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The Properties of Output Frequencies of Nonlinear Volterra Systems

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Abstract: Nonlinear systems usually have complicated output frequencies in the frequency domain. For the class of nonlinear Volterra systems, some interesting properties for system output frequencies are studied in this paper. These properties provide a novel insight into the output frequencies of Volterra systems, i.e., the periodicity of the output frequencies. They also demonstrate several novel frequency characteristics of system output spectrum such as the opposite property, and reveal clearly the nonlinear effects on system output spectrum from different nonlinearities. These new results have significance in the analysis and design of nonlinear systems and nonlinear filters in order to achieve a specific output spectrum in a desired frequency band by taking advantage of nonlinearities, and provide an important guidance to applications of Volterra system theory in practices for analysis and design of nonlinear systems. Examples and discussions are given to illustrate these new results.
Keywords: Volterra systems, Volterra series, Output frequencies, Output spectrum

1 Introduction

The frequency domain analysis of non-linear systems has been studied for many years [1-3]. For a class of nonlinear systems, the frequency domain analysis can be conducted by using the Volterra series theory [3-4]. It is shown in [5-6] that a considerably large class of nonlinear systems, referred to as Volterra systems, have a convergent Volterra series expansion. Extensive studies also show that Volterra systems have large numbers of applications for modelling, identification, control and signal processing in many systems and engineering practices, for example, electrical systems, biological systems, mechanical systems, communication systems, nonlinear filter, image processing, materials engineering, chemical engineering and so on [3-12,18, 26-29]. Based on the theory of Volterra series expansion, the study of nonlinear systems in the frequency domain was initiated by the introduction of the concept of the generalized frequency response functions (GFRFs) [13]. Thereafter, many results have been achieved for the frequency domain analysis of nonlinear systems [3,12,14-19,21-23,27,28].

An important phenomenon for nonlinear systems in the frequency domain is that they always have very complicated output frequencies, for example, super-harmonics, sub-harmonics, inter-modulation, and so on. This usually makes it rather difficult to analyse and design the output frequency response behaviour for nonlinear systems, compared with linear systems. The output frequencies for Volterra systems have been studied by several authors [18-23, 27] by using the frequency domain method above. These results provide algorithms from different viewpoints for the computation and prediction of the output frequencies for nonlinear systems. It can be seen from the previous researches that Volterra systems can effectively be used to account for super-harmonics and inter-modulation in the output spectrum of nonlinear systems.

In this study, some important properties for the output frequencies of Volterra systems are established. They provide a straightforward and complete insight into the super-harmonic and inter-modulation phenomena in the output frequencies of nonlinear systems, especially when the effects from different system nonlinearities are considered. The new properties demonstrate several novel frequency characteristics of the output spectrum for nonlinear systems. They have significance in the analysis and design of nonlinear systems and nonlinear filters in order to achieve a specific output spectrum in a desired frequency band by taking advantage of nonlinearities. These new results can provide an important guidance to modelling, identification, control and signal processing by using the Volterra series theory in practices. Examples and discussions are provided to illustrate the results.

2 Output frequencies for nonlinear Volterra systems

Consider nonlinear systems which has a Volterra series expansion up to order *N* as [3-5]

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ called the *n*th-order Volterra kernel. The *n*th-order GFRF of system (1) is defined as [13]

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{2}$$

The GFRF for a practical system can be obtained by the probing method [3]. The output spectrum of system (1) subject to a general input can be described as [16]

$$\begin{aligned}&Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)\\&Y_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega\end{aligned} \tag{3}$$

where $\int_{\omega_1+\cdots+\omega_n=\omega}(\cdot)d\sigma_\omega$ represents the integration on the super plane $\omega_1+\cdots+\omega_n=\omega$. $Y_n(j\omega)$ is referred to as the nth-order output spectrum. Similarly, when the system is subject to a multi-tone input described by

$$u(t)=\sum_{i=1}^{\bar{K}}|F_i|\cos(\omega_i t+\angle F_i) \tag{4}$$

(where $\bar{K}$ is a positive integer, F_i is a complex number) the system output spectrum can be written as:

$$
\begin{aligned}
&Y(j\omega)=\sum_{n=1}^{N} Y_n(j\omega)\\
&Y_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n})
\end{aligned}
\tag{5}
$$

where $F(\omega_{k_i})$ can be written explicitly as $F(\omega_{k_i})=\left|F_{|k_i|}\right|e^{j\angle F_{|k_i|}\cdot \mathrm{sig1}(k_i)}$ for $k_i\in\{\pm 1,\cdots,\pm\overline{K}\}$, and $\mathrm{sgn1}(a)=\begin{cases}1 & a>0\\ 0 & a=0\\ -1 & a<0\end{cases}$.

As mentioned, nonlinear systems usually have complicated output frequencies, which are quite different from linear systems which have output frequencies completely identical to the input frequencies. From Equations (3) and (5), it can be seen that the output frequencies corresponding to the nth-order output spectrum, denoted by W_{n} and simply referred to as the nth-order output frequencies, are completely determined by

$$\omega=\omega_1+\omega_2+\cdots+\omega_n \text{ or } \omega=\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_n}$$

which produce super-harmonics and inter-modulation in system output frequencies. Consider any continuous and bounded input function $u(t)$ in $t\ge 0$ with Fourier transform $U(j\omega)$ whose input domain is denoted by V, *i.e.*, $\omega\in V$. Note that V can be any closed set in real. Let $\overline{V}$=-$V\cup V$, whose meaning will be discussed later.

