Spatio-temporal Generalised Frequency Response Functions

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Abstract—The concept of generalised frequency response functions (GFRFs), which were developed for nonlinear system identification and analysis, is extended to continuous spatio-temporal dynamical systems normally described by partial differential equations (PDEs). The paper provides the definitions and interpretation of spatio-temporal generalised frequency response functions for linear and nonlinear spatio-temporal systems based on an impulse response procedure. A new probing method is also developed to calculate the GFRFs. Both the Diffusion equation and Fisher's equation are analysed to illustrate the new frequency domain methods.

Index Terms—Generalised frequency response, spatio-temporal systems, Volterra series representation

I. INTRODUCTION

Frequency response analysis is fundamental to and provides important insights into the analysis, stability, and performance characteristics of control, communication, acoustic, and vibration systems, particularly linear time-invariant systems (LTIs). Based on the Volterra series representation of nonlinear relationships, a nonlinear spectral analysis methodology has been developed to overcome the limitations of linear spectral analysis methods applied to nonlinear dynamical systems (Schetzen 1980, Billings and Tsang 1989a,b, Peyton-Jones and Billings 1989, Billings and Peyton-Jones 1990 and the references therein). This methodology characterises nonlinear systems based on the Fourier transforms of the Volterra kernels to produce frequency domain descriptors commonly referred to as generalised frequency response functions (GFRFs). The GFRFs of a nonlinear temporal system provide an intuitive representation of the frequency properties of the system and many nonlinear phenomena can be studied and explained using this framework. Moreover, the GFRFs provide invariant descriptions of the underlying system and are independent of the excitation. The methodology for analyzing and computing generalized frequency response functions for unknown nonlinear systems has also been developed based on a NARMAX description of the nonlinear systems (Billings and Tsang 1989a,b, Peyton-Jones and Billings 1989, Billings and Peyton-Jones 1990). These developments of nonlinear spectral analysis theory have overcome many of the limitations that are associated with a linear or linearised analysis of nonlinear systems. Nonlinear effects such as harmonics, inter-modulation, and energy transfer are just not possible in linear representations and hence linear methods can never fully unravel nonlinear dynamic effects.

Linear spectral analysis has been extended from an analysis of purely temporal dynamical systems to image processing techniques such as Fourier optics. Instead of dealing with temporal frequency effects, Fourier optics makes use of the spatial frequency domain (ξ, η) as the conjugate of the two-dimensional spatial (x,y) domain. The two dimensional point spread function and optical transfer function are the counterparts of the impulse response function and the frequency response function in temporal systems (Goodman 2005). All of these theories and applications show the importance of spectral analysis for both temporal and spatial systems, which motivates this investigation of the frequency domain analysis for spatio-temporal systems.

Spatio-temporal systems are a class of dynamic systems which evolve over both time and space and which are normally described by partial differential equations (PDEs) in the continuous case and coupled map lattices (CMLs) in the discrete case. These models are generally associated with initial and boundary conditions, and this together with the problem of seeking a solution is usually referred to as an initial-boundary value problem. Spatio-temporal systems are different from conventional dynamic systems in many ways. For example, spatio-temporal systems are non-causal with respect to the space variables and the state space is infinite dimensional. They are also different in the way the dynamics and evolution are affected by an external stimulus. For temporal systems there is generally a single input-channel which exerts an influence on the system dynamics, whilst there is a variety of ways that can affect spatio-temporal system dynamics including external control inputs, initial conditions, and boundary conditions. These have been reflected in the control strategies for spatio-temporal systems: distributed control, point-wise control, and

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boundary control. To investigate the system frequency response we should study the (frequency) response of the underlying spatio-temporal system with respect to the above mentioned control inputs. Input-output and frequency approaches for spatio-temporal systems have to date focused on linear and time-invariant spatio-temporal systems based on the derivations of the transfer functions. A number of different descriptions of transfer function models for spatio-temporal systems have been proposed including Curtain and Zwart (1995), Curtain and Morris (2009), Rabenstein and Trautmann (2002), Garcia-Sanz, Huarte, and Asenjo (2007), Billings and Wei (2007), Guo, Billings, Coca, Peng, and Lang (2009), among which there are two most influential models. The first relates to the one-dimensional case, where the control to the system is assumed to be carried out either through boundary or a type of distributed co-location, which means the control input is only dependent on the time variable. The output of the system is generally taken as a measure at a fixed spatial point or an integration of the system over its spatial domain. In this way, the transfer function between the input and output can be derived as follows: initially, Laplace transforming with respect to time t yields an (spatial) ordinary differential equation with s as parameter. By solving this boundary value problem with respect to the spatial variable produces the desired transfer function between input and output (Curtain and Morris 2009). The second important model is called a multi-dimensional transfer function model (Rabenstein and Trautmann 2002) for scalar and vector partial differential equations. Similar to the first method, initially a Laplace transform is applied with respect to time to remove the time derivatives and a Sturm-Liouville transform is applied for the space variable to yield a multidimensional transfer function which is the sum of the responses with respect to the control input variable, the initial conditions, and the boundary conditions. The frequency response of the system can then be evaluated using these transfer functions. It is well known that the transfer functions of purely temporal systems or lumped-parameter systems are rational functions whilst the transfer functions of spatio-temporal systems can be irrational. These frequency approaches reveal the characteristics of linear spatio-temporal systems and provide a basis for the control of this class of systems.

In this paper, the transfer functions and frequency response approaches are extended and developed to deal with nonlinear spatio-temporal systems. This is achieved by adopting a new approach where three types of generalised transfer functions (GTFs) and generalised frequency response functions (GFRFs) are defined based upon unit impulse responses of the system with respect to the external input, the initial conditions, and the boundary conditions. The linear impulse responses are equivalent to Green's functions (Trim 1990) for linear, translation-invariant systems while the nonlinear impulse responses for nonlinear spatio-temporal systems are described using a Volterra series representation. New probing based algorithms are derived to compute the generalised (or nonlinear) frequency response functions for a wide class of nonlinear spatio-temporal systems and several examples are used to illustrate the new methods. We start the investigation with an analysis of the impulse and frequency responses for linear, translation-invariant spatio-temporal systems in Section 2. The formal definitions of the GTFs and GFRFs for nonlinear spatio-temporal systems are then given in Section 3, together with a detailed analysis of these functions. An effective computation method for the calculation of these functions is also included and Section 4 illustrates the proposed methods using the example of diffusion equations and Fisher's equations. Conclusions are drawn in Section 5.

II. SPECTRAL ANALYSIS OF LINEAR TRANSLATION-INVARIANT SPATIO-TEMPORAL SYSTEMS

Traditionally, the transfer function of a spatio-temporal system is defined in a similar way to the transfer function of a temporal dynamical system. The transfer function describes the time response of a spatio-temporal system with respect to the excitation input. It has been shown (Curtain and Zwart 1995) that a linear, translation invariant spatio-temporal system described by the following evolution equation

$$\begin{aligned} \dot{y}(t) &= Ay(t) + Bu(t) \\ v(t) &= Cy(t) + Du(t) \end{aligned} \tag{1}$$

where U, Y, and V are Hilbert spaces and $A \in L(Y), B \in L(U,Y), C \in L(Y,V)$, and $D \in L(U,V)$ are bounded operators, has a transfer function $D + C(sI - A)^{-1}B$ for s with real part larger than the exponential growth bound of the semigroup generated by A if A generates a strongly continuous semigroup. However, the transfer function defined here cannot give a complete description for the system (1) because the relationship between the system evolution and the initial conditions/boundary conditions is not defined and explained. In this section, the transfer relationships between the system output and the excitation input, the initial conditions, and the boundary conditions will be discussed for linear, translation-invariant spatio-temporal systems via impulse responses of the systems.

A. Impulse and Frequency Response of Linear, Translation-invariant Spatio-temporal Systems

In this section, we will consider linear, translation-invariant spatio-temporal systems governed by the following first order evolution equation

$$\begin{aligned} y_t(x,t) + Ay(x,t) &= u(x,t), \\ I\big(y(x,0)\big) &= \varphi(x), x \in \Omega \\ B\big(y(x,t)\big) &= \psi(x,t), x \in \partial\Omega, t > 0 \end{aligned} \quad (2)$$

where x is the space coordinate variable defined on a bounded domain Ω with a boundary $\partial\Omega$ and t is the time variable. A is a bounded linear operator which can, for example, take the form of $Ay(x,t) = a_0 y(x,t) + a_1 y_x(x,t) + a_2 y_{xx}(x,t)$, where $y_t(x,t)$, $y_x(x,t)$, and $y_{xx}(x,t)$ represent the temporal derivative, first and second order spatial derivatives, respectively. I and B are the linear operators for defining the initial and boundary conditions. We assume that $y(x,t)$ and $u(x,t)$ denote the output and the external excitation of the system, respectively. For simplicity, in this initial study, we also restrict our discussion to one spatial dimension and scalar systems, which gives $\Omega \subset R$ as an interval on the real line and $y(x,t), u(x,t) \in R$. The discussion will be given for both cases of $\Omega = (-\infty, +\infty)$ and $\Omega = [a,b]$.

