2m_4 l_{43}\dot\theta_4 r_{43}\sin(\theta_4) + 2m_4 r_{43}\dot\theta_3 l_{43}\sin(\theta_4) + m_4 l_{43} r_{43}\sin(\theta_4)\dot\theta_2\\
&+ 4m_6 r_{63}\dot\theta_6 l_{53}\sin(\theta_6) - 2m_6 l_{33} l_{53}\sin(\theta_3+\theta_5)\dot\theta_2 - 2m_5 l_{33} r_{53}\sin(\theta_3+\theta_5)\dot\theta_2
\end{aligned}
\tag{46}
$$

$$
\mathbf{B}(4,2) = -m_4 r_{43}(2l_{43}\sin(\theta_4)\dot\theta_3 + l_{43}\sin(\theta_4)\dot\theta_2 + l_{33}\sin(\theta_4+\theta_3)\dot\theta_2) \tag{47}
$$

$$
\begin{aligned}
\mathbf{B}(5,2) =& - 2m_6 r_{63} l_{53}\sin(\theta_6)\dot\theta_3 - m_6 r_{63} l_{53}\sin(\theta_6)\dot\theta_2\\
&- 2m_6 r_{63} l_{53}\dot\theta_5\sin(\theta_6) - 2m_6 r_{63}\dot\theta_6 l_{53}\sin(\theta_6) + m_6 l_{33} l_{53}\sin(\theta_3+\theta_5)\dot\theta_2\\
&+ m_5 l_{33} r_{53}\sin(\theta_3+\theta_5)\dot\theta_2
\end{aligned}
\tag{48}
$$

$$
\mathbf{B}(6,2) = m_6 r_{63}(2l_{53}\sin(\theta_6)\dot\theta_3 + l_{53}\sin(\theta_6)\dot\theta_2 + 2l_{53}\sin(\theta_6)\dot\theta_5 + l_{33}\sin(\theta_3+\theta_5+\theta_6)\dot\theta_2) \tag{49}
$$

$$
\begin{aligned}
\mathbf{B}(1,3) =& l_{23}(2m_4 r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot\theta_4 + m_3 r_{33}\sin(\theta_3+\theta_2)\dot\theta_3\\
&+ 2m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot\theta_5 + m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot\theta_3\\
&+ m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot\theta_3 + m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot\theta_3\\
&+ 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot\theta_5 + 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot\theta_6\\
&+ 2m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot\theta_5 + m_4 l_{43}\sin(\theta_3+\theta_2)\dot\theta_3\\
&+ m_4 r_{43}\sin(\theta_4+\theta_3+\theta_2)\dot\theta_3)
\end{aligned}
\tag{50}
$$

$$\begin{aligned}\mathbf{B}(2,3) =& l_{33}(m_4 r_{43}\sin(\theta_4+\theta_3)\dot{\theta}_3 + 2m_4 r_{43}\sin(\theta_4+\theta_3)\dot{\theta}_4 + m_3 r_{33}\sin(\theta_3)\dot{\theta}_3 \\ &+ m_6 r_{63}\dot{\theta}_3\sin(\theta_3+\theta_5+\theta_6) + m_4 l_{43}\sin(\theta_3)\dot{\theta}_3 + m_5 r_{53}\dot{\theta}_3\sin(\theta_3+\theta_5) \\ &+ m_6 l_{53}\dot{\theta}_3\sin(\theta_3+\theta_5) + 2m_6 l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_5 + 2m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_5 \\ &+ 2m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_6 + 2m_5 r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_5)\end{aligned} \tag{51}$$

$$\mathbf{B}(3,3) = 4m_6 r_{63}\dot{\theta}_6 l_{53}\sin(\theta_6) + 2m_4 l_{43}\dot{\theta}_4 r_{43}\sin(\theta_4) + m_4 r_{43}\dot{\theta}_3 l_{43}\sin(\theta_4) \tag{52}$$

$$\mathbf{B}(4,3) = -m_4 r_{43}\dot{\theta}_3 l_{43}\sin(\theta_4) \tag{53}$$

$$\mathbf{B}(5,3) = -m_6 l_{53} r_{63}\sin(\theta_6)(\dot{\theta}_3 + 2\dot{\theta}_5 + 2\dot{\theta}_6) \tag{54}$$

$$\mathbf{B}(6,3) = m_6 l_{53} r_{63}\sin(\theta_6)(\dot{\theta}_3 + 2\dot{\theta}_5) \tag{55}$$

$$\mathbf{B}(1,4) = m_4 l_{23}\dot{\theta}_4 r_{43}\sin(\theta_4+\theta_3+\theta_2) \tag{56}$$

$$\mathbf{B}(2,4) = m_4 l_{33}\dot{\theta}_4 r_{43}\sin(\theta_4+\theta_3) \tag{57}$$

$$\mathbf{B}(3,4) = m_4 l_{43}\dot{\theta}_4 r_{43}\sin(\theta_4) \tag{58}$$

$$\mathbf{B}(4,4) = 0 \tag{59}$$

$$\mathbf{B}(5,4) = 0 \tag{60}$$

$$\mathbf{B}(6,4) = 0 \tag{61}$$

$$\begin{aligned}\mathbf{B}(1,5) =& l_{23}(m_6 l_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_5 + m_5 r_{53}\sin(\theta_3+\theta_2+\theta_5)\dot{\theta}_5 \\ &+ m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_5 + 2m_6 r_{63}\sin(\theta_3+\theta_2+\theta_5+\theta_6)\dot{\theta}_6)\end{aligned} \tag{62}$$

$$\begin{aligned}\mathbf{B}(2,5) =& l_{33}(m_5 r_{53}\sin(\theta_3+\theta_5)\dot{\theta}_5 + m_6 l_{53}\sin(\theta_3+\theta_5)\dot{\theta}_5 + m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_5 \\ &+ 2m_6 r_{63}\sin(\theta_3+\theta_5+\theta_6)\dot{\theta}_6)\end{aligned} \tag{63}$$

$$\mathbf{B}(3,5) = 4m_6 r_{63}\dot{\theta}_6 l_{53}\sin(\theta_6) \tag{64}$$

$$\mathbf{B}(4,5) = 0 \tag{65}$$

$$\mathbf{B}(5,5) = -m_6 l_{53} r_{63}\sin(\theta_6)(\dot{\theta}_5 + 2\dot{\theta}_6) \tag{66}$$

$$\mathbf{B}(6,5) = m_6 r_{63} l_{53}\dot{\theta}_5\sin(\theta_6) \tag{67}$$

$$\mathbf{B}(1,6) = m_6 l_{23} r_{63}\dot{\theta}_6\sin(\theta_3+\theta_2+\theta_5+\theta_6) \tag{68}$$

$$\mathbf{B}(2,6) = m_6 l_{33} r_{63}\dot{\theta}_6\sin(\theta_3+\theta_5+\theta_6) \tag{69}$$

$$\mathbf{B}(3,6) = 2m_6 r_{63}\dot{\theta}_6 l_{53}\sin(\theta_6) \tag{70}$$

$$\mathbf{B}(4,6) = 0 \tag{71}$$

$$\mathbf{B}(5,6) = -m_6 r_{63}\dot{\theta}_6 l_{53}\sin(\theta_6) \tag{72}$$

$$\mathbf{B}(6,6) = 0 \tag{73}$$

$$\mathbf{C}(1,1) = g\sin(\theta_1)(m_5 l_{23} + m_4 l_{23} + m_2 l_{23} + m_3 l_{23} + m_1 r_{13} + m_6 l_{23}) \tag{74}$$

$$\mathbf{C}(2,1) = g(\sin(\theta_2+\theta_1) - \sin(\theta_2))(m_2 r_{23} + m_5 l_{33} + m_6 l_{33} + m_3 l_{33} + m_4 l_{33}) \tag{75}$$

$$\begin{aligned}\mathbf{C}(3,1) =& g(2m_6\sin(\theta_3+\theta_2+\theta_1+\theta_5)l_{53} + 2m_5\sin(\theta_3+\theta_2+\theta_1+\theta_5)r_{53}\\ &+ 2m_6\sin(\theta_3+\theta_2+\theta_1+\theta_5+\theta_6)r_{63} + m_4\sin(\theta_3+\theta_2+\theta_1)l_{43}\\ &+ m_3\sin(\theta_3+\theta_2+\theta_1)r_{33} - 2m_6\sin(\theta_3+\theta_2+\theta_5)l_{53}\\ &- 2m_5\sin(\theta_3+\theta_2+\theta_5)r_{53} - 2m_6\sin(\theta_3+\theta_2+\theta_5+\theta_6)r_{63}\\ &- m_4\sin(\theta_3+\theta_2)l_{43} - m_3\sin(\theta_3+\theta_2)r_{33})\end{aligned} \tag{76}$$

$$\mathbf{C}(4,1) = -m_4 g r_{43}(-\sin(\theta_4+\theta_3+\theta_2+\theta_1) + \sin(\theta_4+\theta_3+\theta_2)) \tag{77}$$

$$\mathbf{C}(5,1) = -g(m_5 r_{53} + m_6 l_{53})(\sin(\theta_3+\theta_2+\theta_1+\theta_5) - \sin(\theta_3+\theta_2+\theta_5)) \tag{78}$$

$$\mathbf{C}(6,1) = -m_6 g r_{63}(\sin(\theta_3+\theta_2+\theta_1+\theta_5+\theta_6) - \sin(\theta_3+\theta_2+\theta_5+\theta_6)) \tag{79}$$

$$\mathbf{C}(1,2) = 0 \tag{80}$$

$$\mathbf{C}(2,2) = g\sin(\theta_2)(m_2 r_{23} + m_5 l_{33} + m_6 l_{33} + m_3 l_{33} + m_4 l_{33}) \tag{81}$$

$$
\begin{aligned}
\mathbf{C}(3,2) = & g(2m_6 \sin(\theta_3+\theta_2+\theta_5)l_{53} + 2m_5 \sin(\theta_3+\theta_2+\theta_5)r_{53} \\
& + 2m_6 \sin(\theta_3+\theta_2+\theta_5+\theta_6)r_{63} + m_4 \sin(\theta_3+\theta_2)l_{43} + m_3 \sin(\theta_3+\theta_2)r_{33} \\
& - 2m_6 \sin(\theta_3+\theta_5)l_{53} - 2m_5 \sin(\theta_3+\theta_5)r_{53} - 2m_6 \sin(\theta_3+\theta_5+\theta_6)r_{63} \\
& - m_4 \sin(\theta_3)l_{43} - m_3 \sin(\theta_3)r_{33})
\end{aligned} \tag{82}
$$

$$
\mathbf{C}(4,2) = m_4 g r_{43}(\sin(\theta_4+\theta_3+\theta_2) - \sin(\theta_4+\theta_3)) \tag{83}
$$

$$
\mathbf{C}(5,2) = -g(m_5 r_{53} + m_6 l_{53})(\sin(\theta_3+\theta_2+\theta_5) - \sin(\theta_3+\theta_5)) \tag{84}
$$

$$
\mathbf{C}(6,2) = m_6 g r_{63}(-\sin(\theta_3+\theta_2+\theta_5+\theta_6) + \sin(\theta_3+\theta_5+\theta_6)) \tag{85}
$$

$$
\mathbf{C}(1,3) = 0 \tag{86}
$$

$$
\mathbf{C}(2,3) = 0 \tag{87}
$$

$$
\begin{aligned}
\mathbf{C}(3,3) = & g(2m_6 \sin(\theta_3+\theta_5)l_{53} + 2m_5 \sin(\theta_3+\theta_5)r_{53} + 2m_6 \sin(\theta_3+\theta_5+\theta_6)r_{63} \\
& + m_4 \sin(\theta_3)l_{43} + m_3 \sin(\theta_3)r_{33} - 2m_6 \sin(\theta_5)l_{53} - 2m_5 \sin(\theta_5)r_{53} \\
& - 2m_6 \sin(\theta_5+\theta_6)r_{63})
\end{aligned} \tag{88}
$$

$$
\mathbf{C}(4,3) = -m_4 g r_{43}(-\sin(\theta_4+\theta_3) + \sin(\theta_4)) \tag{89}
$$

$$
\mathbf{C}(5,3) = -g(\sin(\theta_3+\theta_5) - \sin(\theta_5))(m_5 r_{53} + m_6 l_{53}) \tag{90}
$$

$$
\mathbf{C}(6,3) = -m_6 g r_{63}(\sin(\theta_3+\theta_5+\theta_6) - \sin(\theta_5+\theta_6)) \tag{91}
$$

$$\mathbf{C}(1,4) = 0 \tag{92}$$

$$\mathbf{C}(2,4) = 0 \tag{93}$$

$$\mathbf{C}(3,4) = 0 \tag{94}$$

$$\mathbf{C}(4,4) = m_4 g \sin(\theta_4) r_{43} \tag{95}$$

$$\mathbf{C}(5,4) = 0 \tag{96}$$

$$\mathbf{C}(6,4) = 0 \tag{97}$$

$$\mathbf{C}(1,5) = 0 \tag{98}$$

$$\mathbf{C}(2,5) = 0 \tag{99}$$

$$\mathbf{C}(3,5) = 2g(m_6 \sin(\theta_5) l_{53} + m_5 \sin(\theta_5) r_{53} + m_6 \sin(\theta_5 + \theta_6) r_{63} - m_6 \sin(\theta_6) r_{63}) \tag{100}$$

$$\mathbf{C}(4,5) = 0 \tag{101}$$

$$\mathbf{C}(5,5) = -g \sin(\theta_5)(m_5 r_{53} + m_6 l_{53}) \tag{102}$$

$$\mathbf{C}(6,5) = -m_6 g r_{63}(\sin(\theta_5 + \theta_6) - \sin(\theta_6)) \tag{103}$$

$$\mathbf{C}(1,6) = 0 \tag{104}$$

$$\mathbf{C}(2,6) = 0 \tag{105}$$

$$\mathbf{C}(3,6) = 2m_6 g \sin(\theta_6) r_{63} \tag{106}$$

$$\mathbf{C}(4,6) = 0 \tag{107}$$

$$\mathbf{C}(5,6) = 0 \tag{108}$$

$$\mathbf{C}(6,6) = -m_6 g \sin(\theta_6) r_{63} \tag{109}$$

$$\mathbf{D} = \begin{bmatrix} M_1 - M_2 \\ M_2 - M_3 \\ M_3 - M_4 - M_5 \\ M_4 \\ M_5 - M_6 \\ M_6 \end{bmatrix} \tag{110}$$

4 Kane's Equations

4.1 Introduction

Kane's equations provide an elegant formulation of the dynamical equations of motion. Kane's equations simply state that the sum of the generalized forces (both applied and inertial forces), for each generalized coordinate, is zero. That is, for a mechanical system S having n degrees for freedom, represented by generalized coordinates q_r, Kane's equstions state that [?]

$$F_r + F_r^* = 0 \tag{111}$$

where F_r and F_r^* are generalized applied and inertial forces respectively corresponding to the coordinates q_r $(r = 1, ..., n)$.

4.2 iCub Modelling with Kane's Equations

The angular velocities of the segments are

$$\begin{aligned} \boldsymbol{\omega}_1 &= \begin{bmatrix} 0 \\ \dot{\alpha}_1 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ \dot{\alpha}_2 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_3 = \begin{bmatrix} 0 \\ \dot{\alpha}_3 \\ 0 \end{bmatrix} \\ \boldsymbol{\omega}_4 &= \begin{bmatrix} 0 \\ \dot{\alpha}_4 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_5 = \begin{bmatrix} 0 \\ \dot{\alpha}_5 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_6 = \begin{bmatrix} 0 \\ \dot{\alpha}_6 \\ 0 \end{bmatrix} \end{aligned} \tag{112}$$

where α_i is the absolute angle of the i'th segment. The velocities of the mass centres in the inertial frame (after simplification) are

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} r_{13}\cos(\alpha_1)\dot{\alpha}_1 \\ 0 \\ -r_{13}\sin(\alpha_1)\dot{\alpha}_1 \end{bmatrix} \\ \mathbf{v}_2 &= \begin{bmatrix} l_{23}\cos(\alpha_1)\dot{\alpha}_1 + r_{23}\cos(\alpha_2)\dot{\alpha}_2 \\ 0 \\ -l_{23}\sin(\alpha_1)\dot{\alpha}_1 - r_{23}\sin(\alpha_2)\dot{\alpha}_2 \end{bmatrix} \\ \mathbf{v}_3 &= \begin{bmatrix} l_{23}\cos(\alpha_1)\dot{\alpha}_1 + l_{33}\cos(\alpha_2)\dot{\alpha}_2 + r_{33}\cos(\alpha_3)\dot{\alpha}_3 \\ 0 \\ -l_{23}\sin(\alpha_1)\dot{\alpha}_1 - l_{33}\sin(\alpha_2)\dot{\alpha}_2 - r_{33}\sin(\alpha_3)\dot{\alpha}_3 \end{bmatrix} \\ \mathbf{v}_4 &= \begin{bmatrix} l_{23}\cos(\alpha_1)\dot{\alpha}_1 + l_{33}\cos(\alpha_2)\dot{\alpha}_2 + l_{43}\cos(\alpha_3)\dot{\alpha}_3 + r_{43}\cos(\alpha_4)\dot{\alpha}_4 \\ 0 \\ -l_{23}\sin(\alpha_1)\dot{\alpha}_1 - l_{33}\sin(\alpha_2)\dot{\alpha}_2 - l_{43}\sin(\alpha_3)\dot{\alpha}_3 - r_{43}\sin(\alpha_4)\dot{\alpha}_4 \end{bmatrix} \\ \mathbf{v}_5 &= \begin{bmatrix} l_{23}\cos(\alpha_1)\dot{\alpha}_1 + l_{33}\cos(\alpha_2)\dot{\alpha}_2 + r_{53}\cos(\alpha_5)\dot{\alpha}_5 \\ 0 \\ -l_{23}\sin(\alpha_1)\dot{\alpha}_1 - l_{33}\sin(\alpha_2)\dot{\alpha}_2 - r_{53}\sin(\alpha_5)\dot{\alpha}_5 \end{bmatrix} \\ \mathbf{v}_6 &= \begin{bmatrix} l_{23}\cos(\alpha_1)\dot{\alpha}_1 + l_{33}\cos(\alpha_2)\dot{\alpha}_2 + l_{53}\cos(\alpha_5)\dot{\alpha}_5 + r_{63}\cos(\alpha_6)\dot{\alpha}_6 \\ 0 \\ -l_{23}\sin(\alpha_1)\dot{\alpha}_1 - l_{33}\sin(\alpha_2)\dot{\alpha}_2 - l_{53}\sin(\alpha_5)\dot{\alpha}_5 - r_{63}\sin(\alpha_6)\dot{\alpha}_6 \end{bmatrix} \end{aligned} \tag{113}$$

$$\mathbf{v}_{1\alpha_1} = \begin{bmatrix} r_{13}\cos(\alpha_1) \\ 0 \\ -r_{13}\sin(\alpha_1) \end{bmatrix} \qquad \mathbf{v}_{1\alpha_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 2,\ldots,6$$

$$\mathbf{v}_{2\alpha_1} = \begin{bmatrix} l_{23}\cos(\alpha_1) \\ 0 \\ -l_{23}\sin(\alpha_1) \end{bmatrix} \qquad \mathbf{v}_{2\alpha_2} = \begin{bmatrix} r_{23}\cos(\alpha_2) \\ 0 \\ -r_{23}\sin(\alpha_2) \end{bmatrix}$$

$$\mathbf{v}_{2\alpha_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 3,\ldots,6$$

$$\mathbf{v}_{3\alpha_1} = \begin{bmatrix} l_{23}\cos(\alpha_1) \\ 0 \\ -l_{23}\sin(\alpha_1) \end{bmatrix} \qquad \mathbf{v}_{3\alpha_2} = \begin{bmatrix} l_{33}\cos(\alpha_2) \\ 0 \\ -l_{33}\sin(\alpha_2) \end{bmatrix}$$

$$\mathbf{v}_{3\alpha_3} = \begin{bmatrix} r_{33}\cos(\alpha_3) \\ 0 \\ -r_{33}\sin(\alpha_3) \end{bmatrix} \qquad \mathbf{v}_{3\alpha_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 4,\ldots,6$$

$$\mathbf{v}_{4\alpha_1} = \begin{bmatrix} l_{23}\cos(\alpha_1) \\ 0 \\ -l_{23}\sin(\alpha_1) \end{bmatrix} \qquad \mathbf{v}_{4\alpha_2} = \begin{bmatrix} l_{33}\cos(\alpha_2) \\ 0 \\ -l_{33}\sin(\alpha_2) \end{bmatrix}$$

$$\mathbf{v}_{4\alpha_3} = \begin{bmatrix} l_{43}\cos(\alpha_3) \\ 0 \\ -l_{43}\sin(\alpha_3) \end{bmatrix} \qquad \mathbf{v}_{4\alpha_4} = \begin{bmatrix} r_{43}\cos(\alpha_4) \\ 0 \\ -r_{43}\sin(\alpha_4) \end{bmatrix} \tag{114}$$

$$\mathbf{v}_{4\alpha_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 5, 6$$

$$\mathbf{v}_{5\alpha_1} = \begin{bmatrix} l_{23}\cos(\alpha_1) \\ 0 \\ -l_{23}\sin(\alpha_1) \end{bmatrix} \qquad \mathbf{v}_{5\alpha_2} = \begin{bmatrix} l_{33}\cos(\alpha_2) \\ 0 \\ -l_{33}\sin(\alpha_2) \end{bmatrix}$$

$$\mathbf{v}_{5\alpha_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 3, 4 \text{ and } 6$$

$$\mathbf{v}_{5\alpha_5} = \begin{bmatrix} r_{53}\cos(\alpha_5) \\ 0 \\ -r_{53}\sin(\alpha_5) \end{bmatrix}$$

$$\mathbf{v}_{6\alpha_1} = \begin{bmatrix} l_{23}\cos(\alpha_1) \\ 0 \\ -l_{23}\sin(\alpha_1) \end{bmatrix} \qquad \mathbf{v}_{6\alpha_2} = \begin{bmatrix} l_{33}\cos(\alpha_2) \\ 0 \\ -l_{33}\sin(\alpha_2) \end{bmatrix}$$

$$\mathbf{v}_{6\alpha_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 3, 4 \qquad \mathbf{v}_{6\alpha_5} = \begin{bmatrix} l_{53}\cos(\alpha_5) \\ 0 \\ -l_{53}\sin(\alpha_5) \end{bmatrix}$$

$$\mathbf{v}_{6\alpha_6} = \begin{bmatrix} r_{63}\cos(\alpha_6) \\ 0 \\ -r_{63}\sin(\alpha_6) \end{bmatrix}$$