Therefore, for the general input $U(j\omega)$ defined in V, the nth-order output frequencies are

$$W_n=\left\{\omega=\omega_1+\omega_2+\cdots+\omega_n \,\middle|\, \omega_i\in\overline{V}, i=1,2,...,n\right\} \tag{6a}$$

or for the multi-tone input (4),

$$W_n=\left\{\omega=\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_n} \,\middle|\, \omega_{k_i}\in\overline{V}, i=1,2,...,n\right\} \tag{6b}$$

This is an analytical expression for the super-harmonics and inter-modulations in the *n*th-order output frequencies of nonlinear Volterra systems. Then the whole system output frequencies, denoted by W, can be written as

$$W=W_1\cup W_2\cup\cdots\cup W_N \tag{6c}$$

In Equations (6abc), $\overline{V}$ represents the input frequency range corresponding to the nth-order output spectrum, V is the original input frequency range corresponding to the first order output spectrum and W_1 represents the output frequencies of linear part in the system. For example, V may be a real set [a,b] $\cup$ [c,d], thus $\overline{V}$ =[-d,-c] $\cup$ [-b,-a]$\cup$[a,b]$\cup$[c,d], where $d\ge c\ge b\ge a$>0. Especially, when the system subjects to the multi-tone input (4), then the nth-order input frequency range is $\overline{V}=\{\pm\omega_1,\pm\omega_2,\cdots,\pm\omega_{\overline{K}}\}$, which is obviously a special case of the former one.

3 Fundamental properties and the periodicity property

In this section, some fundamental properties for the output frequencies of system (1) with assumption that V is any closed set of frequency points in real are developed. Especially, the periodicity of the output frequencies is revealed under the general input case. Although the computation of the system output frequencies for the case with V=[a,b] has been studied in [18,19], and for the multi-tone case was also studied in [21, 23, 27], and some of the properties discussed in this section have been partly demonstrated in the previous results for the case V =[a,b] and multi-tone case $V=\{\omega_1,\omega_2,\cdots,\omega_{\bar{K}}\}$, all the properties of this section are established in a uniform manner based on the analytical expressions (6abc) for any input domain V. Let max(.) denote the maximum value of the elements in (.), and min(.) the minimum value.

Property 1. Consider the *n*th-order output frequency W_n,

(a) Expansion, i.e., $W_{n-2} \subseteq W_n$;

(b) Symmetry, i.e., $\forall\Omega \in W_n$, then $-\Omega \in W_n$;

(c) n-multiple, i.e., $\max(W_n) = n \cdot \max(V)$ and $\min(W_n) = -n \cdot \max(V)$.□

The proof is omitted. Property 1 shows that the output frequency range will expand larger and larger with the increase of the nonlinear order, the expansion is symmetric to zero and its rate is n-multiple of the input frequency range. Property 1(a) shows that, the (n-2)th order output frequencies W_{n-2} are completely included in the nth order output frequencies W_n. This property can be used to facilitate the computation of output frequencies for nonlinear systems. That is, only the highest order in odd number and the highest order in even number, to which the corresponding GFRFs are not zero, are needed to be considered in Equation (6c). For example, supposing the system maximum order N=10, only W_{10} and W_9 are needed to be computed if $H_{10}(.)$ and $H_9(.)$ are not zero, and the system output frequencies are $W = W_9 \cup W_{10}$ (in case that $H_9(.)$ is zero, W_9 should be replaced by the output frequencies corresponding to the highest odd order of nonzero GFRFs). For the case that V=[a,b], Property 1(a) has be shown in [19]. Here it is shown to hold for any V.

Properties 1 provide some fundamental characteristics for the output frequencies of system (1) subject to any input frequencies. The following proposition demonstrates a novel property for the output frequencies of Volterra systems, and provides a new insight into the system output frequency characteristics.

Proposition 1 (Periodicity property). The frequencies in W_n can be generated periodically as follows

$$W_n = \bigcup_{i=1}^{\Gamma_n+1} \Pi_i(n) \tag{7a}$$

$$\Pi_i(n) = \left\{\omega = \omega_1 + \omega_2 + \cdots + \omega_n \middle| \omega_j \in \bar{V}, \omega_j < 0 \text{ for } 1 \le j \le i-1, \omega_j > 0 \text{ for } j \ge i\right\} \text{ or} \tag{7b}$$

$$\Pi_i(n) = \left\{\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n} \middle| \omega_{k_j} \in \bar{V}, \omega_{k_j} < 0 \text{ for } 1 \le j \le i-1, \omega_{k_j} > 0 \text{ for } j \ge i\right\} \tag{7c}$$

$$\Gamma_n = n \tag{7d}$$

The above process has the following properties

$$\max(\Pi_i(n)) = -(i-1)\min(V) + (n-i+1)\max(V) \text{ and} \tag{8a}$$

$$\min(\Pi_i(n)) = -(i-1)\max(V) + (n-i+1)\min(V) \tag{8b}$$

$$\max(\Pi_{i-1}(n)) - \max(\Pi_i(n)) = \min(\Pi_{i-1}(n)) - \min(\Pi_i(n)) = \min(V) + \max(V) \tag{8c}$$

$$\Delta(n) = \max(\Pi_i(n)) - \min(\Pi_i(n)) = n \cdot (\max(V) - \min(V)) \tag{8d}$$

Especially, when the system subjects to a general input $U(j\omega)$ defined in [a,b] or the multi-tone input (4) with $\omega_{i+1}-\omega_i = const > 0$ for i=1,…, $\overline{K}$ -1,

$$\Pi_i(n) = \Pi_{i-1}(n) - T \quad \text{for i=2,…, n+1} \tag{8e}$$

where $\Pi_i(n)-T$ is a set whose elements are the elements in $\Pi_i(n)$ minus *T*, *T* $=\min(V)+\max(V)$ is referred to as the frequency generation period, and $\Delta(n)$ is referred to as the frequency span in each period.
Proof. It is omitted. □

Property 2. Consider the *i*th frequency generation period $\Pi_i(n)$ in W_n,

(a) If the system input is the multi-tone function (4), then for any two frequencies Ω and Ω' in $\Pi_i(n)$ and any two frequencies ω and ω' in *V*, min(Ω - Ω') $=\min(\omega-\omega')$.

