Estimation of Generalised Frequency Response Functions

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Abstract—Volterra series theory has a wide application in the representation, analysis, design and control of nonlinear systems. A new method of estimating the Volterra kernels in the frequency domain is introduced based on a non-parametric algorithm. Unlike the traditional non-parametric methods using the DFT transformed input-output data, this new approach uses the time domain measurements directly to estimate the frequency domain response functions.

Index Terms—GFRF's, Non-parametric, OLS, Volterra series.

I. Introduction

THE Volterra series model, first proposed by Volterra [1], is a direct generalization of the linear convolution integral and provides an intuitive representation in a simple and easy to apply way. Volterra theory quickly received a great deal of attention in the field of electrical engineering, mechanical engineering, and later in the biological field, as a powerful approach for modeling nonlinear system behaviors. From the late 1950s, there has been a continuous effort in the application of Volterra series to nonlinear systems theory. Summaries of major contributions in the application of Volterra series modeling for the representation, analysis and design of nonlinear systems can be found in [2]-[5].

The Volterra series is associated with so-called weakly nonlinear systems, which can be well described by the first few kernels with the higher order kernels falling off rapidly. The frequency domain version of the Volterra kernels, called generalized frequency response functions(GFRF's), which can be obtained by taking the multiple Fourier transform of the Volterra kernels, has also been extensively studied and proved to be very powerful in characterizing nonlinear phenomena [6].

Due to the usefulness of the generalized frequency response functions, a number of estimation methods have been proposed. There are generally two classes of methods for the estimation of GFRF -- parametric and non-parametric methods. For the parametric method, the input-output data measurements are used to identify a NARMAX model, from which a NARX model, including purely input-output terms after discarding the terms associated with the noise model, can be derived. Then the probing method [7] can be applied on this NARX model to obtain the GFRF's. The non-parametric method usually makes use of higher order spectral analysis based on the frequency domain Volterra model [8]-[12]. Boyd et al. [13] proposed a non-parametric method of estimating the GFRF's based on the separation of the contribution of each Volterra kernel, using harmonic inputs. Due to the computational complexity associated with non-parametric methods, the Volterra kernels had to be restricted to low orders, for example, up to cubic nonlinearities.

This paper is primarily concerned with the problem of estimating the GFRF's from harmonic input-output data for cubically nonlinear systems. By expanding the algebraic expression of the response analysis in the frequency domain, it is shown that the GFRF's can be estimated directly from input-output measurements.

The paper is organized as follows. Section 2 states the preliminaries. Section 3 studies the estimation of the GFRF's using the simplest Volterra model form, that is, quadratic nonlinear systems. Section 4 discusses the more complex cubically nonlinear system case. In section V the problem of determination of the excitation level is addressed. Finally in section 6, conclusions are given.

II. Preliminaries

Volterra series modeling has been widely studied for the representation, analysis and design of nonlinear systems. For a SISO nonlinear system, where $u(t)$ and $y(t)$ are the input and output respectively, the Volterra series can be expressed as

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \tag{1.a}$$

and $y_n(t)$ is the 'n-th order output' of the system

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \quad n > 0 \tag{1.b}$$

where $h_n(\tau_1, \cdots, \tau_n)$ is called the 'nth-order kernel' or 'nth-order impulse response function'. If $n=1$, this reduces to the familiar linear convolution integral.

The discrete time domain counterpart of the continuous time domain SISO Volterra expression (1) is

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$$y(k) = \sum_{n=1}^{\infty} y_n(k) \tag{2.a}$$

where

$$y_n(k) = \sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(k-\tau_i) \quad n>0, k\in Z \tag{2.b}$$

In practice only the first few kernels are used on the assumption that the contribution of the higher order kernels falls off rapidly. Systems that can be adequately represented by a Volterra series with just a few terms are called weakly nonlinear systems.

For a weakly nonlinear system up to third order Volterra series representation, the frequency domain expression of the discrete time Volterra model (2) is given as

$$\begin{aligned} Y(\omega) &= H_1(\omega)U(\omega) + \sum_{\substack{p,q\\ p+q=\omega}} H_2(p,q)U(p)U(q) \\ &+ \sum_{\substack{l,m,n\\ l+m+n=\omega}} H_3(l,m,n)U(l)U(m)U(n) \end{aligned} \tag{3}$$

where $Y(\omega)$ and $U(\omega)$ are the Fourier Transforms of the output response and input respectively, and $H_n(\omega_1,\cdots,\omega_n)$ is called the nth order Generalized Frequency Response Function (GFRF) which is obtained by taking the multi-dimensional Fourier transform of $h_n(\cdot)$:

$$\begin{aligned} &H_n(\omega_1,\cdots,\omega_n) \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \end{aligned} \tag{4}$$

The generalized frequency response functions represent an inherent and invariant property of the underlying system, and have proved to be an important analysis and design tool for characterizing nonlinear phenomena.

