Monograph:

Li, L.M. and Billings, S.A. (2009) Analysis of nonlinear oscillators using volterra series in the frequency domain Part I : convergence limits. Research Report. ACSE Research Report no. 988 . Automatic Control and Systems Engineering, University of Sheffield

Analysis of Nonlinear Oscillators Using Volterra Series in the Frequency Domain
Part I : Convergence Limits

L.M.Li and S.A.Billings

Department of Automatic Control
and Systems Engineering,
University of Sheffield, Sheffield
Post Box 600 S1 3JD
UK

Research Report No. 988

May 2009

Analysis of Nonlinear Oscillators Using Volterra Series in the Frequency Domain Part I : Convergence Limits

L.M.Li and S.A.Billings*

Department of Automatic Control and Systems Engineering
University of Sheffield
Sheffield S1 3JD
UK
*S.Billings@sheffield.ac.uk

Abstract: The Volterra series representation is a direct generalisation of the linear convolution integral and has been widely applied in the analysis and design of nonlinear systems, both in the time and the frequency domain. The Volterra series is associated with the so-called weakly nonlinear systems, but even within the framework of weak nonlinearity there is a convergence limit for the existence of a valid Volterra series representation for a given nonlinear differential equation. Barrett(1965) proposed a time domain criterion to prove that the Volterra series converges with a given region for a class of nonlinear systems with cubic stiffness nonlinearity. In this paper this time-domain criterion is extended to the frequency domain to accommodate the analysis of nonlinear oscillators subject to harmonic excitation.

1 Introduction

Nonlinear Volterra theory was initially proposed by Volterra(1930). The theory quickly received a great deal of attention in the field of electrical engineering, mechanical engineering, and later in the biological field, as a powerful approach for modelling nonlinear system behaviours. From the late 1950s, there has been a continuous effort in the application of Volterra series to nonlinear systems theory. Summaries of major contributions in the application of Volterra series modelling for the representation, analysis and design of nonlinear systems can be found in Schetzen(1980), Rugh(1981) , Sandberg(1984) and Nam and Powers(1994).

Based on the Volterra series representation, Generalised Frequency Response Functions (GFRF's) have proved to be powerful in the analysis and design of nonlinear systems in the frequency domain(Billings and Tsang, 1989). However this analysis can only be directly applied to so-called weakly nonlinear systems, which usually represent a small subset of the rich characteristics of nonlinear dynamics. It is therefore desirable to have a simple criterion establishing the boundary between weak nonlinearity and severe nonlinearity in the frequency domain.

In part I of this paper a new method is proposed to find the convergence region for a class of nonlinear oscillators with cubic stiffness nonlinearity subject to harmonic excitation. In section 2 the Volterra/frequency modelling for single-input-single-output nonlinear systems is reviewed. In section 3, the extension of the Barrett's time domain method (Barrett,1965) to the frequency domain to accommodate harmonic excitation is presented. In section 4, numerical examples are used to demonstrate the new approach, along with the comparison with previous approaches. Finally in section 5 conclusions are given.

In part II of this paper the well known jump phenomenon associated with the Duffing oscillator is studied.

2 Volterra modelling in the time and frequency domain

Volterra(1930) series modelling has been widely studied for the representation, analysis and design of nonlinear systems. The Volterra model is a direct generalisation of the linear convolution integral, therefore providing an intuitive representation in a simple and easy to apply way. For a SISO nonlinear system, where $u(t)$ and $y(t)$ are the input and output respectively, the Volterra series can be expressed as

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \tag{1.a}$$

and $y_n(t)$ is the 'n-th order output' of the system

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \qquad n > 0 \tag{1.b}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is called the 'nth-order kernel' or 'nth-order impulse response function'. If $n=1$, this reduces to the familiar linear convolution integral.

The discrete time domain counterpart of the continuous time domain SISO Volterra expression (1) is

$$y(k) = \sum_{n=1}^{\infty} y_n(k) \tag{2.a}$$

where

$$y_n(k) = \sum_{-\infty}^{\infty} \cdots \sum_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n) \prod_{i=1}^{n} u(k-\tau_i) \qquad n > 0, k \in \mathrm{Z} \tag{2.b}$$

In practice only the first few kernels are used on the assumption that the contribution of the higher order kernels falls off rapidly. Systems that can be adequately represented by a Volterra series with just a few terms are called a weakly or mildly nonlinear system.

A valid Volterra series representation means valid Generalised Frequency Response Functions(GFRF's). The GFRF's are obtained by taking the multi-dimensional Fourier transform of $h_n(\cdot)$:

$$H_n(\omega_1,\cdots,\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n) \exp(-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)) d\tau_1 \cdots d\tau_n \tag{3}$$

The generalised frequency response functions represent an inherent and invariant property of the underlying system, and have proved to be an important analysis and design tool for characterising nonlinear phenomena. In practice, the GFRF's can be estimated using non-parametric or parametric methods. The parametric method involves mapping a nonlinear differential equation(Billings and Peyton Jones, 1990) or mapping a nonlinear difference equation(Peyton Jones and Billings, 1989) into the frequency domain using the probing method.

3 Derivation of convergence region in the frequency domain

Barrett(1963) was one of the first people to carry out a systematic study of the application of Volterra series to the analysis of nonlinear differential equations and nonlinear feedback systems. A new time domain approach for finding the convergence region of the derived Volterra series representation under arbitrary inputs was later proposed by Barrett(1965). In this paper this time domain approach is extended to the frequency domain to accommodate the case of nonlinear oscillations subject to harmonic excitations.