(b) If $\Delta(n)$ >T, then $\max(\Pi(n)_{i+1}) > \min(\Pi(n)_i)$ for i=1,…, Γ_n . That is, there is overlap between the successive periods of frequencies in W_n. □

The proof is omitted. Proposition 1 and Properties 2 explicitly demonstrate, for the first time, an interesting and useful nature of the output frequencies ------ the periodicity of the output frequencies. This property can not only be used to simplify the computation of the output frequencies for some special cases as stated in Proposition 1 (where only one period of output frequencies need to be computed) but also make light for the computation of the output frequencies in general case. Especially, it reveals a new insight into the output frequency characteristics of nonlinear systems for the understanding of the nonlinear output frequency response behaviour for Volterra systems. Some important issues will be discussed further in the following sections. From Proposition 1, the following corollary is straightforward.

Corollary 1. All the conclusions in Proposition 1 and Properties 1-3 hold for the case that the system subjects to a general input $U(j\omega)$ defined in $\bigcup_{i=1}^{Z}[a+(i-1)\varepsilon, b+(i-1)\varepsilon]$ where $b \ge a, \varepsilon \ge (b-a)$ and *Z* is a positive integer.

Note that when *V* does not satisfy the condition in Corollary 1, the property in Equation (8e) can not hold. Example 1 is given to illustrate the results above.

Example 1.Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^2 + bu^3$$

The input is a multi-tone function u(t)=sin(6t)+sin(7t)+sin(8t). The output spectra are given in Figures 1-2 for different cases. Note that there are only input nonlinearities with order 2 and 3 in the system, thus only the 1st, 2nd and 3rd order GFRFs are not zero and all the other orders of the GFRFs are zero [15]. Hence, the nonlinear output frequencies of the system are the same as the 2nd and 3rd order output frequencies. That is, when *a*=1 and *b*=0, then $W=W_2$; when *a*=0 and *b*=1, then $W=W_3$; and when *a*=1 and *b*=1, then $W = W_2 \cup W_3$. Figures 1-2 demonstrate clearly the results in Properties 3-4 and Proposition 1, and also show that the system output frequencies are simply the accumulation of all the output frequencies corresponding to each order output spectrum when the involved nonlinearities have no crossing effect and no overlap as stated in Property 5. When and how there are crossing effects between different nonlinearities will be discussed in the next section.

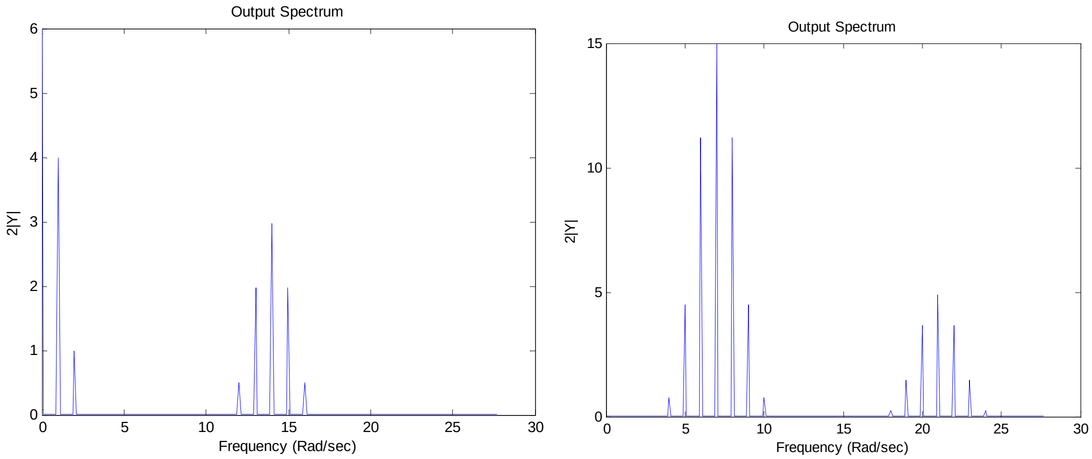

Figure 1. Output frequencies when a=1 and b=0 (left) and when a=0 and b=1 (right)

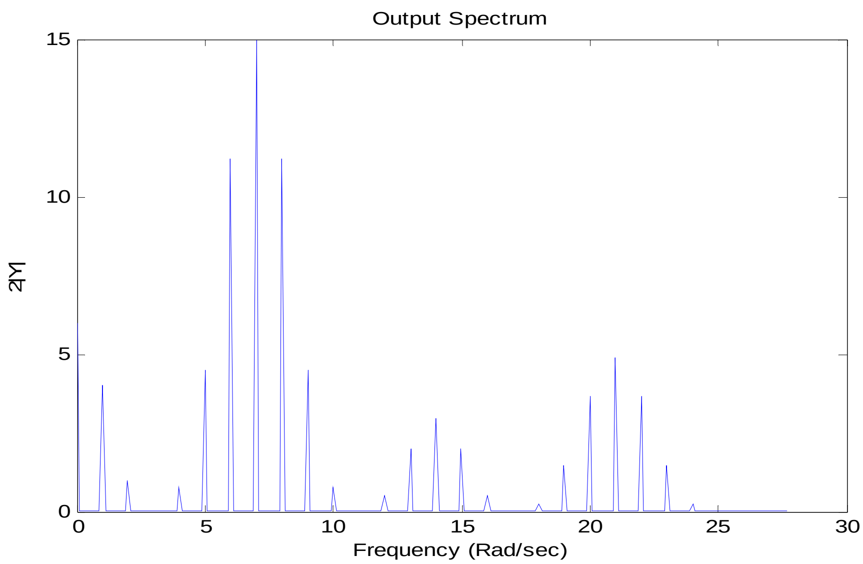

Figure 2. Output frequencies when a=1 and b=1

4 Nonlinear effect in each frequency generation period

The periodicity of output frequencies is revealed and demonstrated in the last section. In this section, the nonlinear effect on system output spectrum in each frequency generation period, and especially the nonlinear interaction between different nonlinearities of the same nonlinear degree and nonlinear type are studied.