In practice, by taking into account the output measurement noise (generally zero-mean white noise), (3) is replaced by

$$\begin{aligned} Y(\omega) &= \hat{Y}(\omega) + \varepsilon(\omega) \\ &= H_1(\omega)U(\omega) + \sum_{\substack{p,q\\ p+q=\omega}} H_2(p,q)U(p)U(q) \\ &+ \sum_{\substack{l,m,n\\ l+m+n=\omega}} H_3(l,m,n)U(l)U(m)U(n) + \varepsilon(\omega) \end{aligned} \tag{5}$$

The excitations used in the application of non-parametric methods can be either Gaussian white noise or non-Gaussian harmonic inputs. For instance, reference [10] investigated the problem of estimation of GFRF's and system identification of a cubically nonlinear system based on (5), subject to a non-Gaussian input. Although (5) is nonlinear between the spectrum of the measured input/output $u(k)$ and $y(k)$, i.e., $U(\omega)$ and $Y(\omega)$, it is linear between $Y(\omega)$ and the unknown GFRF's $H_1(\cdot)$, $H_2(\cdot)$ and $H_3(\cdot)$, and the standard Least Squares type algorithms can be readily applied to obtain different orders of transfer functions. The possible disadvantages of the above purely frequency domain based approaches are that large data sets are often needed and also the frequency domain noisy term $\varepsilon(\omega)$, obtained from time domain white Gaussian noise $e(k)$, may no longer be white, potentially resulting in bias in the estimates obtained from a Least Square procedure.

Alternatively, the steady-state response of the nonlinear system that can be adequately represented by up to third order Volterra kernels, excited by a harmonic signal at frequency ω, is given [2] by

$$\begin{aligned} y(k) &= \hat{y}(k) + e(k) \\ &= A\,\mathrm{Re}\{H_1(\omega)e^{j\omega k}\} + 2(\frac{A}{2})^2\,\mathrm{Re}\{H_2(\omega,\omega)e^{j2\omega k}\} \\ &+ 2(\frac{A}{2})^2\,\mathrm{Re}\{H_2(\omega,-\omega)\} + 2(\frac{A}{2})^3\,\mathrm{Re}\{H_3(\omega,\omega,\omega)e^{j3\omega k}\} \\ &+ 6(\frac{A}{2})^3\,\mathrm{Re}\{H_3(\omega,\omega,-\omega)e^{j\omega k}\} + e(k) \end{aligned} \tag{6}$$

where A is the amplitude of the input signal and $e(k)$ is a zero-mean Gaussian white noise. 'Re' represents the real part of a complex number.

Equation (6) forms the basis of the current study which suggests an alternative non-parametric approach of estimating generalized frequency response functions directly from noisy time domain data. The study begins with quadratic nonlinear systems in Section 3, followed by cubic nonlinear systems in Section 4.

III. Estimation of Frequency Response Functions of Quadratically Nonlinear Systems

A. Formation of the Estimator

Because it is the simplest special case of the finite Volterra series model, the quadratic Volterra model class has been studied fairly extensively.

By considering the first two Volterra kernels in (6), the output response $y(k)$ can be expressed as

$$\begin{aligned} y(k) &= \hat{y}(k) + e(k) \\ &= A\,\mathrm{Re}\{H_1(\omega)e^{j\omega k}\} + 2(\frac{A}{2})^2\,\mathrm{Re}\{H_2(\omega,\omega)e^{j2\omega k}\} \\ &+ 2(\frac{A}{2})^2\,\mathrm{Re}\{H_2(\omega,-\omega)e^{j20k}\} + e(k) \end{aligned} \tag{7}$$

Defining

$$H_1(\omega) = R_1(\omega) + jI_1(\omega) \tag{8}$$

where $R_1(\omega)$ and $I_1(\omega)$ are the real and imaginary parts of $H_1(\omega)$ respectively. For simplicity, $R_1(\omega)$ and $I_1(\omega)$ will be written in abbreviated form as R_1 and I_1.