The partial velocities of the mass centres are shown in Equation (114)

The applied agents (forces/torques) contributing to the generalized active forces are the weight or gravity forces and the applied joint torques. The generalized applied forces F_{α_1} through F_{α_6} are (after simplification)

$$\begin{aligned} F_{\alpha_1} = \; & m_1 g \sin(\alpha_1) r_{13} + M_{12} - M_{22} + m_2 g \sin(\alpha_1) l_{23} + m_3 g \sin(\alpha_1) l_{23} \\ & + m_4 g \sin(\alpha_1) l_{23} + m_5 g \sin(\alpha_1) l_{23} + m_6 g \sin(\alpha_1) l_{23} \end{aligned} \tag{115}$$

$$\begin{aligned} F_{\alpha_2} = \; & m_2 g \sin(\alpha_2) r_{23} + M_{22} - M_{32} + m_3 g \sin(\alpha_2) l_{33} + m_4 g \sin(\alpha_2) l_{33} \\ & + m_5 g \sin(\alpha_2) l_{33} + m_6 g \sin(\alpha_2) l_{33} \end{aligned} \tag{116}$$

$$F_{\alpha_3} = m_3 g \sin(\alpha_3) r_{33} + M_{32} - M_{42} - M_{52} + m_4 g \sin(\alpha_3) l_{43} \tag{117}$$

$$F_{\alpha_4} = m_4 g \sin(\alpha_4) r_{43} + M_{42} \tag{118}$$

$$F_{\alpha_5} = m_5 g \sin(\alpha_5) r_{53} + M_{52} - M_{62} + m_6 g \sin(\alpha_5) l_{53} \tag{119}$$

$$F_{\alpha_6} = m_6 g \sin(\alpha_6) r_{63} + M_{62} \tag{120}$$

$$\begin{aligned} F^*_{\alpha_1} = \; & -m_6 l_{23}^2 \ddot{\alpha}_1 - m_1 r_{13}^2 \ddot{\alpha}_1 - m_5 l_{23}^2 \ddot{\alpha}_1 - m_3 l_{23}^2 \ddot{\alpha}_1 - m_2 l_{23}^2 \ddot{\alpha}_1 - m_4 l_{23}^2 \ddot{\alpha}_1 - I_{12} \ddot{\alpha}_1 \\ & - m_6 l_{23} \ddot{\alpha}_6 r_{63} \cos(\alpha_1 - \alpha_6) - m_3 l_{23} \dot{\alpha}_2^2 l_{33} \sin(\alpha_1 - \alpha_2) \\ & + m_5 l_{23} \dot{\alpha}_5^2 r_{53} \sin(\alpha_5 - \alpha_1) - m_2 l_{23} \dot{\alpha}_2^2 r_{23} \sin(\alpha_1 - \alpha_2) \\ & + m_3 l_{23} \dot{\alpha}_3^2 r_{33} \sin(\alpha_3 - \alpha_1) - m_4 l_{23} \dot{\alpha}_4^2 r_{43} \sin(\alpha_1 - \alpha_4) \\ & - m_5 l_{23} \dot{\alpha}_2^2 l_{33} \sin(\alpha_1 - \alpha_2) - m_6 l_{23} \dot{\alpha}_2^2 l_{33} \sin(\alpha_1 - \alpha_2) \\ & + m_4 l_{23} \dot{\alpha}_3^2 l_{43} \sin(\alpha_3 - \alpha_1) - m_2 l_{23} \ddot{\alpha}_2 r_{23} \cos(\alpha_1 - \alpha_2) \\ & - m_3 l_{23} \ddot{\alpha}_3 r_{33} \cos(\alpha_3 - \alpha_1) - m_5 l_{23} \ddot{\alpha}_5 r_{53} \cos(\alpha_5 - \alpha_1) \\ & - m_6 l_{23} \ddot{\alpha}_2 l_{33} \cos(\alpha_1 - \alpha_2) - m_4 l_{23} \ddot{\alpha}_3 l_{43} \cos(\alpha_3 - \alpha_1) \\ & - m_5 l_{23} \ddot{\alpha}_2 l_{33} \cos(\alpha_1 - \alpha_2) - m_3 l_{23} \ddot{\alpha}_2 l_{33} \cos(\alpha_1 - \alpha_2) \\ & - m_4 l_{23} \ddot{\alpha}_2 l_{33} \cos(\alpha_1 - \alpha_2) - m_4 l_{23} \dot{\alpha}_2^2 l_{33} \sin(\alpha_1 - \alpha_2) \\ & - m_4 l_{23} \ddot{\alpha}_4 r_{43} \cos(\alpha_1 - \alpha_4) - m_6 l_{23} \ddot{\alpha}_5 l_{53} \cos(\alpha_5 - \alpha_1) \\ & + m_6 l_{23} \dot{\alpha}_5^2 l_{53} \sin(\alpha_5 - \alpha_1) - m_6 l_{23} \dot{\alpha}_6^2 r_{63} \sin(\alpha_1 - \alpha_6) \end{aligned} \tag{121}$$

$$
\begin{aligned}
F^*_{\alpha_2} = \; & -m_5 l_{33}^2 \ddot{\alpha}_2 - m_3 l_{33}^2 \ddot{\alpha}_2 - m_6 l_{33}^2 \ddot{\alpha}_2 - m_3 l_{33} \ddot{\alpha}_3 r_{33} \cos(\alpha_3 - \alpha_2) \\
& - m_4 l_{33} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_2) - m_6 l_{33} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_2) - I_{22} \ddot{\alpha}_2 \\
& + m_3 l_{33} \dot{\alpha}_3^2 r_{33} \sin(\alpha_3 - \alpha_2) + m_6 l_{33} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_2) \\
& - m_4 l_{33} \ddot{\alpha}_3 l_{43} \cos(\alpha_3 - \alpha_2) + m_4 l_{33} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_2) \\
& - m_5 l_{33} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_2) - m_4 l_{33} \dot{\alpha}_4^2 r_{43} \sin(\alpha_2 - \alpha_4) \\
& + m_5 l_{33} \dot{\alpha}_5^2 r_{53} \sin(-\alpha_2 + \alpha_5) + m_4 l_{33} \dot{\alpha}_3^2 l_{43} \sin(\alpha_3 - \alpha_2) \\
& - m_6 l_{33} \ddot{\alpha}_6 r_{63} \cos(\alpha_2 - \alpha_6) - m_3 l_{33} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_2) \\
& - m_2 r_{23} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_2) + m_5 l_{33} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_2) \\
& - m_5 l_{33} \ddot{\alpha}_5 r_{53} \cos(-\alpha_2 + \alpha_5) - m_4 l_{33}^2 \ddot{\alpha}_2 - m_2 r_{23}^2 \ddot{\alpha}_2 \\
& + m_6 l_{33} \dot{\alpha}_5^2 l_{53} \sin(-\alpha_2 + \alpha_5) - m_4 l_{33} \ddot{\alpha}_4 r_{43} \cos(\alpha_2 - \alpha_4) \\
& - m_6 l_{33} \ddot{\alpha}_5 l_{53} \cos(-\alpha_2 + \alpha_5) + m_3 l_{33} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_2) \\
& - m_6 l_{33} \dot{\alpha}_6^2 r_{63} \sin(\alpha_2 - \alpha_6) + m_2 r_{23} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_2)
\end{aligned} \tag{122}
$$

$$
\begin{aligned}
F^*_{\alpha_3} = \; & -I_{32} \ddot{\alpha}_3 - m_3 r_{33} \ddot{\alpha}_1 l_{23} \cos(\alpha_3 - \alpha_1) \\
& - m_4 l_{43} \ddot{\alpha}_1 l_{23} \cos(\alpha_3 - \alpha_1) - m_4 l_{43} \dot{\alpha}_2^2 l_{33} \sin(\alpha_3 - \alpha_2) - m_4 l_{43}^2 \ddot{\alpha}_3 - m_3 r_{33}^2 \ddot{\alpha}_3 \\
& - m_4 l_{43} \dot{\alpha}_4^2 r_{43} \sin(\alpha_3 - \alpha_4) - m_3 r_{33} \dot{\alpha}_1^2 l_{23} \sin(\alpha_3 - \alpha_1) \\
& - m_4 l_{43} \ddot{\alpha}_2 l_{33} \cos(\alpha_3 - \alpha_2) - m_4 l_{43} \ddot{\alpha}_4 r_{43} \cos(\alpha_3 - \alpha_4) \\
& - m_3 r_{33} \dot{\alpha}_2^2 l_{33} \sin(\alpha_3 - \alpha_2) - m_4 l_{43} \dot{\alpha}_1^2 l_{23} \sin(\alpha_3 - \alpha_1) \\
& - m_3 r_{33} \ddot{\alpha}_2 l_{33} \cos(\alpha_3 - \alpha_2)
\end{aligned} \tag{123}
$$

$$
\begin{aligned}
F^*_{\alpha_4} = \; & -m_4 r_{43}^2 \ddot{\alpha}_4 - I_{42} \ddot{\alpha}_4 + m_4 r_{43} \dot{\alpha}_3^2 l_{43} \sin(\alpha_3 - \alpha_4) - m_4 r_{43} \ddot{\alpha}_3 l_{43} \cos(\alpha_3 - \alpha_4) \\
& + m_4 r_{43} \dot{\alpha}_2^2 l_{33} \sin(\alpha_2 - \alpha_4) - m_4 r_{43} \ddot{\alpha}_2 l_{33} \cos(\alpha_2 - \alpha_4) \\
& + m_4 r_{43} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_4) - m_4 r_{43} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_4)
\end{aligned} \tag{124}
$$

$$
\begin{aligned}
F^*_{\alpha_5} = \; & -m_5 r_{53} \dot{\alpha}_1^2 l_{23} \sin(\alpha_5 - \alpha_1) - m_6 l_{53} \dot{\alpha}_6^2 r_{63} \sin(\alpha_5 - \alpha_6) \\
& - m_5 r_{53} \ddot{\alpha}_1 l_{23} \cos(\alpha_5 - \alpha_1) - I_{52} \ddot{\alpha}_5 - m_6 l_{53} \ddot{\alpha}_6 r_{63} \cos(\alpha_5 - \alpha_6) \\
& - m_6 l_{53} \ddot{\alpha}_2 l_{33} \cos(-\alpha_2 + \alpha_5) - m_6 l_{53}^2 \ddot{\alpha}_5 - m_6 l_{53} \dot{\alpha}_1^2 l_{23} \sin(\alpha_5 - \alpha_1) \\
& - m_5 r_{53} \ddot{\alpha}_2 l_{33} \cos(-\alpha_2 + \alpha_5) - m_6 l_{53} \dot{\alpha}_2^2 l_{33} \sin(-\alpha_2 + \alpha_5) \\
& - m_5 r_{53} \dot{\alpha}_2^2 l_{33} \sin(-\alpha_2 + \alpha_5) - m_6 l_{53} \ddot{\alpha}_1 l_{23} \cos(\alpha_5 - \alpha_1) - m_5 r_{53}^2 \ddot{\alpha}_5
\end{aligned} \tag{125}
$$

$$
\begin{aligned}
F^*_{\alpha_6} = \; & -m_6 r_{63} \ddot{\alpha}_2 l_{33} \cos(\alpha_2 - \alpha_6) + m_6 r_{63} \dot{\alpha}_5^2 l_{53} \sin(\alpha_5 - \alpha_6) \\
& + m_6 r_{63} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_6) - I_{62} \ddot{\alpha}_6 - m_6 r_{63} \ddot{\alpha}_5 l_{53} \cos(\alpha_5 - \alpha_6) \\
& + m_6 r_{63} \dot{\alpha}_2^2 l_{33} \sin(\alpha_2 - \alpha_6) - m_6 r_{63}^2 \ddot{\alpha}_6 - m_6 r_{63} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_6)
\end{aligned} \tag{126}
$$

The six final equation's governing the rigid body dynamics of the six segment biped robot in the sagittal plane are then obtained according to Equation (111) through summing

Equations (115)-(120) to the corresponding Equations (121) -(126) and then equating to zero.

$$\begin{aligned}
-I_{12}\ddot{\alpha}_1 + m_3 g\sin(\alpha_1)l_{23} + m_1 g\sin(\alpha_1)r_{13} - m_3 l_{23}\dot{\alpha}_2^2 l_{33}\sin(\alpha_1-\alpha_2)\\
-m_5 l_{23}\ddot{\alpha}_5 r_{53}\cos(\alpha_5-\alpha_1) - m_6 l_{23}\ddot{\alpha}_2 l_{33}\cos(\alpha_1-\alpha_2) - m_6 l_{23}^2\ddot{\alpha}_1\\
-m_1 r_{13}^2\ddot{\alpha}_1 - m_2 l_{23}^2\ddot{\alpha}_1 - m_3 l_{23}^2\ddot{\alpha}_1 - m_4 l_{23}^2\ddot{\alpha}_1 + m_5 l_{23}\dot{\alpha}_5^2 r_{53}\sin(\alpha_5-\alpha_1)\\
-m_6 l_{23}\dot{\alpha}_2^2 l_{33}\sin(\alpha_1-\alpha_2) + m_6 l_{23}\dot{\alpha}_5^2 l_{53}\sin(\alpha_5-\alpha_1) - m_5 l_{23}^2\ddot{\alpha}_1\\
+m_6 g\sin(\alpha_1)l_{23} + m_4 g\sin(\alpha_1)l_{23} - m_4 l_{23}\dot{\alpha}_4^2 r_{43}\sin(\alpha_1-\alpha_4)\\
-m_2 l_{23}\dot{\alpha}_2^2 r_{23}\sin(\alpha_1-\alpha_2) - m_4 l_{23}\ddot{\alpha}_4 r_{43}\cos(\alpha_1-\alpha_4)\\
-m_3 l_{23}\ddot{\alpha}_3 r_{33}\cos(\alpha_3-\alpha_1) - m_2 l_{23}\ddot{\alpha}_2 r_{23}\cos(\alpha_1-\alpha_2)\\
-m_4 l_{23}\ddot{\alpha}_3 l_{43}\cos(\alpha_3-\alpha_1) + m_3 l_{23}\dot{\alpha}_3^2 r_{33}\sin(\alpha_3-\alpha_1) + m_5 g\sin(\alpha_1)l_{23}\\
-m_4 l_{23}\ddot{\alpha}_2 l_{33}\cos(\alpha_1-\alpha_2) - m_5 l_{23}\ddot{\alpha}_2 l_{33}\cos(\alpha_1-\alpha_2)\\
+m_4 l_{23}\dot{\alpha}_3^2 l_{43}\sin(\alpha_3-\alpha_1) - m_6 l_{23}\ddot{\alpha}_6 r_{63}\cos(\alpha_1-\alpha_6)\\
-m_4 l_{23}\dot{\alpha}_2^2 l_{33}\sin(\alpha_1-\alpha_2) - m_6 l_{23}\dot{\alpha}_6^2 r_{63}\sin(\alpha_1-\alpha_6)\\
-m_5 l_{23}\dot{\alpha}_2^2 l_{33}\sin(\alpha_1-\alpha_2) - m_6 l_{23}\ddot{\alpha}_5 l_{53}\cos(\alpha_5-\alpha_1) + m_2 g\sin(\alpha_1)l_{23}\\
-m_3 l_{23}\ddot{\alpha}_2 l_{33}\cos(\alpha_1-\alpha_2) + M_{12} - M_{22} &= 0
\end{aligned} \tag{127}$$

$$\begin{aligned}
m_3 g\sin(\alpha_2)l_{33} + m_5 l_{33}\dot{\alpha}_5^2 r_{53}\sin(\alpha_5-\alpha_2) - m_5 l_{33}\ddot{\alpha}_5 r_{53}\cos(\alpha_5-\alpha_2)\\
-m_4 l_{33}\dot{\alpha}_4^2 r_{43}\sin(\alpha_2-\alpha_4) - m_3 l_{33}\ddot{\alpha}_1 l_{23}\cos(\alpha_1-\alpha_2)\\
-m_6 l_{33}\ddot{\alpha}_5 l_{53}\cos(\alpha_5-\alpha_2) - m_5 l_{33}\ddot{\alpha}_1 l_{23}\cos(\alpha_1-\alpha_2)\\
+m_3 l_{33}\dot{\alpha}_1^2 l_{23}\sin(\alpha_1-\alpha_2) - m_6 l_{33}\dot{\alpha}_6^2 r_{63}\sin(\alpha_2-\alpha_6)\\
-m_3 l_{33}\ddot{\alpha}_3 r_{33}\cos(\alpha_3-\alpha_2) - m_2 r_{23}\ddot{\alpha}_1 l_{23}\cos(\alpha_1-\alpha_2) + m_5 g\sin(\alpha_2)l_{33}\\
+m_6 l_{33}\dot{\alpha}_5^2 l_{53}\sin(\alpha_5-\alpha_2) + m_3 l_{33}\dot{\alpha}_3^2 r_{33}\sin(\alpha_3-\alpha_2)\\
-m_6 l_{33}\ddot{\alpha}_6 r_{63}\cos(\alpha_2-\alpha_6) - m_4 l_{33}\ddot{\alpha}_1 l_{23}\cos(\alpha_1-\alpha_2)\\
-m_4 l_{33}\ddot{\alpha}_3 l_{43}\cos(\alpha_3-\alpha_2) - m_6 l_{33}\ddot{\alpha}_1 l_{23}\cos(\alpha_1-\alpha_2) - I_{22}\ddot{\alpha}_2\\
+m_4 l_{33}\dot{\alpha}_1^2 l_{23}\sin(\alpha_1-\alpha_2) + m_4 g\sin(\alpha_2)l_{33} + m_6 g\sin(\alpha_2)l_{33}\\
+m_2 r_{23}\dot{\alpha}_1^2 l_{23}\sin(\alpha_1-\alpha_2) - m_5 l_{33}^2\ddot{\alpha}_2 - m_6 l_{33}^2\ddot{\alpha}_2 - m_2 r_{23}^2\ddot{\alpha}_2 - m_4 l_{33}^2\ddot{\alpha}_2\\
-m_3 l_{33}^2\ddot{\alpha}_2 + m_6 l_{33}\dot{\alpha}_1^2 l_{23}\sin(\alpha_1-\alpha_2) + m_4 l_{33}\dot{\alpha}_3^2 l_{43}\sin(\alpha_3-\alpha_2)\\
-m_4 l_{33}\ddot{\alpha}_4 r_{43}\cos(\alpha_2-\alpha_4) + m_5 l_{33}\dot{\alpha}_1^2 l_{23}\sin(\alpha_1-\alpha_2) + m_2 g\sin(\alpha_2)r_{23}\\
+M_{22} - M_{32} &= 0
\end{aligned} \tag{128}$$

$$\begin{aligned}
-m_3 r_{33}^2\ddot{\alpha}_3 - m_4 l_{43}\dot{\alpha}_2^2 l_{33}\sin(\alpha_3-\alpha_2) - m_3 r_{33}\ddot{\alpha}_1 l_{23}\cos(\alpha_3-\alpha_1)\\
-m_4 l_{43}\ddot{\alpha}_2 l_{33}\cos(\alpha_3-\alpha_2) + M_{32} - M_{42} - M_{52} - m_4 l_{43}\dot{\alpha}_1^2 l_{23}\sin(\alpha_3-\alpha_1)\\
-m_3 r_{33}\dot{\alpha}_2^2 l_{33}\sin(\alpha_3-\alpha_2) - m_4 l_{43}\ddot{\alpha}_1 l_{23}\cos(\alpha_3-\alpha_1) + m_4 g\sin(\alpha_3)l_{43}\\
-m_3 r_{33}\ddot{\alpha}_2 l_{33}\cos(\alpha_3-\alpha_2) - m_4 l_{43}\dot{\alpha}_4^2 r_{43}\sin(\alpha_3-\alpha_4) - I_{32}\ddot{\alpha}_3\\
-m_3 r_{33}\dot{\alpha}_1^2 l_{23}\sin(\alpha_3-\alpha_1) - m_4 l_{43}^2\ddot{\alpha}_3 - m_4 l_{43}\ddot{\alpha}_4 r_{43}\cos(\alpha_3-\alpha_4)\\
+m_3 g\sin(\alpha_3)r_{33} &= 0
\end{aligned} \tag{129}$$

$$
\begin{aligned}
m_4 g \sin(\alpha_4) r_{43} - m_4 r_{43}^2 \ddot{\alpha}_4 + m_4 r_{43} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_4) \\
-m_4 r_{43} \ddot{\alpha}_3 l_{43} \cos(\alpha_3 - \alpha_4) + M_{42} - m_4 r_{43} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_4) \\
+m_4 r_{43} \dot{\alpha}_3^2 l_{43} \sin(\alpha_3 - \alpha_4) + m_4 r_{43} \dot{\alpha}_2^2 l_{33} \sin(\alpha_2 - \alpha_4) - I_{42} \ddot{\alpha}_4 \\
-m_4 r_{43} \ddot{\alpha}_2 l_{33} \cos(\alpha_2 - \alpha_4) &= 0
\end{aligned}
\tag{130}
$$

$$
\begin{aligned}
m_5 g \sin(\alpha_5) r_{53} - m_5 r_{53} \dot{\alpha}_1^2 l_{23} \sin(\alpha_5 - \alpha_1) - m_6 l_{53} \dot{\alpha}_2^2 l_{33} \sin(\alpha_5 - \alpha_2) \\
-m_6 l_{53} \ddot{\alpha}_1 l_{23} \cos(\alpha_5 - \alpha_1) - m_5 r_{53} \dot{\alpha}_2^2 l_{33} \sin(\alpha_5 - \alpha_2) \\
-m_6 l_{53} \ddot{\alpha}_2 l_{33} \cos(\alpha_5 - \alpha_2) + m_6 g \sin(\alpha_5) l_{53} - m_5 r_{53} \ddot{\alpha}_2 l_{33} \cos(\alpha_5 - \alpha_2) \\
-m_5 r_{53} \ddot{\alpha}_1 l_{23} \cos(\alpha_5 - \alpha_1) - m_6 l_{53}^2 \ddot{\alpha}_5 - m_5 r_{53}^2 \ddot{\alpha}_5 - m_6 l_{53} \dot{\alpha}_6^2 r_{63} \sin(\alpha_5 - \alpha_6) \\
-I_{52} \ddot{\alpha}_5 - m_6 l_{53} \dot{\alpha}_1^2 l_{23} \sin(\alpha_5 - \alpha_1) - m_6 l_{53} \ddot{\alpha}_6 r_{63} \cos(\alpha_5 - \alpha_6) + M_{52} - M_{62} &= 0
\end{aligned}
\tag{131}
$$

$$
\begin{aligned}
-m_6 r_{63} \ddot{\alpha}_5 l_{53} \cos(\alpha_5 - \alpha_6) - I_{62} \ddot{\alpha}_6 + m_6 r_{63} \dot{\alpha}_2^2 l_{33} \sin(\alpha_2 - \alpha_6) \\
+m_6 g \sin(\alpha_6) r_{63} - m_6 r_{63} \ddot{\alpha}_2 l_{33} \cos(\alpha_2 - \alpha_6) + m_6 r_{63} \dot{\alpha}_5^2 l_{53} \sin(\alpha_5 - \alpha_6) \\
+m_6 r_{63} \dot{\alpha}_1^2 l_{23} \sin(\alpha_1 - \alpha_6) - m_6 r_{63}^2 \ddot{\alpha}_6 - m_6 r_{63} \ddot{\alpha}_1 l_{23} \cos(\alpha_1 - \alpha_6) + M_{62} &= 0
\end{aligned}
\tag{132}
$$

Equations (127)-(132) are the final set of dynamic equations which can easily be rearranged in the form of Equation (1).