From (3) and (5), it can be seen that the operators $\int_{\omega_1+\cdots+\omega_n=\omega}(\cdot)d\sigma_\omega$ and $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ have an important and fundamental role in the frequency characteristics of the nth order output spectrum in each frequency generation period. The following property can be obtained.

Property 3. For $\omega\in\Pi_i(n)$ $(1\le i\le\lceil(n+1)/2\rceil)$, $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1$ reaches its maximum at the central frequency $(\max(\Pi_i(n))+\min(\Pi_i(n)))/2$ or around the central frequency if the central frequency is not available, and has its minimum value at frequencies $\max(\Pi_i(n))$ and $\min(\Pi_i(n))$, i.e., $\min_{\omega\in\Pi_i(n)}\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1\right)=\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\max(\Pi_i(n))}1=\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\min(\Pi_i(n))}1=C_n^{i-1}$. Moreover,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1>\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=<\omega+T>}1\quad\text{for }\omega\in\Pi_i(n)\ \ (2\le i\le\lceil(n+1)/2\rceil)$$

Especially, for the multi-tone input case with $\omega_{i+1}-\omega_i=const>0$ for i=1,…, $\overline{K}$ -1,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\max(\Pi_i(n))-k'\cdot const}1=\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\min(\Pi_i(n))+k'\cdot const}1\quad\text{for }0\le k'\le T/const$$

where, $\lceil(n+1)/2\rceil$ is the smallest integer which is not less than $(n+1)/2$, $<\omega+T>$ is the frequency in $\Pi_{i-1}(n)$ which is the most approximate to $\omega+T$. The similar results also hold for the general input case defined in Corollary 1 by replacing $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1$ as $\int_{\omega_1+\cdots+\omega_n=\omega}1d\sigma_\omega$. □

The proof is omitted. Property 3 shows that in each frequency generation period, the effect of the operator $\int_{\omega_1+\cdots+\omega_n=\omega}(\cdot)d\sigma_\omega$ and $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ on system output spectrum tends naturally to be more complicated at the central frequency. That is, there is only one case for the operator $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ at the two boundary frequency of each period, it reaches the maximum at the central frequency of the same period and tends to be decreased in different period with the frequency increasing. These can be regarded as the natural characteristics of the output frequencies that can not be changed (This can be verified by Figure 3 in Example 2).

Note that different nonlinearities may have quite different effect on system output spectrum. In order to study the nonlinear effect between different nonlinearities of the same nonlinear degree and kind, consider the nonlinear Volterra systems which are described by a nonlinear differential equation (NDE) model as follows:

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{l_1,l_{p+q}=0}^{K}c_{p,q}(l_1,\cdots,l_{p+q})\prod_{i=1}^{p}\frac{d^{l_i}y(t)}{dt^{l_i}}\prod_{i=p+1}^{p+q}\frac{d^{l_i}u(t)}{dt^{l_i}}=0\tag{9}$$

where $\left.\frac{d^l x(t)}{dt^l}\right|_{l=0}=x(t)$, $p+q=m$, $\sum_{l_1,l_{p+q}=0}^{K}(\cdot)=\sum_{l_1=0}^{K}\cdots\sum_{l_{p+q}=0}^{K}(\cdot)$, M is the maximum degree of nonlinearity in terms of y(t) and u(t), and K is the maximum order of the derivative. In this model, the parameters such as $c_{0,1}(.)$ and $c_{1,0}(.)$ are referred to as linear parameters, which correspond to the linear terms in the model, *i.e.*, $\frac{d^l y(t)}{dt^l}$ and $\frac{d^l u(t)}{dt^l}$ for k=0,1,…,K, and $c_{p,q}(\cdot)$ for $p+q>1$ are nonlinear parameters corresponding to nonlinear terms in the model of the form $\prod_{i=1}^{p}\frac{d^{l_i}y(t)}{dt^{l_i}}\prod_{i=p+1}^{p+q}\frac{d^{l_i}u(t)}{dt^{l_i}}$, *e.g.*, $y(t)^p u(t)^q$. $p+q$ is called the nonlinear degree of the nonlinear parameter $c_{p,q}(\cdot)$. Similar results discussed in this study can also be established immediately for a discrete nonlinear model known as NARX model.

When different kinds and degrees of nonlinearities exist in the system, there will be much crossing effects at the same frequency from different nonlinearities. This will make the output spectrum at the interested frequency to be enhanced or suppressed. For example, different nonlinearities of the same order and the same kind bring the same output frequencies from [15]. However, the effect from different nonlinearities at the same frequency generation period may counteract with each other such that the output spectrum may be suppressed in some periods and others enhanced. Clearly, this property is of great significance in the design of nonlinear systems for suppressing output vibration [25].