Then the first term in the right-hand side of (7) can be expanded as

$$\begin{aligned} &A\,\mathrm{Re}\{H_1(\omega)e^{j\omega k}\} \\ &= A\,\mathrm{Re}\{[R_1 + jI_1][\cos(\omega k) + j\sin(\omega k)]\} \\ &= AR_1\cos(\omega k) - AI_1\sin(\omega k) \end{aligned} \tag{9}$$

Similarly by defining

$$H_2(\omega,\omega) = R_2(\omega,\omega) + jI_2(\omega,\omega) \tag{10}$$

and noting that $H_2(\omega,-\omega) = R_0$ is a constant, the second and the third terms on the right-hand side of (7) can be expanded as

$$2(\frac{A}{2})^2 \operatorname{Re}\{H_2(\omega,\omega)e^{j2\omega k}\} + 2(\frac{A}{2})^2 \operatorname{Re}\{H_2(\omega,-\omega)e^{j20k}\}$$
$$= 2(\frac{A}{2})^2 R_2 \cos(2\omega k) - 2(\frac{A}{2})^2 I_2 \sin(2\omega k) + 2(\frac{A}{2})^2 R_0 \qquad (11)$$

Combining (9) and (11) yields

$$\begin{aligned} y(k) &= R_1[A\cos(\omega k)] + I_1[-A\sin(\omega k)] \\ &+ R_2[2(\frac{A}{2})^2 \cos(2\omega k)] + I_2[-2(\frac{A}{2})^2 \sin(2\omega k)] \\ &+ R_0 2(\frac{A}{2})^2 + e(k) \end{aligned} \qquad (12)$$

For k=1 to N, (12) can be arranged in the matrix form as

$$\mathbf{Y} = \mathbf{X\theta} + \mathbf{E} \qquad (13)$$

where $\mathbf{Y} = [y(N) \cdots y(1)]^T$, $\boldsymbol{\theta} = [R_1\ I_1\ R_2\ I_2\ R_0]^T$, $\mathbf{E} = [e(N) \cdots e(1)]$, and $\mathbf{X} = [\mathbf{x_1}\ \mathbf{x_2}\ \mathbf{x}_3\ \mathbf{x}_4\ \mathbf{x}_5]$, with

$$\begin{aligned} \mathbf{x_1} &= [A\cos(\omega N) \cdots A\cos(\omega)]^T \\ \mathbf{x}_2 &= [-A\sin(\omega N) \cdots -A\sin(\omega)]^T \\ \mathbf{x}_3 &= [2(\frac{A}{2})^2 \cos(2\omega N) \cdots 2(\frac{A}{2})^2 \cos(2\omega)]^T \\ \mathbf{x}_4 &= [-2(\frac{A}{2})^2 \sin(2\omega N) \cdots -2(\frac{A}{2})^2 \sin(2\omega)]^T \\ \mathbf{x}_5 &= [2(\frac{A}{2})^2 \cdots 2(\frac{A}{2})^2]^T \end{aligned} \qquad (14)$$

The estimation of the unknown frequency response functions $\hat{\boldsymbol{\theta}} = [R_1\ I_1\ R_2\ I_2\ R_0]^T$ can now be derived from (13) using a standard least square procedure. If the truncation error is sufficiently small, on the assumption that third and higher orders of Volterra kernels make a negligible contribution to the output, then the estimation $\hat{\boldsymbol{\theta}}$ will be unbiased. Unlike the previous complex estimator based on (5), the new estimator is in the real domain based on time domain measurements.

B. Simulation Example and Discussion

Consider a system described as

$$\ddot{y} + a\dot{y} + by + cy^2 = u(t) \qquad (15)$$

where $u(t) = A\cos(\omega t)$.

In the continuous time domain, the GFRF's can be derived [14] as

$$\begin{aligned} H_1(s)\big|_{s=j\omega} &= \frac{1}{s^2 + as + b} \\ H_2(s_1, s_2)\big|_{\substack{s_1=j\omega_1 \\ s_2=j\omega_1}} &= -cH_1(s_1)H_1(s_2)H_1(s_1+s_2) \end{aligned} \qquad (16)$$

This system (15), which has a quadratic nonlinear term y^2, is not a quadratically but infinitely nonlinear system in terms of the Volterra series representation because the nonlinearity is on the output. However for a considerable range of input level the system can be adequately represented by up to second order Volterra kernels, and importantly all the coefficients of the underlying system (15) can be fully characterized by the first two orders of Volterra kernels or the associated frequency response functions.

Once the estimates of $\hat{H}_1$ and $\hat{H}_2$ are available, by using (16), the original continuous time system parameters can be extracted from $\hat{H}_1$ and $\hat{H}_2$ as

$$\begin{aligned} \hat{a} &= \frac{1}{\omega} \operatorname{Im}[\frac{1}{\hat{H}_1(\omega)}] \\ \hat{b} &= \operatorname{Re}[\frac{1}{\hat{H}_1(\omega)}] + \omega^2 \\ \hat{c} &= \frac{-\hat{H}_2(\omega,-\omega)}{\hat{H}_1(\omega)\hat{H}_1(-\omega)\hat{H}_1(0)} = \frac{-\hat{b}\hat{H}_2(\omega,-\omega)}{\hat{H}_1(\omega)\hat{H}_1(-\omega)} \end{aligned} \qquad (17)$$

To illustrate the new GFRF estimation algorithm, system (15) was excited by a single sinusoidal input at A=2 and $\omega = 1.6\,\mathrm{rad/sec}$ for $a = 0.2, b = 1$ and $c = 0.1$. A total amount of 2000 input and output data were collected at a sampling time $T_s = 0.01\mathrm{sec}$ and a zero-mean white noise was added to the output with SNR =40 dB.