Systems evolving over the entire real line $\Omega = (-\infty, +\infty)$

When the spatial domain of the system (2) is the entire real line this indicates the problem (2) has open boundary conditions. There is no need to prescribe the boundary conditions in order to study the solution of (2). Due to the linearity, the problem (2) can be split into the following two subproblems

- Inhomogeneous equation with zero initial conditions

$$\begin{aligned} y_t(x,t) + Ay(x,t) &= u(x,t), \\ y(x,0) &= 0,\ x \in (-\infty,+\infty), t > 0 \end{aligned} \quad (3)$$

and

- homogeneous equation with nonzero initial conditions

$$\begin{aligned} y_t(x,t) + Ay(x,t) &= 0, \\ y(x,0) &= \varphi(x),\ \ x \in (-\infty,+\infty), t > 0 \end{aligned} \quad (4)$$

If $y_1(x,t)$ and $y_2(x,t)$ are the solutions to (3) and (4) respectively, then the sum of these two solutions $y(x,t) = y_1(x,t) + y_2(x,t)$ is the solution to (2).

In general, Green's function of the (initial) boundary problem is defined as the solution of the problem in response to a unit impulse input signal, that is, the Dirac delta function. It follows that Green's function $h^e(x,t;\xi,\tau)$ of the system with respect to the external excitation is the solution to the problem (3) with $u(x,t) = \delta(x-\xi, t-\tau)$

$$h_t^e(x,t;\xi,\tau) + Ah^e(x,t;\xi,\tau) = \delta(x-\xi,t-\tau),$$

$$h^e(x,0;\xi,\tau) = 0,\ x,\ \xi \in (-\infty,+\infty), t,\tau > 0 \quad (5)$$

Similarly, Green's function $h^i(x,t;\xi,\tau)$ of the system with respect to the initial conditions is the solution to the problem (4) with $y(x,0) = \varphi(x) = \delta(x-\xi)$

$$\begin{aligned} h_t^i(x,t;\xi) + Ah^i(x,t;\xi) &= 0, \\ h^i(x,0;\xi) &= \delta(x-\xi),\ x,\ \xi \in (-\infty,+\infty), t > 0 \end{aligned} \quad (6)$$

Note that considering the causality of the temporal system, it is a general requirement that $h(x,t;\xi,\tau) = 0$ for $t < \tau$. Following the definition of Green's function and the superposition of the solutions, the general solution to (2) with an external excitation $u(x,t)$ and inhomogeneous initial conditions $\varphi(x)$ can be obtained as

$$y(x,t) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(x,t;\xi,\tau)u(\xi,\tau)\,d\xi d\tau + \int_{-\infty}^{+\infty} h^i(x,t;\xi)\varphi(\xi)\,d\xi \quad (7)$$

In order to introduce the concepts of the unit impulse response and the frequency response functions, some extra conditions are required. These are defined below.

Assumption 1. It is assumed that Green's functions for the problem defined in (3) and (4) exist and are unique.

Assumption 2. The underlying spatio-temporal system is time and spatially translation invariant.

Under the assumptions 1 and 2, Green's functions have the following invariance property

$$h(x+\alpha, t+\beta; \xi+\alpha, \tau+\beta) = h(x,t;\xi,\tau) \quad (8)$$

under any translations (α,β) with respect to the coordinates (x,t). It follows that

$$h(x,t;\xi,\tau) = h(x-\xi, t-\tau) \quad (9)$$

So that (7) takes the following convolution form

$$\begin{aligned} y(x,t) &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(x-\xi,t-\tau)\,u(\xi,\tau)\,d\xi d\tau + \int_{-\infty}^{+\infty} h^i(x-\xi,t)\,\varphi(\xi)d\xi \\ &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(\xi,\tau)\,u(x-\xi,t-\tau)\,d\xi d\tau + \int_{-\infty}^{+\infty} h^i(\xi,t)\,\varphi(x-\xi)d\xi \end{aligned}$$

(10)

Consider the case where the input function is $u(x,t) = e^{\kappa x}e^{st}$ and the initial conditions are $y(x,0) = \varphi(x) = e^{\kappa x}$, the output is then given by

$$\begin{aligned}
y(x,t) &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(x-\xi,t-\tau)\,u(\xi,\tau)\,d\xi d\tau \\
&\qquad + \int_{-\infty}^{+\infty} h^i(x-\xi,t)\,\varphi(\xi)d\xi \\
&= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(x-\xi,t-\tau)\,e^{\kappa\xi}\,e^{s\tau}\,d\xi d\tau \\
&\qquad + \int_{-\infty}^{+\infty} h^i(x-\xi,t)\,e^{\kappa\xi}\,d\xi \\
&= \int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(\xi,\tau)\,e^{\kappa(x-\xi)}e^{s(t-\tau)}\,d\xi d\tau \\
&\qquad + \int_{-\infty}^{+\infty} h^i(\xi,t)\,e^{\kappa(x-\xi)}d\xi \\
&= e^{\kappa x}e^{st}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} h^e(\xi,\tau)\,e^{-\kappa\xi}\,e^{-s\tau}\,d\xi d\tau \\
&\qquad + e^{\kappa x}\int_{-\infty}^{+\infty} h^i(\xi,t)\,e^{-\kappa\xi}\,d\xi \\
&= e^{\kappa x}e^{st}H^e(\kappa,s) + e^{\kappa x}H^i(\kappa,t)
\end{aligned} \tag{11}$$

Clearly, $H^e(\kappa,s) = \int_{-\infty}^{+\infty}\int_0^{+\infty} h^e(\xi,\tau)\,e^{-\kappa\xi}\,e^{-s\tau}\,d\xi d\tau$ is the (two-sided) Laplace transform of the function $h^e(\xi,\tau)$, and $H^i(\kappa,t) = \int_{-\infty}^{+\infty} h^i(\xi,t)\,e^{-\kappa\xi}\,d\xi$ is the two-sided Laplace transform of the function $h^i(\xi,t)$ with respect to the variable x. In this paper, the functions $h^e(\xi,\tau)$ and $h^i(\xi,t)$ will be called the impulse response functions, and $H^e(\kappa,s)$ and $H^i(\kappa,t)$ will be called the transfer functions of the system with respect to the external excitation and initial conditions, respectively. The Fourier transform version of the transfer functions is called the frequency response function of the system (2).

Remark 1. Note that the impulse response function, the transfer function, and the frequency response function with respect to the initial conditions contain a time index t which indicates that they are defined at that time instant.

Remark 2. From (11), it can be observed that for linear, translation-invariant spatio-temporal systems, the system's response is the sum of the scaled versions of the inputs. Under zero initial conditions, the systems have eigenfunctions $e^{\kappa x}e^{st}$ and the corresponding eigenvalues $H^e(\kappa,s)$. If there is no external excitation, at each time t the systems have eigenfunctions $e^{\kappa x}$ and eigenvalues $H^i(\kappa,t)$. This is similar to conventional linear time invariant purely temporal systems.

Systems evolving over a finite interval on the real line $\Omega = [a,b]$

In this case, certain boundary conditions are required to obtain a unique solution to (2). The general form of the boundary conditions is

$$B\big(y(x_b,t)\big) = \psi(x_b,t),\ x_b = a,b \tag{12}$$

which includes three commonly used boundary conditions: Dirichlet ($B\big(y(x_b,t)\big) = y(x_b,t)$), Newmann ($B\big(y(x_b,t)\big) = y_x(x_b,t)$), and Robin ($B\big(y(x_b,t)\big) = b_1 y(x_b,t) + b_2 y_x(x_b,t)$), $x_b = a,b$. The initial-boundary value problem is then given as

$$\begin{gathered}
y_t(x,t) + Ay(x,t) = u(x,t), \\
y(x,0) = \varphi(x), \qquad x \in \Omega, \\
B\big(y(a,t)\big) = \psi(a,t), B(y(b,t)) = \psi(b,t),\ t > 0
\end{gathered} \tag{13}$$

which can be split into four sub-problems

- Inhomogeneous equation with zero initial conditions and homogeneous boundary conditions

$$\begin{gathered}
y_t(x,t) + Ay(x,t) = u(x,t), \\
y(x,0) = 0, \qquad x \in [a,b], \\
B\big(y(a,t)\big) = 0, B(y(b,t)) = 0, t > 0
\end{gathered} \tag{14}$$

- homogeneous equation with nonzero initial conditions and homogeneous boundary conditions

$$\begin{gathered}
y_t(x,t) + Ay(x,t) = 0, \\
y(x,0) = \varphi(x), x \in [a,b], \\
B\big(y(a,t)\big) = 0, B\big(y(b,t)\big) = 0, t > 0
\end{gathered} \tag{15}$$