Considering the following system with cubic stiffness nonlinearity

$$L(\frac{d}{dt})y(t) + k_3 y^3(t) = u(t) \tag{4}$$

where k_3 is a small constant and $L(q) = q^p + a_1 q^{p-1} + \ldots + a_p$ with the assumption that $L(q) = 0$ has roots with negative real parts.

Initially, the time domain convergence analysis proposed by Barrett(1965) will be briefly illustrated.

The nonlinear system (4) has a Volterra series representation which includes the 1st and 3rd order kernels(Barrett, 1965)

$$y(t) = \int_{-\infty}^{\infty} h\ (t-\tau)u(\tau)d\tau - k_3 \int\int\int_{-\infty}^{\infty} h_3(t-\tau_1, t-\tau_2, t-\tau_3)u(\tau_1)u(\tau_2)u(\tau_3)d\tau_1 d\tau_2 d\tau_3 \tag{5}$$

where

$$h_3(t-\tau_1, t-\tau_2, t-\tau_3) = \int_{-\infty}^{\infty} h_1(t-\tau)h_1(t-\tau_1)h_1(t-\tau_2)h_1(t-\tau_3)d\tau \tag{6}$$

Denoting

$$\|u\| = \sup_{-\infty<t<\infty} |u(t)| \ ,\ \Theta = \int_0^{\infty} |h(t)| dt$$

For a convergent Volterra expression, each of the terms in (5) needs to be bounded. For the first term,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \right| &\le \int_{-\infty}^{\infty} |h(t-\tau)||u(\tau)|d\tau \\ &\le \Theta \|u\| \end{aligned} \tag{7}$$

Similarly for the second term, by using (6),

$$
\begin{aligned}
&\left|\iiint_{-\infty}^{\infty} h_3(t-\tau_1, t-\tau_2, t-\tau_3)u(\tau_1)u(\tau_2)u(\tau_3)d\tau_1 d\tau_2 d\tau_3\right| \\
&\le \iiint_{-\infty}^{\infty} \left|h_3(t_1,t_2,t_3)\right| dt_1 dt_2 dt_3 \cdot \|u\|^3 \\
&= \iiint_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} h_1(t')h_1(t_1-t')h_1(t_2-t')h_1(t_3-t')dt'\right| dt_1 dt_2 dt_3 \cdot \|u\|^3 \\
&\le \iiint \int_{-\infty}^{\infty} \left|h_1(t')\right|\left|h_1(t_1-t')\right|\left|h_1(t_2-t')\right|\left|h_1(t_3-t')\right| dt' dt_1 dt_2 dt_3 \cdot \|u\|^3 \\
&\le \int_{-\infty}^{\infty} \left|h_1(t')\right| dt' \int_{-\infty}^{\infty} \left|h_1(t_1-t')\right| dt_1 \int_{-\infty}^{\infty} \left|h_1(t_2-t')\right| dt_2 \int_{-\infty}^{\infty} \left|h_1(t_3-t')\right| dt_3 \cdot \|u\|^3 \\
&= \left(\int_{-\infty}^{\infty} \left|h_1(t)\right| dt\right)^4 \cdot \|u\|^3 \\
&= \Theta^4 \|u\|^3
\end{aligned}
\tag{8}
$$

Therefore each term of the Volterra series representation in (5) does not exceed, in absolute magnitude, the corresponding term of a certain power series

$$Y = \Phi(U) = \Theta U + |k_3|\Theta^4 U^3 \tag{9}$$

provided that $\|u\| < U$.

In fact the solution of (9) is a series solution of the equation

$$Y - |k_3|\Theta Y^3 = \Theta U \tag{10}$$

by the method of successive approximation.

Barrett(1965) proved that the Volterra series (5) is convergent if $\|u\| < U_1$ where

$$U_1 = \frac{2}{\sqrt{|k_3|}(3\Theta)^{3/2}} \tag{11}$$

Nonlinear systems described by (4) represent a large number of single-degree-of-freedom physical systems with nonlinear stiffness and widely exist in circuits, aircraft and marine engineering etc. Most of the dynamics of these nonlinear systems, such as hysteresis, limit cycles, bifurcations and chaos, were built in the framework of nonlinear oscillation and vibration subject to harmonic excitations. It is therefore desirable to extend the purely time domain criterion (11) into the frequency domain by probing the system using harmonics excitation.

Assume the excitation is in the harmonic form

$$u(t) = Ae^{j\omega t} \qquad \text{with } A>0$$

The first term in (5) is bounded by

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau\right| &= A\left|\int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau\right| \\
&= A\left|\int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau\right| \\
&= A\left|e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right| \\
&= A\left|e^{j\omega t}H_1(\omega)\right| \\
&\le A\left|e^{j\omega t}\right|\left|H_1(\omega)\right| \\
&\le A\left|H_1(\omega)\right|
\end{aligned}
\tag{12}
$$

using the definition

$$H_1(\omega) = \int_{-\infty}^{\infty} h(\tau)\exp(-j\omega\tau)d\tau$$

Similarly the second term in (5) is bounded by

$$\left|\int\int\int_{-\infty}^{\infty} h_3(t-\tau_1, t-\tau_2, t-\tau_3)u(\tau_1)u(\tau_2)u(\tau_3)d\tau_1 d\tau_2 d\tau_3\right| \le A^3 |H_1(\omega)|^4 \quad (13)$$