Before proceeding using the new algorithm, the traditional parametric NARMAX procedure was applied. First the ideal situation, noise-free input /output data [15], were considered in the NARX modeling. The resulting model was obtained as

$$\begin{aligned} y(k) = &\ 1.9979\,y(k-1) - 0.9980\,y(k-2) \\ &+ 9.99e\text{-}05\,u(k-1) - 9.989e\text{-}06\ y^2(k-1) \end{aligned} \qquad (18)$$

It can be easily verified that the GFRF's from the discrete time model (18) are exactly the same as the GFRF's from the original continuous time model in (16) , along the entire frequency axes. This suggests that the NARX modeling procedure is the simplest and most efficient choice for the GFRF's estimation in the noise-free situation.

When noise is present, however, the identification of a NARMAX model that can provide satisfactory frequency domain estimation is not a trivial task. For example, a NARMAX model whose terms are selected from a pool of candidate terms, using the noisy measurements, is given as

$$\begin{aligned} y(k) = &\ 0.1432\,y(k-1) + 0.2774\,y(k-2) - 4.7275\,u(k-1) \\ &+ 4.3761\,u(k-2) - 0.2564u^2(k-2) - 0.6544\ y^2(k-1) \\ &- 0.8029\,y(k-1)u(k-2) + \Theta_e + e(k) \end{aligned} \qquad (19)$$

where Θ_e represents the noise model terms.

Equation (19) has a very good model predicted output (MPO) and model validity test [16]—[19] shown in Fig. 1.

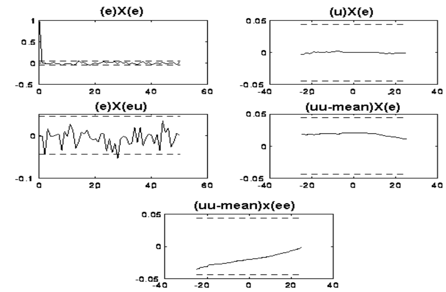

Fig. 1. Correlation test of the NARMAX modelling of (19)

The estimation of H_1 and H_2 from (19) at $\omega = 1.6$ is shown in Table I, compared with the true values from the original system (15), and the reconstructed parameters of continuous time model (15) from Table 1 using (17) are shown in Table II.

TABLE I
ESTIMATION OF H_1 AND H_2 BY THE NARMAX MODEL (19)

	From Eqn (15) –True	From Eqn (19) --Estimates
$H_1(\omega)$	-0.6151 - 0.1262j	-0.6110 - 0.0993j
$H_2(\omega,\omega)$	0.0038 + 0.0019 j	-0.0040 + 0.0014 j
$H_2(\omega,-\omega) = R_0$	-0.0394	-0.0310

TABLE II
ESTIMATION OF THE PARAMETERS OF THE ORIGINAL CONTINUOUS TIME SYSTEM (15) USING (17)

	a	b	c
True value	0.2	1.0	0.1
Estimates from GFRF's by (19)	0.1620	0.9654	0.0780

It is clear that there is bias on the reconstructed estimates of the systems parameters in Table II. This suggests that, though the NARMAX model (19) is a very good fit in the time domain as the MPO and correlation based model validity tests indicate, its ability of capturing the frequency domain features of the underlying system is not always reliable, as indicated in this simple example. Improved frequency response function estimation is possible if a larger pool of candidate model terms is fed into the NARMAX model estimator. The difficulty is that without a frequency domain validity guideline, it is sometimes not easy to know when to stop the search for a model that is not only valid in the time domain but also in the frequency domain. Now, the proposed new procedure (12)—(14) is applied to obtain the $\hat{H}_1$ and $\hat{H}_2$ using a Least Squares estimator. The results are shown in Table III which are very satisfactory.

TABLE III
Estimation of H_1 and H_2 of system (15) using the new approach

	From Eqn (15) –True	From new approach
$H_1(\omega) = R_1 + jI_1$	-0.6151 - 0.1262j	-0.6100 - 0.1250j
$H_2(\omega,\omega) = R_2 + jI_2$	0.0038 + 0.0019 j	0.0043 + 0.0022j
$H_2(\omega,-\omega) = R_0$	-0.0394	-0.0397

The reconstructed parameters of the continuous time model (15) from Table III using (17) are shown in Table IV, which is a significant improvement in accuracy compared with the results by the parametric method in Table II.