- homogeneous equation with zero initial conditions and inhomogeneous boundary conditions at a

$$\begin{gathered}
y_t(x,t) + Ay(x,t) = 0, \\
y(x,0) = 0, x \in [a,b], \\
B\big(y(a,t)\big) = \psi(a,t), B(y(b,t)) = 0, t > 0
\end{gathered} \tag{16}$$

- homogeneous equation with zero initial conditions and inhomogeneous boundary conditions at b

$$\begin{gathered}
y_t(x,t) + Ay(x,t) = 0, \\
y(x,0) = 0, x \in [a,b], \\
B\big(y(a,t)\big) = 0, B(y(b,t)) = \psi(b,t),\ t > 0
\end{gathered} \tag{17}$$

If $y_1(x,t)$, $y_2(x,t)$, $y_3(x,t)$,and $y_4(x,t)$ are the solutions to (14)-(17), then superposition shows that the solution to (13) is the sum of these three solutions $y(x,t) = y_1(x,t) + y_2(x,t) + y_3(x,t) + y_4(x,t)$. The Green's functions of these four initial-boundary value problems $h^e(x,t;\xi,\tau)$ $h^i(x,t;\xi)$, $h^a(x,t;\tau)$, and $h^b(x,t;\tau)$ can be obtained in a similar way to the previous discussion. It follows from the translation invariance that the solution to (2) and (12) is given as follows

$$
\begin{aligned}
y(x,t) = \int_a^b \int_0^{+\infty} & h^e(\xi,\tau)\, u(x-\xi, t-\tau)\, d\xi d\tau \\
& + \int_a^b h^i(\xi,t)\, \varphi(x-\xi) d\xi \\
& + \int_0^{+\infty} h^a(x,\tau)\, \psi(a, t-\tau) d\tau \\
& + \int_0^{+\infty} h^b(x,\tau)\, \psi(b, t-\tau) d\tau
\end{aligned}
\tag{18}
$$

Consider the case for the input functions $u(x,t) = e^{\kappa x} e^{st}$, initial conditions $y(x,0) = \varphi(x) = e^{\kappa x}$, and the boundary conditions $\psi(x_b, t) = e^{st}$, $x_b = a, b$, the output is then

$$
\begin{aligned}
y(x,t) = \int_a^b \int_0^{+\infty} & h^e(\xi,\tau)\, e^{\kappa(x-\xi)} e^{s(t-\tau)}\, d\xi d\tau \\
& + \int_a^b h^i(\xi,t)\, e^{\kappa(x-\xi)} d\xi \\
& + \int_0^{+\infty} h^a(x,\tau)\, e^{s(t-\tau)} d\tau \\
& + \int_0^{+\infty} h^b(x,\tau)\, e^{s(t-\tau)} d\tau \\
= e^{\kappa x} e^{st} \int_a^b \int_0^{+\infty} & h^e(\xi,\tau)\, e^{-\kappa\xi}\, e^{-s\tau}\, d\xi d\tau \\
& + e^{\kappa x} \int_a^b h^i(\xi,t)\, e^{-\kappa\xi}\, d\xi \\
& + e^{st} \int_0^{+\infty} h^a(x,\tau)\, e^{-s\tau} d\tau \\
& + e^{st} \int_0^{+\infty} h^b(x,\tau)\, e^{-s\tau} d\tau \\
= e^{\kappa x} e^{st} H^e(\kappa,s) & + e^{\kappa x} H^i(\kappa,t) \\
& + e^{st} H^a(x,s) + e^{st} H^b(x,s)
\end{aligned}
\tag{19}
$$

Again, the functions $h^e(\xi,\tau)$, $h^i(\xi,t)$, $h^a(x,\tau)$, and $h^b(x,\tau)$ will be called the impulse response functions, and $H^e(\kappa,s)$, $H^i(\kappa,t)$, $H^a(x,s)$ and $H^b(x,s)$ will be called the transfer functions of the systems with respect to the external excitation, initial conditions, and boundary conditions, respectively. The Fourier complex domain version $H^e(jk,j\omega)$, $H^i(jk,t)$, $H^a(x,j\omega)$, and $H^b(x,j\omega)$ of the transfer functions is called the frequency response function of the system (2) with the boundary conditions (12), where k,ω are the spatial frequency and frequency, respectively.

Remark 3. It can be observed that the transfer functions $H^a(x,s)$ and $H^b(x,s)$ of the system with respect to the boundary conditions are functions of x. An interpretation that can be given is that the boundary control for the spatio-temporal systems is realised via spatial interactions.

Remark 4. Note that our output measurement is set to be the state $y(x,t)$. Actually, the output measurement using the output $z = g(y(x,t))$ can also be accommodated. For example, the output can be measured at a fixed point x_0 on the spatial domain: $z(t) = y(x_0,t)$ or the average over the spatial domain: $z(t) = \int_a^b y(x,t)w(x)dx$. As special cases, a study for the transfer function with respect to these outputs can be found in Curtain and Morris (2009).

B. Computation of the Linear Frequency Response Functions

In the last section, it was shown that there are three different types of frequency response functions for linear, translation invariant spatio-temporal systems corresponding to the three channels: excitation input-output, initial conditions-output, and boundary conditions-output. This is consistent with the way in which the spatio-temporal systems can be controlled. In this section, examples will be used to illustrate how to compute these frequency response functions using a development of the probing method (Billings and Tsang 1989a, Peyton-Jones and Billings 1989, Billings and Peyton-Jones 1990, Bedrosian and Rice 1971).

Consider a linear system described by the following first order evolution equation with a Neumann boundary condition

$$
\begin{gathered}
y_t(x,t) + a_0 y(x,t) + a_1 y_x(x,t) + a_2 y_{xx}(x,t) \\
= u(x,t), \\
y(x,0) = \varphi(x), \qquad x \in [a,b], \\
y_x(a,t) = 0, \quad y_x(b,t) = \psi(t), \qquad t > 0
\end{gathered}
\tag{20}
$$

where $a_i, i = 0,1,2$ are constants. The frequency response functions to be calculated are $H^e(k,\omega)$, $H^i(k,t)$, and $H^b(x,\omega)$, corresponding to the problems (21), (22), and (23),

- Inhomogeneous equation with zero initial conditions and homogeneous boundary conditions

$$\begin{aligned} y_t(x,t) + a_0 y(x,t) + a_1 y_x(x,t) + a_2 y_{xx}(x,t) \\ = u(x,t), \\ y(x,0) = 0, \qquad x \in [a,b], \\ y_x(a,t) = 0, \quad y_x(b,t) = 0, \qquad t > 0 \end{aligned} \tag{21}$$

- homogeneous equation with nonzero initial conditions and homogeneous boundary conditions

$$\begin{aligned} y_t(x,t) + a_0 y(x,t) + a_1 y_x(x,t) + a_2 y_{xx}(x,t) = 0, \\ y(x,0) = \varphi(x), \qquad x \in [a,b], \\ y_x(a,t) = 0, \quad y_x(b,t) = 0, \qquad t > 0 \end{aligned} \tag{22}$$

- homogeneous equation with zero initial conditions and inhomogeneous boundary conditions at b

$$\begin{aligned} y_t(x,t) + a_0 y(x,t) + a_1 y_x(x,t) + a_2 y_{xx}(x,t) = 0, \\ y(x,0) = 0, \qquad x \in [a,b], \\ y_x(a,t) = 0, \quad y_x(b,t) = \psi(t), \qquad t > 0 \end{aligned} \tag{23}$$

To calculate $H^e(k,\omega)$, assume the input to (21) is $u(x,t) = e^{jkx} e^{j\omega t}$, from (19) the output and the associated temporal and spatial derivatives of (21) are then

$$\begin{aligned} y(x,t) = H^e(k,\omega) e^{jkx} e^{j\omega t} \\ y_t(x,t) = j\omega H^e(k,\omega) e^{jkx} e^{j\omega t} \\ y_x(x,t) = jk H^e(k,\omega) e^{jkx} e^{j\omega t} \\ y_{xx}(x,t) = -k^2 H^e(k,\omega) e^{jkx} e^{j\omega t} \end{aligned} \tag{24}$$

Substituting equation (24) and $u(x,t) = e^{jkx} e^{j\omega t}$ into equation (21) yields

$$\begin{aligned} j\omega H^e(k,\omega) e^{jkx} e^{j\omega t} + a_0 H^e(k,\omega) e^{jkx} e^{j\omega t} \\ + a_1 jk H^e(k,\omega) e^{jkx} e^{j\omega t} \\ - a_2 k^2 H^e(k,\omega) e^{jkx} e^{j\omega t} = e^{jkx} e^{j\omega t} \end{aligned} \tag{25}$$