5 Lagrange's Dynamic Equations for iCub Robot

5.1 Introduction

Lagrange's equations are probably the most widely used equations for studying systems with several degrees of freedom. Unlike Kane's equations, Lagrange's equations are primarily restricted to holonomic systems.

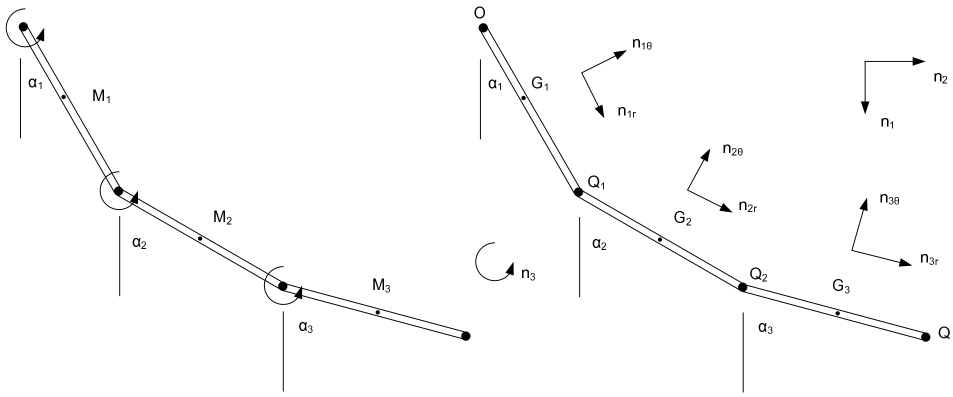
Consider a holonomic mechanical system S with n degrees of freedom represented by the coordinates q_r $(r = 1, , n)$. Lagrange's equation of motion is given by

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial q_r}\right) - \frac{\partial L}{\partial q_r} = F_r \, (r = 1, ..., n) \tag{133}
$$

where F_r is the generalised force, and the Lagrangian L is the difference between the kinetic energy K and potential energy P.

$$
L \triangleq K - P \tag{134}
$$

5.2 Triple-Rod Pendulum



(a) A triple-rod pendulum (b) Unit vector geometry

Figure 3: A triple-rod pendulum

Consider the triple-rod pendulum shown in Figure 3. It is assumed that three rods be identical, each having length l and mass m. Let the rods be connected by frictionless pins. Meanwhile, let there be moments M_1, M_2 and M_3 on and between the rods. Also, let there be a point mass Q (with mass M) at the end of the third rod. This system has three degrees of freedom as represented by the absolute angles α_1, α_2 and α_3. G_1, G_2 and G_3 shown in Figure 3b are the mass centres of the rods, O, Q_1 and Q_2 are points on the pin axes, and B_1, B_2 and B_3 are the rods themselves.

Using the notations defined above, the angular velocities of the rods are

$$\begin{aligned}
\boldsymbol{\omega}^{B_1} &= \dot{\alpha}_1 \boldsymbol{n}_3 \\
\boldsymbol{\omega}^{B_2} &= \dot{\alpha}_2 \boldsymbol{n}_3 \\
\boldsymbol{\omega}^{B_3} &= \dot{\alpha}_3 \boldsymbol{n}_3
\end{aligned} \tag{135}$$

Also, the velocities of the mass centre are

$$\begin{aligned}
\boldsymbol{v}^{G_1} &= (l/2)\, \dot{\alpha}_1 \boldsymbol{n}_{1\alpha} \\
\boldsymbol{v}^{G_2} &= l\dot{\alpha}_1 \boldsymbol{n}_{1\alpha} + (l/2)\, \dot{\alpha}_2 \boldsymbol{n}_{2\alpha} \\
\boldsymbol{v}^{G_3} &= l\dot{\alpha}_1 \boldsymbol{n}_{1\alpha} + l\dot{\alpha}_2 \boldsymbol{n}_{2\alpha} + (l/2)\, \dot{\alpha}_3 \boldsymbol{n}_{3\alpha}
\end{aligned} \tag{136}$$

and the velocities of the pin are

$$
\begin{aligned}
\boldsymbol{v}^{Q_1} &= l\dot{\alpha}_1 \boldsymbol{n}_{1\alpha} \\
\boldsymbol{v}^{Q_2} &= l\dot{\alpha}_1 \boldsymbol{n}_{1\alpha} + l\dot{\alpha}_2 \boldsymbol{n}_{2\alpha} \\
\boldsymbol{v}^{Q_3} &= l\dot{\alpha}_1 \boldsymbol{n}_{1\alpha} + l\dot{\alpha}_2 \boldsymbol{n}_{2\alpha} + l\dot{\alpha}_3 \boldsymbol{n}_{3\alpha}
\end{aligned}
\tag{137}
$$

The partial angular velocities of the rods are then

$$
\begin{array}{lll}
\boldsymbol{\omega}^{B_1}_{\dot{\alpha}_1} = \boldsymbol{n}_3 & \boldsymbol{\omega}^{B_1}_{\dot{\alpha}_2} = 0 & \boldsymbol{\omega}^{B_1}_{\dot{\alpha}_3} = 0 \\
\boldsymbol{\omega}^{B_2}_{\dot{\alpha}_1} = 0 & \boldsymbol{\omega}^{B_2}_{\dot{\alpha}_2} = \boldsymbol{n}_3 & \boldsymbol{\omega}^{B_2}_{\dot{\alpha}_3} = 0 \\
\boldsymbol{\omega}^{B_3}_{\dot{\alpha}_1} = 0 & \boldsymbol{\omega}^{B_3}_{\dot{\alpha}_2} = 0 & \boldsymbol{\omega}^{B_3}_{\dot{\alpha}_3} = \boldsymbol{n}_3
\end{array}
\tag{138}
$$

Similarly, the partial velocities of the mass centres and of point Q are

$$
\begin{array}{lll}
\boldsymbol{v}^{G_1}_{\dot{\alpha}_1} = (l/2)\,\boldsymbol{n}_{1\alpha} & \boldsymbol{v}^{G_1}_{\dot{\alpha}_2} = 0 & \boldsymbol{v}^{G_1}_{\dot{\alpha}_3} = 0 \\
\boldsymbol{v}^{G_2}_{\dot{\alpha}_1} = l\boldsymbol{n}_{1\alpha} & \boldsymbol{v}^{G_2}_{\dot{\alpha}_2} = (l/2)\,\boldsymbol{n}_{2\alpha} & \boldsymbol{v}^{G_2}_{\dot{\alpha}_3} = 0 \\
\boldsymbol{v}^{G_3}_{\dot{\alpha}_1} = l\boldsymbol{n}_{1\alpha} & \boldsymbol{v}^{G_3}_{\dot{\alpha}_2} = l\boldsymbol{n}_{2\alpha} & \boldsymbol{v}^{G_3}_{\dot{\alpha}_3} = (l/2)\,\boldsymbol{n}_{3\alpha}
\end{array}
\tag{139}
$$

and

$$
\boldsymbol{v}^{Q}_{\dot{\alpha}_1} = l\boldsymbol{n}_{1\alpha} \quad \boldsymbol{v}^{Q}_{\dot{\alpha}_2} = l\boldsymbol{n}_{2\alpha} \quad \boldsymbol{v}^{Q}_{\dot{\alpha}_3} = l\boldsymbol{n}_{3\alpha}
\tag{140}
$$

The applied forces that contribute to the generalized forces are weight forces $mg\boldsymbol{n}_1$ through the mass centres, the weight force $Mg\boldsymbol{n}_1$ through Q and the moments at the pin joints. Hence, the generalized forces become

$$
\begin{aligned}
F_{\alpha_1} = {} & mg\boldsymbol{n}_1 \cdot \boldsymbol{v}^{G_1}_{\dot{\alpha}_1} + mg\boldsymbol{n}_1 \cdot \boldsymbol{v}^{G_2}_{\dot{\alpha}_1} + mg\boldsymbol{n}_1 \cdot \boldsymbol{v}^{G_3}_{\dot{\alpha}_1} + Mg\boldsymbol{n}_1 \cdot \boldsymbol{v}^{Q}_{\dot{\alpha}_1} \\
& + (M_1 - M_2)\boldsymbol{n}_3 \cdot \boldsymbol{\omega}^{B_1}_{\dot{\alpha}_1} + (M_2 - M_3)\boldsymbol{n}_3 \cdot \boldsymbol{\omega}^{B_2}_{\dot{\alpha}_1} + M_3\boldsymbol{n}_3 \cdot \boldsymbol{\omega}^{B_3}_{\dot{\alpha}_1}
\end{aligned}
\tag{141}
$$

By substituting from Equation (138), (139) and (140), F_{α_1} becomes

$$
F_{\alpha_1} = -mg\left(\frac{5l}{2}\right)\sin\alpha_1 + M_1 - M_2 - Mgl\sin\alpha_1
\tag{142}
$$

Similarly, F_{α_2} and F_{α_3} are

$$F_{\alpha_2} = -mg\left(\frac{3l}{2}\right)\sin\alpha_2 + M_2 - M_3 - Mgl\sin\alpha_2 \tag{143}$$

$$F_{\alpha_3} = -mg\left(\frac{l}{2}\right)\sin\alpha_3 + M_3 - Mgl\sin\alpha_3 \tag{144}$$

The kinetic energy K of the system may be expressed as

$$\begin{aligned} K = &\left(\frac{1}{2}\right) m\left(\boldsymbol{v}^{G_1}\right)^2 + \left(\frac{1}{2}\right)\boldsymbol{I}\left(\boldsymbol{\omega}^{B_1}\right)^2 + \left(\frac{1}{2}\right) m\left(\boldsymbol{v}^{G_2}\right)^2 + \left(\frac{1}{2}\right)\boldsymbol{I}\left(\boldsymbol{\omega}^{B_2}\right)^2 \\ &+ \left(\frac{1}{2}\right) m\left(\boldsymbol{v}^{G_3}\right)^2 + \left(\frac{1}{2}\right)\boldsymbol{I}\left(\boldsymbol{\omega}^{B_3}\right)^2 + \left(\frac{1}{2}\right)\boldsymbol{M}\left(\boldsymbol{v}^{Q}\right)^2 \end{aligned} \tag{145}$$

Using Equations (135) and (136), K becomes

$$\begin{aligned} K = &\left(\frac{1}{2}\right) ml^2\left(\left(\frac{7}{3}\right)\dot\alpha_1^2 + \left(\frac{4}{3}\right)\dot\alpha_2^2 + \left(\frac{1}{3}\right)\dot\alpha_3^2 + 3\dot\alpha_1\dot\alpha_2\cos(\alpha_2-\alpha_1)\right. \\ &\left. + \dot\alpha_2\dot\alpha_3\cos(\alpha_3-\alpha_2) + \dot\alpha_1\dot\alpha_3\cos(\alpha_3-\alpha_1)\right) \\ &+ \left(\frac{1}{2}\right) Ml^2\left(\dot\alpha_1^2 + \dot\alpha_2^2 + \dot\alpha_3^2 + 2\dot\alpha_1\dot\alpha_2\cos(\alpha_2-\alpha_1)\right. \\ &\left. + 2\dot\alpha_2\dot\alpha_3\cos(\alpha_3-\alpha_2) + 2\dot\alpha_1\dot\alpha_3\cos(\alpha_3-\alpha_1)\right) \end{aligned} \tag{146}$$

By differentiating in Equation (146) the following terms will be obtained

$$\begin{aligned} \frac{\partial K}{\partial\alpha_1} = &\left(\frac{1}{2}\right) ml^2\left(3\dot\alpha_1\dot\alpha_2\sin(\alpha_2-\alpha_1) - \dot\alpha_1\dot\alpha_3\sin(\alpha_1-\alpha_3)\right) \\ &+ \left(\frac{1}{2}\right) Ml^2\left(2\dot\alpha_1\alpha_2\sin(\alpha_2-\alpha_1) - 2\alpha_1\dot\alpha_3\sin(\alpha_1-\alpha_3)\right) \end{aligned} \tag{147}$$

$$\begin{aligned} \frac{\partial K}{\partial\alpha_2} = &\left(\frac{1}{2}\right) ml^2\left(-3\dot\alpha_1\dot\alpha_2\sin(\alpha_2-\alpha_1) - \dot\alpha_2\dot\alpha_3\sin(\alpha_3-\alpha_2)\right) \\ &+ \left(\frac{1}{2}\right) Ml^2\left(-2\dot\alpha_1\alpha_2\sin(\alpha_2-\alpha_1) + 2\alpha_3\dot\alpha_2\sin(\alpha_3-\alpha_2)\right) \end{aligned} \tag{148}$$

$$\frac{\partial K}{\partial \alpha_3} = \left(\frac{1}{2}\right) ml^2 \left(-\dot{\alpha}_2\dot{\alpha}_3 \sin(\alpha_3 - \alpha_2) + \dot{\alpha}_1\dot{\alpha}_3 \sin(\alpha_1 - \alpha_3)\right) \\ + \left(\frac{1}{2}\right) Ml^2 \left(-2\dot{\alpha}_2\alpha_3 \sin(\alpha_3 - \alpha_2) + 2\dot{\alpha}_1\dot{\alpha}_3 \sin(\alpha_1 - \alpha_3)\right) \tag{149}$$

$$\frac{\partial K}{\partial \dot{\alpha}_1} = \left(\frac{1}{2}\right) ml^2 \left(\left(\frac{14}{3}\right) \dot{\alpha}_1 + 3\dot{\alpha}_2 \cos(\alpha_2 - \alpha_1) + \dot{\alpha}_3 \cos(\alpha_1 - \alpha_3)\right) \\ + \left(\frac{1}{2}\right) Ml^2 \left(2\dot{\alpha}_1 + 2\dot{\alpha}_2 \cos(\alpha_2 - \alpha_1) + 2\dot{\alpha}_3 \cos(\alpha_1 - \alpha_3)\right) \tag{150}$$

$$\frac{\partial K}{\partial \dot{\alpha}_2} = \left(\frac{1}{2}\right) ml^2 \left(\left(\frac{8}{3}\right) \dot{\alpha}_2 + 3\dot{\alpha}_1 \cos(\alpha_2 - \alpha_1) + \dot{\alpha}_3 \cos(\alpha_3 - \alpha_2)\right) \\ + \left(\frac{1}{2}\right) Ml^2 \left(2\dot{\alpha}_2 + 2\dot{\alpha}_1 \cos(\alpha_2 - \alpha_1) + 2\dot{\alpha}_3 \cos(\alpha_3 - \alpha_2)\right) \tag{151}$$

$$\frac{\partial K}{\partial \dot{\alpha}_3} = \left(\frac{1}{2}\right) ml^2 \left(\left(\frac{2}{3}\right) \dot{\alpha}_3 + \dot{\alpha}_2 \cos(\alpha_3 - \alpha_2) + \dot{\alpha}_1 \cos(\alpha_1 - \alpha_3)\right) \\ + \left(\frac{1}{2}\right) Ml^2 \left(2\dot{\alpha}_3 + 2\dot{\alpha}_2 \cos(\alpha_3 - \alpha_2) + 2\dot{\alpha}_1 \cos(\alpha_1 - \alpha_3)\right) \tag{152}$$

By substituting from Equations into Lagrange's equations, the governing equations are

$$\left(\frac{7}{3}\right) \ddot{\alpha}_1 + \left(\frac{3}{2}\right) \ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + \left(\frac{1}{2}\right) \ddot{\alpha}_3 \cos(\alpha_1 - \alpha_3) + \left(\frac{3}{2}\right) \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) \\ + \left(\frac{1}{2}\right) \dot{\alpha}_3 \sin(\alpha_1 - \alpha_3) + \left(\frac{M}{m}\right) \Big(\ddot{\alpha}_1 + \ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + \ddot{(\alpha)}_3 \cos(\alpha_1 - \alpha_3) \\ + \dot{\alpha}_1^2 \sin(\alpha_2 - \alpha_1) + \dot{\alpha}_3^2 \sin(\alpha_2 - \alpha_3)\Big) + \left(\frac{3}{2}\right) \left(\frac{g}{l}\right) \sin\alpha_1 + \frac{(M_2 + M_3)}{ml^2} + \frac{Mg}{ml} \sin\alpha_1 = 0 \tag{153}$$

$$\left(\frac{4}{3}\right) \ddot{\alpha}_2 + \left(\frac{3}{2}\right) \ddot{\alpha}_1 \cos(\alpha_2 - \alpha_1) + \left(\frac{1}{2}\right) \ddot{\alpha}_3 \cos(\alpha_2 - \alpha_3) + \left(\frac{3}{2}\right) \dot{\alpha}_2 \sin(\alpha_2 - \alpha_1) \\ + \left(\frac{1}{2}\right) \dot{\alpha}_3 \sin(\alpha_2 - \alpha_3) + \left(\frac{M}{m}\right) \Big(\ddot{\alpha}_2 + \ddot{\alpha}_1 \cos(\alpha_2 - \alpha_1) + \ddot{(\alpha)}_3 \cos(\alpha_2 - \alpha_3) \\ + \dot{\alpha}_1^2 \sin(\alpha_2 - \alpha_1) + \dot{\alpha}_3^2 \sin(\alpha_2 - \alpha_3)\Big) + \left(\frac{3}{2}\right) \left(\frac{g}{l}\right) \sin\alpha_2 + \frac{(M_2 + M_3)}{ml^2} + \frac{Mg}{ml} \sin\alpha_2 = 0 \tag{154}$$

$$
\left(\frac{1}{3}\right)\ddot{\alpha}_3 + \left(\frac{1}{2}\right)\ddot{\alpha}_1\cos(\alpha_3-\alpha_1) + \left(\frac{1}{2}\right)\ddot{\alpha}_2\cos(\alpha_3-\alpha_2) + \left(\frac{1}{2}\right)\dot{\alpha}_1\sin(\alpha_3-\alpha_1)
+ \left(\frac{1}{2}\right)\dot{\alpha}_2\sin\alpha_2 + \left(\frac{M}{m}\right)\Big(\ddot{\alpha}_3 + \ddot{\alpha}_1\cos(\alpha_3-\alpha_1) + \ddot{(\alpha)}_2\cos(\alpha_3-\alpha_2)
+ \dot{\alpha}_1^2\sin(\alpha_3-\alpha_1) + \dot{\alpha}_2^2\sin(\alpha_3-\alpha_2)\Big) + \left(\frac{1}{2}\right)\left(\frac{g}{l}\right)\sin\alpha_3 + \frac{M_3}{ml^2} + \frac{Mg}{ml}\sin\alpha_3 = 0 \tag{155}
$$

5.3 iCub Robot

The iCub robot model is developed based on the procedure shown in the Triple-Rod Pendulum example but in a larger scale.