Hence the response of the Volterra series representation (5) will not exceed, in absolute magnitude,

$$\hat{Y} = A|H_1(\omega)| + |k_3| A^3 |H_1(\omega)|^4 \quad (14)$$

with $\hat{Y}$ is the successive approximation of

$$\hat{Y} - |k_3| A^3 |H_1(\omega)|^4 = A|H_1(\omega)| \quad (15)$$

The maximum amplitude $\tilde{A}$ of excitation allowed to have a convergent Volterra series at different excitation frequency ω can be obtained by solving $dA/d\hat{Y} = 0$ in (15) to give

$$\tilde{A}(\omega) = \frac{2}{\sqrt{|k_3|}(3|H_1(\omega)|)^{3/2}} \quad (16)$$

Note that

$$\begin{aligned} |H_1(\omega)| &= \left|\int_0^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau\right| \\ &\le \int_0^{\infty} |h(\tau)|\left|e^{j\omega(t-\tau)}\right|d\tau \\ &\le \int_0^{\infty} |h(\tau)|d\tau = \Theta \end{aligned} \quad (17)$$

Therefore the following result holds between the time domain criterion (11) and the frequency domain criterion (16):

$$U_1 \le \min_{0<\omega<\infty}(\tilde{A}(\omega)) \quad (18)$$

(18) reflects the fact that Barrett's time domain criterion, which deals with arbitrary excitation and with harmonic excitation as a special case, represents the worst case and is therefore a conservative result.

4 Numerical illustrations and discussions

By setting $p = 2$ with $u(t)$ in sinusoidal format, (4) reduces to

$$\ddot{y} + a_1\dot{y} + a_2 y + k_3 y^3 = A\cos(\omega t) \quad (19)$$

which is the well-know Duffing's oscillator, introduced by Duffing in 1918 to describe a mechanical problem under periodic forces. It is one of the most common examples in the study of nonlinear oscillations. Typically in a mechanical format Duffing's oscillator is described as

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = A\cos(\omega t) \quad (20)$$

where m is the mass, c is the damping, k_1 is proportional to the stiffness of the spring, and k_3 is the cubic stiffness. $k_3 > 0$ models a hardening nonlinearity, and $k_3 < 0$

models a softening nonlinearity. The upper limit of excitation level which allows a valid Volterra series representation under both hardening and softening nonlinearity situations will be discussed next. A valid Volterra series representation means valid GFRF's from which the steady-state estimation of the response can be generated. Because the Duffing equation (20) contains a cubic nonlinear term y^3, all even orders of GFRF's are zero and make no contribution to the system response. Therefore the steady-state response using with nonlinearities up to the 7th order truncation is (Bedrosian and Rice, 1971)

$$y(t) \approx \sum_{n=1}^{5} y_n(t) = y_1(t) + y_3(t) + y_5(t) + y_7(t) \tag{21}$$

where the response for the various orders are

$$y_1(t) = 2(\frac{A}{2})\operatorname{Re}\{H_1(j\omega)e^{j\omega t}\} \tag{22a}$$

$$\begin{aligned} y_3(t) = {} & 2(\frac{A}{2})^3 \operatorname{Re}\{H_3(j\omega, j\omega, j\omega)e^{3j\omega t}\} \\ & + 6(\frac{A}{2})^3 \operatorname{Re}\{H_3(j\omega, j\omega, -j\omega)e^{j\omega t}\} \end{aligned} \tag{22b}$$

$$\begin{aligned} y_5(t) = {} & 2(\frac{A}{2})^5 \operatorname{Re}\{H_5(j\omega, j\omega, j\omega, j\omega, j\omega)e^{5j\omega t}\} \\ & + 10(\frac{A}{2})^5 \operatorname{Re}\{H_5(j\omega, j\omega, j\omega, j\omega, -j\omega)e^{3j\omega t}\} \\ & + 20(\frac{A}{2})^5 \operatorname{Re}\{H_5(j\omega, j\omega, j\omega, -j\omega, -j\omega)e^{j\omega t}\} \end{aligned} \tag{22c}$$

$$\begin{aligned} y_7(t) = {} & 2(\frac{A}{2})^7 \operatorname{Re}\{H_7(j\omega, j\omega, j\omega, j\omega, j\omega, j\omega, j\omega)e^{7j\omega t}\} \\ & + 14(\frac{A}{2})^7 \operatorname{Re}\{H_7(j\omega, j\omega, j\omega, j\omega, j\omega, j\omega, -j\omega)e^{5j\omega t}\} \\ & + 42(\frac{A}{2})^7 \operatorname{Re}\{H_7(j\omega, j\omega, j\omega, j\omega, j\omega, -j\omega, -j\omega)e^{3j\omega t}\} \\ & + 70(\frac{A}{2})^7 \operatorname{Re}\{H_7(j\omega, j\omega, j\omega, j\omega, -j\omega, -j\omega, -j\omega)e^{j\omega t}\} \end{aligned} \tag{22d}$$

4.1 Hardening nonlinearity

A number of criteria to find the upper limits of the magnitude of harmonic excitation in Duffing's oscillator with $k_3 > 0$ have been proposed (Tomlinson *et al* , 1996;Chatterjee and Vyas, 2000; Peng and Lang, 2007). Peng and Lang(2007) pointed out that the criterion proposed by Chatterjee and Vyas(2000) based on a ratio test procedure was essentially the same as Tomlinson's(Tomlinson *et al* , 1996) with different computational approaches. Therefore only Tomlinson's and Peng and Lang's result are presented here for discussion.