In this study, consider there are only input nonlinearities in the NDE model above with $c_{p,q}(.)=0$ for all $p+q>1$ and $p>0$. In this case, the GFRFs can be written as [17]

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{1}{L_n(j\omega_1+\cdots+j\omega_n)}\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_1)^{l_1}\cdots(j\omega_n)^{l_n} \tag{10}$$

where

$$L_n(j\omega_1+\cdots+j\omega_n)=-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1} \tag{11}$$

From (10-11) and (5), the nth-order output spectrum under the multi-tone input (4) can be obtained

$$\begin{aligned}Y_n(j\omega)&=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\left(\frac{F(\omega_{k_1})\cdots F(\omega_{k_n})}{L_n(j\omega_{k_1}+\cdots+j\omega_{k_n})}\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\\&=\frac{1}{2^n L_n(j\omega)}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\left(F(\omega_{k_1})\cdots F(\omega_{k_n})\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\end{aligned} \tag{12}$$

To reveal the nonlinear effect from input nonlinearities in each frequency generation period, the following results can be obtained.

Definition 1 (Opposite property). Considering two input nonlinear terms of the same degree with coefficients $c_{0,n}(l_1,\cdots,l_n)$ and $c_{0,n}(l'_1,\cdots,l'_n)$, if there exist two positive real number c_1 and c_2 satisfying $c_{0,n}(l_1,\cdots,l_n)=c_1$ and $c_{0,n}(l'_1,\cdots,l'_n)=c_2$, such that at a given frequency $\Omega>0$,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega}\Big(F(\omega_{k_1})\cdots F(\omega_{k_n})\Big(c_1(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}+c_2(j\omega_{k_1})^{l'_1}\cdots(j\omega_{k_n})^{l'_n}\Big)\Big)=0$$

with respect to a multi-tone input (4), then the two terms are referred to as opposite at frequency Ω, whose effects in frequency domain counteract with each other at Ω.

Note that the concept of the opposite property can be defined similarly for the other kinds of nonlinearities. The following result can be concluded for the opposite property of two input nonlinear terms.

Proposition 2 (Opposite of input nonlinearity). Consider nonlinear systems with only input nonlinearities subject to multi-tone input, in which there are two nonlinear terms with coefficients $c_{0,n}(l_1,\cdots,l_n)$ and $c_{0,n}(l_1',\cdots,l_n')$. If there exists a non-negative integer $m \le \lceil (n+1)/2 \rceil$-1 such that for a real Ω>0, sgn($F(\omega_{k_1})\cdots F(\omega_{k_n})$) is constant with respect to all the combinations of $\omega_{k_1},\cdots,\omega_{k_n} \in \{\pm\omega_1,\cdots,\pm\omega_{\overline{K}}\}$ satisfying $\omega_{k_1}+\cdots+\omega_{k_n} \in \Pi_{m+1}(n)$, and

$$\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}\left(c_{0,n}(l_1\cdots l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\right) = -\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}\left(c_{0,n}(l_1'\cdots l_n')(j\omega_{k_1})^{l_1'}\cdots(j\omega_{k_n})^{l_n'}\right)\right) \quad (13)$$

i.e.,

$$\begin{aligned}&\mathrm{sgn\,1}(c_{0,n}(l_1\cdots l_n))\,\mathrm{sgn\,1}\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{l_1}\cdots(\omega_{k_n})^{l_n}\Big)\,\mathrm{sgn}(j^{l_1+\cdots+l_n}) \\ &= -\mathrm{sgn\,1}(c_{0,n}(l_1'\cdots l_n'))\,\mathrm{sgn\,1}\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{l_1'}\cdots(\omega_{k_n})^{l_n'}\Big)\,\mathrm{sgn}(j^{l_1'+\cdots+l_n'})\end{aligned} \quad (14)$$

then the two nonlinear terms are opposite in the (m+1)th frequency generation period $\Pi_{m+1}(n)$. That is, the frequency domain effects from the two nonlinear terms counteract with each other in $\Pi_{m+1}(n)$. Here, $\mathrm{sgn}(a+bj)=[\mathrm{sgn\,1}(a),\mathrm{sgn\,1}(b)]$. □

The proof is omitted. From Equation (12), it can be seen that the magnitude of $Y_n(j\omega)$ is dependent on three terms: $L_n(j\omega)$ and $F(\omega_{k_1})\cdots F(\omega_{k_n})\sum_{l_1,l_n=1}^{K} c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}$, and the function operator $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$. $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ represents the system natural effect which can not be changed as mentioned. The first term $L_n(j\omega)$ represents the effect from the linear part of the system and the second term represents the nonlinear effect from input nonlinearities. These two terms can be designed purposely in practices. Therefore, the results in Proposition 2 provide guidance to the design of input nonlinearities to achieve a specific output spectrum. Similar results can also be established for the other kinds of nonlinearities. For paper limitation, this will be discussed in detail in a future publication. The following example illustrates the result in Proposition 2.

Example 2. Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^5 + bu^3\dot{u}^2$$

The input is a multi-tone function u(t)=0.8sin(7t)+0.8sin(8t)+sin(9t), which can be written as u(t)=0.8cos(7t-90^0)+0.8cos(8t-90^0)+cos(9t-90^0). Therefore, F($\omega_{\pm1}$)= ∓ 0.8j, F($\omega_{\pm2}$)=∓ 0.8j and F($\omega_{\pm3}$)=∓ j. It can be verified that, sgn($F(\omega_{k_1})\cdots F(\omega_{k_5})$) is constant in each period $\Pi_i(5)$ (i= 1,..,6). This satisfies the condition in Proposition 2. The output spectrum under different parameter values are provided in Figures 3-4. It can be verified that the two nonlinear terms are opposite at the second frequency generation period. For the nonlinear term au^5,

$$\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}\left(c_{0,n}(l_1\cdots l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\right)=\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(5-2)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(a)\right)=\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=3\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}1\right)=[1,0]$$

For the nonlinear term $bu^3\ddot{u}^2$,

$$\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}\left(c_{0,n}(l_1'\cdots l_n')(j\omega_{k_1})^{l_1'}\cdots(j\omega_{k_n})^{l_n'}\right)\right)=\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(5-2)\cdot\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}\left(b(j\omega_1)(j\omega_2)\right)\right)$$

$$=\mathrm{sgn}\left(-\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=3\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_1\omega_2)\right)$$

Note that there are five combinations for $\omega_{k_1}+\cdots+\omega_{k_n}=3\Omega,\omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}$, i.e., $-\Omega,\Omega,\Omega,\Omega,\Omega;\Omega,-\Omega,\Omega,\Omega,\Omega;\Omega,\Omega,-\Omega,\Omega,\Omega;\Omega,\Omega,\Omega,-\Omega,\Omega;\Omega,\Omega,\Omega,\Omega,-\Omega;$

Therefore $\mathrm{sgn}\left(-\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=3\Omega \\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_1\omega_2)\right)=[-1,0]$. Equation (13) or (14) is satisfied.