TABLE IV
ESTIMATION OF THE PARAMETERS OF THE ORIGINAL CONTINUOUS TIME SYSTEM (15) USING (17)

	a	b	c
True value	0.2	1.0	0.1
Estimates from new approach	0.2015	0.9867	0.1011

IV. Estimation of Frequency Response Functions of Cubically Nonlinear Systems

The literature associated with the cubic Volterra series model is substantially smaller compared with that associated with the quadratic Volterra model due to the significantly greater complexity induced. In terms of frequency response function estimation for up to third order Volterra representation, this procedure is not as straightforward as the quadratic case in Section 3. In fact, the main complication is that, unlike the quadratic Volterra model case where the first harmonics in the output are all due to the linear response function, for a cubic Volterra model both first and third order frequency response functions make contributions to the first harmonics. Therefore additional efforts are needed to solve the separation of contributions between H_1 and H_3.

First, by defining

$$H_3(\omega,\omega,\omega) = R_3(\omega,\omega,\omega) + jI_3(\omega,\omega,\omega) \quad (20)$$

the fourth term in (6) can be expanded as

$$\begin{aligned} &2(\frac{A}{2})^3 \operatorname{Re}\{H_3(\omega,\omega,\omega)e^{j3\omega k}\} \\ &= 2(\frac{A}{2})^3 R_3 \cos(3\omega k) - 2(\frac{A}{2})^3 I_3 \sin(3\omega k) \end{aligned} \quad (21)$$

which makes a contribution purely to the third harmonics (3ω) in the response.

The problem arises with the fifth term in (6). By defining

$$H_3(\omega,\omega,-\omega) = R_{31}(\omega,\omega,-\omega) + jI_{31}(\omega,\omega,-\omega) \quad (22)$$

the fifth term in (6) can be expanded as

$$\begin{aligned} &6(\frac{A}{2})^3 \operatorname{Re}\{H_3(\omega,\omega,-\omega)e^{j3\omega k}\} \\ &= 6(\frac{A}{2})^3 R_{31} \cos(\omega k) - 6(\frac{A}{2})^3 I_{31} \sin(\omega k) \end{aligned} \quad (23)$$

which makes a contribution to the first harmonics, mixed up with the contribution from the linear kernel, shown in (9).

The overall expansion of (6) for the first three Volterra kernel terms, by combining (12), (21) and (23), is given as

$$\begin{aligned}
y(k) &= [AR_1 + 6(\frac{A}{2})^3 R_{31}]\cos(\omega k) \\
&+ [AI_1 + 6(\frac{A}{2})^3 I_{31}][-\sin(\omega k)] + R_2[2(\frac{A}{2})^2 \cos(2\omega k)] \\
&+ I_2[-2(\frac{A}{2})^2 \sin(2\omega k)] + R_0 2(\frac{A}{2})^2 + 2(\frac{A}{2})^3 R_3 \cos(3\omega k) \\
&- 2(\frac{A}{2})^3 I_3 \sin(3\omega k) + e(k)
\end{aligned} \tag{24}$$

It is possible to define

$$\begin{aligned}
AR_1 + 6(\frac{A}{2})^3 R_{31} &= R \\
AI_1 + 6(\frac{A}{2})^3 I_{31} &= I
\end{aligned} \tag{25}$$

Then for $k = 1 \text{ to } N$, (24) can be arranged in the matrix form as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\theta} + \mathbf{E} \tag{26}$$

where $\mathbf{Y} = [y(N) \cdots y(1)]^T$, $\boldsymbol{\theta} = [R\ I\ R_2\ I_2\ R_0\ R_3\ I_3]^T$, $\mathbf{E} = [e(N) \cdots e(1)]$, and $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4\ \mathbf{x}_5\ \mathbf{x}_6\ \mathbf{x}_7]$, with

$$\begin{aligned}
\mathbf{x}_1 &= [\cos(\omega N) \cdots \cos(\omega)]^T \\
\mathbf{x}_2 &= [-\sin(\omega N) \cdots -\sin(\omega)]^T \\
\mathbf{x}_3 &= [2(\frac{A}{2})^2 \cos(2\omega N) \cdots 2(\frac{A}{2})^2 \cos(2\omega)]^T \\
\mathbf{x}_4 &= [-2(\frac{A}{2})^2 \sin(2\omega N) \cdots -2(\frac{A}{2})^2 \sin(2\omega)]^T \\
\mathbf{x}_5 &= [2(\frac{A}{2})^2 \cdots 2(\frac{A}{2})^2]^T \\
\mathbf{x}_6 &= [2(\frac{A}{2})^3 \cos(3\omega N) \cdots 2(\frac{A}{2})^3 \cos(3\omega)]^T \\
\mathbf{x}_7 &= [-2(\frac{A}{2})^3 \sin(3\omega N) \cdots -2(\frac{A}{2})^3 \sin(3\omega)]^T
\end{aligned} \tag{27}$$