Tomlinson's criterion is defined as

$$\tilde{A}_T(\omega) < \left[\tfrac{2}{3}(k_3 |H_1(\omega)|^3\right]^{-\frac{1}{2}} \tag{23}$$

which is similar to the extended Barrett's result (16) but differs in magnitude, and Peng and Lang's criterion is defined as

$$\tilde{A}_p(\omega) < \frac{1}{|H_1(\omega)|\sqrt{\lambda k_3}} \quad (24)$$

where

$$\lambda = \max_{k=1,\ldots,\infty}\left(|H_1((2k-1)\omega|\right) \quad (25)$$

The coefficients of the Duffing oscillator used in the numerical study are

$$m = 1, c = 1.5, k_1 = 0.5, k_3 = 0.1 \quad (26)$$

The frequency domain criteria by (16), (23) and (24) are shown in Figure 1.

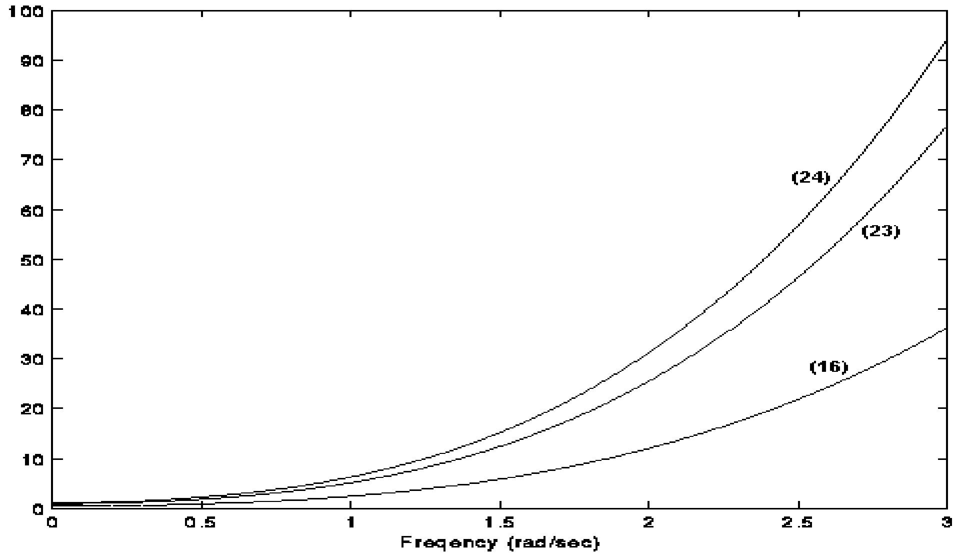


Figure 1. Frequency domain criterion by (16), (23) and (24)

Barrett's time domain result by (11), in this example, is exactly the minimum value from the new frequency domain criterion (16), that is

$$U_1 = \min_{0<\omega<\infty}(\tilde{A}(\omega)) = 0.43$$

which occurs at $\omega = 0$.

It can be seen from Figure 1 that in the low frequency range the results of the different criterion are not significantly different, but as the frequency increases the system can endure higher and higher levels of excitation with a valid Volterra series solution, and the difference between different criteria becomes more and more significant. For example, at $\omega = 2.5\,rad/sec$, the result by criterion (23) is $\tilde{A}_T(\omega)\big|_{\omega=2.5} = 46.44$. Exciting the Duffing system (26) at level $A = 45$ which is within the region set out by $\tilde{A}_T(\omega)$, the resulting real response and the synthesized response up to 7th order GFRF's using (21)-(22) are compared in Figure 2.

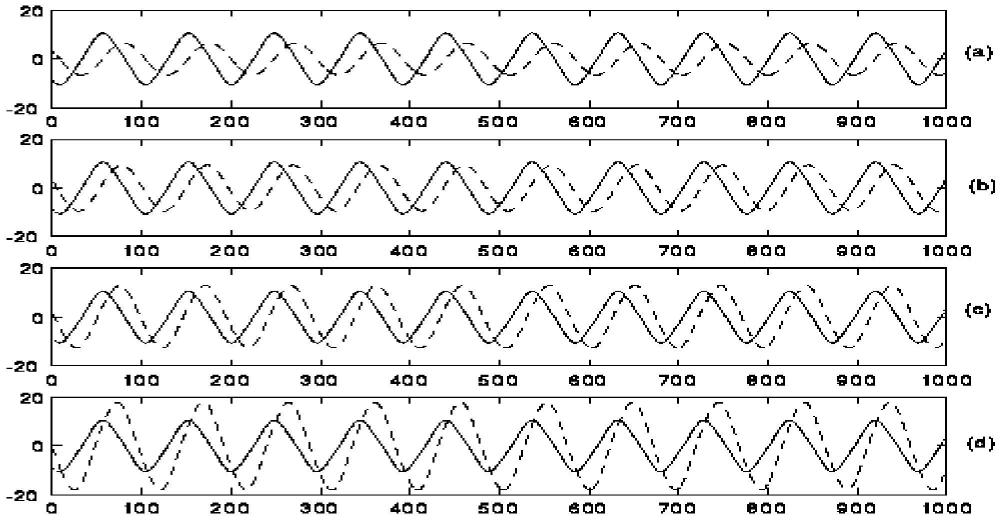


Figure 2 (a) First order output response, (b) up to the 3rd order response, (c) up to the 5th order response, and (d) up to the 7th order response Dashed— synthesized output by GFRF's from (22); Solid--simulated original output from (26)

A clearer numerical measure of the closeness of fit between the synthesized response and the real response can be obtained by using the Normalised Root Mean Square error defined as

$$NMSE = \sqrt{\frac{\sum (y_{syn}(t) - y_{real}(t))^2}{\sum (y_{real}(t) - y_{mean}(t))^2}} \tag{27}$$

where $y_{syn}(t)$ is the synthesized response and $y_{mean}(t)$ is the mean value of the real data set $y(t)$. Table 1 shows the NMSE in different approximation orders.