It follows that

$$H^e(k,\omega) = \frac{1}{-a_2 k^2 + a_1 jk + a_0 + j\omega} \tag{26}$$

subject to the homogeneous boundary conditions. The homogeneous Neumann conditions are

$$\begin{aligned} y_x(a,t) = jk H^e(k,\omega) e^{jka} e^{j\omega t} = 0 \\ y_x(b,t) = jk H^e(k,\omega) e^{jkb} e^{j\omega t} = 0, t > 0 \end{aligned} \tag{27}$$

which amounts to

$$\begin{aligned} jk H^e(k,\omega) e^{jka} \\ = |k H^e(k,\omega)| e^{j\left(2m\pi + \frac{\pi}{2} + ka + \arg\left(H^e(k,\omega)\right)\right)} = 0 \\ jk H^e(k,\omega) e^{jkb} = |k H^e(k,\omega)| e^{j\left(2l\pi + \frac{\pi}{2} + kb + \arg\left(H^e(k,\omega)\right)\right)} \\ = 0 \end{aligned} \tag{28}$$

It follows that either

$$|k H^e(k,\omega)| = 0 \tag{29}$$

or

$$\begin{aligned} e^{j\left(2m\pi + \frac{\pi}{2} + ka + \arg\left(H^e(k,\omega)\right)\right)} = 0 \\ e^{j\left(2l\pi + \frac{\pi}{2} + kb + \arg\left(H^e(k,\omega)\right)\right)} = 0 \end{aligned} \tag{30}$$

Multiplying both sides of (30) by $e^{-j\left(2m\pi + \frac{\pi}{2} + ka + \arg\left(H^e(k,\omega)\right)\right)}$ yields

$$\begin{aligned} e^{j0} = 0 \\ e^{j2(l-m)\pi + jk(b-a)} = 0 \end{aligned} \tag{31}$$

which means $\sin\left(k(b-a)\right) = 0$, that is $k = \frac{n\pi}{b-a}, n = 1, 2, \cdots$ Therefore, the frequency response function can be given by

$$H^e(k,\omega) = \begin{cases} \dfrac{1}{-a_2 k^2 + a_1 jk + a_0 + j\omega}, & k = \dfrac{n\pi}{b-a}, n = 1, 2, \cdots \\ 0, & otherwise \end{cases} \tag{32}$$

This indicates that the spatial frequency spectrum of the system (21) is actually discrete.

To calculate $H^i(k,t)$, suppose the input to (22) is $\varphi(x) = e^{jkx}$, from (19) the output and the associated temporal and spatial derivatives of (22) are then

$$\begin{aligned} y(x,t) = H^i(k,t) e^{jkx} \\ y_t(x,t) = H_t^i(k,t) e^{jkx} \\ y_x(x,t) = jk H^i(k,\omega) e^{jkx} \\ y_{xx}(x,t) = -k^2 H^i(k,\omega) e^{jkx} \end{aligned} \tag{33}$$

Substituting equation (33) and $\varphi(x) = e^{jkx}$ into equation (22) yields

$$\begin{aligned} H_t^i(k,t) e^{jkx} + a_0 H^i(k,t) e^{jkx} + a_1 jk H^i(k,t) e^{jkx} \\ - a_2 k^2 H^i(k,t) e^{jkx} \\ = H_t^i(k,t) e^{jkx} + (a_0 + a_1 jk \\ - a_2 k^2) H^i(k,t) e^{jkx} = 0 \\ H^i(k,0) e^{jkx} = e^{jkx}, \qquad x \in [a,b], \\ H^i(k,t) e^{jka} = 0, \quad H^i(k,t) e^{jkb} = 0, \ t > 0 \end{aligned} \tag{34}$$

The frequency response function $H^i(k,t)$ can be obtained as the solution to the initial value problem (the first two equations of (34)) and the last equation in (34) indicates the spectrum is discrete $k = \frac{n\pi}{b-a}, n = 0, 1, 2, \cdots$ as discussed earlier

$$H^i(k,t) = \begin{cases} e^{(a_2k^2 - a_1jk - a_0)t}, & k = \dfrac{n\pi}{b-a}, n = 0,1,2,\cdots \\ 0, & otherwise \end{cases}, \quad t > 0 \tag{35}$$

To calculate $H^b(x,\omega)$, suppose the input to (23) is $\psi(t) = e^{j\omega t}$, from (19) the output and the associated temporal and spatial derivatives of (23) are then

$$\begin{gathered} y(x,t) = H^b(x,\omega)e^{j\omega t} \\ y_t(x,t) = j\omega H^b(x,\omega)e^{j\omega t} \\ y_x(x,t) = H_x^b(x,\omega)e^{j\omega t} \\ y_{xx}(x,t) = H_{xx}^b(x,\omega)e^{j\omega t} \end{gathered} \tag{36}$$

Substituting equation (36) and $\psi(t) = e^{j\omega t}$ into equation (23) yields

$$\begin{gathered} j\omega H^b(x,\omega)e^{j\omega t} + a_0 H^b(x,\omega)e^{j\omega t} + a_1 H_x^b(x,\omega)e^{j\omega t} \\ - a_2 H_{xx}^b(x,\omega)e^{j\omega t} = 0 \\ H^b(a,\omega)e^{j\omega t} = 0, \quad H^b(b,\omega)e^{j\omega t} = e^{j\omega t}, t > 0 \\ H^b(x,\omega)e^{j\omega 0} = 0, \qquad x \in [a,b] \end{gathered} \tag{37}$$

The frequency response function $H^b(x,\omega)$ can be obtained as the solution to the boundary value problem (the first two equations in (37)) with the following characteristic equation

$$p(k) = a_2k^2 - a_1k - (a_0 + j\omega) \tag{38}$$

III. SPECTRAL ANALYSIS OF NONLINEAR SPATIO-TEMPORAL SYSTEMS

In this section, we extend the idea of impulse response functions, transfer functions, and frequency response functions for linear, translation invariant spatio-temporal systems to the nonlinear cases using a Volterra series representation of nonlinear relationships. Our discussions will focus on the finite interval $x \in [a,b]$ because the infinite case is much simpler than this case.

A. Generalised Transfer Functions of Nonlinear Spatio-temporal Systems

Consider the following first order nonlinear evolution equation with initial conditions I and boundary conditions B

$$\begin{gathered} y_t(x,t) + Ay(x,t) = u(x,t), \\ I\big(y(x,0)\big) = \varphi(x), x \in [a,b], \\ B\big(y(a,t)\big) = \psi(a,t), B\big(y(b,t)\big) = \psi(b,t), \qquad t > 0 \end{gathered} \tag{39}$$

where A is a bounded nonlinear operator which can, for example, take a form of $Ay(x,t) = a_0 y(x,t)y_x(x,t) + a_2 y_{xx}(x,t)$. As in the linear case, define $y(x,t)$ and $u(x,t)$to be the output and the external excitation of the system, respectively. We will investigate the impulse responses of the following equations derived from equation (39)

- Inhomogeneous equation with homogeneous initial conditions and homogeneous boundary conditions

$$\begin{gathered} y_t(x,t) + Ay(x,t) = u(x,t), \\ I\big(y(x,0)\big) = 0, x \in [a,b], \\ B\big(y(a,t)\big) = 0, \ B\big(y(b,t)\big) = 0, \qquad t > 0 \end{gathered} \tag{40}$$

- homogeneous equation with inhomogeneous initial conditions and homogeneous boundary conditions

$$\begin{gathered} y_t(x,t) + Ay(x,t) = 0, \\ I\big(y(x,0)\big) = \varphi(x), x \in [a,b], \\ B\big(y(a,t)\big) = 0, \ B\big(y(b,t)\big) = 0, \qquad t > 0 \end{gathered} \tag{41}$$

- homogeneous equation with homogeneous initial conditions and inhomogeneous boundary conditions at a

$$\begin{gathered} y_t(x,t) + Ay(x,t) = 0, \\ I\big(y(x,0)\big) = 0, x \in [a,b], \\ B\big(y(a,t)\big) = \psi(a,t), B\big(y(b,t)\big) = 0, \quad t > 0 \end{gathered} \tag{42}$$

- homogeneous equation with homogeneous initial conditions and inhomogeneous boundary conditions at a

$$\begin{gathered} y_t(x,t) + Ay(x,t) = 0, \\ I\big(y(x,0)\big) = 0, \qquad x \in [a,b], \\ B\big(y(a,t)\big) = 0, B\big(y(b,t)\big) = \psi(b,t), \quad t > 0 \end{gathered} \tag{43}$$

Remark 5. Due to the nonlinearity of the operators involved, the solutions $y_1(x,t), y_2(x,t)$, $y_3(x,t)$, $y_4(x,t)$ of (40), (41), (42), and (43) may not be expressed according to the linear convolution between the impulse response functions and the inputs because there are nonlinear (dynamical) relations between the input and output. The Green's functions of nonlinear operators have been developed to describe the above mentioned nonlinear dynamical relationships based on slack products of the nonlinear operators (Schwartz 1997, Qiao and Ruda 2004). In this paper, we take a different approach to deal with this problem. More specifically, we will us a Volterra series representation, which is capable of describing a more general class of nonlinear dynamical systems.