First of all, the angular velocities of each limbs are

$$
\begin{aligned}
\boldsymbol{\omega}^1 &= \dot{\alpha}_1\boldsymbol{n}_y, \boldsymbol{\omega}^2 = \dot{\alpha}_2\boldsymbol{n}_y \\
\boldsymbol{\omega}^3 &= \dot{\alpha}_3\boldsymbol{n}_y, \boldsymbol{\omega}^4 = \dot{\alpha}_4\boldsymbol{n}_y \\
\boldsymbol{\omega}^5 &= \dot{\alpha}_5\boldsymbol{n}_y, \boldsymbol{\omega}^6 = \dot{\alpha}_6\boldsymbol{n}_y
\end{aligned} \tag{156}
$$

The velocities of mass centre after simplification are (all in reference frame)

$$
\begin{aligned}
\boldsymbol{v}^1 &= -r_{13}(\sin\alpha_1)\dot{\alpha}_1\boldsymbol{n}_z \\
\boldsymbol{v}^2 &= -l_{23}(\sin\alpha_1)\dot{\alpha}_1\boldsymbol{n}_z - r_{23}(\sin\alpha_2)\dot{\alpha}_2\boldsymbol{n}_z \\
\boldsymbol{v}^3 &= -l_{23}(\sin\alpha_1)\dot{\alpha}_1\boldsymbol{n}_z - l_{33}(\sin\alpha_2)\dot{\alpha}_2\boldsymbol{n}_z - r_{23}(\sin\alpha_2)\dot{\alpha}_2\boldsymbol{n}_z \\
\boldsymbol{v}^4 &= -l_{23}(\sin\alpha_1)\dot{\alpha}_1\boldsymbol{n}_z - l_{33}(\sin\alpha_2)\dot{\alpha}_2\boldsymbol{n}_z - l_{43}(\sin\alpha_3)\dot{\alpha}_3\boldsymbol{n}_z - r_{43}(\sin\alpha_4)\dot{\alpha}_4\boldsymbol{n}_z \\
\boldsymbol{v}^5 &= -l_{23}(\sin\alpha_1)\dot{\alpha}_1\boldsymbol{n}_z - l_{33}(\sin\alpha_2)\dot{\alpha}_2\boldsymbol{n}_z - r_{53}(\sin\alpha_5)\dot{\alpha}_5\boldsymbol{n}_z \\
\boldsymbol{v}^6 &= -l_{23}(\sin\alpha_1)\dot{\alpha}_1\boldsymbol{n}_z - l_{33}(\sin\alpha_2)\dot{\alpha}_2\boldsymbol{n}_z - l_{63}(\sin\alpha_5)\dot{\alpha}_5\boldsymbol{n}_z - r_{63}(\sin\alpha_6)\dot{\alpha}_6\boldsymbol{n}_z
\end{aligned} \tag{157}
$$

The partial velocities of the mass centres are

$$
\begin{aligned}
v^1_{\dot{\alpha}_1} &= -r_{13}(\sin\alpha_1)\boldsymbol{n}_z, & v^1_{\dot{\alpha}_2} &= 0 \\
v^1_{\dot{\alpha}_3} &= 0, & v^1_{\dot{\alpha}_4} &= 0 \\
v^1_{\dot{\alpha}_5} &= 0, & v^1_{\dot{\alpha}_6} &= 0
\end{aligned} \tag{158}
$$

$$\begin{aligned}
&v^2_{\dot\alpha_1} = -l_{23}\,(\sin\alpha_1)\,\boldsymbol{n}_z, && v^2_{\dot\alpha_2} = -r_{23}\,(\sin\alpha_2)\,\boldsymbol{n}_z\\
&v^2_{\dot\alpha_3} = 0, && v^2_{\dot\alpha_4} = 0\\
&v^2_{\dot\alpha_5} = 0, && v^2_{\dot\alpha_6} = 0
\end{aligned} \tag{159}$$

$$\begin{aligned}
&v^3_{\dot\alpha_1} = -l_{23}\,(\sin\alpha_1)\,\boldsymbol{n}_z, && v^3_{\dot\alpha_2} = -l_{33}\,(\sin\alpha_2)\,\boldsymbol{n}_z\\
&v^3_{\dot\alpha_3} = -r_{33}\,(\sin\alpha_3)\,\boldsymbol{n}_z, && v^3_{\dot\alpha_4} = 0\\
&v^3_{\dot\alpha_5} = 0, && v^3_{\dot\alpha_6} = 0
\end{aligned} \tag{160}$$

$$\begin{aligned}
&v^4_{\dot\alpha_1} = -l_{23}\,(\sin\alpha_1)\,\boldsymbol{n}_z, && v^4_{\dot\alpha_2} = -l_{33}\,(\sin\alpha_2)\,\boldsymbol{n}_z\\
&v^4_{\dot\alpha_3} = -l_{43}\,(\sin\alpha_3)\,\boldsymbol{n}_z, && v^4_{\dot\alpha_4} = -r_{43}\,(\sin\alpha_4)\,\boldsymbol{n}_z\\
&v^4_{\dot\alpha_5} = 0, && v^4_{\dot\alpha_6} = 0
\end{aligned} \tag{161}$$

$$\begin{aligned}
&v^5_{\dot\alpha_1} = -l_{23}\,(\sin\alpha_1)\,\boldsymbol{n}_z, && v^5_{\dot\alpha_2} = -l_{33}\,(\sin\alpha_2)\,\boldsymbol{n}_z\\
&v^5_{\dot\alpha_3} = 0, && v^5_{\dot\alpha_4} = 0\\
&v^5_{\dot\alpha_5} = -r_{53}\,(\sin\alpha_5)\,\boldsymbol{n}_z, && v^5_{\dot\alpha_6} = 0
\end{aligned} \tag{162}$$

$$\begin{aligned}
&v^6_{\dot\alpha_1} = -l_{23}\,(\sin\alpha_1)\,\boldsymbol{n}_z, && v^6_{\dot\alpha_2} = -l_{33}\,(\sin\alpha_2)\,\boldsymbol{n}_z\\
&v^6_{\dot\alpha_3} = 0, && v^6_{\dot\alpha_4} = 0\\
&v^6_{\dot\alpha_5} = -l_{63}\,(\sin\alpha_5)\,\boldsymbol{n}_z, && v^6_{\dot\alpha_6} = -r_{63}\,(\sin\alpha_6)\,\boldsymbol{n}_z
\end{aligned} \tag{163}$$

The applied forces F_{α_1} to F_{α_6} will be (after simplification)

$$\begin{aligned}
F_{\alpha_1} = &- g(-m_1\sin(\alpha_1)r_{13} - m_2\sin(\alpha_1)l_{23} - m_3\sin(\alpha_1)l_{23} - m_4\sin(\alpha_1)l_{23}\\
&- m_5\sin(\alpha_1)l_{23} - m_6\sin(\alpha_1)l_{23}) + M_1 - M_2
\end{aligned} \tag{164}$$

$$\begin{aligned}
F_{\alpha_2} = &- g(-m_2 sin(\alpha_2)r_{23} - m_3 sin(\alpha_2)l_{33} - m_4 sin(\alpha_2)l_{33} - m_5 sin(\alpha_2)\\
&l_{33} - m_6 sin(\alpha_2)l_{33}) + M_2 - M_3
\end{aligned} \tag{165}$$

$$F_{\alpha_3} = gm_3\sin(\alpha_3)r33 + gm_4\sin(\alpha_3)l_{43} + M_3 - M_4 - M_5 \tag{166}$$

$$F_{\alpha_4} = gm_4\sin(\alpha_4)r_{43} + M_4 \tag{167}$$

$$F_{\alpha_5} = gm_5\sin(\alpha_5)r_{53} + gm_6\sin(\alpha_5)l_{63} + M_5 - M_6 \tag{168}$$

$$F_{\alpha_6} = gm_6\sin(\alpha_6)r_{63} + M_6 \tag{169}$$

The kinetic energy K of the system may be expressed as

$$\begin{aligned}
K =& \frac{1}{2}m_1\cos(\alpha_1)^2\dot{\alpha_1}^2r_{13}^2 + \frac{1}{2}m_1\sin(\alpha_1)^2\dot{\alpha_1}^2r_{13}^2 + \frac{1}{2}m_2\left(\cos(\alpha_2)\dot{\alpha_2}r_{23} + \cos(\alpha_1)\dot{\alpha_1}l_{23}\right)^2 \\
&+ \frac{1}{2}m_2(-\sin(\alpha_2)\dot{\alpha_2}r_{23} - \sin(\alpha_1)\dot{\alpha_1}l_{23})^2 \\
&+ \frac{1}{2}m_3(\cos(\alpha_3)\dot{\alpha_3}r_{33} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33})^2 \\
&+ \frac{1}{2}m_3(-\sin(\alpha_3)\dot{\alpha_3}r_{33} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33})^2 \\
&+ \frac{1}{2}m_4(\cos(\alpha_4)\dot{\alpha_4}r_{43} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_3)\dot{\alpha_3}l_{43})^2 \\
&+ \frac{1}{2}m_4(-\sin(\alpha_4)\dot{\alpha_4}r_{43} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_3)\dot{\alpha_3}l_{43})^2 \\
&+ \frac{1}{2}m_5(\cos(\alpha_5)\dot{\alpha_5}r_{53} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33})^2 \\
&+ \frac{1}{2}m_5(-\sin(\alpha_5)\dot{\alpha_5}r_{53} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33})^2 \\
&+ \frac{1}{2}m_6(\cos(\alpha_6)\dot{\alpha_6}r_{63} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_5)\dot{\alpha_5}l_{63})^2 \\
&+ \frac{1}{2}m_6(-\sin(\alpha_6)\dot{\alpha_6}r_{63} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_5)\dot{\alpha_5}l_{63})^2 \\
&+ \frac{1}{2}\dot{\alpha_1}^2I_{12} + \frac{1}{2}\dot{\alpha_2}^2I_{22} + \frac{1}{2}\dot{\alpha_3}^2I_{32} + \frac{1}{2}\dot{\alpha_4}^2I_{42} + \frac{1}{2}\dot{\alpha_5}^2I_{52} + \frac{1}{2}\dot{\alpha_6}^2I_{62}
\end{aligned} \tag{170}$$

By differentiating in Equation (170) the following terms will be obtained

$$
\begin{aligned}
&\frac{\partial K}{\partial \alpha_1} = \\
&- m_2(\cos(\alpha_2)\dot{\alpha}_2 r_{23} + \cos(\alpha_1)\dot{\alpha}_1 l_{23}) \sin(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_2(-\sin(\alpha_2)\dot{\alpha}_2 r_{23} - \sin(\alpha_1)\dot{\alpha}_1 l_{23}) \cos(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_3(\cos(\alpha_3)\dot{\alpha}_3 r_{33} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33}) \sin(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_3(-\sin(\alpha_3)\dot{\alpha}_3 r_{33} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33}) \cos(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_4(\cos(\alpha_4)\dot{\alpha}_4 r_{43} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33} + \cos(\alpha_3)\dot{\alpha}_3 l_{43}) \sin(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha}_4 r_{43} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33} - \sin(\alpha_3)\dot{\alpha}_3 l_{43}) \cos(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_5(\cos(\alpha_5)\dot{\alpha}_5 r_{53} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33}) \sin(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_5(-\sin(\alpha_5)\dot{\alpha}_5 r_{53} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33}) \cos(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_6(\cos(\alpha_6)\dot{\alpha}_6 r_{63} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33} + \cos(\alpha_5)\dot{\alpha}_5 l_{63}) \sin(\alpha_1)\dot{\alpha}_1 l_{23} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha}_6 r_{63} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33} - \sin(\alpha_5)\dot{\alpha}_5 l_{63}) \cos(\alpha_1)\dot{\alpha}_1 l_{23}
\end{aligned}
\tag{171}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \alpha_2} = \\
&- m_2(\cos(\alpha_2)\dot{\alpha}_2 r_{23} + \cos(\alpha_1)\dot{\alpha}_1 l_{23}) \sin(\alpha_2)\dot{\alpha}_2 r_{23} \\
&- m_2(-\sin(\alpha_2)\dot{\alpha}_2 r_{23} - \sin(\alpha_1)\dot{\alpha}_1 l_{23}) \cos(\alpha_2)\dot{\alpha}_2 r_{23} \\
&- m_3(\cos(\alpha_3)\dot{\alpha}_3 r_{33} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33}) \sin(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_3(-\sin(\alpha_3)\dot{\alpha}_3 r_{33} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33}) \cos(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_4(\cos(\alpha_4)\dot{\alpha}_4 r_{43} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33} + \cos(\alpha_3)\dot{\alpha}_3 l_{43}) \sin(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha}_4 r_{43} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33} - \sin(\alpha_3)\dot{\alpha}_3 l_{43}) \cos(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_5(\cos(\alpha_5)\dot{\alpha}_5 r_{53} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33}) \sin(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_5(-\sin(\alpha_5)\dot{\alpha}_5 r_{53} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33}) \cos(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_6(\cos(\alpha_6)\dot{\alpha}_6 r_{63} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33} + \cos(\alpha_5)\dot{\alpha}_5 l_{63}) \sin(\alpha_2)\dot{\alpha}_2 l_{33} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha}_6 r_{63} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33} - \sin(\alpha_5)\dot{\alpha}_5 l_{63}) \cos(\alpha_2)\dot{\alpha}_2 l_{33}
\end{aligned}
\tag{172}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \alpha_3} = \\
&- m_3(\cos(\alpha_3)\dot{\alpha}_3 r_{33} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33}) \sin(\alpha_3)\dot{\alpha}_3 r_{33} \\
&- m_3(-\sin(\alpha_3)\dot{\alpha}_3 r_{33} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33}) \cos(\alpha_3)\dot{\alpha}_3 r_{33} \\
&- m_4(\cos(\alpha_4)\dot{\alpha}_4 r_{43} + \cos(\alpha_1)\dot{\alpha}_1 l_{23} + \cos(\alpha_2)\dot{\alpha}_2 l_{33} + \cos(\alpha_3)\dot{\alpha}_3 l_{43}) \sin(\alpha_3)\dot{\alpha}_3 l_{43} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha}_4 r_{43} - \sin(\alpha_1)\dot{\alpha}_1 l_{23} - \sin(\alpha_2)\dot{\alpha}_2 l_{33} - \sin(\alpha_3)\dot{\alpha}_3 l_{43}) \cos(\alpha_3)\dot{\alpha}_3 l_{43}
\end{aligned}
\tag{173}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \alpha_4} = \\
&- m_4(\cos(\alpha_4)\dot{\alpha_4} r_{43} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33} + \cos(\alpha_3)\dot{\alpha_3} l_{43}) \sin(\alpha_4)\dot{\alpha_4} r_{43} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha_4} r_{43} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33} - \sin(\alpha_3)\dot{\alpha_3} l_{43}) \cos(\alpha_4)\dot{\alpha_4} r_{43}
\end{aligned}
\tag{174}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \alpha_5} = \\
&- m_5(\cos(\alpha_5)\dot{\alpha_5} r_{53} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33}) \sin(\alpha_5)\dot{\alpha_5} r_{53} \\
&- m_5(-\sin(\alpha_5)\dot{\alpha_5} r_{53} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33}) \cos(\alpha_5)\dot{\alpha_5} r_{53} \\
&- m_6(\cos(\alpha_6)\dot{\alpha_6} r_{63} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33} + \cos(\alpha_5)\dot{\alpha_5} l_{63}) \sin(\alpha_5)\dot{\alpha_5} l_{63} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha_6} r_{63} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33} - \sin(\alpha_5)\dot{\alpha_5} l_{63}) \cos(\alpha_5)\dot{\alpha_5} l_{63}
\end{aligned}
\tag{175}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \alpha_6} = \\
&- m_6(\cos(\alpha_6)\dot{\alpha_6} r_{63} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33} + \cos(\alpha_5)\dot{\alpha_5} l_{63}) \sin(\alpha_6)\dot{\alpha_6} r_{63} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha_6} r_{63} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33} - \sin(\alpha_5)\dot{\alpha_5} l_{63}) \cos(\alpha_6)\dot{\alpha_6} r_{63}
\end{aligned}
\tag{176}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \dot{\alpha_1}} = \\
&m_1 \cos(\alpha_1)^2 \dot{\alpha_1} r_{13}^2 + m_1 \sin(\alpha_1)^2 \dot{\alpha_1} r_{13}^2 \\
&+ m_2(\cos(\alpha_2)\dot{\alpha_2} r_{23} + \cos(\alpha_1)\dot{\alpha_1} l_{23}) \cos(\alpha_1) l_{23} \\
&- m_2(-\sin(\alpha_2)\dot{\alpha_2} r_{23} - \sin(\alpha_1)\dot{\alpha_1} l_{23}) \sin(\alpha_1) l_{23} \\
&+ m_3(\cos(\alpha_3)\dot{\alpha_3} r_{33} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33}) \cos(\alpha_1) l_{23} \\
&- m_3(-\sin(\alpha_3)\dot{\alpha_3} r_{33} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33}) \sin(\alpha_1) l_{23} \\
&+ m_4(\cos(\alpha_4)\dot{\alpha_4} r_{43} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33} + \cos(\alpha_3)\dot{\alpha_3} l_{43}) \cos(\alpha_1) l_{23} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha_4} r_{43} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33} - \sin(\alpha_3)\dot{\alpha_3} l_{43}) \sin(\alpha_1) l_{23} \\
&+ m_5(\cos(\alpha_5)\dot{\alpha_5} r_{53} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33}) \cos(\alpha_1) l_{23} \\
&- m_5(-\sin(\alpha_5)\dot{\alpha_5} r_{53} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33}) \sin(\alpha_1) l_{23} \\
&+ m_6(\cos(\alpha_6)\dot{\alpha_6} r_{63} + \cos(\alpha_1)\dot{\alpha_1} l_{23} + \cos(\alpha_2)\dot{\alpha_2} l_{33} + \cos(\alpha_5)\dot{\alpha_5} l_{63}) \cos(\alpha_1) l_{23} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha_6} r_{63} - \sin(\alpha_1)\dot{\alpha_1} l_{23} - \sin(\alpha_2)\dot{\alpha_2} l_{33} - \sin(\alpha_5)\dot{\alpha_5} l_{63}) \sin(\alpha_1) l_{23} \\
&+ \dot{\alpha_1} I_{12}
\end{aligned}
\tag{177}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \dot{\alpha_2}} = \\
&m_2(\cos(\alpha_2)\dot{\alpha_2}r_{23} + \cos(\alpha_1)\dot{\alpha_1}l_{23})\cos(\alpha_2)r_{23} \\
&- m_2(-\sin(\alpha_2)\dot{\alpha_2}r_{23} - \sin(\alpha_1)\dot{\alpha_1}l_{23})\sin(\alpha_2)r_{23} \\
&+ m_3(\cos(\alpha_3)\dot{\alpha_3}r_{33} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33})\cos(\alpha_2)l_{33} \\
&- m_3(-\sin(\alpha_3)\dot{\alpha_3}r_{33} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33})\sin(\alpha_2)l_{33} \\
&+ m_4(\cos(\alpha_4)\dot{\alpha_4}r_{43} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_3)\dot{\alpha_3}l_{43})\cos(\alpha_2)l_{33} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha_4}r_{43} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_3)\dot{\alpha_3}l_{43})\sin(\alpha_2)l_{33} \\
&+ m_5(\cos(\alpha_5)\dot{\alpha_5}r_{53} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33})\cos(\alpha_2)l_{33} \\
&- m_5(-\sin(\alpha_5)\dot{\alpha_5}r_{53} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33})\sin(\alpha_2)l_{33} \\
&+ m_6(\cos(\alpha_6)\dot{\alpha_6}r_{63} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_5)\dot{\alpha_5}l_{63})\cos(\alpha_2)l_{33} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha_6}r_{63} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_5)\dot{\alpha_5}l_{63})\sin(\alpha_2)l_{33} \\
&+ \dot{\alpha_2}I_{22}
\end{aligned} \tag{178}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \dot{\alpha_3}} = \\
&m_3(\cos(\alpha_3)\dot{\alpha_3}r_{33} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33})\cos(\alpha_3)r_{33} \\
&- m_3(-\sin(\alpha_3)\dot{\alpha_3}r_{33} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33})\sin(\alpha_3)r_{33} \\
&+ m_4(\cos(\alpha_4)\dot{\alpha_4}r_{43} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_3)\dot{\alpha_3}l_{43})\cos(\alpha_3)l_{43} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha_4}r_{43} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_3)\dot{\alpha_3}l_{43})\sin(\alpha_3)l_{43} \\
&+ \dot{\alpha_3}I_{32}
\end{aligned} \tag{179}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \dot{\alpha_4}} = \\
&m_4(\cos(\alpha_4)\dot{\alpha_4}r_{43} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_3)\dot{\alpha_3}l_{43})\cos(\alpha_4)r_{43} \\
&- m_4(-\sin(\alpha_4)\dot{\alpha_4}r_{43} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_3)\dot{\alpha_3}l_{43})\sin(\alpha_4)r_{43} \\
&+ \dot{\alpha_4}I_{42}
\end{aligned} \tag{180}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \dot{\alpha_5}} = \\
&m_5(\cos(\alpha_5)\dot{\alpha_5}r_{53} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33})\cos(\alpha_5)r_{53} \\
&- m_5(-\sin(\alpha_5)\dot{\alpha_5}r_{53} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33})\sin(\alpha_5)r_{53} \\
&+ m_6(\cos(\alpha_6)\dot{\alpha_6}r_{63} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_5)\dot{\alpha_5}l_{63})\cos(\alpha_5)l_{63} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha_6}r_{63} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_5)\dot{\alpha_5}l_{63})\sin(\alpha_5)l_{63} \\
&+ \dot{\alpha_5}I_{52}
\end{aligned} \tag{181}
$$

$$
\begin{aligned}
&\frac{\partial K}{\partial \dot{\alpha_6}} = \\
&m_6(\cos(\alpha_6)\dot{\alpha_6}r_{63} + \cos(\alpha_1)\dot{\alpha_1}l_{23} + \cos(\alpha_2)\dot{\alpha_2}l_{33} + \cos(\alpha_5)\dot{\alpha_5}l_{63})\cos(\alpha_6)r_{63} \\
&- m_6(-\sin(\alpha_6)\dot{\alpha_6}r_{63} - \sin(\alpha_1)\dot{\alpha_1}l_{23} - \sin(\alpha_2)\dot{\alpha_2}l_{33} - \sin(\alpha_5)\dot{\alpha_5}l_{63})\sin(\alpha_6)r_{63} \\
&+ \dot{\alpha_6}I_{62}
\end{aligned} \tag{182}
$$