	1st order synthesis	Up to 3rd order synthesis	Up to 5th order synthesis	Up to 7th order synthesis
NMSE	1.1646	1.1944	1.2586	1.4729

Table 1. Comparison of NMSE between the real response and synthesized response

The results in Figure 2 and Table 1 suggest that the Volterra representation at this excitation level is not convergent.

In addition, the power spectrum of the real response is illustrated in Figure 3, showing an overwhelmingly dominant first harmonic presence, compared with other higher order harmonics. Whereas in Figure 1(a) there is a large bias in the first order harmonic estimation, in terms of both amplitude and phase. It can be argued that even theoretically there is a convergent Volterra series representation around this level of excitation, this convergence can only be achieved at the high cost of employing extremely high order GFRF's in order to compensate the bias in the first order harmonic. This makes the Volterra series representation or the application of GFRF's using the original system description (26) uneconomic in terms of computational burden and expression, and therefore less practical.

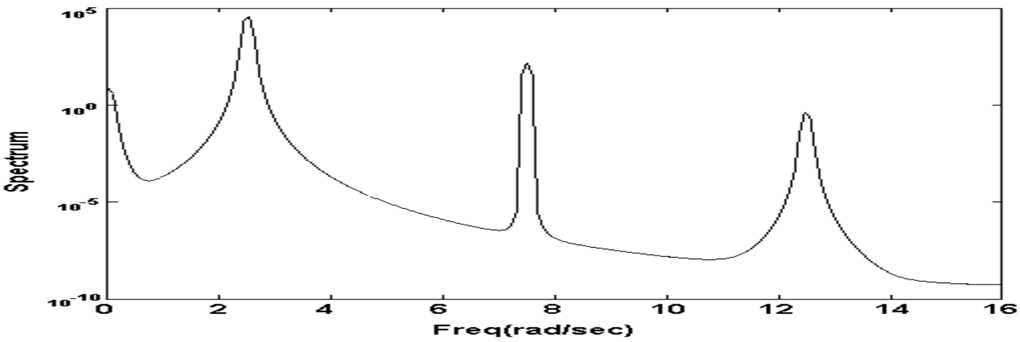


Figure 3. Power spectrum of the real response from system (26) at A**=45**

Consider another example. By denoting $\omega_0 = \sqrt{k_1/m}$, $\mu = c/(2\omega_0)$, $\varepsilon = k_3/k_1$, $A_0 = (1/m)A$, (20) can be transferred to the other commonly used form

$$\ddot{y} + 2\mu\omega_0\dot{y} + \omega_0^2 y + \varepsilon\omega_0^2 y^3 = A_0\cos(\omega t) \tag{28}$$

which has been studied by Peng and Lang(2007).

A comparison of the criteria is shown in Figure 4 using (16), (23) and (24) respectively. Again in the low frequency range, there are small differences between all the criteria (16), (23) and (24) for locating the upper limits of the Volterra series representation under harmonic excitation. In particular, the criterion (24) by Peng and Lang(2007) can provide a more accurate estimation of the excitation limit at the $\frac{1}{3}\omega_0$ sub-resonant frequency. While as the frequency increases to greater than the natural resonance frequency ω_0, as the system experiences more complicated dynamics, the differences between the criteria become more and more significant. The frequency dynamics of the underlying system can be more easily studied by a diagram called the Response Spectrum Map(RSM), which was proposed by Billings and Boaghe(2001) as a frequency domain alternative to the traditional bifurcation diagram. One such RSM is shown in Figure 5 at $\omega = 2.1\omega_0$, with $A_1 = 24.23, A_2 = 51.42$ and $A_3 = 62.97$ in the figure corresponding to the criteria (16), (23) and (24) respectively. The line H1 represents the first order harmonics ω (fundamental frequency) and the line H3 is the third order harmonics 3ω. It is very clear from Figure 5 that for $38 < A < 50$, the system has $\frac{1}{3}$ subharmonics in the response at most frequencies, and for $50 < A < 65$ the system experiences $\frac{1}{5}$ subharmonics. It is well known that the Volterra series can not be used to directly model the system that exhibits subharmonics, this means that in this case, the values A_2 and A_3, which are well into the subharmonic zone, are overestimated. It can also be easily examined that the real convergent upper limit for this case is around $A = 21$, which is close to the new result $A_1 = 24.23$ by (16).