Following the general nonlinear system and Volterra series representation theory (Schetzen 1980), the four solutions can be expressed as the Volterra series representations

$$
\begin{aligned}
y_1(x,t) &= \sum_{n=1}^{\infty} y_n^e(x,t) \\
&= \sum_{n=1}^{\infty} \int_a^b \cdots \int_0^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n) \prod_{i=1}^{n} u(x-\xi_i,t-\tau_i)d\xi_i d\tau_i \\
y_2(x,t) &= \sum_{n=1}^{\infty} y_n^i(x,t) \\
&= \sum_{n=1}^{\infty} \int_a^b \cdots \int_a^b h_n^i(\xi_1,\cdots,\xi_n;t) \prod_{i=1}^{n} \varphi(x-\xi_i)d\xi_i \\
y_3(x,t) &= \sum_{n=1}^{\infty} y_n^a(x,t) \\
&= \sum_{n=1}^{\infty} \int_0^{+\infty} \cdots \int_0^{+\infty} h_n^a(x;\tau_1,\cdots\tau_n) \prod_{i=1}^{n} \psi(a,t-\tau_i)d\tau_i \\
y_4(x,t) &= \sum_{n=1}^{\infty} y_n^b(x,t) \\
&= \sum_{n=1}^{\infty} \int_0^{+\infty} \cdots \int_0^{+\infty} h_n^b(x;\tau_1,\cdots\tau_n) \prod_{i=1}^{n} \psi(b,t-\tau_i)d\tau_i
\end{aligned} \tag{44}
$$

where $y_n^e(x,t)$, $y_n^i(x,t)$, $y_n^a(x,t)$, and $y_n^b(x,t)$ are the nth order outputs of the system with

$$
\begin{aligned}
y_n^e(x,t) &= \int_a^b \cdots \int_0^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n) \prod_{i=1}^{n} u(x-\xi_i,t-\tau_i)d\xi_i d\tau_i \\
y_n^i(x,t) &= \int_a^b \cdots \int_a^b h_n^i(\xi_1,\cdots,\xi_n;t) \prod_{i=1}^{n} \varphi(x-\xi_i)d\xi_i \\
y_n^a(x,t) &= \int_0^{+\infty} \cdots \int_0^{+\infty} h_n^a(x;\tau_1,\cdots\tau_n) \prod_{i=1}^{n} \psi(a,t-\tau_i)d\tau_i \\
y_n^b(x,t) &= \int_0^{+\infty} \cdots \int_0^{+\infty} h_n^b(x;\tau_1,\cdots\tau_n) \prod_{i=1}^{n} \psi(b,t-\tau_i)d\tau_i
\end{aligned} \tag{45}
$$

Define the functions $h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)$, $h_n^i(\xi_1,\cdots,\xi_n;t)$, and $h_n^a(x;\tau_1,\cdots\tau_n)$ and $h_n^b(x;\tau_1,\cdots\tau_n)$ as the n^{th} order generalised impulse response functions of the system with respect to external signals, initial conditions, and boundary conditions, respectively. The associated Laplace transforms $H_n^e(\kappa_1,\cdots,\kappa_n;s_1,\cdots s_n)$, $H_n^i(\kappa_1,\cdots,\kappa_n;t)$, and $H_n^a(x;s_1,\cdots s_n)$ and $H_n^b(x;s_1,\cdots s_n)$ and Fourier transforms $H_n^e(k_1,\cdots,k_n;\omega_1,\cdots\omega_n)$, $H_n^i(k_1,\cdots,k_n;t)$, and $H_n^a(x;\omega_1,\cdots\omega_n)$ and $H_n^b(x;\omega_1,\cdots\omega_n)$ are called the n^{th} order generalised transfer functions and frequency response functions of the system with respect to external signals, initial conditions, and boundary conditions, respectively.

Remark 6. The definitions given here are based on the assumption that the Laplace transform and Fourier transform of the impulse response functions exist. Moreover, because the impulse response functions are defined on a finite interval with respect to spatial variable, the Laplace or Fourier transforms should be considered to apply to an extension of the functions into the half or entire real line.

Remark 7. Note that in general the solution of (39) may not be the sum of the four solutions in (44) because the operators A, I, and B are nonlinear. This solution could take the following general form

$$
y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) \tag{46}
$$

with $y_n(x,t) = f_n(y_n^e(x,t), y_n^i(x,t), y_n^a(x,t), y_n^b(x,t))$, where f_n is a nonlinear map.

Remark 8. It is well known that a nonlinear relationship can be described as a Volterra series with different orders of Volterra kernels which can be visualised as nonlinear impulse response functions (Marmarelis and Marmarelis 1978, Schetzen 1980). Here based on these results, we develop these concepts for nonlinear spatio-temporal systems, which is in consistent with the linear cases (see section 2) and conventional temporal dynamical systems.

Assumption 3. In this paper, it is assumed that the generalised impulse functions and the corresponding frequency response functions are symmetric with respect to all the time frequency and all the spatial frequency variables.

According to the above definition, taking the multiple Fourier transform of the n^{th} order generalised impulse response function $h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)$ with respect to the external excitation $u(x,t)$ yields the following n^{th} order generalised frequency response function

$$
\begin{aligned}
H_n^e(k_1,\cdots,k_n;\omega_1,\cdots\omega_n) = \\
\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots,\tau_n) \cdot \\
e^{-j(k_1\xi_1+\cdots+k_n\xi_n)} e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)} d\xi_1\cdots d\xi_n d\tau_1\cdots d\tau_n
\end{aligned} \tag{47}
$$

Note that because of the causality with respect to time, we can write the integration for time from $-\infty$ to $+\infty$ with $h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots,\tau_n) = 0$ for any $\tau_i < 0$. Conversely, the n^{th} order generalised impulse response function $h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)$ with respect to the external excitation $u(x,t)$ can be obtained by the inverse Fourier transform

$$h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots,\tau_n) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} H_n^e(k_1,\cdots,k_n;\omega_1,\cdots\omega_n)\cdot e^{j(k_1\xi_1+\cdots+k_n\xi_n)}e^{j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}dk_1\cdots dk_n d\omega_1\cdots d\omega_n \tag{48}$$

When assuming homogeneous initial and boundary conditions, the nth order output is then

$$y_n(x,t) = y_n^e(x,t) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)\prod_{i=1}^{n}u(x-\xi_i,t-\tau_i)d\xi_i d\tau_i \tag{49}$$

Substituting (48) into (49) and carrying out the multiple integrals on $\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n$ gives the following relation

$$y_n(x,t) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} H_n^e(k_1,\cdots,k_n;\omega_1,\cdots\omega_n)\cdot \prod_{i=1}^{n}U(k_i,\omega_i)e^{jk_ix}e^{j\omega_it}\,dk_1\cdots dk_n d\omega_1\cdots d\omega_n \tag{50}$$

where the input spectrum is given by

$$U(k,\omega) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}u(\xi,\tau)e^{-jk\xi}\,e^{-j\omega\tau}\,d\xi d\tau \tag{51}$$

with k,ω the spatial and time frequency respectively.

Suppose the input functions $(x,t)=\sum_{l=1}^{L}e^{jk_lx}e^{j\omega_lt}$, then from (49) the nth output of the system is, due to the symmetric property of assumption 3, is given by

$$y_n(x,t) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)\cdot \prod_{i=1}^{n}\sum_{l=1}^{L}e^{jk_l(x-\xi_i)}e^{j\omega_l(t-\tau_i)}d\xi_i d\tau_i$$

$$= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)\cdot \prod_{i=1}^{n}\sum_{l=1}^{L}e^{jk_lx}e^{-jk_l\xi_i}e^{j\omega_lt}e^{-j\omega_l\tau_i}\,d\xi_i d\tau_i$$

$$= \sum_{l_1=1}^{L}\cdots\sum_{l_n=1}^{L}\prod_{i=1}^{n}e^{jk_{l_i}x}e^{j\omega_{l_i}t}\cdot \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h_n^e(\xi_1,\cdots,\xi_n;\tau_1,\cdots\tau_n)\prod_{i=1}^{n}e^{-jk_{l_i}\xi_i}e^{-j\omega_{l_i}\tau_i}d\xi_i d\tau_i \tag{52}$$

Substituting from equation (47) yields

$$y_n(x,t) = \sum_{l_1=1}^{L}\cdots\sum_{l_n=1}^{L}H_n^e\left(k_{l_1},\cdots,k_{l_n};\omega_{l_1},\cdots\omega_{l_n}\right)\prod_{i=1}^{n}e^{jk_{l_i}x}e^{j\omega_{l_i}t} \tag{53}$$

Similarly, for the spatio-temporal generalised frequency response with respect to the initial conditions $\varphi(x)=\sum_{l=1}^{L}e^{jk_lx}$

$$y_n(x,t) = \sum_{l_1=1}^{L}\cdots\sum_{l_n=1}^{L}H_n^i\left(k_{l_1},\cdots,k_{l_n};t\right)\prod_{i=1}^{n}e^{jk_{l_i}x} \tag{54}$$