From which the unknown frequency response function data $\hat{\boldsymbol{\theta}} = [R\ I\ R_2\ I_2\ R_0\ R_3\ I_3]^T$ can be derived using a standard Least Square procedure. In order to separate in (25) the R_1 and R_{31} from R, and the I_1 and I_{31} from I, two tests at different input levels, denoted by $A^{(1)}$ and $A^{(2)}$, at the same frequency, are required to obtain two sets of estimates, $R^{(1)}$, $R^{(2)}$, and $I^{(1)}$ and $I^{(2)}$ respectively. Then the final estimates of $\hat{H}_1 = \hat{R}_1 + j\hat{I}_1$ and $\hat{H}_3(\omega,\omega,-\omega) = \hat{R}_{31} + j\hat{I}_{31}$ can be calculated from (25) as

$$\begin{aligned}
\hat{R}_1 &= \frac{6(\frac{A^{(2)}}{2})^3 R^{(1)} - 6(\frac{A^{(1)}}{2})^3 R^{(2)}}{A^{(1)} 6(\frac{A^{(2)}}{2})^3 - A^{(2)} 6(\frac{A^{(1)}}{2})^3} \\
\hat{R}_{31} &= \frac{A^{(2)} R^{(1)} - A^{(1)} R^{(2)}}{A^{(2)} 6(\frac{A^{(1)}}{2})^3 - A^{(1)} 6(\frac{A^{(2)}}{2})^3}
\end{aligned} \tag{28a}$$

$$\begin{aligned}
\hat{I}_1 &= \frac{6(\frac{A^{(2)}}{2})^3 I^{(1)} - 6(\frac{A^{(1)}}{2})^3 I^{(2)}}{A^{(1)} 6(\frac{A^{(2)}}{2})^3 - A^{(2)} 6(\frac{A^{(1)}}{2})^3} \\
\hat{I}_{31} &= \frac{A^{(2)} I^{(1)} - A^{(1)} I^{(2)}}{A^{(2)} 6(\frac{A^{(1)}}{2})^3 - A^{(1)} 6(\frac{A^{(2)}}{2})^3}
\end{aligned} \tag{28b}$$

The above procedure will be illustrated using the well-known Duffing oscillator.

Consider a Duffing oscillator, with cubic nonlinearity, subject to a sinusoidal excitation as

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = A\cos(\omega t) \tag{29}$$

where m, c, k_1 and k_3 are the mass, the damping, the linear stiffness and nonlinear stiffness respectively. The nonlinear stiffness parameter k_3 in (29) needs to stay small in order to be 'weakly' nonlinear for the existence of the Volterra series representation. The corresponding GFRF's from (29) are

$$\begin{aligned}
H_1(s)\big|_{s=j\omega} &= \frac{1}{s^2 + cs + k_1} \\
H_2(s_1, s_2)\big|_{\substack{s_1=j\omega_1 \\ s_2=j\omega_2}} &= 0 \\
H_3(s_1, s_2, s_3)\big|_{\substack{s_1=j\omega_1 \\ s_2=j\omega_2 \\ s_3=j\omega_3}} &= -k_3 H_1(s_1) H_1(s_2) H_1(s_3) \\
&\cdot H_1(s_1 + s_2 + s_3)
\end{aligned} \tag{30}$$

The system (29) was excited at $\omega = 1.5$ rad/sec and $T_s = 0.02$sec with the parameters

$$m = 1, c = 1.5, k_1 = 0.5, k_3 = 0.1 \tag{31}$$

The excitation is chosen as $A^{(i)}\cos(\omega t), i = 1,2$ where $A^{(1)} = 2$ and $A^{(2)} = 3$. A zero-mean white noise was added to each of the outputs with a SNR =40 dB. The length of the data was 2000. The estimation results for $\hat{H}_1$ and $\hat{H}_3$ using the new algorithm are given in Table V.

TABLE V
ESTIMATION OF H_1 AND H_3 FOR THE DUFFING EQUATION (29) USING THE NEW APPROACH

	Eqn (29) –True	From new approach
$H_1(\omega) = R_1 + jI_1$	-0.2154- 0.2769 j	-0.2159 - 0.2776 j
$H_3(\omega,\omega,\omega) = R_3 + jI_3$	2.0616e-04 -1.7152e-05 j	2.011e-04 -1.8124e-04 j
$H_3(\omega,\omega,-\omega) = R_{01} + jI_{31}$	0.3729e-03-0.1468e-02 j	0.4595e-03-0.1579e-02 j

By using (30), the original continuous time system parameters can be extracted from $\hat{H}_1$ and $\hat{H}_2$ as

$$\hat{c} = \frac{1}{\omega}\mathrm{Im}[\frac{1}{\hat{H}_1(\omega)}]$$

$$\hat{k}_1 = \mathrm{Re}[\frac{1}{\hat{H}_1(\omega)}] + \omega^2 \tag{32}$$

$$\hat{k}_3 = \frac{-\hat{H}_3(\omega,\omega,-\omega)}{\hat{H}_1(\omega)\hat{H}_1(\omega)\hat{H}_1(\omega)\hat{H}_1(-\omega)}$$

It is clear from Table VI that the reconstructed continuous time system parameters, comparing with the true system parameters in (31), are very satisfactory.