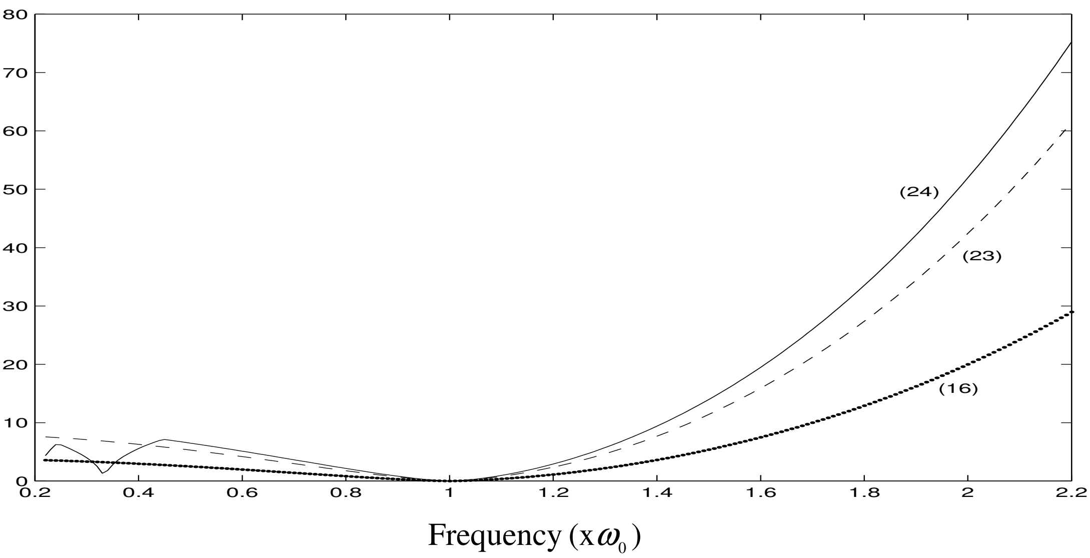
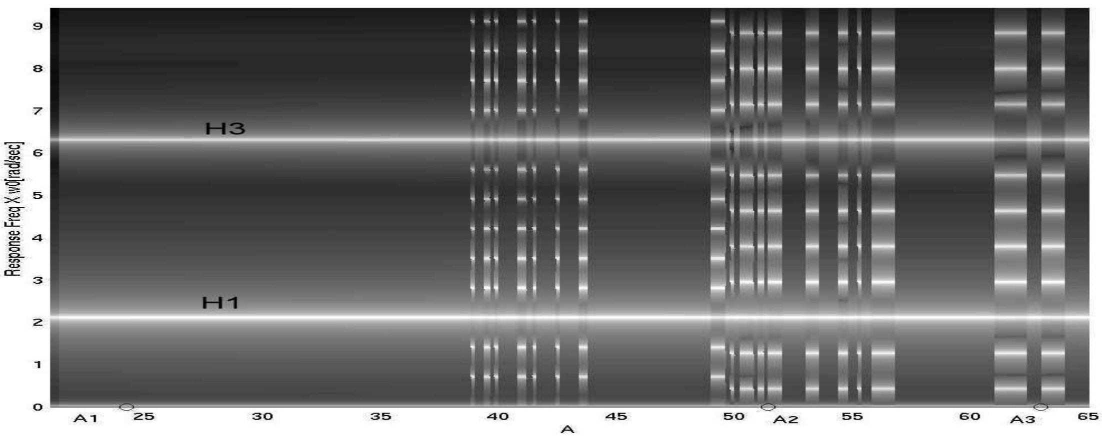
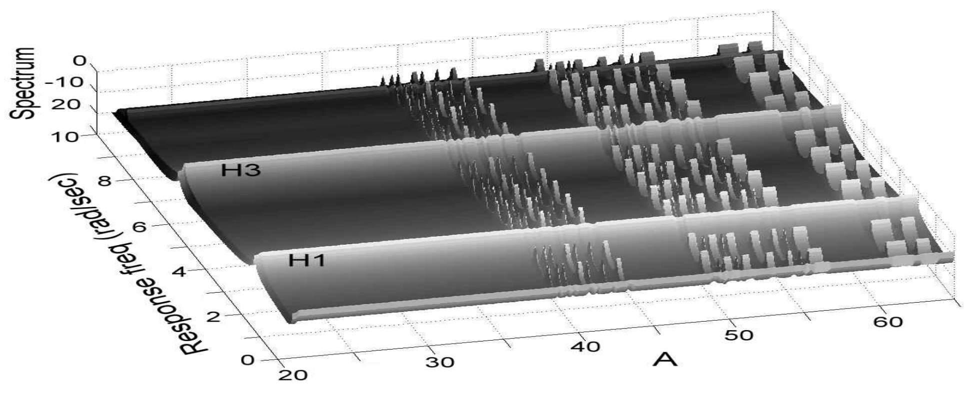


Figure 4. The criteria by (16)-dot, (23)-dashed and (24)-solid



(a)



(b)

Figure 5. Response Spectrum Map of Duffing oscillator (28) at $\omega = 2.1\omega_0$: (a) 2D view and (b) 3D view

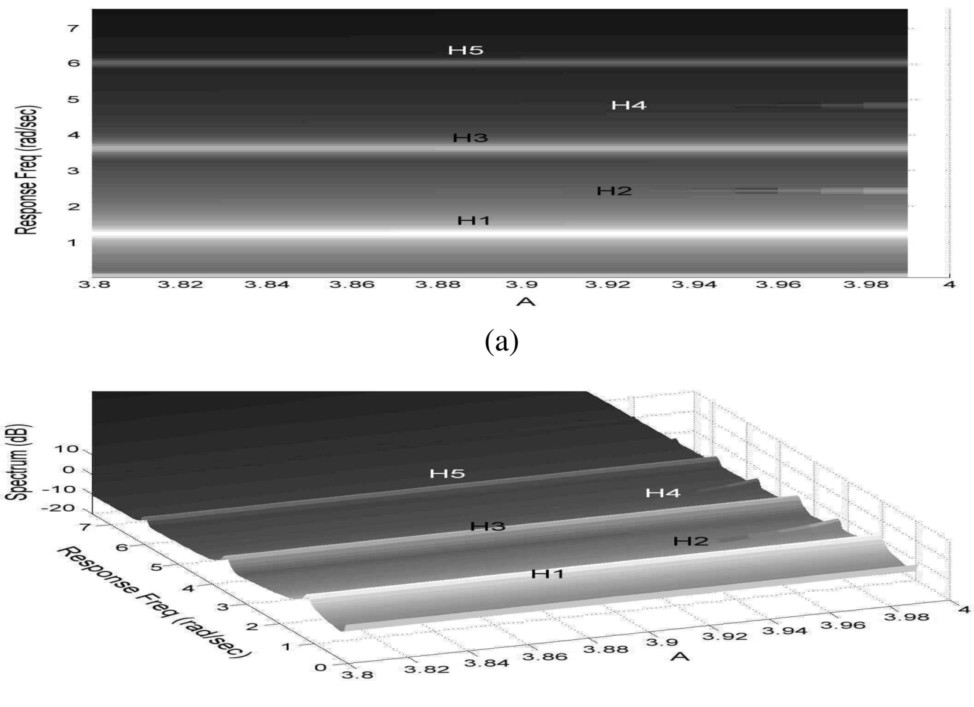
4.2 Softening nonlinearity case

The other frequency domain criteria mentioned in section 4.1 excludes the situation when the cubic stiffness is less than zero. Generally speaking when $k_3 < 0$, the Duffing oscillator has a more restricted convergence region than that for $k_3 > 0$.