For the spatio-temporal generalised frequency response with respect to the boundary conditions $\psi(x_b,t)=\sum_{l=1}^{L}e^{j\omega_lt}$, $x_b=a,b$

$$y_n(x,t) = \sum_{l_1=1}^{L}\cdots\sum_{l_n=1}^{L}H_n^a\left(x;\omega_{l_1},\cdots\omega_{l_n}\right)\prod_{i=1}^{n}e^{j\omega_{l_i}t} \tag{55}$$

Finally

$$y_n(x,t) = \sum_{l_1=1}^{L}\cdots\sum_{l_n=1}^{L}H_n^b\left(x;\omega_{l_1},\cdots\omega_{l_n}\right)\prod_{i=1}^{n}e^{j\omega_{l_i}t} \tag{56}$$

B. The Calculation of Spatio-temporal Generalised Frequency Response Functions

Consider an example nonlinear system described by the following first order evolution equation with Dirichlet boundary condition

$$\begin{aligned} &y_t(x,t)+a_1y(x,t)y_x(x,t)+a_2y_{xx}(x,t)=u(x,t),\\ &y(x,0)=\varphi(x), \qquad x\in[a,b],\\ &y(a,t)=0,\quad y(b,t)=\psi(t), \qquad t>0 \end{aligned} \tag{57}$$

where $a_i, i=1,2$ are constants. The frequency response functions to be calculated are $H_n^e(k_1,\cdots,k_n;\omega_1,\cdots\omega_n)$, $H_n^i(k_1,\cdots,k_n;t)$, and $H_n^a(x;\omega_1,\cdots\omega_n)$ and $H_n^b(x;\omega_1,\cdots\omega_n)$, corresponding to the problems (58), (59), (60), and (61)

- Inhomogeneous equation with homogeneous initial conditions and homogeneous boundary conditions

$$y_t(x,t) + a_1 y(x,t) y_x(x,t) + a_2 y_{xx}(x,t) = u(x,t),$$
$$y(x,0) = 0, \quad x \in [a,b],$$
$$y(a,t) = 0, \quad y(b,t) = 0, \qquad t > 0 \tag{58}$$

- homogeneous equation with inhomogeneous initial conditions and homogeneous boundary conditions

$$y_t(x,t) + a_1 y(x,t) y_x(x,t) + a_2 y_{xx}(x,t) = 0,$$
$$y(x,0) = \varphi(x), \qquad x \in [a,b],$$
$$y(a,t) = 0, \quad y(b,t) = 0, \qquad t > 0 \tag{59}$$

- homogeneous equation with homogeneous initial conditions and inhomogeneous boundary conditions at a

$$y_t(x,t) + a_1 y(x,t) y_x(x,t) + a_2 y_{xx}(x,t) = 0,$$
$$y(x,0) = 0, \;\; x \in [a,b],$$
$$y(a,t) = 0, \quad y(b,t) = 0, \;\; t > 0 \tag{60}$$

- homogeneous equation with homogeneous initial conditions and inhomogeneous boundary conditions at a

$$y_t(x,t) + a_1 y(x,t) y_x(x,t) + a_2 y_{xx}(x,t) = 0,$$
$$y(x,0) = 0, \;\; x \in [a,b],$$
$$y(a,t) = 0, \quad y(b,t) = \psi(t), \;\; t > 0 \tag{61}$$

The calculation of nth spatio-temporal generalised transfer functions can be calculated in a similar way to the linear translation invariant spatio-temporal systems. For instance, to calculate $H_1^e(k,\omega)$, suppose the input to (58) is $u(x,t) = e^{jkx} e^{j\omega t}$, from (53) the output and the associated temporal and spatial derivatives are then

$$\begin{gathered} y(x,t) = H_1^e(k,\omega) e^{jkx} e^{j\omega t} \\ y_t(x,t) = j\omega H_1^e(k,\omega) e^{jkx} e^{j\omega t} \\ y_x(x,t) = jk H_1^e(k,\omega) e^{jkx} e^{j\omega t} \\ y_{xx}(x,t) = -k^2 H_1^e(k,\omega) e^{jkx} e^{j\omega t} \end{gathered} \tag{62}$$

Substituting equation (62) and $u(x,t) = e^{jkx} e^{j\omega t}$ into equation (58) yields

$$\begin{aligned} &j\omega H_1^e(k,\omega) e^{jkx} e^{j\omega t} \\ &\quad + a_1 jk H_1^e(k,\omega)^2 e^{jkx} e^{j\omega t} e^{jkx} e^{j\omega t} \\ &\quad - a_2 k^2 H_1^e(k,\omega) e^{jkx} e^{j\omega t} = e^{jkx} e^{j\omega t} \end{aligned} \tag{63}$$

It follows, by equating the coefficients of the term $e^{jkx} e^{j\omega t}$, that the first order spatio-temporal generalised frequency response is

$$H_1^e(k,\omega) = \frac{1}{-a_2 k^2 + j\omega} \tag{64}$$

To calculate $H_2^e(k_1,k_2;\omega_1,\omega_2)$, suppose the input is $u(x,t) = e^{jk_1x} e^{j\omega_1 t} + e^{jk_2x} e^{j\omega_2 t}$, again from (53) the output is then

$$\begin{aligned} &y(x,t) \\ &= H_1^e(k_1,\omega_1) e^{jk_1x} e^{j\omega_1 t} + H_1^e(k_2,\omega_2) e^{jk_2x} e^{j\omega_2 t} \\ &+ 2H_2^e(k_1,k_2;\omega_1,\omega_2) e^{j(k_1+k_2)x} e^{j(\omega_1+\omega_2)t} \\ &+ H_2^e(k_1,k_1;\omega_1,\omega_1) e^{j2k_1x} e^{j2\omega_1 t} \\ &+ H_2^e(k_2,k_2;\omega_2,\omega_2) e^{j2k_2x} e^{j2\omega_2 t} \end{aligned} \tag{65}$$

Substituting (65) and the associated temporal derivative $y_t(x,t)$ and spatial derivatives $y_x(x,t)$, $y_{xx}(x,t)$ and equating the coefficients of the term $e^{j(k_1+k_2)x} e^{j(\omega_1+\omega_2)t}$ yields

$$\begin{aligned} &H_2^e(k_1,k_2;\omega_1,\omega_2) \\ &\qquad = \frac{ja_1(k_1+k_2) H_1^e(k_1,\omega_1) H_1^e(k_2,\omega_2)}{j2(\omega_1+\omega_2) - 2a_2(k_1+k_2)^2} \end{aligned} \tag{66}$$

The solution should be subjected to the homogeneous Dirichlet boundary conditions, that is

$$\begin{aligned} &y(a,t) \\ &= H_1^e(k_1,\omega_1) e^{jk_1a} e^{j\omega_1 t} + H_1^e(k_2,\omega_2) e^{jk_2a} e^{j\omega_2 t} \\ &+ 2H_2^e(k_1,k_2;\omega_1,\omega_2) e^{j(k_1+k_2)a} e^{j(\omega_1+\omega_2)t} \\ &+ H_2^e(k_1,k_1;\omega_1,\omega_1) e^{j2k_1a} e^{j2\omega_1 t} \\ &+ H_2^e(k_2,k_2;\omega_2,\omega_2) e^{j2k_2a} e^{j2\omega_2 t} = 0 \\ &y(b,t) \\ &= H_1^e(k_1,\omega_1) e^{jk_1b} e^{j\omega_1 t} + H_1^e(k_2,\omega_2) e^{jk_2b} e^{j\omega_2 t} \\ &+ 2H_2^e(k_1,k_2;\omega_1,\omega_2) e^{j(k_1+k_2)b} e^{j(\omega_1+\omega_2)t} \\ &+ H_2^e(k_1,k_1;\omega_1,\omega_1) e^{j2k_1b} e^{j2\omega_1 t} \\ &+ H_2^e(k_2,k_2;\omega_2,\omega_2) e^{j2k_2b} e^{j2\omega_2 t} = 0, \qquad t > 0 \end{aligned} \tag{67}$$

It follows from the orthogonality of Fourier basis $\{e^{jv_1t} e^{jv_2t}, v_1, v_2 \in R\}$ that

$$\begin{gathered} H_1^e(k_1,\omega_1) e^{jk_1a} = 0 \\ H_1^e(k_1,\omega_1) e^{jk_1b} = 0 \\ H_2^e(k_1,k_2;\omega_1,\omega_2) e^{j(k_1+k_2)a} = 0 \\ H_2^e(k_1,k_2;\omega_1,\omega_2) e^{j(k_1+k_2)b} = 0 \end{gathered} \tag{68}$$

From the first equation in (68), it can be derived that either $H_1^e(k_1,\omega_1) = 0$ or $\sin\big(k_1(b-a)\big) = 0$, that is $k_1 = \frac{n\pi}{b-a}, n = 1,2,\cdots$ From the second equation in (68), it can be shown that either $H_2^e(k_1,k_2;\omega_1,\omega_2) = 0$ or $\sin\big((k_1+k_2)(b-a)\big) = 0$, that is $k_1 + k_2 = \frac{m\pi}{b-a}, m = 1,2,\cdots$ Therefore, the first and second order generalised frequency response functions can be given by

$$H_1^e(k,\omega) = \begin{cases} \dfrac{1}{-a_2k^2 + j\omega}, & k = \dfrac{n\pi}{b-a}, n = 1,2,\cdots \\ 0, & \text{otherwise} \end{cases}$$

And

$$H_2^e(k_1,k_2;\omega_1,\omega_2) = \begin{cases} \dfrac{ja_1(k_1+k_2)H_1^e(k_1,\omega_1)H_1^e(k_2,\omega_2)}{j2(\omega_1+\omega_2)-2a_2(k_1+k_2)^2}, & k_1+k_2 = \dfrac{m\pi}{b-a} \\ 0, & \text{otherwise} \end{cases} \tag{69}$$

where $m = 1,2,\cdots$

The other order spatio-temporal generalised frequency response functions can be calculated following the same procedure.