TABLE VI
ESTIMATION OF PARAMETERS OF ORIGINAL CONTINUOUS TIME SYSTEM (29) USING (32)

	c	k_1	k_3
True value from (31)	1.5	0.5	0.1
Estimates from (32)	1.4960	0.5037	0.0991

V. THE SELECTION OF LEVEL OF EXCITATION USING OLS

The new algorithm introduced in the previous sections is based on the assumption that the underlying system is weakly nonlinear in the sense that it can be well described by the first few Volterra kernels, with the higher order kernels fading off rapidly. The accuracy of the results of the new procedure is largely dependant on this assumption as are all other non-parametric methods, and the significance of the nonlinearity of the system, reflected by the Volterra kernel order, is associated with the level of the excitation. Either under-excitation or over-excitation may result in inaccuracy in the estimates. It is therefore essential to have a measure that can provide an indication of the order of the Volterra kernels under certain levels of excitation before the final application of the new algorithm. One possible measure is based on using the orthogonal least squares method (OLS) [20]—[21], which is briefly reviewed below.

Consider a system expressed by the linear-in-the-parameters model

$$z = \sum_{i=1}^{M}\theta_i p_i + \varepsilon \tag{33}$$

where $\theta_i, i = 1,\ldots,M$ are unknown parameters.

Reformulating equation (33) in the form of an auxiliary model yields

$$z = \sum_{i=1}^{M} g_i w_i + \varepsilon \tag{34}$$

where $g_i, i = 1,\ldots,M$ are the auxiliary parameters and $w_i, i = 1,\ldots,M$ are constructed to be orthogonal over the data record such that

$$\sum_{t=1}^{N} w_j(t) w_{k+1}(t) = 0, \qquad j = 0,1,\ldots,k \tag{35}$$

where N is the length of the data record.

Multiplying the auxiliary model (34) by itself, using the orthogonal property (35) and taking the time average gives

$$\frac{1}{N}\sum_{t=1}^{N} z^2(t) = \frac{1}{N}\sum_{t=1}^{N}\left\{\sum_{i=0}^{M} g_i^2 w_i^2(t)\right\} + \frac{1}{N}\sum_{t=1}^{N}\varepsilon^2(t) \tag{36}$$

Finally define

$$\mathrm{ERR}_i = \frac{\sum_{t=1}^{N} g_i^2 w_i^2(t)}{\sum_{t=1}^{N} z^2(t) - \frac{1}{N}\left\{\sum_{t=1}^{N} z(t)\right\}^2} \times 100 \tag{37}$$

for $i = 1,2,\ldots,M$. The quantity ERR_i is called the Error Reduction Ratio and provides an indication of which terms should be included in the model in accordance with the contribution each term makes to the energy of the dependent variable. Terms with associated ERR values which are less than a pre-defined threshold value can be considered to be insignificant and negligible.

For simplicity, the quadratic system (15), with $a = 0.2, b = 1$ and $c = 0.1$, was used as an example to illustrate the use of OLS in the selection of the amplitude of excitation A. Two tests were conducted at different excitation amplitudes at frequency $\omega = 2\,\mathrm{rad/sec}$. First, the amplitude of the input was chosen at A=0.3 and the response was corrupted by a zero-mean white noise with SNR =40 dB. The OLS was applied to obtain the estimates of linear and quadratic frequency response functions, together with the values ERR_i, shown in Table VII.

TABLE VII
ESTIMATION OF H_1 AND H_2 OF SYSTEM (15) AT $A = 0.3$ USING OLS

	Result by Eqn (15) –True	Result by OLS	ERR_i (%)
R_1	-0.3275	-0.3276	97.3370
I_1	-0.04367	-0.04349	1.6999
R_2	6.902e-04	-9.519e-04	1.867e-05
I_2	2.275e-04	1.980e-03	7.8622e-05
R_0	-0.01092	-0.01459	8.704e-03

It can be seen from Table VII that at this level of excitation, the contributions by the quadratic term H_2, i.e., R_2, I_2 and R_0, are extremely small compared with the contributions from the linear term. This means that the quadratic kernel contribution has a very small SNR, consequently the LS estimation results for the quadratic terms are more affected by the presence of noise, leading to unreliability in the estimates. The almost negligible sum of ERR of the quadratic terms, i.e., from terms R_2, I_2 and R_0, suggests that the system can be regarded as linear at this level of excitation, therefore this can be considered as under-excited.