From Figure 4 it can be seen that, the counteraction between the effects from the two input nonlinear terms results in suppression of the output spectrum in the second period and enhancement for the first and third periods, compared with the output spectrum under single nonlinear term au^5.

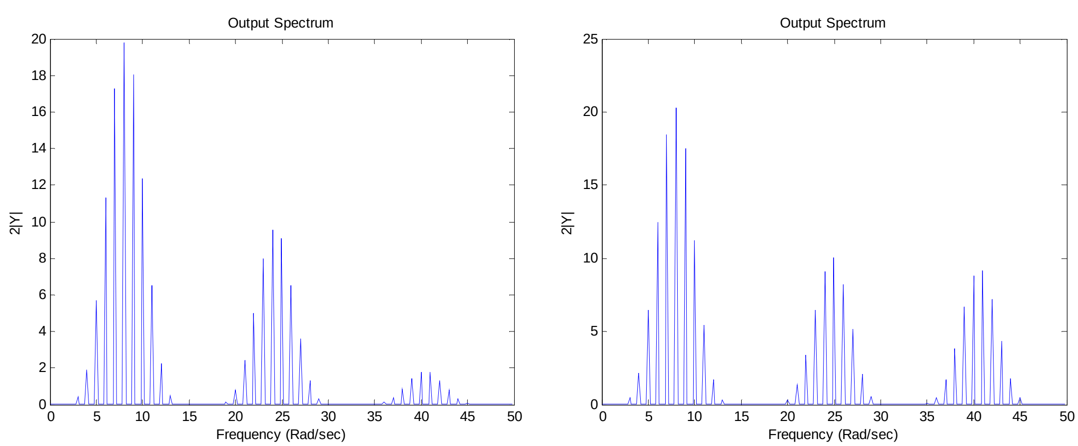

Figure 3. Output spectrum when a=1.3 b=0 (left) and a=0,b=0.1(right)

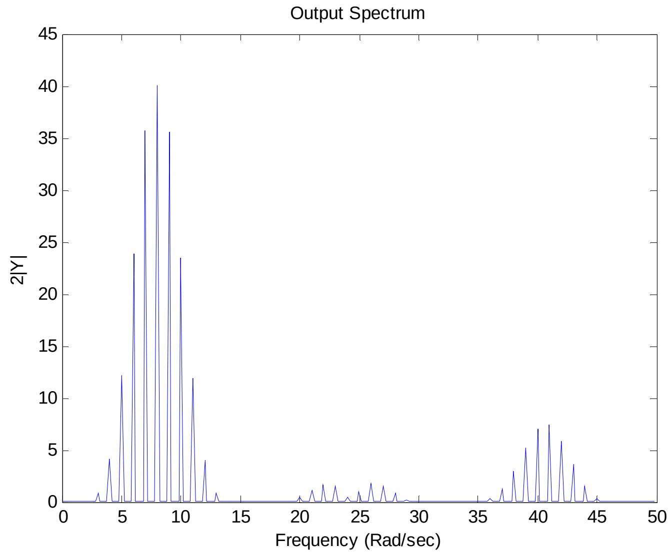

Figure 4. Output spectrum when a=1.3 b=0.1

Moreover, it is obvious that given system model and input function, the system output spectrum can be analytically determined from (3-5). Contrarily, given system model in the multi-tone input case, the input function can be obtained from the output spectrum at a specific frequency generation period for example $\Pi_1(n)$. Because each output frequency in $\Pi_i(n)$ can be explicitly determined, thus a series of equations can be obtained in terms of $F(\omega_{k_1})\cdots F(\omega_{k_n})$, and then $F(\omega_1),\cdots,F(\omega_n)$ can be solved. That is, the original input signal can be recovered from the received signal in a specific frequency generation period. This is another interesting property based on the periodicity and is under study now.

5 Parametric characteristic of the output frequencies

There are three kinds of nonlinearities in model (9): input nonlinearity with coefficient $c_{0,q}(.)$ (q>1), output nonlinearity with coefficient $c_{p,0}(.)$ (p>1), and input output cross nonlinearity with coefficient $c_{p,q}(.)$ (p+q>1 and p>0) (where p and q are integers). Different kind and degree of nonlinearity in a system can bring different output frequencies to the system. How a nonlinear term affects system output frequencies and what the effect is for Volterra systems are a very interesting and important topic. However, few results have been reported for this. This section provides some useful results for this topic based on the properties developed above.