The coefficients of the Duffing oscillator used in the softening nonlinearity case are

$$m = 1, c = 1.5, k_1 = 0.5, k_3 = -0.1 \tag{29}$$

A Response Spectrum Map for system (29) at $\omega = 1.2$ is shown in Figure 6. It can be seen from Figure 6 that, before the system becomes unstable at $A = 3.99$, it experiences a very brief window in which the response contains, in addition to the odd order harmonics, even order harmonics components 2ω and 4ω (very weak though) over $3.93 < A < 3.99$. Theoretically this means that the existence of a Volterra series representation for this amplitude range is ruled out. For $A < 3.93$ the response contains standard odd order harmonics, that is, ω, 3ω and 5ω, etc. Therefore $A = 3.93$ can be regarded as the decision point between the existence and non-existence of a valid Volterra series representation for the Duffing oscillator (29) at $\omega = 1.2$. This decision value is close to the result given by the new criterion (16) at $\tilde{A}(\omega)|_{\omega=1.2} = 3.52$. It is interesting to analyse the frequency domain synthesis of the response using GFRF's, as shown in Figure 7, at the decision point $\tilde{A} = 3.93$. It is clear from Figure 7 that a fast convergence can be achieved by using only the first and 3rd orders of Volterra kernels. This suggests that a quick convergence of Volterra series representation as assumed from the derivations of (16) has practical grounds.



(a)

(b)

Figure 6. Response Spectrum Map for system (29) at $\omega = 1.2$: (a) 2D view and (b) 3D view

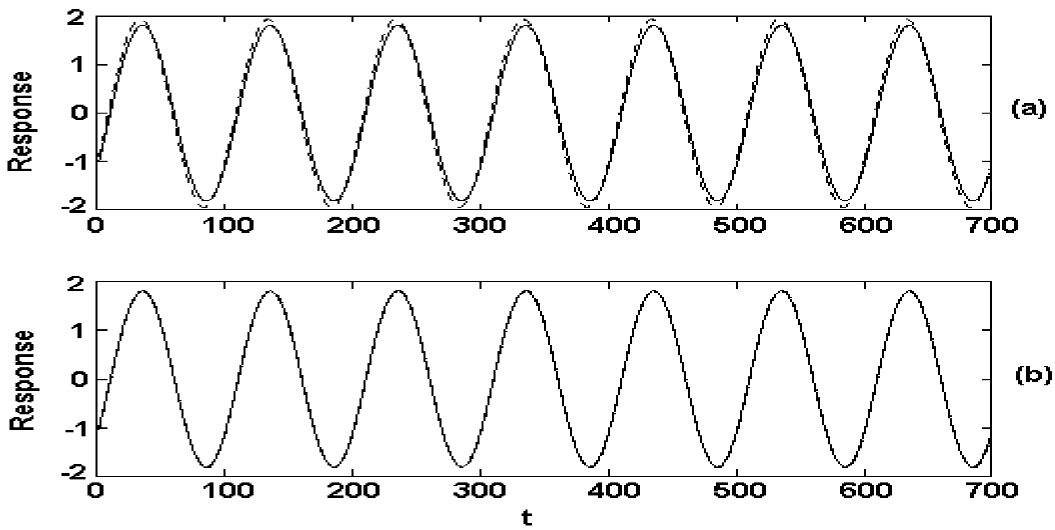


Figure 7. (a) First order output response, (b) up to the 3rd order response, Dashed—synthesized output by GFRF's; Solid--simulated original output from (29)

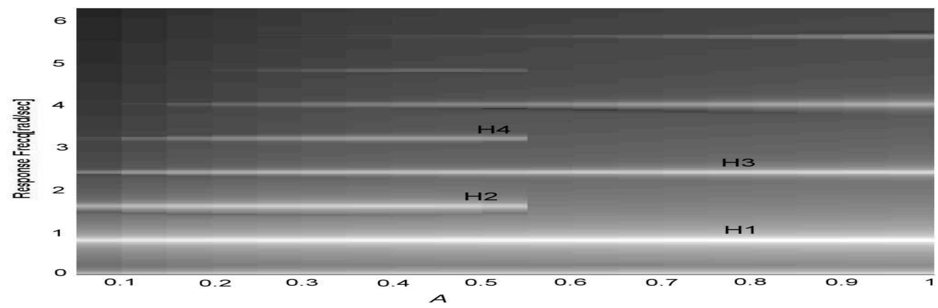
4.3 The case when L(p) has unstable roots

The criterion (16) is based on the assumption that the linear part of the nonlinear oscillator is stable, that is, the roots of $L(q)=0$ in (4) have negative real parts.