By substituting Equations (164) to (169) and (171) to (182) , the governing equations are

$$
\begin{aligned}
&m_1\ddot{\alpha_1}r_{13}^2 + m_6l_{23}\ddot{\alpha_6}r_{63}\cos(\alpha_1 - \alpha_6) + \ddot{\alpha_1}I_{12} + m_2l_{23}^2\ddot{\alpha_1} - M_1 + M_2 + m_5l_{23}^2\ddot{\alpha_1} \\
&+ m_3l_{23}^2\ddot{\alpha_1} + m_5l_{23}\ddot{\alpha_5}r_{53}\cos(\alpha_1 - \alpha_5) + m_5l_{23}\ddot{\alpha_2}l_{33}\cos(\alpha_1 - \alpha_2) \\
&+ m_6l_{23}\ddot{\alpha_2}l_{33}\cos(\alpha_1 - \alpha_2) + m_6l_{23}\ddot{\alpha_5}l_{63}\cos(\alpha_1 - \alpha_5) + m_4l_{23}\dot{\alpha_2}^2l_{33}\sin(\alpha_1 - \alpha_2) \\
&- gm_4\sin(\alpha_1)l_{23} - gm_5\sin(\alpha_1)l_{23} - gm_6\sin(\alpha_1)l_{23} + m_4l_{23}\ddot{\alpha_2}l_{33}\cos(\alpha_1 - \alpha_2) \\
&+ m_2l_{23}\ddot{\alpha_2}r_{23}\cos(\alpha_1 - \alpha_2) - gm_2\sin(\alpha_1)l_{23} + m_2l_{23}\dot{\alpha_2}^2r_{23}\sin(\alpha_1 - \alpha_2) \\
&- gm_3\sin(\alpha_1)l_{23} - gm_1\sin(\alpha_1)r_{13} + m_4l_{23}\dot{\alpha_4}^2r_{43}\sin(\alpha_1 - \alpha_4) \\
&+ m_4l_{23}\ddot{\alpha_3}l_{43}\cos(\alpha_1 - \alpha_3) + m_3l_{23}\ddot{\alpha_2}l_{33}\cos(\alpha_1 - \alpha_2) + m_4l_{23}^2\ddot{\alpha_1} \\
&+ m_5l_{23}\dot{\alpha_5}^2r_{53}\sin(\alpha_1 - \alpha_5) + m_4l_{23}\dot{\alpha_3}^2l_{43}\sin(\alpha_1 - \alpha_3) + m_3l_{23}\ddot{\alpha_3}r_{33}\cos(\alpha_1 - \alpha_3) \\
&+ m_6l_{23}\dot{\alpha_6}^2r_{63}\sin(\alpha_1 - \alpha_6) + m_3l_{23}\dot{\alpha_3}^2r_{33}\sin(\alpha_1 - \alpha_3) + m_4l_{23}\ddot{\alpha_4}r_{43}\cos(\alpha_1 - \alpha_4) \\
&+ m_3l_{23}\dot{\alpha_2}^2l_{33}\sin(\alpha_1 - \alpha_2) + m_6l_{23}^2\ddot{\alpha_1} + m_5l_{23}\dot{\alpha_2}^2l_{33}\sin(\alpha_1 - \alpha_2) \\
&+ m_6l_{23}\dot{\alpha_2}^2l_{33}\sin(\alpha_1 - \alpha_2) + m_6l_{23}\dot{\alpha_5}^2l_{63}\sin(\alpha_1 - \alpha_5) = 0
\end{aligned} \tag{183}
$$

$$
\begin{aligned}
&m_3l_{33}\ddot{\alpha_3}r_{33}\cos(\alpha_2 - \alpha_3) + m_3l_{33}^2\ddot{\alpha_2} + m_6l_{33}^2\ddot{\alpha_2} + m_2r_{23}^2\ddot{\alpha_2} + m_5l_{33}^2\ddot{\alpha_2} \\
&+ m_6l_{33}\ddot{\alpha_6}r_{63}\cos(\alpha_2 - \alpha_6) + m_3l_{33}\ddot{\alpha_1}l_{23}\cos(\alpha_1 - \alpha_2) + m_4l_{33}\ddot{\alpha_1}l_{23}\cos(\alpha_1 - \alpha_2) \\
&+ m_4l_{33}\ddot{\alpha_3}l_{43}\cos(\alpha_2 - \alpha_3) + m_4l_{33}\ddot{\alpha_4}r_{43}\cos(\alpha_2 - \alpha_4) + m_6l_{33}\ddot{\alpha_1}l_{23}\cos(\alpha_1 - \alpha_2) \\
&- m_3l_{33}\dot{\alpha_1}^2l_{23}\sin(\alpha_1 - \alpha_2) - m_6l_{33}\dot{\alpha_1}^2l_{23}\sin(\alpha_1 - \alpha_2) + m_4l_{33}^2\ddot{\alpha_2} - M_2 + M_3 \\
&+ m_6l_{33}\dot{\alpha_6}^2r_{63}\sin(\alpha_2 - \alpha_6) + m_2r_{23}\ddot{\alpha_1}l_{23}\cos(\alpha_1 - \alpha_2) + m_6l_{33}\dot{\alpha_5}^2l_{63}\sin(\alpha_2 - \alpha_5) \\
&- m_5l_{33}\dot{\alpha_1}^2l_{23}\sin(\alpha_1 - \alpha_2) - gm_2\sin(\alpha_2)r_{23} - gm_3\sin(\alpha_2)l_{33} - gm_4\sin(\alpha_2)l_{33} \\
&- gm_5\sin(\alpha_2)l_{33} - gm_6\sin(\alpha_2)l_{33} - m_4l_{33}\dot{\alpha_1}^2l_{23}\sin(\alpha_1 - \alpha_2) \\
&+ m_5l_{33}\dot{\alpha_5}^2r_{53}\sin(\alpha_2 - \alpha_5) + m_5l_{33}\ddot{\alpha_5}r_{53}\cos(\alpha_2 - \alpha_5) + m_6l_{33}\ddot{\alpha_5}l_{63}\cos(\alpha_2 - \alpha_5) \\
&+ m_4l_{33}\dot{\alpha_3}^2l_{43}\sin(\alpha_2 - \alpha_3) + m_3l_{33}\dot{\alpha_3}^2r_{33}\sin(\alpha_2 - \alpha_3) + m_5l_{33}\ddot{\alpha_1}l_{23}\cos(\alpha_1 - \alpha_2) \\
&+ \ddot{\alpha_2}I22 + m_4l_{33}\dot{\alpha_4}^2r_{43}\sin(\alpha_2 - \alpha_4) - m_2r_{23}\dot{\alpha_1}^2l_{23}\sin(\alpha_1 - \alpha_2) = 0
\end{aligned} \tag{184}
$$

$$
\begin{aligned}
&- M_3 + M_4 + M_5 - m_3 r_{33} \dot{\alpha_2}^2 l_{33} \sin(\alpha_2 - \alpha_3) + m_3 r_{33}^2 \ddot{\alpha_3} + m_4 l_{43}^2 \ddot{\alpha_3} \\
&- m_4 l_{43} \dot{\alpha_1}^2 l_{23} \sin(\alpha_1 - \alpha_3) + m_4 l_{43} \ddot{\alpha_1} l_{23} \cos(\alpha_1 - \alpha_3) - g m_3 \sin(\alpha_3) r_{33} \\
&- g m_4 \sin(\alpha_3) l_{43} + \ddot{\alpha_3} I32 + m_3 r_{33} \ddot{\alpha_1} l_{23} \cos(\alpha_1 - \alpha_3) + m_4 l_{43} \ddot{\alpha_4} r_{43} \cos(\alpha_3 - \alpha_4) \\
&- m_3 r_{33} \dot{\alpha_1}^2 l_{23} \sin(\alpha_1 - \alpha_3) - m_4 l_{43} \dot{\alpha_2}^2 l_{33} \sin(\alpha_2 - \alpha_3) \\
&+ m_4 l_{43} \ddot{\alpha_2} l_{33} \cos(\alpha_2 - \alpha_3) + m_3 r_{33} \ddot{\alpha_2} l_{33} \cos(\alpha_2 - \alpha_3) + m_4 l_{43} \dot{\alpha_4}^2 r_{43} \sin(\alpha_3 - \alpha_4) = 0
\end{aligned} \tag{185}
$$

$$
\begin{aligned}
&- M_4 - m_4 r_{43} \dot{\alpha_1}^2 l_{23} \sin(\alpha_1 - \alpha_4) + \ddot{\alpha_4} I_{42} \\
&+ m_4 r_{43}^2 \ddot{\alpha_4} + m_4 r_{43} \ddot{\alpha_1} l_{23} \cos(\alpha_1 - \alpha_4) - m_4 r_{43} \dot{\alpha_2}^2 l_{33} \sin(\alpha_2 - \alpha_4) \\
&- g m_4 \sin(\alpha_4) r_{43} + m_4 r_{43} \ddot{\alpha_2} l_{33} \cos(\alpha_2 - \alpha_4) - m_4 r_{43} \dot{\alpha_3}^2 l_{43} \sin(\alpha_3 - \alpha_4) \\
&+ m_4 r_{43} \ddot{\alpha_3} l_{43} \cos(\alpha_3 - \alpha_4) = 0
\end{aligned} \tag{186}
$$

$$
\begin{aligned}
&- m_6 l_{63} \dot{\alpha_2}^2 l_{33} \sin(\alpha_2 - \alpha_5) - M_5 + M_6 + \ddot{\alpha_5} I_{52} + m_6 l_{63} \ddot{\alpha_6} r_{63} \cos(\alpha_5 - \alpha_6) \\
&- m_5 r_{53} \dot{\alpha_1}^2 l_{23} \sin(\alpha_1 - \alpha_5) + m_6 l_{63} \ddot{\alpha_2} l_{33} \cos(\alpha_2 - \alpha_5) + m_5 r_{53}^2 \ddot{\alpha_5} \\
&+ m_6 l_{63}^2 \ddot{\alpha_5} - m_5 r_{53} \dot{\alpha_2}^2 l_{33} \sin(\alpha_2 - \alpha_5) - m_6 l_{63} \dot{\alpha_1}^2 l_{23} \sin(\alpha_1 - \alpha_5) \\
&+ m_5 r_{53} \ddot{\alpha_1} l_{23} \cos(\alpha_1 - \alpha_5) - g m_5 \sin(\alpha_5) r_{53} - g m_6 \sin(\alpha_5) l_{63} \\
&+ m_5 r_{53} \ddot{\alpha_2} l_{33} \cos(\alpha_2 - \alpha_5) + m_6 l_{63} \dot{\alpha_6}^2 r_{63} \sin(\alpha_5 - \alpha_6) + m_6 l_{63} \ddot{\alpha_1} l_{23} \cos(\alpha_1 - \alpha_5) = 0
\end{aligned} \tag{187}
$$

$$
\begin{aligned}
&- M_6 - m_6 r_{63} \dot{\alpha_5}^2 l_{63} \sin(\alpha_5 - \alpha_6) - m_6 r_{63} \dot{\alpha_1}^2 l_{23} \sin(\alpha_1 - \alpha_6) - g m_6 \sin(\alpha_6) r_{63} \\
&+ m_6 r_{63}^2 \ddot{\alpha_6} + \ddot{\alpha_6} I62 + m_6 r_{63} \ddot{\alpha_1} l_{23} \cos(\alpha_1 - \alpha_6) + m_6 r_{63} \ddot{\alpha_5} l_{63} \cos(\alpha_5 - \alpha_6) \\
&+ m_6 r_{63} \ddot{\alpha_2} l_{33} \cos(\alpha_2 - \alpha_6) - m_6 r_{63} \dot{\alpha_2}^2 l_{33} \sin(\alpha_2 - \alpha_6) = 0
\end{aligned} \tag{188}
$$

Finally, by substituting the relationship of relative angle and absolute angle, and apply the approximations, the system matrix will be

$$
\begin{cases}
\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0_{6\times6} & I_{6\times6} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0_{6\times6} \\ B_2 \end{bmatrix} u \\
y = \begin{bmatrix} I_{6\times6} & 0_{6\times6} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}
\end{cases} \tag{189}
$$

where

$$A_{21} = 10^3 \times 5 \begin{bmatrix} 0.0546 & -0.9140 & 0.0228 & -0.0003 & 0 & 0 \\ -0.641 & 1.8252 & -0.1362 & 0.0017 & 0 & 0 \\ 0.0101 & -0.9667 & 0.3413 & -0.0172 & 0 & 0 \\ -0.0006 & 0.0575 & -0.2360 & 0.0520 & 0 & 0 \\ 0.0238 & -0.0323 & 0.2512 & -0.0157 & 0.0232 & -0.0286 \\ -0.0291 & -0.0291 & -0.0291 & 0 & -0.0291 & 0.0914 \end{bmatrix} \tag{190}$$

$$A_{22} = \begin{bmatrix} -0.1529 & -0.1529 & -0.1529 & 0 & -0.1529 & -0.0450 \\ 0.9124 & 0.9124 & 0.9124 & 0 & 0.9124 & 0.2684 \\ -1.8188 & -1.8188 & -1.8188 & 0 & -1.8188 & -0.5349 \\ 1.0968 & 1.0968 & 1.0968 & 0 & 1.0968 & 0.3226 \\ 1.3077 & 0.1756 & -1.0593 & 0 & -1.0593 & -0.3116 \\ -2.9638 & -1.5463 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{191}$$

$$B_2 = \begin{bmatrix} 20.1259 & -39.2563 & 20.2962 & -1.2071 & -1.1659 & 0 \\ -39.2563 & 79.3072 & -47.0063 & 7.2013 & 6.9554 & 0 \\ 20.2962 & -47.0063 & 44.7942 & -20.5280 & -18.0841 & 0 \\ -1.2071 & 7.2013 & -20.5280 & 21.9475 & 14.5338 & 0 \\ 1.1659 & -6.9554 & 18.0841 & -14.5338 & -29.7005 & -5.6563 \\ 0 & 0 & 0 & 0 & 40.4680 & 33.2261 \end{bmatrix} \tag{192}$$

6 Impact model

The overall model for the biped robot walking is a hybrid model; the model for the single support phase described above associated with the impact model. An impact occurs when the swing leg contacts the ground. The impact is modelled as a contact between two rigid bodies. There are many rigid impact models in the literature, and all of them can be used to obtain an expression for the generalized velocity just after the impact of the swing leg with the walking surface in terms of the generalized velocity and position just before the impact. In this work, the model from [4] is used. The model assumes the following hypotheses:

1. an impact results from the contact of the swing leg end with the ground
2. the impact is instantaneous
3. the impact results in no rebound and no slipping of the swing leg;
4. in the case of walking, at the moment of impact, the stance leg lifts from the ground without interaction, 7 while in the case of running, at the moment of impact, the former stance leg is not in contact with the ground;

5. the externally applied forces during the impact can be represented by impulses;
6. the actuators cannot generate impulses and hence can be ignored during impact; and
7. the impulsive forces may result in an instantaneous change in the robot's velocities, but there is no instantaneous change in the configuration

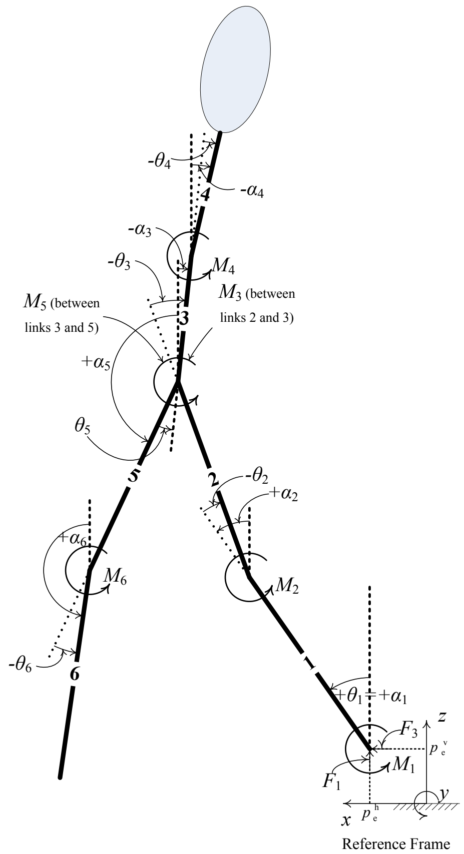


Figure 4: Sign conventions for the generalized coordinates and torques/forces used in the unpinned, 8 DOF model. α_i and θ_i are the absolute and relative angle of the i'th joint respectively. The fixed point on the body of the robot is chosen to be positioned at the support ankle

The development of the impact model involves the reaction forces at the leg ends, and thus requires the unpinned or $(N+2)$-DOF model of the robot (Figure 4). Let q_s be the generalized coordinates used in the single support model and complete these to a set of generalized coordinates for the unpinned model by letting $p_\mathrm{e} = (p_\mathrm{e}^\mathrm{h}; p_\mathrm{e}^\mathrm{v})$ be the Cartesian coordinates of some fixed point on the robot or its center of mass. With the generalized coordinates $q_\mathrm{e} = (q_\mathrm{s}; p_\mathrm{e})$, the set of equations of motion can be reached as before using Kane's or Lagrange's equations in the following form

$$\mathbf{A}_{\mathrm{e}}(q_{\mathrm{e}})\ddot{\mathbf{q}}_{\mathrm{e}} + \mathbf{B}_{\mathrm{e}}(q_{\mathrm{e}}, \dot{q}_{\mathrm{e}})\dot{\mathbf{q}}_{\mathrm{e}} + \mathbf{G}_{\mathrm{e}}(\dot{q}_{\mathrm{e}}) + \mathbf{C}_{\mathrm{e}}(q_{\mathrm{e}}) = \mathbf{D}_{\mathrm{e}} + \delta\mathbf{F}_{\mathrm{ext}} \tag{193}$$

where δF_{ext} represents the vector of external forces acting on the robot due to the contact between the swing leg end and the ground. From hypothesis 5, these forces are impulsive, hence the notation δF_{ext}. Applying the hypotheses and 'conservation of momentum', the impact model can be obtained as

$$\begin{bmatrix} \dot{q}_{\mathrm{e}}^{+} \\ F_2 \end{bmatrix} = \begin{bmatrix} \bar{\Delta}_{\dot{q}_{\mathrm{e}}}(q_{\mathrm{s}}^{-}) \\ \Delta_{F_2}(q_{\mathrm{s}}^{-}) \end{bmatrix} \dot{q}_{\mathrm{s}}^{-} \tag{194}$$

where the superscript $+/-$ implies just after/before the impact,

$$\Delta_{F_2} = -(E_2 \mathbf{A}_{\mathrm{e}}^{-1} E_2')^{-1} E_2 \tag{195}$$

and

$$\bar{\Delta}_{\dot{q}_{\mathrm{e}}} = \mathbf{A}_{\mathrm{e}}^{-1} E_2' \Delta_{F_2} + \begin{bmatrix} I_{N\times N} \\ \frac{\partial}{\partial q_{\mathrm{s}}} \Upsilon_{\mathrm{e}} \end{bmatrix} \tag{196}$$

Υ_{e} is such that during the single support phase, p_{e}, the Cartesian coordinate added to the robot's body, can be determined from q_{s}; denote this by $p_{\mathrm{e}} = \Upsilon_{\mathrm{e}}(q_{\mathrm{s}})$. In this work, the fixed point on the robot's body was chosen to be at the ankle joint of the support leg, rendering $\frac{\partial}{\partial q_{\mathrm{s}}}\Upsilon_{\mathrm{e}}$ a 2 by 6 null matrix in Equations (195) and (196).

Further, $E_2(q_{\mathrm{e}}) = \frac{\partial}{\partial q_{\mathrm{e}}} p_2(q_{\mathrm{e}})$, $p_2(q_{\mathrm{e}})$ denoting the position of the end of the swing leg with respect to the inertial frame. $F_2 = (F_2^{\mathrm{T}}; F_2^{\mathrm{N}})$ is the vector of forces acting at the end of the swing leg.

The first N rows of (194) thus constitute the new set of generalized velocities following the impact and hence be used to reinitialize the biped model (Figure 5).

Besides $\frac{\partial}{\partial q_{\mathrm{s}}}\Upsilon_{\mathrm{e}}$, evaluating Equation (194) requires evaluating $\mathbf{A}_{\mathrm{e}}^{-1}$ and E_2. For the considered model, these have been found to be as follows:

$$\begin{aligned} \mathbf{A}_{\mathrm{e}}(1,1) = \; & -m_4 l_{23}^2 - m_3 l_{23}^2 - m_1 r_{13}^2 - m_5 l_{23}^2 - m_2 l_{23}^2 - m_6 l_{23}^2 - I_{12} \\ & + m_6 l_{23} r_{63} \cos(\theta_2 + \theta_3 - \theta_5 - \theta_6) + m_5 l_{23} r_{53} \cos(\theta_2 + \theta_3 - \theta_5) \\ & - m_3 l_{23} r_{33} \cos(\theta_3 + \theta_2) + m_6 l_{23} l_{53} \cos(\theta_2 + \theta_3 - \theta_5) - m_6 l_{23} l_{33} \cos(\theta_2) \\ & - m_4 l_{23} l_{43} \cos(\theta_3 + \theta_2) - m_4 l_{23} l_{33} \cos(\theta_2) - m_5 l_{23} l_{33} \cos(\theta_2) \\ & - m_2 l_{23} r_{23} \cos(\theta_2) - m_4 l_{23} r_{43} \cos(\theta_4 + \theta_3 + \theta_2) - m_3 l_{23} l_{33} \cos(\theta_2) \end{aligned} \tag{197}$$

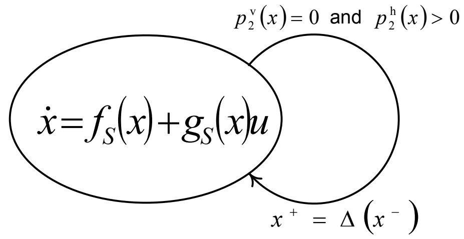


Figure 5: Hybrid model of walking. The continuous dynamics of the single support phase, written in state-space form as $\dot{x} = f_{\mathrm{s}}(x) + g_s(x)u$, the switching or impact condition, $p_2^{\mathrm{v}}(q) = 0$, $p_2^{\mathrm{h}}(q) > 0$, which detects when the height of the swing leg above the walking surface is zero and the swing leg is in front of the stance leg, and the reinitialization rule coming from the impact map, Δ