IV. NUMERICAL EXAMPLES

A. Linear Spatio-temporal Systems -- Diffusion Equation

Consider the following diffusion equation (Debnath 2005) with Dirichlet boundary conditions

$$\begin{gathered} y_t(x,t) - Dy_{xx}(x,t) = u(x,t), \\ y(x,0) = \varphi(x), \qquad x \in [a,b], \\ y(a,t) = \psi(t), y(b,t) = 0, \quad t > 0 \end{gathered} \tag{70}$$

where D is the diffusion coefficient. According to the analysis in section 2, system (70) can be divided into three subsystems with the corresponding frequency response functions $H^e(k,\omega)$, $H^i(k,t)$, and $H^a(x,\omega)$.

To calculate the frequency response $H^e(k,\omega)$ with respect to the external excitation, we consider the problem

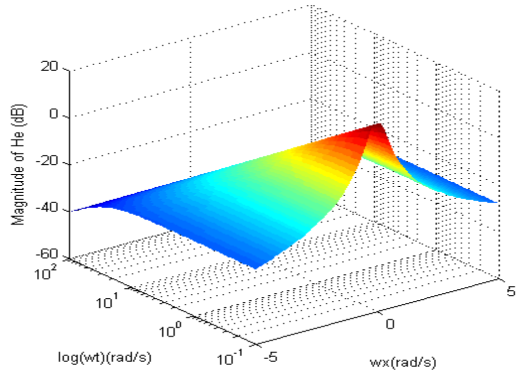
$$\begin{gathered} y_t(x,t) - Dy_{xx}(x,t) = u(x,t), \\ y(x,0) = 0, \qquad x \in [a,b], \\ y(a,t) = 0, \quad y(b,t) = 0, \qquad t > 0 \end{gathered} \tag{71}$$

Suppose the input to (71) is $u(x,t) = e^{jkx}e^{j\omega t}$, then the probing method gives the frequency response function as

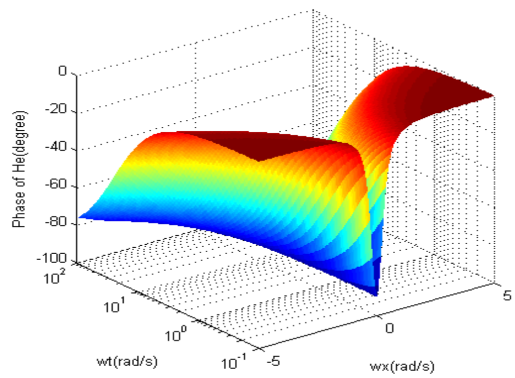
$$H^e(k,\omega) = \begin{cases} \dfrac{1}{j\omega + Dk^2}, & k = \dfrac{n\pi}{b-a}, n = 1,2,\cdots \\ 0, & otherwise \end{cases} \tag{72}$$

The magnitude and phase of (72) with $D = 1$ are shown in Fig. 1, where Fig. 1 (a) shows that system (71) works as a low-pass filter with respect to both space and time frequencies. The frequency domain response depends on both space and time frequencies. these interact with each other. For example, for a certain spatial frequency k_0, $H^e(k_0,\omega)$ is a first order linear system and the corner frequency of the first order system increases with the increase of k_0.

The frequency response $H^i(k,t)$ is related to the following problem



(a) $|H^e(k_0,\omega)|$



(b) $\angle H^e(k_0,\omega)$

Fig. 1 $H^e(k_0,\omega)$

$$\begin{gathered} y_t(x,t) - Dy_{xx}(x,t) = 0, \\ y(x,0) = \varphi(x), x \in [a,b], \\ y(a,t) = 0, \quad y(b,t) = 0, t > 0 \end{gathered} \tag{73}$$

Suppose the input to (73) is $\varphi(x) = e^{jkx}$, the output and the associated temporal and spatial derivatives are then

$$\begin{gathered} y(x,t) = H^i(k,t)e^{jkx} \\ y_t(x,t) = H_t^i(k,t)e^{jkx} \\ y_{xx}(x,t) = -k^2H^i(k,\omega)e^{jkx} \end{gathered} \tag{74}$$

Substituting (74) and $\varphi(x) = e^{jkx}$ into equation (73) yields

$$H_t^i(k,t)e^{jkx} + Dk^2H^i(k,t)e^{jkx} = 0 \tag{75}$$

The frequency response function $H^i(k,t)$ can be obtained as the solution to the initial value problem (75)

$$H^i(k,t) = \begin{cases} e^{-Dk^2 t}, & k = \dfrac{n\pi}{b-a}, n = 0,1,2,\cdots \\ 0, & otherwise \end{cases} \tag{76}$$

Fig. 2 shows that the response excited by initial conditions declines with elapsing time. An initial condition with a high frequency sharply drops to zero while a low frequency initial condition declines with a relatively lower speed.

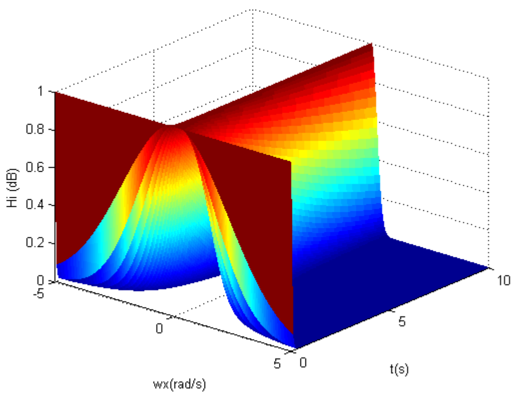


Fig. 2 $H^i(k,t)$ with $D = 1$

To calculate the boundary frequency response $H^a(x,\omega)$, suppose an input of $\psi(t) = e^{j\omega t}$, then

$$j\omega H^a(x,\omega)e^{j\omega t} - DH^a_{xx}(x,\omega)e^{j\omega t} = 0 \tag{77}$$

whose solution is given by

$$H^a(x,\omega) = \frac{e^{\sqrt{\frac{j\omega x}{D}}} - e^{\sqrt{\frac{j\omega b}{D}}}}{e^{\sqrt{\frac{j\omega a}{D}}} - e^{\sqrt{\frac{j\omega b}{D}}}} \tag{78}$$

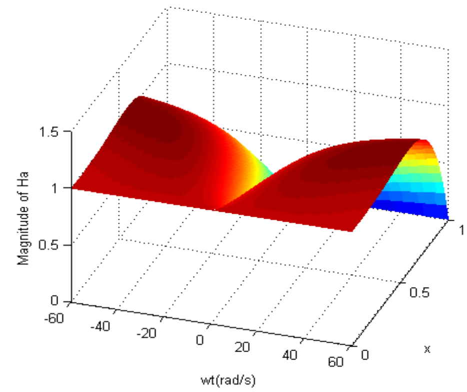
Fig. 3 shows that the response excited by a boundary condition depends on both the frequency of the boundary condition and the spatial coordinate. The low frequency boundary conditions drop much faster than the high frequency boundary conditions do. Obviously, the result of example 1 discussed in Curtain and Morris (2009) is a special case of the result here.