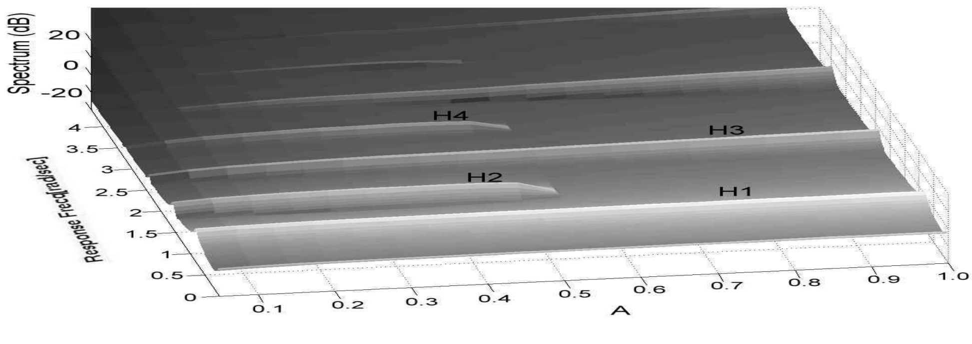
This restriction was not explicitly imposed by the previous criteria (23) and (24), etc. The necessity of this restriction can be made evident in the following simulation. Figure 8 shows the RSM of a Duffing-Holmes oscillator with the parameters

$$m = 1, c = 1.5, k_1 = -0.2, k_3 = 1 \tag{30}$$

and excitation frequency $\omega = 0.8\, rad/\sec$.



(a)



(b)

Figure 8. RSM for Duffing-Holmes oscillator (30): (a) 2D view and (b) 3D view

It is clear from Figure 8 that for the lower amplitude range $A < 0.57$ there are $0, \omega, 2\omega, 3\omega, 4\omega, \ldots$ harmonics, which immediately rules out the existence of a valid Volterra series representation. For $A > 0.57$ valid Volterra series representations can be derived from (30) until the excitation level A reaches around 1.6. Therefore this essentially means that the original system (30) can only possibly produce a convergent Volterra series representation over a narrow band of external excitation levels, rather than within a radius of convergence, making any of the previous mentioned criteria inapplicable. Further work is needed to solve this class of problem.

5 Conclusions

The Volterra series representation has been extensively studied and applied in the modelling, analysis and control of nonlinear systems. However it has a limited convergence. It is therefore desirable to establish the convergence radius. A new criterion has been derived in this article by extending the previous time domain work to periodically excited nonlinear differential equations based on a truncated Volterra series representation. It has been shown that in many cases this new criterion can produce a more accurate estimation of the upper limit of the convergence region than the previously proposed criteria. This criterion is especially useful from the practical viewpoint for systems that require quick convergence in the Volterra domain.

The advantage of the new criterion will be more evident in Part II of this paper in which it is applied to successfully predict the upper limits of Volterra series representation for Duffing oscillators that exhibit a common type of severe nonlinearity called a shock jump in the response.

Acknowledgement: The authors gratefully acknowledge that this work was supported by the Engineering and Physical Sciences Research Council(EPSRC) UK, and a European Research Council Advanced Investigator Award.

References:

Barrett, J.F., 1963, The use of functionals in the analysis of nonlinear physical systems, Journal of Electronics and Control, Vol 15 pp.567-615.

Barrett, J.F., 1965, The use of Volterra Series to Find Region of Stability of a Non-linear Differential Equation, Int. J. Control, Vol. 1, pp.209-216.

Bedrosian, E. and Rice, S.O., 1971, The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputs, Proc. IEEE, Vol 59, pp.1688-1707.

Billings, S.A. and Boaghe, O.M., 2001, The response spectrum map, a frequency domain equivalent to the bifurcation diagram, Int. J. of Bifurcation and Chaos, Vol.11, No.7, pp.1961-1975.

Billings, S.A. and Peyton Jones, J.C., 1990, Mapping non-linear integro-differential equations into the frequency domain, Int.J.Control, Vol. 52, No. 4,pp.863-879.

Billings, S.A and Tsang, K.M., 1989, Spectral analysis for nonlinear systems, Part II—interpretation of nonlinear frequency response functions, *J. Mech Systems and Signal Processing*, Vol. 3, No. 4, pp341-359.

Chatterjee, A and Vyas, N.S., 2000, Convergence analysis of Volterra series response of nonlinear systems subjected to harmonic excitation, Journal of Sound and Vibration, Vol. 236, No. 2, pp. 339–358.

Nam, S.W and Powers, E.J (1994). Application of Higher Order Spectral Analysis to Cubically Non-linear System Identification, IEEE Trans. on Signal Processing, Vol. 42, No.7, pp.1746-1766.

Peng, Z.K. and Lang, Z.Q., 2007, On the convergence of the Volterra-series representation of the Duffing's oscillators subjected to harmonic excitations, J. of Sound and Vibration, Vol. 305, pp.322-332.

Peyton Jones, J.C. and Billings, S.A., 1989, A Recursive algorithm for computing the frequency response of a class of non-linear difference equation models, Int.J.Control, Vol.50, No.5, pp.1925-1940.

Rugh, W.J., 1981, *Nonlinear System Theory—The Volterra/Wiener approach*, The John Hopkins University Press.

Schetzen, M., 1980, *The Volterra and Wiener Theories of Non-linear System*, New York: Wiley.

Sandberg, W., 1984, A Perspective on System Theory, *IEEE Transactions on Circuits and Systems*, Vol. CAS-31, No. 1, pp.88-103.

Tomlinson, G.R., Manson, G. and Lee, G.M., 1996, A simple criterion for establishing an upper limit to the harmonic excitation level of the Duffing oscillator using the Volterra Series, J. of Sound and Vibration, Vol. 190, No. 5, pp.751-762.

Volterra, V., 1930, *Theory of Functionals*, Blackie and Sons.