Consider nonlinear Volterra systems described by the NDE model in (9). What model parameters contribute to a specific order GFRF and how model parameters affect the GFRFs can be revealed by using the parametric characteristic analysis in [15]. From Equations (3, 5), it can be seen that the nth-order output frequencies W_n are also determined by the nth order GFRF. If the nth order GFRF is zero, then W_n=[]. It is known that the nth order GFRF is dependent on its parametric characteristics [15], thus the nth-order output frequencies are also determined by the parametric characteristics of the nth-order GFRF. That is, Equations (6a-b) can be written as

$$W_n = \left\{ \omega = (\omega_1 + \omega_2 + \cdots + \omega_n) \cdot (1 - \delta(CE(H_n(\omega_1, \cdots, \omega_n)))) \middle| \omega_i \in \bar{V}, i = 1,2,...,n \right\} \quad (15a)$$

and

$$W_n = \left\{ \omega = (\omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}) \cdot (1 - \delta(CE(H_n(\omega_{k_1}, \cdots, \omega_{k_n})))) \middle| \omega_{k_i} \in \bar{V}, i = 1,2,...,n \right\} \quad (15b)$$

where $\delta(x) = \begin{cases} 1 & x = 0 \text{ or } 1 \\ 0 & \text{else} \end{cases}$, *CE*(.) is a coefficient extraction operator defined in [15], and $CE(H_n(.))$ can be recursively determined from the nonlinear parameters of model (9) [15]. In Equations (11ab), suppose W_n is empty when $\delta(CE(H_n(.)))=1$.

Equations (15a-b) demonstrate the parametric characteristics of the output frequencies for Volterra systems described by (9) and (10), by which the effect on the system output frequencies from different nonlinearities can be studied. Since negative output frequencies are symmetrical with positive output frequencies with respect to zero (Property 2(b)), thus for convenience only non-negative output frequencies are considered in what follows.

Property 4. Regarding nonlinearities of odd and even degrees,

(a) when there are no nonlinearities of even degrees, the output frequencies brought by the nonlinearities with odd degrees happen at central frequencies $(2l+1)T/2$ for $l=0,1,2,\ldots$ with certain frequency span;

(b) when there are only input nonlinearities of even degrees, the output frequencies happen at central frequencies $l \cdot T$ for $l=0,1,2,\ldots$ with certain frequency span;

(c) in other cases, the output frequencies happen at central frequencies $l \cdot T/2$ for $l=0,1,2,\ldots$ with certain frequency span.

The frequency span is $\Delta(n)$ corresponding to the nth order output frequencies if applicable. □

The proof is omitted. Property 4 shows that odd degrees of nonlinearities bring quite different output frequencies to the system from those brought by even degrees of nonlinearities.

Property 5. Regarding different kinds of nonlinearities,

(a) when there are only input nonlinearities of the largest nonlinear degree n, the non-negative output frequencies are in the closed set $[0,\ n \cdot \max(V)\ ]$;

(b) in other cases, the output frequencies span to infinity. □

The proof is omitted. Input nonlinearities of finite nonlinear degree can independently bring output frequencies to the system in a finite band width.

Property 6. Regarding different kinds and degrees of nonlinearities,

(a) when there are only input nonlinearities, a nonlinear term of degree *n* can only bring output frequencies W_n, and there are no crossing effect on output frequencies between different degrees of input nonlinearities;

(b) in other cases, a nonlinear term of degree n contributes to not only output frequencies W_n but also some high order output frequencies W_m for $m > n$ due to crossing effect with other nonlinearities. □

The proof is omitted. From Property 6, the crossing effect happens easily between the nonlinearities from output nonlinearities and input-output cross nonlinearities.

Properties 4-6 provide some novel and interesting results about the output frequencies for nonlinear systems when the effects from different nonlinearities are considered, based on the results from parametric characteristic analysis in [15]. Property 4 shows that odd degrees of nonlinearities have quite different effect on system output frequencies from even degrees of nonlinearities. Especially, it is shown from the properties above that input nonlinearities have special effect on system output frequencies compared with the other kinds of nonlinearities. That is, input nonlinearities can move the input frequencies to higher frequency bands without interference between different frequency generation periods. These properties may have significance in design of nonlinear systems for some special purposes in practices. For example, some proper input nonlinearities can be used to design a nonlinear filter such that input frequencies are moved to a place of higher frequency or lower frequency as discussed in [24]. The results in this section have also significance in modelling and identification of nonlinear systems. For example, if a Volterra system has only output frequencies which are odd multiples of the input frequency when subject to a sinusoidal input, the system may have only nonlinearities of odd degree according to Property 4. Obviously, the results in this section provide a useful guidance to the structure determination and parameter selection for the design of novel nonlinear filters and also for system modelling or identification.

Example 3. Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^5 - by^3 - cy^2$$

The input is a multi-tone function u(t)=sin(6t)+sin(7t)+sin(8t). The output spectra under different parameter values are given in Figures 5-7, which demonstrate the results in Properties 4-6. For the input nonlinearity, the readers can also refer to Figures 1-2.

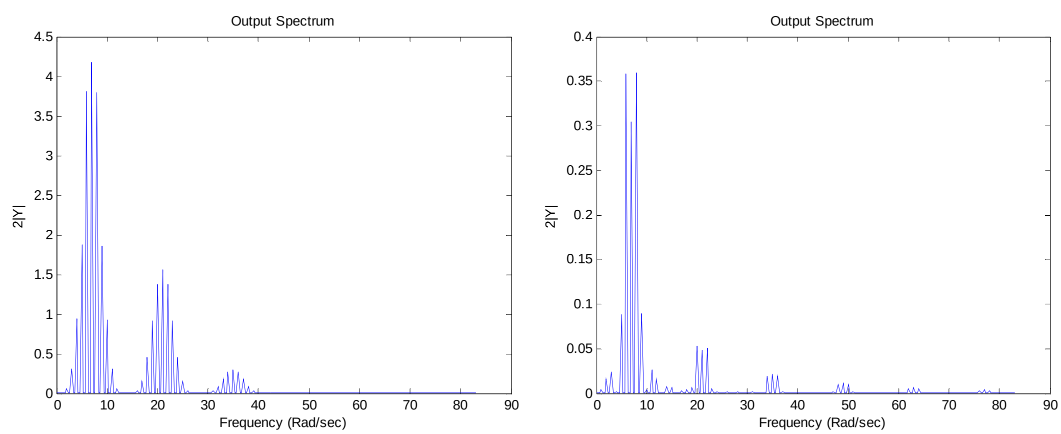

Figure 5. Output frequencies when *a*=0.1, *b*=0, *c*=0 (left) and *a*=0, *b*=5, *c*=0 (right)