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(2,1) = \; & m_6 l_{33} l_{53}\cos(\theta_5-\theta_3) - m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3) \\ & + m_5 l_{33} r_{53}\cos(\theta_5-\theta_3) - m_6 l_{23} l_{33}\cos(\theta_2) - m_4 l_{23} l_{33}\cos(\theta_2) \\ & - m_5 l_{23} l_{33}\cos(\theta_2) - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) - m_2 l_{23} r_{23}\cos(\theta_2) \\ & - m_3 l_{23} l_{33}\cos(\theta_2) + m_6 l_{33} r_{63}\cos(\theta_5-\theta_3+\theta_6) - m_2 r_{23}^2 - m_6 l_{33}^2 \\ & - m_3 l_{33}^2 - m_4 l_{33}^2 - m_5 l_{33}^2 - I_{22}\end{aligned} \tag{198}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(3,1) = \; & -m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3) - m_3 l_{23} r_{33}\cos(\theta_3+\theta_2) \\ & - m_4 l_{23} l_{43}\cos(\theta_3+\theta_2) - m_3 r_{33}^2 - m_4 l_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) - I_{32}\end{aligned} \tag{199}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(4,1) = \; & -m_4 l_{23} r_{43}\cos(\theta_4+\theta_3+\theta_2) - m_4 r_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) - I_{42} \\ & - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3)\end{aligned} \tag{200}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(5,1) = \; & m_6 l_{33} l_{53}\cos(\theta_5-\theta_3) + m_5 l_{23} r_{53}\cos(\theta_2+\theta_3-\theta_5) \\ & + m_5 l_{33} r_{53}\cos(\theta_5-\theta_3) + m_6 l_{23} l_{53}\cos(\theta_2+\theta_3-\theta_5) - m_6 l_{53}^2 \\ & - m_6 l_{53} r_{63}\cos(\theta_6) - m_5 r_{53}^2 - I_{52}\end{aligned} \tag{201}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(6,1) = \; & m_6 l_{23} r_{63}\cos(\theta_2+\theta_3-\theta_5-\theta_6) + m_6 l_{33} r_{63}\cos(\theta_5-\theta_3+\theta_6) \\ & - m_6 l_{53} r_{63}\cos(\theta_6) - m_6 r_{63}^2 - I_{62}\end{aligned} \tag{202}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(1,2) = \; & -l_{23}(m_5 l_{33}\cos(\theta_2) + m_2 r_{23}\cos(\theta_2) + m_3 l_{33}\cos(\theta_2) + m_6 l_{33}\cos(\theta_2) \\ & + m_4 l_{33}\cos(\theta_2) + m_4 l_{43}\cos(\theta_3+\theta_2) - m_6 r_{63}\cos(\theta_2+\theta_3-\theta_5-\theta_6) \\ & + m_3 r_{33}\cos(\theta_3+\theta_2) - m_5 r_{53}\cos(\theta_2+\theta_3-\theta_5) \\ & + m_4 r_{43}\cos(\theta_4+\theta_3+\theta_2) - m_6 l_{53}\cos(\theta_2+\theta_3-\theta_5))\end{aligned} \tag{203}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(2,2) = \quad & m_5 l_{33} r_{53}\cos(\theta_5-\theta_3) + m_6 l_{33} l_{53}\cos(\theta_5-\theta_3) - m_4 l_{33} l_{43}\cos(\theta_3) \\ & -m_3 l_{33} r_{33}\cos(\theta_3) + m_6 l_{33} r_{63}\cos(\theta_5-\theta_3+\theta_6) \\ & -m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) - m_4 l_{33}^2 - m_3 l_{33}^2 - m_5 l_{33}^2 - m_6 l_{33}^2 - m_2 r_{23}^2 \\ & -I_{22}\end{aligned} \tag{204}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(3,2) = \quad & -m_4 l_{43}^2 - m_3 r_{33}^2 - m_3 l_{33} r_{33}\cos(\theta_3) - m_4 l_{43} r_{43}\cos(\theta_4) \\ & -m_4 l_{33} l_{43}\cos(\theta_3) - I_{32}\end{aligned} \tag{205}$$

$$\mathbf{A}_{\mathrm{e}}(4,2) = -m_4 r_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) - I_{42} - m_4 l_{33} r_{43}\cos(\theta_4+\theta_3) \tag{206}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(5,2) = \quad & -m_6 l_{53}^2 - m_5 r_{53}^2 - m_6 l_{53} r_{63}\cos(\theta_6) - I_{52} + m_5 l_{33} r_{53}\cos(\theta_5-\theta_3) \\ & +m_6 l_{33} l_{53}\cos(\theta_5-\theta_3)\end{aligned} \tag{207}$$

$$\mathbf{A}_{\mathrm{e}}(6,2) = m_6 l_{33} r_{63}\cos(\theta_5-\theta_3+\theta_6) - m_6 l_{53} r_{63}\cos(\theta_6) - I_{62} - m_6 r_{63}^2 \tag{208}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(1,3) = \quad & l_{23}(m_6 l_{53}\cos(\theta_2+\theta_3-\theta_5) + m_6 r_{63}\cos(\theta_2+\theta_3-\theta_5-\theta_6) \\ & -m_4 l_{43}\cos(\theta_3+\theta_2) - m_3 r_{33}\cos(\theta_3+\theta_2) + m_5 r_{53}\cos(\theta_2+\theta_3-\theta_5) \\ & -m_4 r_{43}\cos(\theta_4+\theta_3+\theta_2))\end{aligned} \tag{209}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(2,3) = \quad & l_{33}(-m_3 r_{33}\cos(\theta_3) - m_4 l_{43}\cos(\theta_3) + m_6 r_{63}\cos(\theta_5-\theta_3+\theta_6) \\ & -m_4 r_{43}\cos(\theta_4+\theta_3) + m_5 r_{53}\cos(\theta_5-\theta_3) + m_6 l_{53}\cos(\theta_5-\theta_3))\end{aligned} \tag{210}$$

$$\mathbf{A}_{\mathrm{e}}(3,3) = -I_{32} - m_3 r_{33}^2 - m_4 l_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) \tag{211}$$

$$\mathbf{A}_{\mathrm{e}}(4,3) = -m_4 r_{43}^2 - m_4 l_{43} r_{43}\cos(\theta_4) - I_{42} \tag{212}$$

$$\mathbf{A}_{\mathrm{e}}(5,3) = -I_{52} - m_6 l_{53} r_{63}\cos(\theta_6) - m_5 r_{53}^2 - m_6 l_{53}^2 \tag{213}$$

$$\mathbf{A}_{\mathrm{e}}(6,3) = -m_6 l_{53} r_{63} \cos(\theta_6) - I_{62} - m_6 r_{63}^2 \tag{214}$$

$$\mathbf{A}_{\mathrm{e}}(1,4) = -m_4 l_{23} r_{43} \cos(\theta_4 + \theta_3 + \theta_2) \tag{215}$$

$$\mathbf{A}_{\mathrm{e}}(2,4) = -m_4 l_{33} r_{43} \cos(\theta_4 + \theta_3) \tag{216}$$

$$\mathbf{A}_{\mathrm{e}}(3,4) = -m_4 l_{43} r_{43} \cos(\theta_4) \tag{217}$$

$$\mathbf{A}_{\mathrm{e}}(4,4) = -m_4 r_{43}^2 - I_{42} \tag{218}$$

$$\mathbf{A}_{\mathrm{e}}(5,4) = 0 \tag{219}$$

$$\mathbf{A}_{\mathrm{e}}(6,4) = 0 \tag{220}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(1,5) = \; & -l_{23}(m_6 l_{53} \cos(\theta_2 + \theta_3 - \theta_5) + m_6 r_{63} \cos(\theta_2 + \theta_3 - \theta_5 - \theta_6) \\ & + m_5 r_{53} \cos(\theta_2 + \theta_3 - \theta_5))\end{aligned} \tag{221}$$

$$\begin{aligned}\mathbf{A}_{\mathrm{e}}(2,5) = \; & -l_{33}(m_6 r_{63} \cos(\theta_5 - \theta_3 + \theta_6) + m_5 r_{53} \cos(\theta_5 - \theta_3) \\ & + m_6 l_{53} \cos(\theta_5 - \theta_3))\end{aligned} \tag{222}$$

$$\mathbf{A}_{\mathrm{e}}(3,5) = 0 \tag{223}$$

$$\mathbf{A}_{\mathrm{e}}(4,5) = 0 \tag{224}$$

$$\mathbf{A}_{\mathrm{e}}(5,5) = I_{52} + m_5 r_{53}^2 + m_6 l_{53}^2 + m_6 l_{53} r_{63} \cos(\theta_6) \tag{225}$$

$$\mathbf{A}_{\mathrm{e}}(6,5) = m_6 l_{53} r_{63} \cos(\theta_6) + I_{62} + m_6 r_{63}^2 \tag{226}$$

$$\mathbf{A}_{\mathrm{e}}(1,6) = -m_6 l_{23} r_{63} \cos(\theta_2 + \theta_3 - \theta_5 - \theta_6) \tag{227}$$

$$\mathbf{A}_{\mathrm{e}}(2,6) = -m_6 l_{33} r_{63} \cos(\theta_5 - \theta_3 + \theta_6) \tag{228}$$

$$\mathbf{A}_{\mathrm{e}}(3,6) = 0 \tag{229}$$

$$\mathbf{A}_{\mathrm{e}}(4,6) = 0 \tag{230}$$

$$\mathbf{A}_{\mathrm{e}}(5,6) = m_6 l_{53} r_{63} \cos(\theta_6) \tag{231}$$

$$\mathbf{A}_{\mathrm{e}}(6,6) = I_{62} + m_6 r_{63}^2 \tag{232}$$

and

$$E_2' = \begin{bmatrix} -l_{23}\sin(\theta_1) & l_{23}\cos(\theta_1) \\ -l_{33}\sin(\theta_2+\theta_1) & l_{33}\cos(\theta_2+\theta_1) \\ 0 & 0 \\ 0 & 0 \\ l_{53}\sin(\theta_1+\theta_2+\theta_3-\theta_5) & -l_{53}\cos(\theta_1+\theta_2+\theta_3-\theta_5) \\ l_{63}\sin(\theta_1+\theta_2+\theta_3-\theta_5-\theta_6) & -l_{63}\cos(\theta_1+\theta_2+\theta_3-\theta_5-\theta_6) \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{233}$$

7 iCub Biped Walking Pattern and Its Control System

7.1 Introduction

In this section, the control system for iCub biped locomotion is addressed. The basic idea of the controller is based on Medrano-Cerda & Eldukhri (1997).

For iCub biped walking, independent MIMO control systems are used for each plane (sagittal and lateral plane). Each control system includes position and velocity feedback, integral action (included to help overcome some stiction in the gearbox) and feedforward terms. These controllers ensure the stability of the biped robot for tracking any given trajectories. Any torque disturbances and measurement errors should be sufficiently rejected, this will make the control system quite robust. Besides the two MIMO controllers, there are two SISO controllers needed. These are responsible for keeping the swing foot parallel to ground (zero absolute angles) in both planes, they can be achieved by utilising simple position feedback.

In this section, the design of the control system for sagittal plane is fully addressed, the design procedure is exactly the same for lateral plane which is neglected here.

The structure of the controller is shown in Figure 6

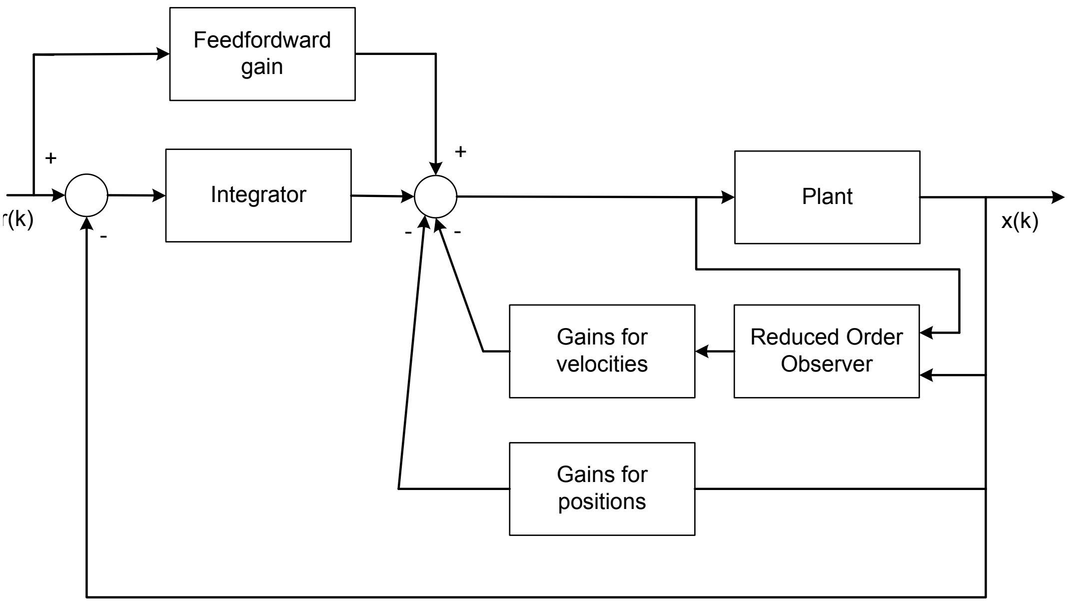


Figure 6: Controller structure for iCub biped walking

The rest of this chapter is organised as follows. The design of the reduced order observer will be addressed in section 1.2, then in section 1.3, the design of integrator and calculation of three gains (feedfordward gains, gains for velocities, gains for positions) will be given. Following the walking pattern generation in section 1.4, the simulation results will be shown in section 1.5. Finally, the chapter is finished by the conclusion in section 1.6.

7.2 Design of Reduced Order Observer

Only relative angles of the iCub joints are measured, in order to estimate the corresponding relative angular velocities, the design of reduced order observer is necessary. In the design of the observer, torque disturbances and simple disturbances are neglected. This negligence in the observer design produces errors in the estimated angular velocities. However, despite of state estimation errors, the performance of an observer-based controller can exhibit a good degree of robustness with respect to plant parameter variations. Here, obtaining a robust observer-based controller is more focused instead of the design of an observer with minimum estimation errors.

The linear discrete time model of the plant is

$$\begin{cases} x_s(k+1) = \Phi_s x_s(k) + \Gamma_s u_s(k) \\ y_s(k) = C_s x_s(k) \end{cases} \tag{234}$$

with sampling time $T_s = 1ms$, and Φ_s, Γ_s, C_s are matrices with appropriate dimensions.

The structure of the observer is given below for the sagittal plane (similar as lateral plane)

$$\begin{cases} z_s(k+1) = F_s z_s(k) + E_s x_{s1}(k) + H_s u_s(k) \\ \hat{x}_{s2}(k) = z_s(k) + K_s x_{s1}(k) \end{cases} \tag{235}$$

Here z_s is the state vector of the observer and $\hat{x}_{s2}$ is the vector of estimated relative angular velocities in the sagittal plane.

The observer state matrix F_s and remaining observer parameters are computed from the following relations

$$\Phi_s = \begin{bmatrix} \Phi_{s11} & \Phi_{s12} \\ \Phi_{s21} & \Phi_{s22} \end{bmatrix} \quad \Gamma_s = \begin{bmatrix} \Gamma_{s1} \\ \Gamma_{s2} \end{bmatrix}$$

$$K_s = (\Phi_{s22} - F_s)\Phi_{s12}^{-1}$$

$$H_s = \Gamma_{s2} - K_s \Gamma_{s1}$$

$$E_s = (\Phi_{s21} - K_s \Phi_{s11}) + F_s K_s$$

$F_s = 0.9 I_{66}$ (where I_{66} is a 6 by 6 identity matrix)

The selection of these observer parameters is a compromise between the speed of response of the observer and a reduction of relative stability.

7.3 Design of Control System

Linear Quadratic Regulator (LQR) technique is used to calculate the optimal state feedback gains. The plant model of both planes are controllable, therefore, the closed loop poles of the system can be placed any location using state feedback gains.

The structure of the control system is

$$u_s(k) = -L_s^2 x_{as}(k) - L_s^{11} x_{s1}(k) - L_s^{12} \hat{x}_{s2}(k) + L_s^{ff} r_s(k) \tag{236}$$

where L_s^{11} is the gain associated with the relative angles x_{s1}, L_s^{12} is for the relative angular velocities $\hat{x}_{s2}(k)$ and L_s^2 is for the integral action, x_{as} is the state vector for the integrators,

r_s is the vector of filtered reference signals. The feedforward gain L_s^{ff} is set equal to L_s^{11}.

$x_s(k) = \begin{bmatrix} x_{s1} \\ x_{s2} \end{bmatrix}$, x_{s1} is the first 6 and x_{s2} is the remaining 6 elements of the state vector associate with relative positions and velocities respectively.

The state space representation of the integral action is

$$\begin{cases} x_{as}(k+1) = \Phi_{as} x_{as}(k) + \Gamma_{as}(k) \\ u_{as}(k) = r_s(k) - x_{s1}(k) \end{cases} \tag{237}$$

Here x_{as} is the state vector for the integrators and r_s is the vector of reference signals, which is set to zero in the LQR design. Φ_{as} and Γ_{as} are 6×6 identity matrices, x_{as}, r_s and x_{s1} are 6×1 vectors. The design of the optimal state feedback matrix starts with the specification of a quadratic performance index J and the constraint equation. The objective is to minimise J around the nominal operating point. The performance index is in the form

$$J_s = \sum_{k=0}^{\infty} \left(x_{ds}(k)^T Q_{ds} x_{ds}(k) + u_s(k)^T R_{ds} u_s(k) \right) \tag{238}$$

where Q_{ds} and R_{ds} are chosen as diagonal matrices with positive entries. The constraint equation is (ignoring torque disturbances)

$$x_{ds}(k+1) = \Phi_{ds} x_{ds}(k) + \Gamma_{ds} u_s(k) \tag{239}$$

where $x_{ds} = \begin{bmatrix} x_s \\ x_{as} \end{bmatrix}$, $\Phi_{ds} = \begin{bmatrix} \Phi_s & 0_{12\times12} \\ -\Gamma_{as} C_s & \Gamma_{as} \end{bmatrix}$ and $\Gamma_{ds} = \begin{bmatrix} \Gamma_s \\ 0_{6\times6} \end{bmatrix}$.

The selection of Q_{ds} and R_{ds} was investigated in MATLAB simulations and during experiments. The aims were to achieve fast response with little or no overshoot and to maintain the control signal within the power supplies limitation. The chosen values for Q_{ds} and R_{ds} are

$$\begin{aligned} Q_{ds} &= diag(\begin{bmatrix} Q_{sp} & Q_{sv} & Q_{su} \end{bmatrix}) \\ Q_{sp} &= diag(\begin{bmatrix} 10^6 & 10^6 & 10^4 & 10^6 & 10^4 & 10^6 \end{bmatrix}) \\ Q_{sv} &= 10^{-4} I_{6\times6} \\ Q_{su} &= 10^{-1} I_{6\times6} \\ R_{ds} &= I_{6\times6} \end{aligned}$$

Here Q_{sp} is the matrix penalising angular positions, Q_{sv} is for the velocities and Q_{su} is for the integral actions.

In selection of Q_{sp}, it is wanted as low as possible to prevent demands for large control actions. Low Q_{sp} values increase relative stability and reduce sensitivity to noise. However, to track reference signals with small errors and quickly attenuate torque disturbances, relatively high values of Q_{sp} are needed. Hence, the final value of Q_{sp} is a compromise between relative stability margins and the good tracking of reference signals. It is suggested that due to the fact that most of the oscillations originated in the hip joints, it is better to reduce the gains further in the hip joints by lowering the corresponding values for hip in Q_{sp} in order to dampen any oscillations.

In order to minimise the control efforts, Q_{sv} is set to a low value. In the sagittal plane, gains for integral actions are kept to minimum. High integral gains tend to cause oscillations in the system, mainly due to the presence of backlash.

The value of state feedback gains calculated are shown in the Appendix.

7.4 Generation of Locomotion Trajectories

This section presents the generation of locomotion trajectories for static walking. The locomotion trajectories are obtained by solving kinematic equations that define the location of the centre of gravity and the relative distance between the feet.

For the robot to be in static balance, in the single leg support phase, the projection of the centre of gravity (COG) must be inside the support foot.

During double and single support phases, the swing foot is kept parallel to the ground at all times. In this way, the front and the rear of the swing foot is at the same height from the ground. This requirement is achieved by simply setting the absolute angles of the swing ankle joints to zero in both planes. Therefore, only set points for six joints in the sagittal plane and for four joints in the lateral plane are needed.

In this report, the set points given in Table 1 are used in the simulation. Those set points are gathered via experimental work.

Table 1: Sagittal plane relative joint angles

Set point numbers	Support ankle	Support knee	Support hip	Trunk	Swing hip	Swing knee
1	-0.0980	-0.0081	0.0905	-0.0020	-0.1138	-0.0530
2	-0.0734	-0.0029	0.0907	-0.0022	-0.1162	-0.0520
3	-0.0386	-0.0138	0.0902	-0.0022	-0.1162	-0.0541
4	-0.0038	-0.0247	0.0898	-0.0027	-0.1207	-0.0563
5	-0.0025	-0.0165	0.0599	-0.0018	-0.0805	-0.0375
6	-0.0013	-0.0082	0.0299	-0.0009	-0.0402	-0.0188
7	0	0	0	0	0	0
8	0.0117	-0.0041	-0.0415	0.0003	0.0389	-0.0156
9	0.0234	-0.0081	-0.0830	0.0006	0.0779	-0.0311
10	0.0351	-0.0122	-0.1245	0.0009	0.1168	-0.0467
11	0.0561	-0.0006	-0.1255	0.0014	0.1221	-0.0488
12	0.0908	-0.0109	-0.1275	0.0009	0.1184	-0.0551
13	0.1263	-0.0224	-0.1295	0.0005	0.1141	-0.0617

The center of gravity (COG) equation is calculated as mass i $(i = 1, 2, ..., 6)$ times its position vector form R. COG equation in R is

$$\begin{aligned} COG_s = &(m_1 r_{13} \sin\alpha_1 + m_2(l_{23}\sin\alpha_1 + r_{23}\sin\alpha_2) \\ &+ m_3(l_{23}\sin\alpha_1 + l_{33}\sin\alpha_2 + r_{33}\sin\alpha_3) \\ &+ m_4(l_{23}\sin\alpha_1 + l_{33}\sin\alpha_2 + l_{42}\sin\alpha_3 + l_{43}\sin\alpha_3 + r_{43}\sin\alpha_4) \\ &+ m_5(l_{23}\sin\alpha_1 + l_{33}\sin\alpha_2 + l_{53}\sin\alpha_3 - r_{53}\sin\alpha_5) \\ &+ m_6(l_{23}\sin\alpha_1 + l_{33}\sin\alpha_2 + l_{53}\sin\alpha_3 - l_{63}\sin\alpha_5 + r_{63}\sin\alpha_6)) \\ &/(m_1 + m_2 + m_3 + m_4 + m_5 + m_6) \end{aligned}$$

Therefore, the COG can be computed via equation

$$COG_s = \begin{bmatrix} 0.4484 & 0.4 & -0.0639 & 0.0092 & 0.1707 & 0.0797 \end{bmatrix} x_s \tag{240}$$

where $x_s = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 \end{bmatrix}^T$

Each set point is completed in unequal time periods (given in Table 2)

Table 2: Time taken to complete the transition between set points (total time = $12.57s$)

Set point Number	1-2	2-3	3-4	4-5	5-6	6-7
Times(s)	1.19	1.38	1.29	0.84	1.25	1.27
Set point Number	7-8	8-9	9-10	10-11	11-12	12-13
Time(s)	1.28	1.27	1.26	70.18	0.71	0.65

7.5 Simulation results

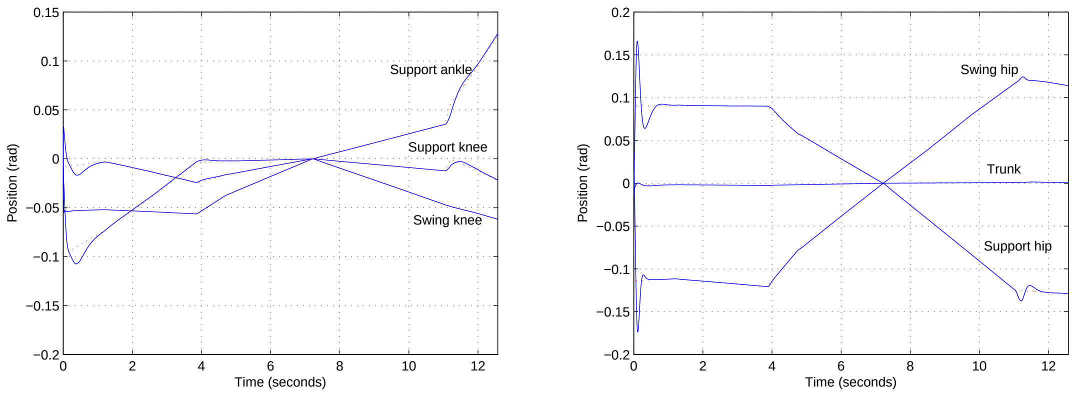


Figure 7: Tracking Performance of the iCub robot in sagittal plane

Figure 7 shows the tracking performance of the iCub robot. Red dotted lines are the reference signals and the blue solid lines represent actual joint orientations. The simulation results indicate that reference signals are tracked quite closely.

7.6 Conclusions

In this chapter, the design of the control systems for iCub biped walking is presented. The purpose of the controller is to stabilise the body of the robot while tracking the reference joint angles. The reference joint angles and COG are generated for static walking. From the simulation results, the designed control system demonstrates proper tracking performance. Due to the fact of the static walking trajectories, the walking is in a very low speed.