B. Nonlinear Spatio-temporal Systems – *Fisher's* Equation

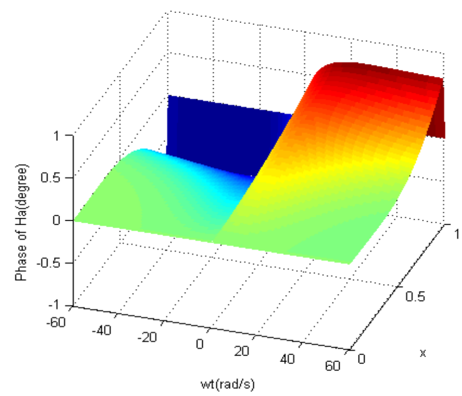
Consider the following Fisher's equation in dimensionless form (Debnath 2005)

$$\begin{aligned} y_t(x,t) - Dy_{xx}(x,t) - y(x,t)\big(1 - y(x,t)\big) = 0, \\ y(x,0) = \varphi(x), \qquad x \in (-\infty,+\infty), t > 0 \end{aligned} \tag{79}$$

where D is the diffusion coefficient. In this example, only the first and second order generalized frequency responses will be calculated. An initial condition $\varphi(x) = e^{jkx}$ yields



(a) Magnitude of $H^a(x,\omega)$



(b) Phase of $H^a(x,\omega)$

Fig. 3 $H^a(x,\omega)$

$$\begin{aligned} H^i_{1t}(k,t)e^{jkx} + Dk^2 H^i_1(k,t)e^{jkx} \\ - H^i_1(k,t)e^{jkx}\left(1 - H^i_1(k,t)e^{jkx}\right) \\ = 0 \end{aligned} \tag{80}$$

Equating the coefficients of e^{jkx} on both sides yields

$$H^i_{1t}(k,t) + (Dk^2 - 1)H^i_1(k,t) = 0 \tag{81}$$

so that the first order generalised frequency response is

$$H^i_1(k,t) = e^{-(Dk^2-1)t} \tag{82}$$

In order to calculate $H^i_2(k_1,k_2;t)$, suppose the input is $\varphi(x) = e^{jk_1 x} + e^{jk_2 x}$, the corresponding output is

$$y(x,t) = H_1^i(k_1,t)e^{jk_1x} + H_1^i(k_2,t)e^{jk_2x} + 2H_2^i(k_1,k_2;t)e^{j(k_1+k_2)x} + H_2^i(k_1,k_1;t)e^{j2k_1x} + H_2^i(k_2,k_2;t)e^{j2k_2x} \tag{83}$$

The probing method gives

$$2H_{2t}^i(k_1,k_2;t) + (2D(k_1+k_2)^2 - 2)H_2^i(k_1,k_2;t) + 2H_1^i(k_1,t)H_1^i(k_2,t) = 0 \tag{84}$$

Substituting (82) into (84) yields

$$H_{2t}^i(k_1,k_2;t) + (D(k_1+k_2)^2 - 1)H_2^i(k_1,k_2;t) = -e^{-(D(k_1^2+k_2^2)-2)t} \tag{85}$$

The general solution of (85) can be represented as

$$H_2^i(k_1,k_2;t) = Ce^{-(D(k_1+k_2)^2-1)t} - \frac{1}{1+2Dk_1k_2}e^{-(D(k_1^2+k_2^2)-2)t} \tag{86}$$

According the initial condition $y(x,0) = \varphi(x)$

$$y(x,0) = H_1^i(k_1,0)e^{jk_1x} + H_1^i(k_2,0)e^{jk_2x} + H_2^i(k_1,k_2;0)e^{j(k_1+k_2)x} = e^{jk_1x} + e^{jk_2x} \tag{87}$$

The generalised frequency response with respect to initial conditions is given by

$$H_2^i(k_1,k_2;t) = \frac{1}{1+2Dk_1k_2}(e^{-(D(k_1+k_2)^2-1)t} - e^{-(D(k_1^2+k_2^2)-2)t}) \tag{88}$$

Figs. 4 and 5 show the spatio-temporal generalised frequency response functions $H_1^i(k,t)$ and $H_2^i(k_1,k_2;t)$, respectively.

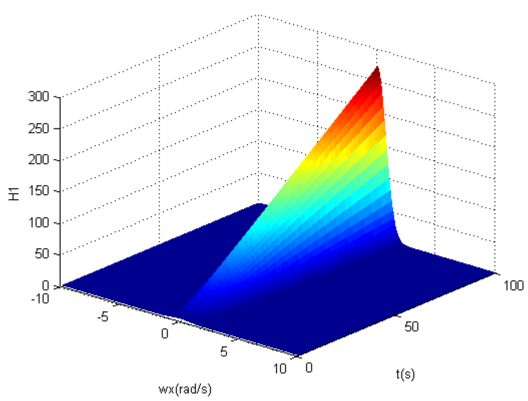


Fig. 4 $H_1^i(k,t)$ with $D = 1$

Figs. 4 and 5 show that the magnitude for both $H_1^i(k,t)$ and $H_2^i(k_1,k_2;t)$ increase with elapsing time for low frequency initial conditions while they decrease with elapsing time for high frequency initial conditions. The stability condition for $H_1^i(k,t)$ is $|t| > 1$ and the stability condition for $H_2^i(k_1,k_2;t)$ is

$\{(k_1,k_2) \in C^2: (k_1+k_2)^2 > 1 \; and \; k_1^2 + k_2^2 > 2\}$

which have been shown in Fig. 6.

V. CONCLUSIONS

The concept of classical transfer functions and frequency responses have been extended to both linear and nonlinear spatio-temporal systems. It has been shown, through a theoretical analysis and numerical examples, that the proposed generalised transfer functions and frequency response functions are consistent with the classical definitions. A new method for identifying and computing the generalised frequency response functions for spatio-temporal systems has also been presented. The definitions and methodology introduced in this paper provide a solid basis and powerful tools for further investigations of the spectral analysis and properties of spatio-temporal systems.

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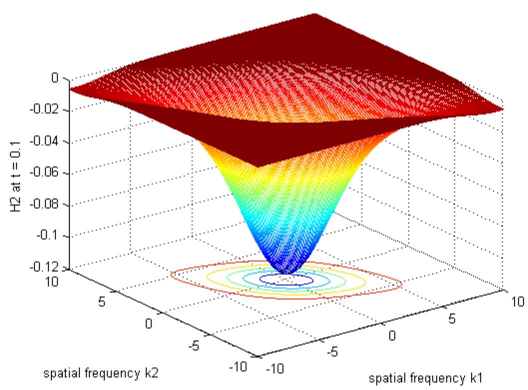
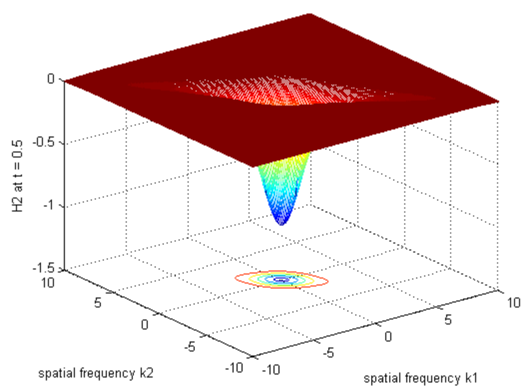
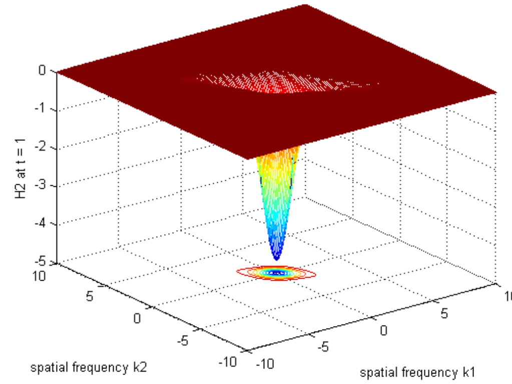


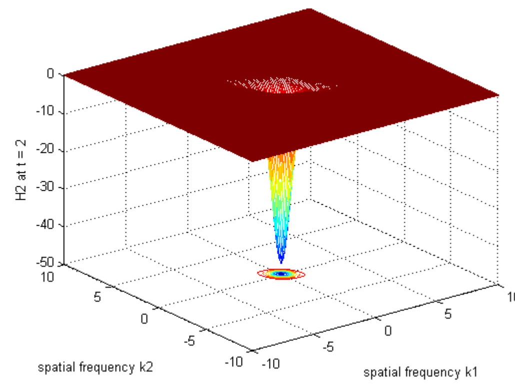
(a)



(b)



(c)



(d)

Fig. 5 $H_2^i(k_1, k_2; t)$ with $D = 1$ (a) $t = 0.1$, (b) $t = 0.5$, (c) $t = 1$, (d) $t = 2$

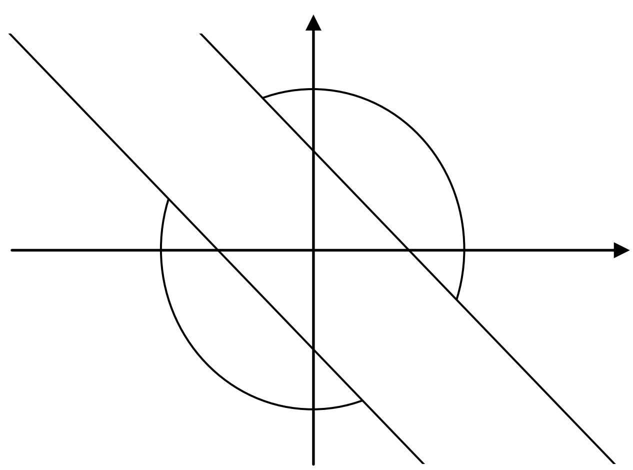


Fig. 6 the stable region of $H_2^i(k_1, k_2; t), t > 0$