Now the amplitude of input was chosen at $A = 5$ and again a zero-mean white noise was added to the response with SNR =40 dB. The OLS results are given in Table VIII.

Table VIII
Estimation of H_1 and H_2 of system (15) at $A=5$ using OLS

	Result by Eqn (15) –True	Result by OLS	ERR_i (%)
R_1	-0.3275	-0.3246	96.117
I_1	-0.04367	-0.04315	1.5499
R_2	6.902e-04	5.753e-04	1.892e-03
I_2	2.275e-04	2.485e-03	3.437e-04

It is clear by looking at the ERR_i values from Table VIII that the overall contributions by the quadratic terms, especially the R_0 term are no longer negligible. As a result, the accuracy of the estimates of the quadratic response functions becomes very satisfactory. Therefore in terms of GFRF's estimation for system (15) using the new approach, the amplitude used in the second test, $A=5$, is a much better choice, following the suggestions of ERR_i. In addition, the sum of the values ERR_i of all the linear and quadratic terms is 99.0214%, indicating that this system at this amplitude level can be sufficiently described by up to second order Volterra kernels.

It needs to be pointed out that the approach presented in this paper can be extended to admit 2-tone or multi-tone sinusoidal inputs. For example, if the input is a 2-tone signal in the form

$$u(k)=\sum_{i=1}^{2} A_i \cos(\omega_i k) \tag{38}$$

and the system can be sufficiently described by first and quadratic Volterra kernels, then the single-tone response expression (7) can be extended to

$$\begin{aligned} y(k) &= \sum_{i=1}^{2} A_i \operatorname{Re}\{H_1(\omega_i)e^{j\omega_i k}\} + \sum_{i=1}^{2} 2(\frac{A_i}{2})^2 \operatorname{Re}\{H_2(\omega_i,\omega_i)e^{j2\omega_i k}\} \\ &+ \sum_{i=1}^{2} 2(\frac{A_i}{2})^2 \operatorname{Re}\{H_2(\omega_i,-\omega_i)\} + 4(\frac{A_1}{2}\frac{A_2}{2})\operatorname{Re}\{H_2(\omega_1,\omega_2)e^{j2(\omega_1+\omega_2)k}\} \\ &+ 4(\frac{A_1}{2}\frac{A_2}{2})\operatorname{Re}\{H_2(\omega_1,-\omega_2)e^{j2(\omega_1-\omega_2)k}\} + e(k) \end{aligned} \tag{39}$$

Equation (39) can be further expanded to a form similar to (12) involving real and imaginary parts of the individual GFRF's, from which $H_1(\omega_1)$, $H_1(\omega_2)$, $H_2(\omega_1,\omega_1)$, $H_2(\omega_2,\omega_2)$ and $H_2(\omega_1,\omega_2)$ etc can be estimated using the LS procedure illustrated in Section 3. The only concern for the quadratic system under a 2-tone sinusoidal input is the separation of the contributions by $H_2(\omega_1,-\omega_1)$ and $H_2(\omega_2,-\omega_2)$, which are two d.c. components. This can be dealt with by applying the 2-tone signal twice at 2 levels for each frequency.

VI. CONCLUSIONS

A new non-parametric algorithm, which directly uses the time domain input-output measurements but avoids the direct differentiation of these data, has been derived to estimate up to the third order generalised frequency response functions and subsequently identify the associated continuous time model. This is achieved by expanding the algebraic expression in the analysis of the output response using the real and imaginary parts of each order of Volterra kernels. The new algorithm has the advantage of admitting smaller data sets than the traditional spectral analysis based frequency domain non-parametric methods.

Like many other general parameter estimation problems, the accuracy of the estimation of the GFRF's in this new approach depends on the fact that the level of excitation is appropriate. That is, the relevant order of nonlinearity is adequately excited. The OLS method has superior numerical properties compared with the ordinary LS method, in the sense that it can provide vital information on the suitability of the excitation level, indirectly from the contribution indicators (ERR) for each Volterra order.

Although the new approach was illustrated using single-tone sinusoidal inputs, the basic idea can be readily extended to accommodate 2-tone or even multi-tone inputs, in which case the complexity of the procedure will inevitably grow dramatically as the number of tones in the input and the order of Volterra kernels are increased.

The continuous time model can be reconstructed from the estimated GFRF's. A direct extraction of the continuous time model for the simple low dynamic order system is possible from the GFRF's at one single frequency point, as illustrated in this paper. When the continuous time model structure is not known a priori, or is in a more complicated form, this new procedure can be repeated at different excitation frequencies until a sufficient number of points have been collected, from which the identification of the general form of continuous time model can be derived using the approach in [22].

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