8 Second Order Design Method for iCub Bipedal Locomotion

8.1 Introduction

The control scheme described in section 7 is very complicated. In this section, a novel second order design method is developed. Compared to the previous algorithm, this design method is much more systematic, and it is simple, easy to understand and implement.

The second order design method addressed here can be separated into two stages. In Stage 1, a Proportional and Derivative (PD) compensator is chosen to let the plant possess desired dynamics. After that, in stage 2, by adding outer loop controllers, better trajectory tracking and stability properties can be achieved.

8.2 Design stage 1

Consider a simple negative feedback control scheme shown in figure 8, where r is the desired walking gait, and y is the actual iCub walking gait.

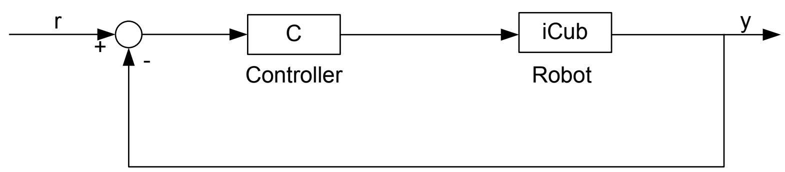


Figure 8: Simple feedback control structure for iCub bipedal locomotion

The constructed linearised iCub dynamic model in section 3 can be expressed as a second order system, i.e.,

$$G^{-1} = A_0 s^2 + A_1 s + A_2 \tag{241}$$

Therefore, according to the control structure shown in figure 8, the closed loop transfer function H_{cl} will be

$$\begin{aligned} H_{cl} &= (I + GC)^{-1} GC \\ &= \left(G^{-1} + C\right)^{-1} C \end{aligned} \tag{242}$$

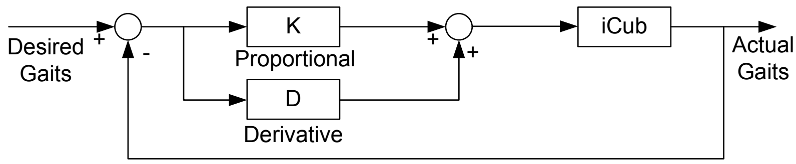
Now, assume that controller C is a simple PD controller with transfer function $C = K_p + K_d s$, substitute into equation (242) together with equation (241), hence,

$$\begin{aligned} H_{cl} &= \left(A_0 s^2 + A_1 s + A_2 + K_p + K_d s\right)^{-1} C \\ &= \left(s^2 + sA_0^{-1}(A_1 + K_d) + A_0^{-1}(A_2 + K_p)\right)^{-1} A_0^{-1} C \end{aligned} \tag{243}$$

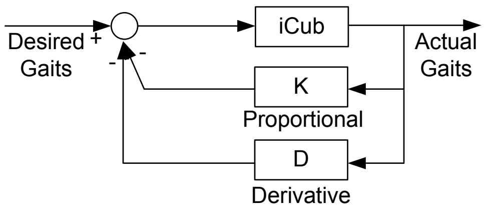
It is easy to see that the characteristic equation of this closed loop system has the same form as the characteristic equation of a standard second order system, $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. Finally, with specified damping ratio ζ and natural frequency ω_n, the parameter of PD controller K_p and K_d can be determined by,

$$\begin{aligned} K_p &= \omega_n^2 A_0 - A_2 \\ K_d &= 2\zeta\omega_n A_0 - A_1 \end{aligned} \tag{244}$$

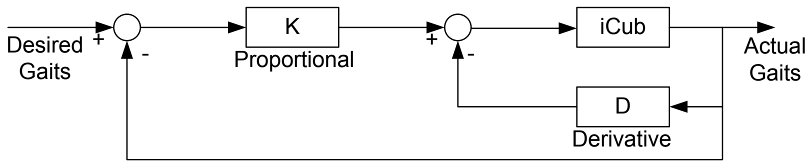
Take a further investigation of the location of the PD controller, there are three possible places for the PD controller as shown in figure 9



(a) PD controller in forward path



(b) PD controller in feedback path



(c) Inner loop PD controller

Figure 9: Three different locations for PD controller

The closed loop transfer function for each structure shown above will be

For structure shown in figure 9a

$$H_{cl}1 = \left(s^2 I + sA_0^{-1}(A_1 + Kd) + A_0^{-1}(A_2 + K_p)\right)^{-1} A_0^{-1}(K_p + K_d s) \tag{245}$$

For structure shown in figure 9b

$$H_{cl}2 = \left(s^2 I + sA_0^{-1}(A_1 + Kd) + A_0^{-1}(A_2 + K_p)\right)^{-1} A_0^{-1} \tag{246}$$

For structure shown in figure 9c

$$H_{cl}3 = \left(s^2 I + sA_0^{-1}(A_1 + Kd) + A_0^{-1}(A_2 + K_p)\right)^{-1} A_0^{-1} K_p \tag{247}$$

Comparing these three different structures and their transfer functions, although all of them have the same dynamics, it is clear that the 9b and 9c are better than 9a. Moreover, 9c has more advantages than 9b.

First of all, the proportional gain in the forward path will give the system fast response to the error signal, and the derivative term in the minor feedback loop will reduce the overshoot. Secondly, if substitute $K_p = \omega_n^2 A_0 - A_2$, the numerator of equation (247) will become $\omega_n^2 - A_0^{-1} A_2$, if the square of the natural frequency $\omega_n^2 >> A_0^{-1} A_2$, the effect of the interaction of the entire closed loop system can be ignored. Also, the designer can decouple the entire closed loop system by adding another compensator $K_p^{-1} A_0 \omega_n^2$.

It should be notice that without the integral term, constant steady state error in inevitable. In order to eliminate the steady state error, an additional controller is necessary.

8.3 Design stage 2

The two major functions of the controller for iCub robot biped locomotion are:

1. Tracking the desired trajectories as accurate as possible;

2. Stabilised the entire system, avoiding iCub tip over or falling down.

The walking progress can be estimated as a periodic process, then the trajectory for different joint can be seen as a periodic signal. Therefore, Repetitive Control (RC) is a suitable control algorithm for this purpose.

In this design stage, the RC controller is added. RC controller is work as an integral term to eliminate the steady state error. The overall control structure is shown in figure 10

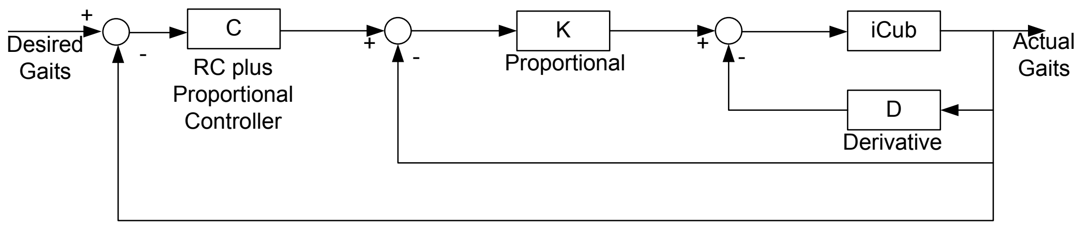


Figure 10: Novel Second Order Design iCub bipedal locomotion control structure

There are several design process and design structures for RC controller, only the joint based multivariable RC control algorithm is utilised here. For tracking N-periodic signal, the discretised control law is

$$u(k) = q^{-N}u(k) + q^{-N}\beta G(q^{-1})^{*}e(k) \tag{248}$$

where $G(q^{-1})^{*}$ is the adjoint system of $G(q)$, β is the learning gain. For more information about adjoint based RC algorithm, please refer to [2].

8.4 Static walking pattern simulation

The static walking pattern gathered from thesis [1] is very slow as shown in figure 11. The diagrams show the trajectory of each joint for one strike. Each strike is separated into 13 points. It can be seen that for each strike, it needs 12.57 seconds to complete and that is due to the consideration of the response time of the drive motors. Now assume that the drive motors are ideal, and reduce the interval of each point to 0.1 second, the new faster walking gaits can be generated, as shown in figure 12.

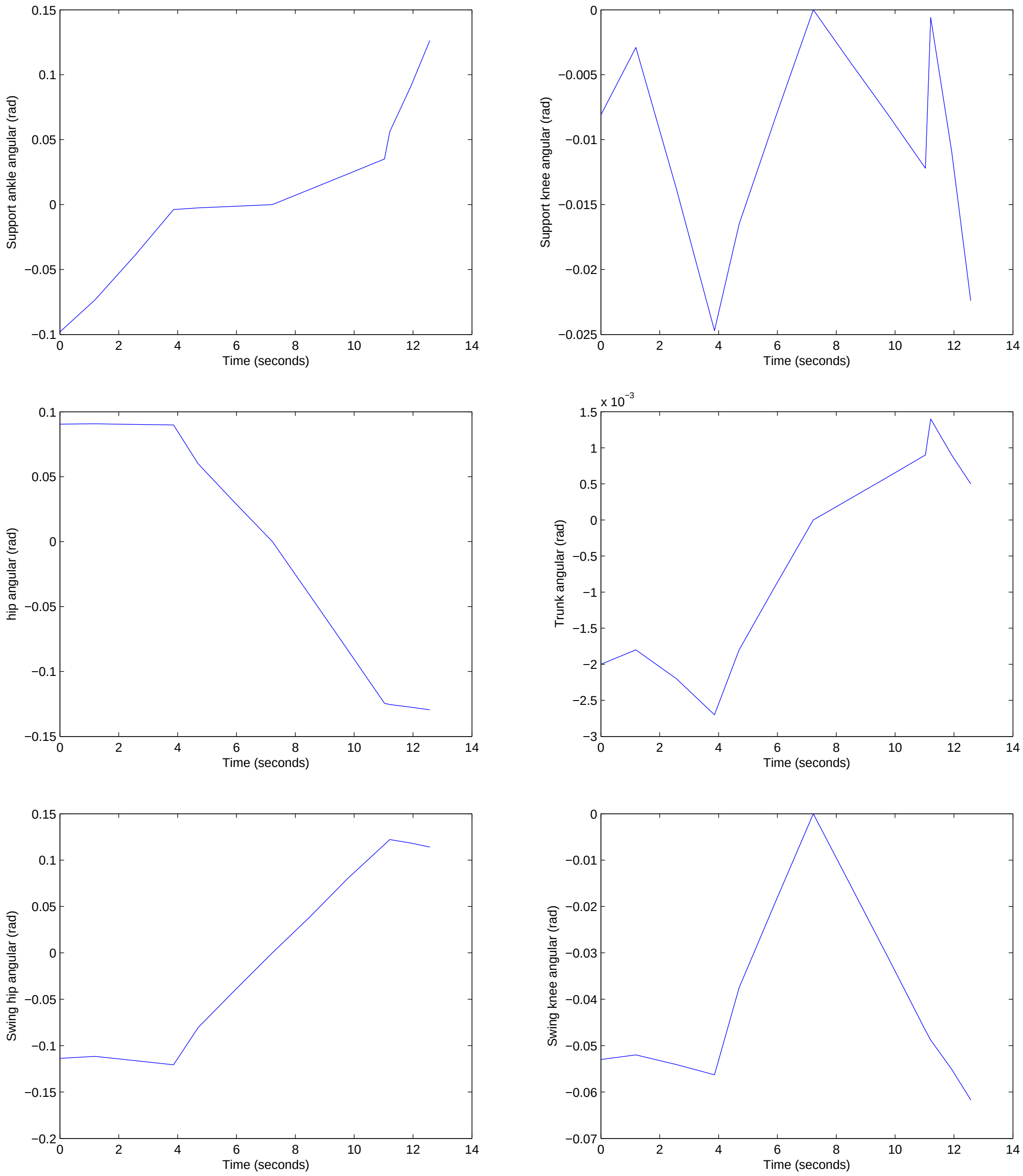


Figure 11: The relative joint angular of static walking

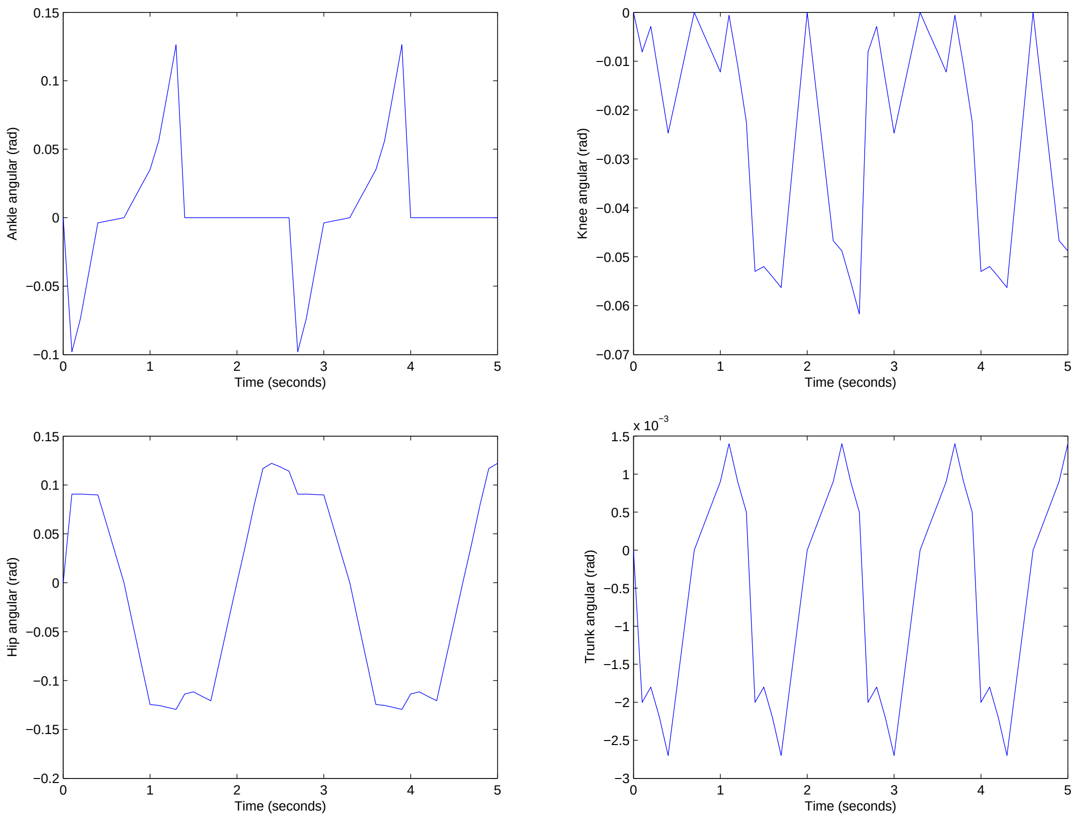


Figure 12: The relative joint angular of fast static walking

Using second order design method, choose $\omega_n = 50$ and $\zeta = 0.9$, the tracking errors are shown in figure 13 and the torques to the icub robot joints are shown in figure 14. Despite the steady state error, the tracking performance is acceptable. The diagram of torques shows a lot of harsh spikes, which are coherence with the spikes in the reference signal. That's because of the fact that the reference signal is not very smooth, at each set points, sudden change of acceleration is unavoidable which requires large torque to achieve that. That explains those harsh spikes in the diagram of torques. In order to eliminate those spikes, a smooth trajectories is needed.

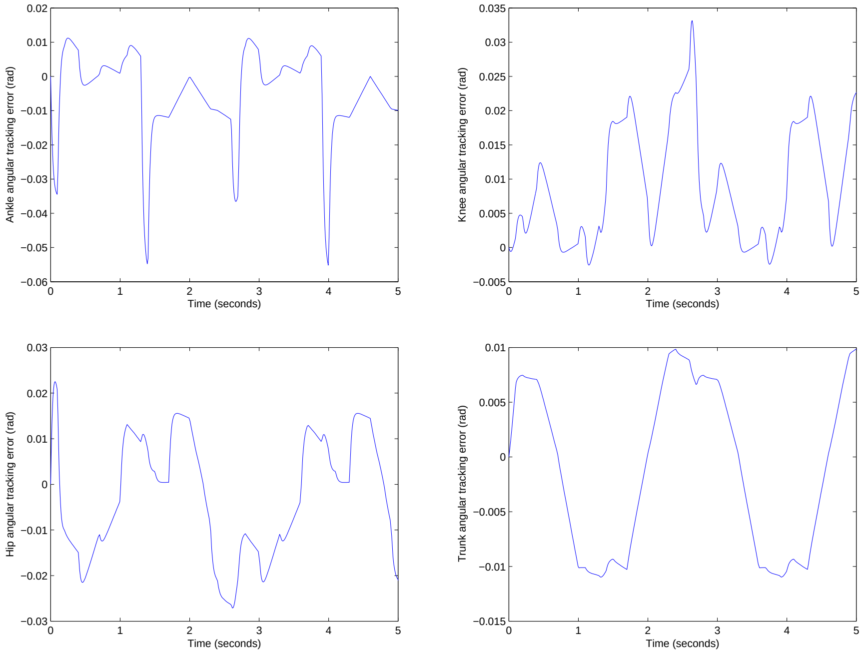


Figure 13: Tracking error of static walking pattern with PD controller

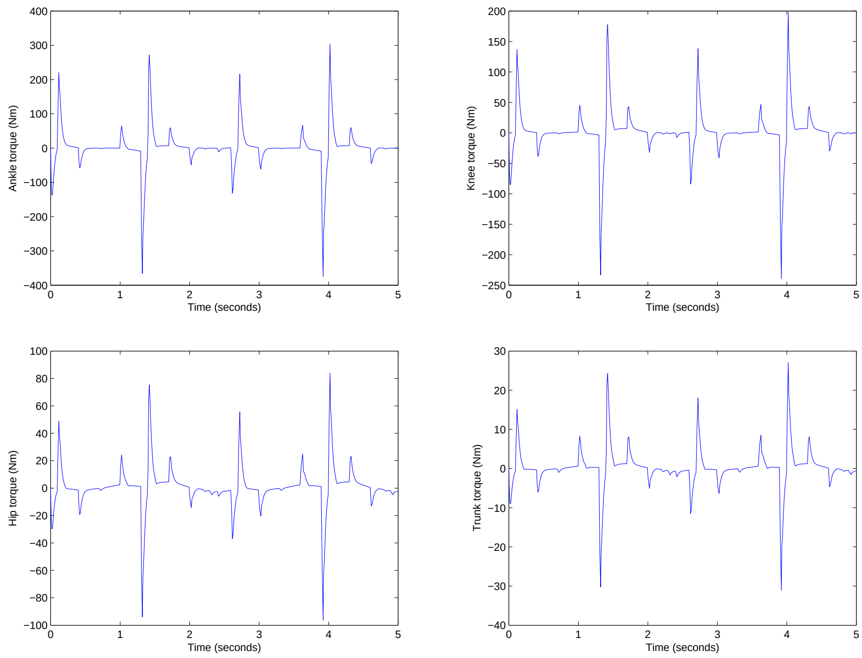


Figure 14: Torques of static walking pattern with PD controller

8.5 Human walking pattern simulation

A more realistic reference gaits adopted here is obtained from book [5]. Compared to the static state walking gaits used before, these gaits are much more smooth. Figure 15 shows the reference signals of different joints. One of the disadvantages is that the book only provides data for one and half period, so only one entire period data is gathered and duplicated for simulation, this may cause some problems. Another disadvantage is that the book only provides relative joint angles for ankle, knee and hip, but the simulation requires the angle of trunk. Refer to the static walking patter, the angle of trunk is much small compared with other joints, hence, it is reasonable to assume that the relative joint angle for trunk is always zero.

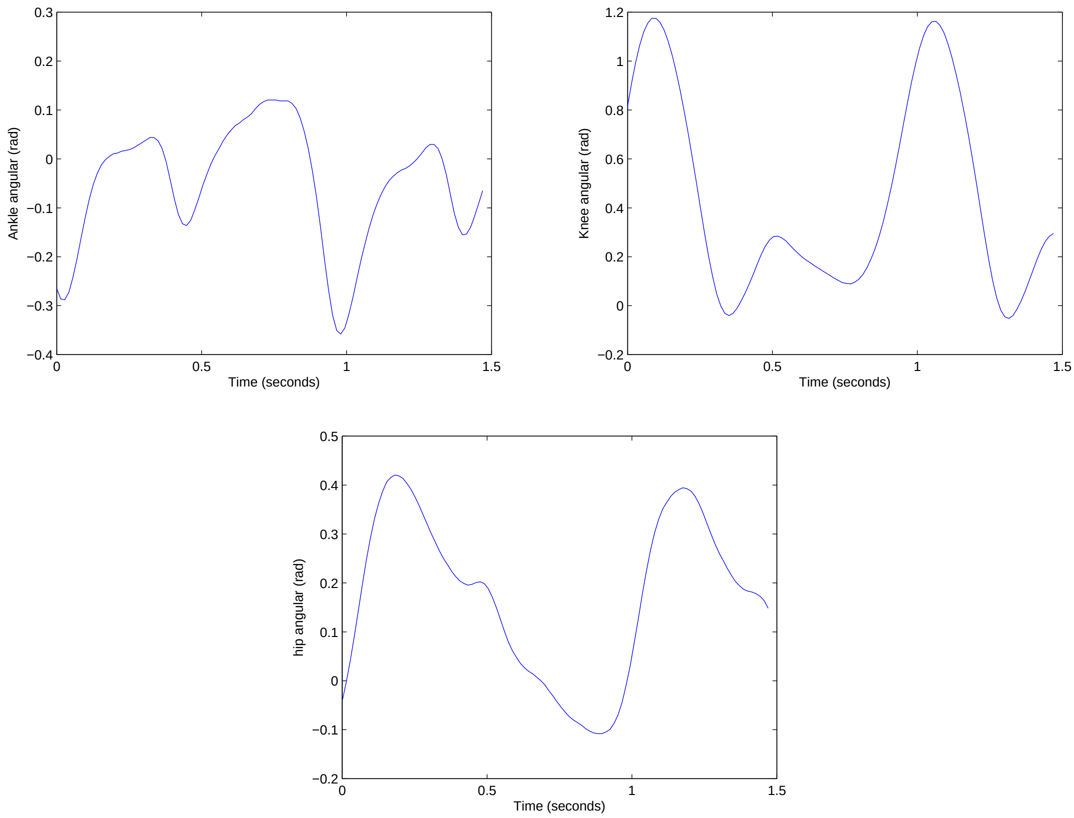


Figure 15: Human walking relative joint angular – Ankle, Knee, and Hip

The reference signals are filtered in order to eliminate the sudden changes between each point.

To determined the value of ω_n, the spectrum of the reference signals are shown in figure 16.

From those plots, it is clear to see that setting $\omega_n = 50 rad/s$ is adequate for the tracking performance. Furthermore, $\zeta = 0.9$ is also chosen.

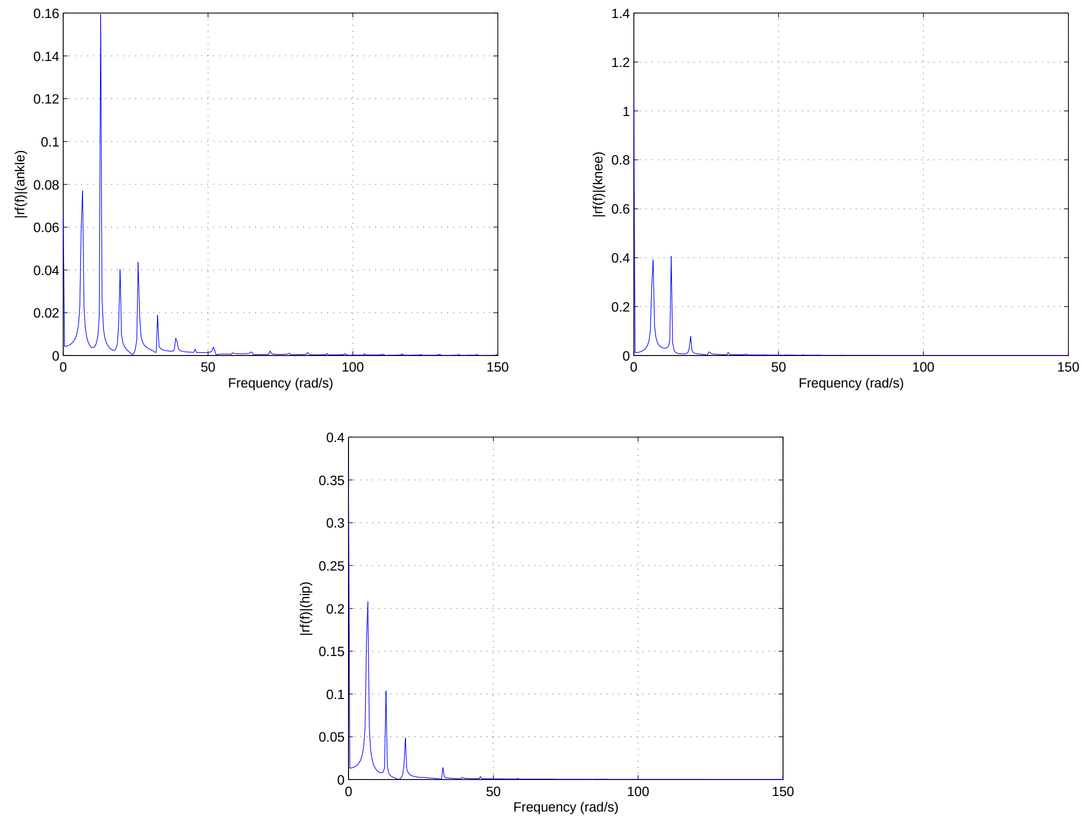


Figure 16: Spectrum of filtered reference signal – ankle, knee, and hip

The tracking performance and the torques of each joint with PD controller are shown in figure 17 and 18.

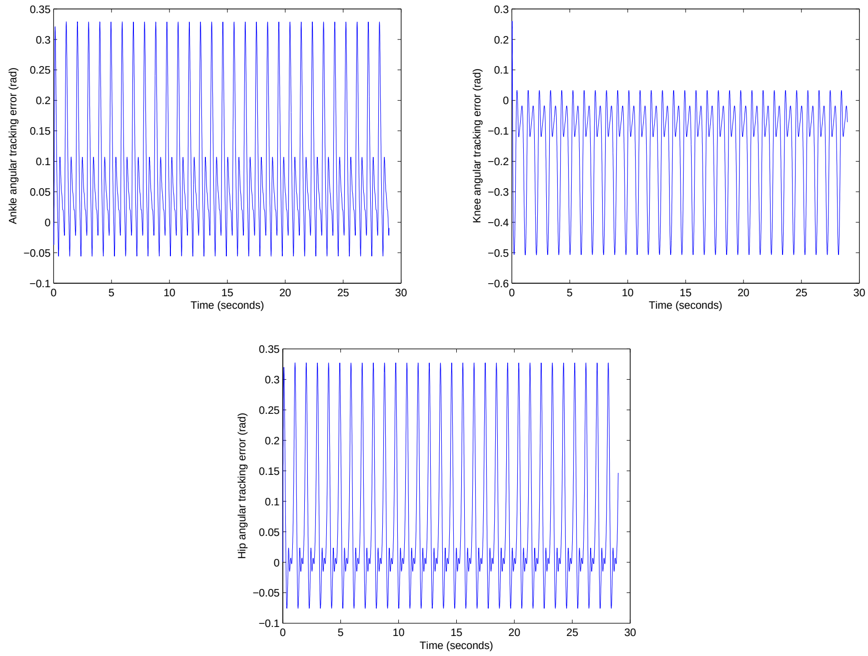


Figure 17: Tracking error of human walking pattern with PD controller

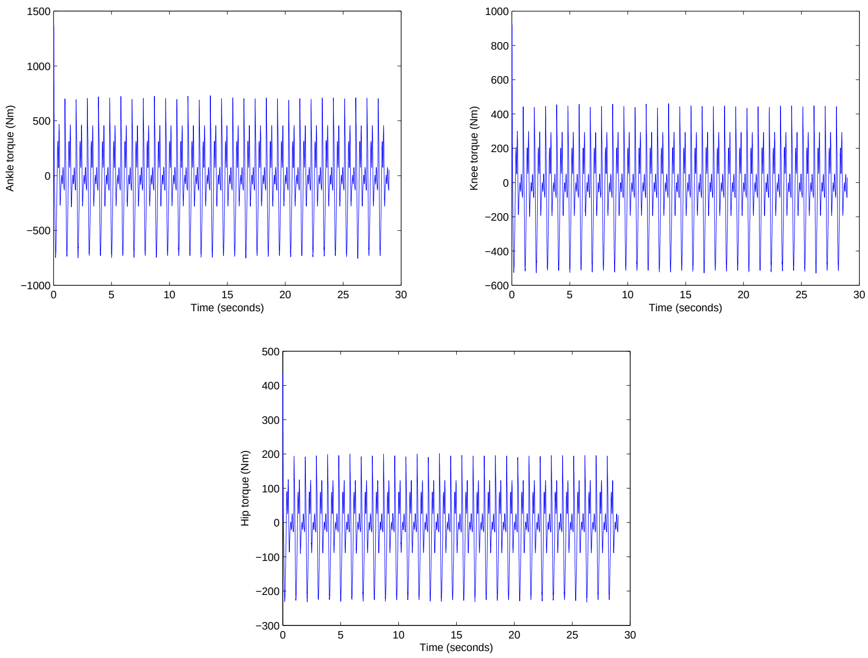


Figure 18: Torques of human walking pattern with PD controller

The tracking performance is good only with some phase shift. By shifting the phase of actual gaits, as shown in figure 19, it can be seen that PD controller performs substantial tracking. Therefore, the PD controller is adequate if phase shifted reference tracking is acceptable.

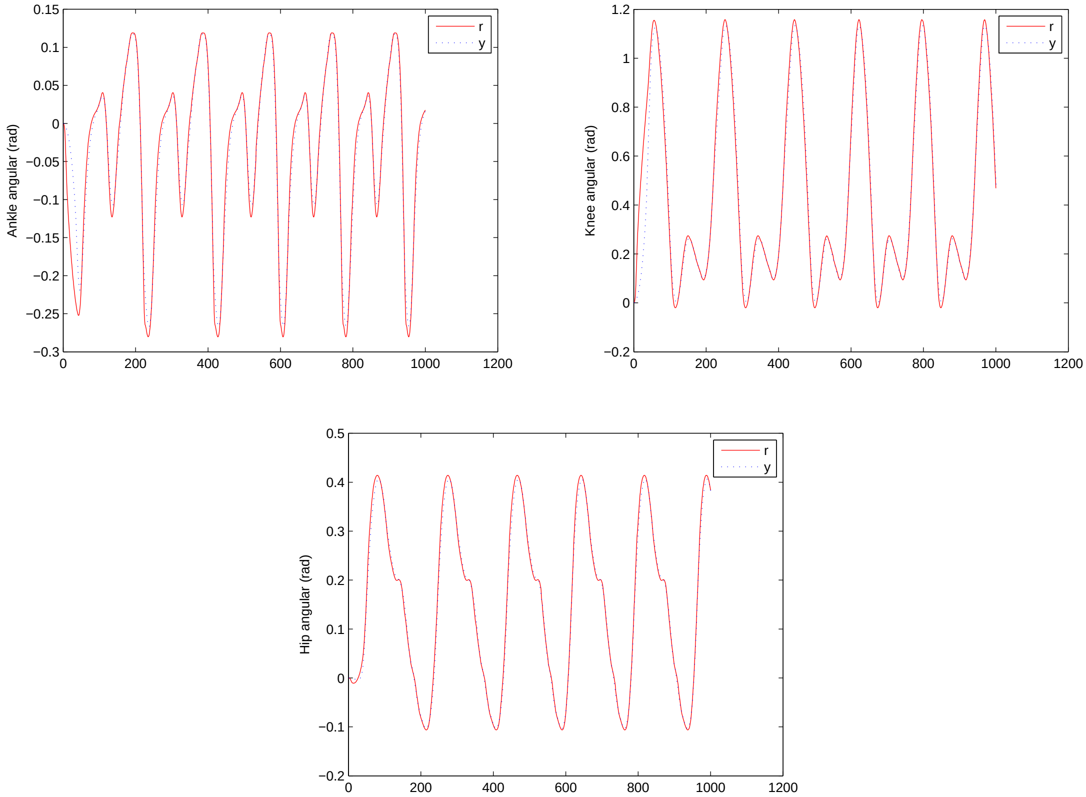


Figure 19: Phase shift of actual gait and reference

By applying RC control algorithm, with $\beta = 1$, the tracking performance and the torques of the joins are shown in figure 20 and 21.

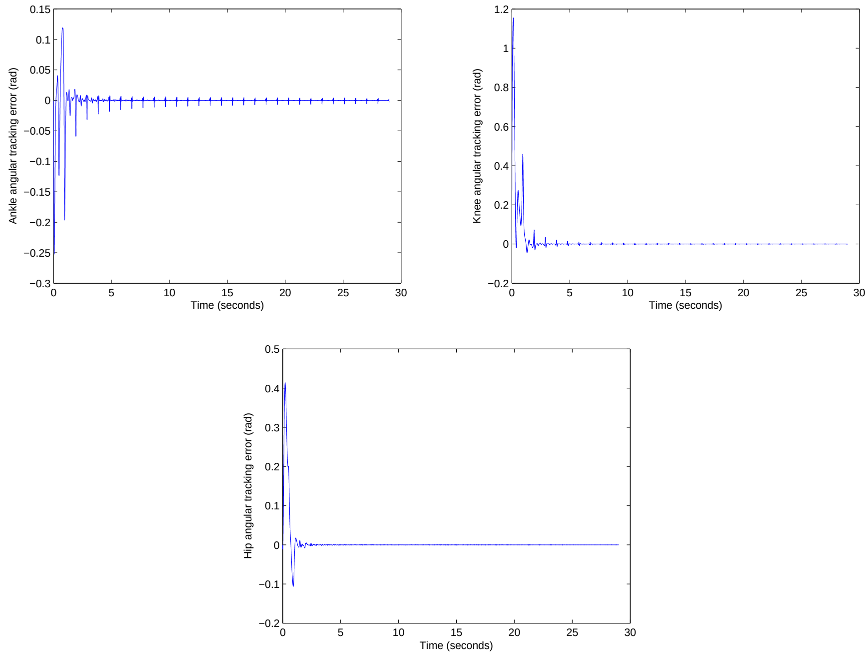


Figure 20: Tracking error of human walking patter with PD and RC controller

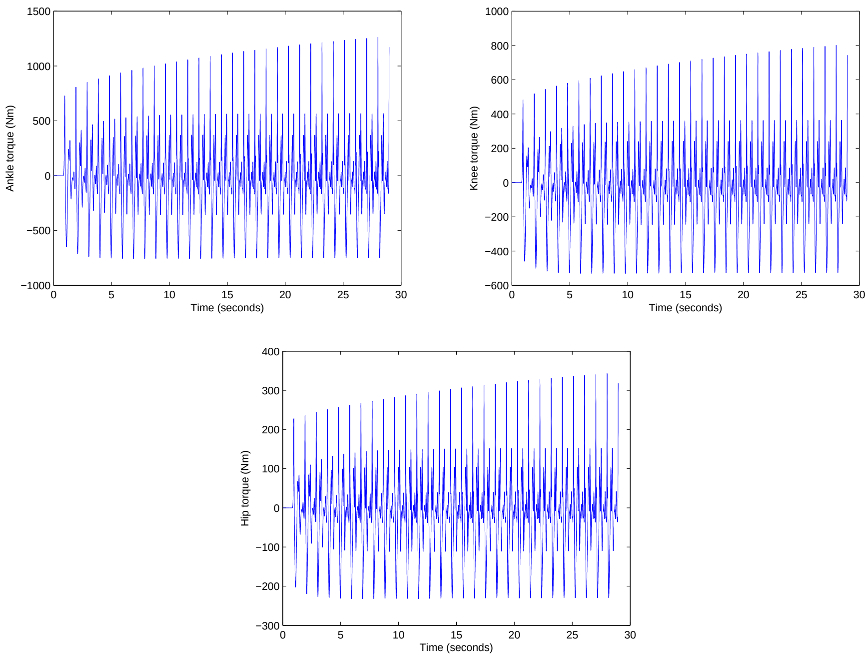


Figure 21: Torques of human walking pattern with PD and RC controller

The tracking performance of RC and PD controller are excellent. After one or two steps, the tracking error goes down to zero. The reason is twofold. First of all, the initial condition of the iCub robot for walking is unknown, hence, it may need some time to achieve that condition. Secondly, the RC controller needs gather required information to start work, and it starts working after one step.

Compared figure 21 with figure 18, the average value of torques are almost the same, for ankle, it is around 500Nm, for knee, it is about 400Nm, and for hip, it is 150Nm. Those high peaks of torques happened in RC control only occured between each period, that maybe because the connection of each period is not so smooth as stated before.

8.6 Conclusion and outstanding issues

From the simulation results shown above, it can be seen that the novel second order design method together with RC control can achieve great tracking performance. The only concerned is extremely high value of torques, one of the reasons is that the reference trajectory is not a continuous signal, it is just a single step data chopped from human walking data that repeats again and again, which means that the connections between each period is not smooth, despite the fact that in actual human walking, each step is different. By manually changing the value of ankle joints at the end of each period, Figure 22 shows the difference between original ankle angle and the modified ankle angle. It can be seen that the spike in the original ankle reference signal has been smooth. By utilizing the second order PD control algorithm shown above, the simulation results are shown in figure 23, compare to the original simulation results, it can be seen that there is a significant drop of value of input torques.

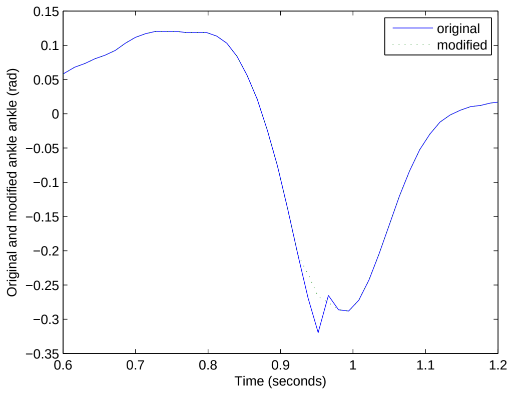


Figure 22: Original and modified ankle angle

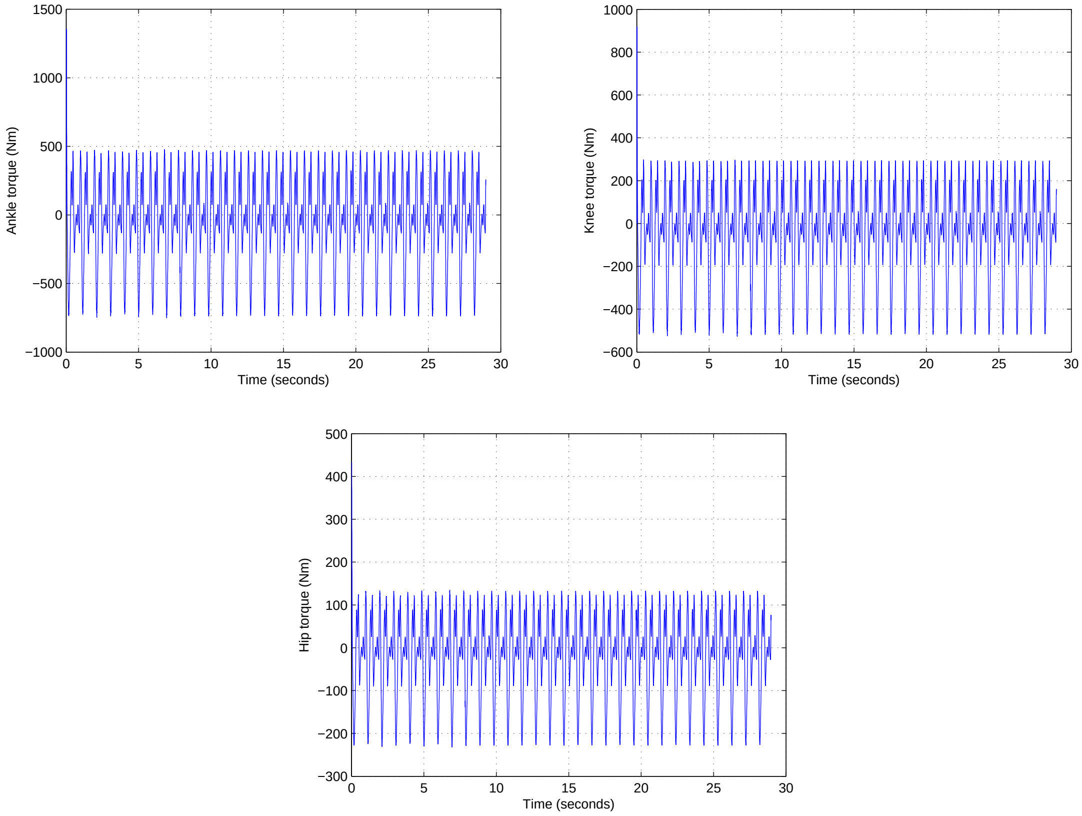


Figure 23: Simulation results with modified ankle angle

Therefore, a better reference signal that suitable for iCub robot is very essential in order to achieve a realistic input to the motor.

Appendices

A State feedback gains

$$L_s^{11} = 10^3 \times \begin{bmatrix} 1.4474 & 0.3582 & 0.1886 & 0.1645 & -0.0062 & -0.0159 \\ 0.2265 & 1.1652 & 0.2433 & -0.0340 & -0.0213 & -0.0003 \\ 0.1581 & -0.1815 & 0.3597 & -0.2706 & -0.0462 & 0.0740 \\ -0.1032 & -0.0158 & 0.1483 & 0.9566 & 0.0455 & -0.0854 \\ -0.0307 & 0.0045 & 0.0156 & 0.2085 & -0.1674 & 0.6909 \\ 0.0319 & 0.0209 & -0.0302 & -0.0622 & 0.1825 & 0.7225 \end{bmatrix}$$

$$L_s^{12} = \begin{bmatrix} 132.9243 & 81.3261 & 31.1248 & 9.7708 & -0.8608 & -0.3183 \\ 81.0360 & 58.8629 & 26.4513 & 8.2451 & -1.8652 & -0.1927 \\ 33.8263 & 27.1121 & 22.1690 & 7.9772 & -3.1641 & 0.1844 \\ 6.6599 & 7.6070 & 9.5856 & 14.8924 & 1.0589 & -0.5723 \\ -1.4662 & 0.1707 & 2.2126 & -0.1587 & -5.4733 & 2.7121 \\ 1.2664 & -0.5099 & -2.8514 & 0.6786 & 6.4154 & 6.3061 \end{bmatrix}$$

$$L_s^{2} = \begin{bmatrix} -0.3107 & -0.0246 & -0.0040 & -0.0421 & 0.0028 & 0.0050 \\ 0.0171 & -0.2965 & -0.0907 & 0.0196 & 0.0059 & 0.0003 \\ -0.0147 & 0.0878 & -0.2836 & 0.0947 & 0.0189 & -0.0228 \\ 0.0394 & 0.0132 & -0.1024 & -0.2861 & -0.0603 & 0.0264 \\ 0.0076 & -0.0020 & 0.0075 & -0.0661 & 0.2165 & -0.2153 \\ -0.0082 & -0.0064 & 0.0092 & 0.0203 & -0.2201 & -0.2219 \end{bmatrix}$$

B List of MATLAB programmes

Programmed by M Saiful Huq (For more detail about the programmes, please contact Saif)

Package name	Comments
Kanes Approach	Programme package for iCub model creation using Kane's equation
Kanes Approach Impact	Programme package for iCub Impact model creation

Programmed by F Zhang

Filename	Comments
Building iCub model	
icub_motor.mat	iCub motor model
icub_symbolic_sagittal_lagrange.m	Creating iCub model using Lagrange's equation
LQR control algorithm	
icub_sagittal_model_0609_torque.mat	iCub model with torque as input
StaticGait.mat	Static walking trajectories
general_sagittal_biped_toque.m	Control system design using LQR method
sagblock.mdl	simulation model for LQR controller
Second Order Design Mehtod	
irlss.mdl	Impulse response for RC controller design
RCgaitMRG.mat	Human walking trajectories
SecOrderMIMOpRCsm.m	m-file for Second Order Design method
SecOrderMIMOpRCs.mdl	Simulation model for second order PD plus RC

References

[1] D. Akdas. *Modelling, control and locomotion of a twelve degree of freedom biped robot.* PhD thesis, University of Salford, 2001.

[2] J. Hätönen, F. Zhang, S. Daley, and Owens D.H. Adjoint multivariable repetitive control algorithm for active control of vibration. In proceedings of Third IFAC Workshop on Periodic Control Systems, 2007.

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[4] E.R. Westervelt, J.W. Grizzle, C. Chevallereau, J.H. Choi, and B. Morris. *Feedback control of dynamic bipedal robot locomotion.* CRC Press, 2007.

[5] D.A. Winter. *Biomechanics and motor control of human movement.* Wiley, 2009.