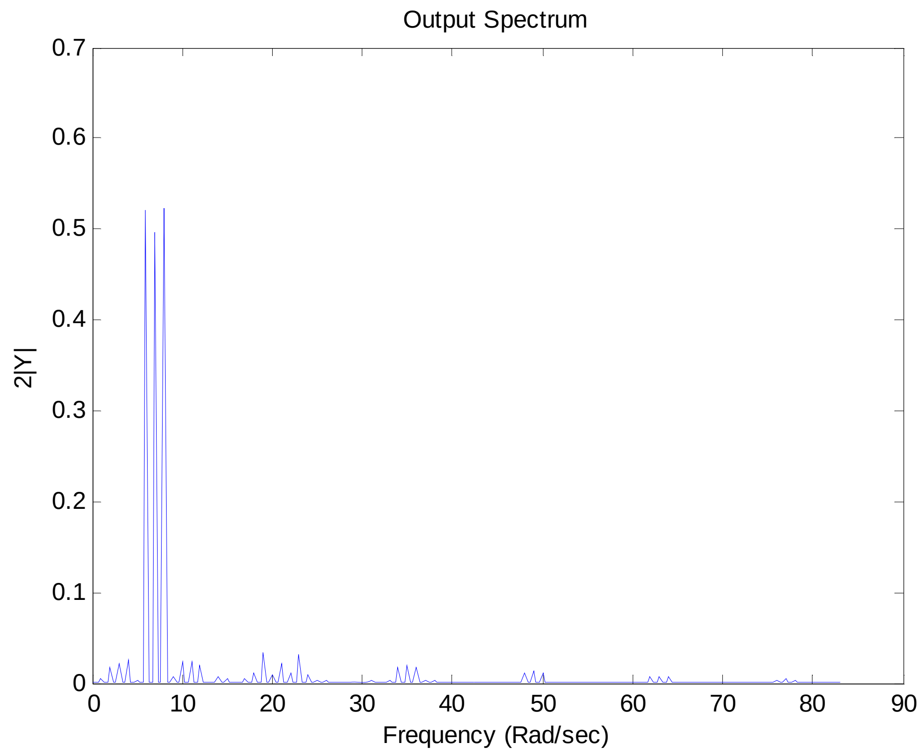

Figure 6. Output frequencies when a=0.1, b=5, c=0

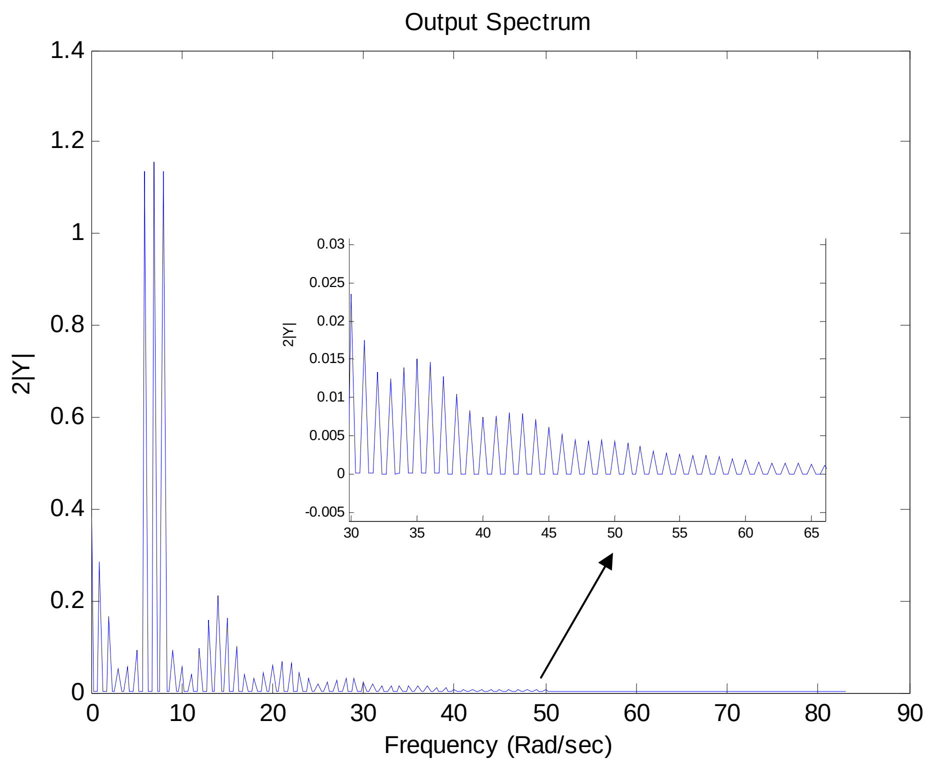

Figure 7. Output frequencies when a=0, b=0, c=0.09

When there are only odd nonlinearities, the output frequencies happen at around central frequencies 7*(2k+1). When there are even nonlinearities, the output frequencies appear at around central frequencies 7*k. The input nonlinearities only bring independently a finite band width of output frequencies to the system. The periodicity of the output frequencies can also be seen clearly from these figures.

Especially, it is worthy pointing out from Figures 1, 2 and 5 that there can be no crossing effects between proper chosen input nonlinearities as mentioned before, which can not be realized by the other kinds of nonlinearities. Thus the input frequencies can be moved to higher frequency periodically without interference between different periods and then decoded by using some methods. This property may have significance when a system is designed to achieve a special output spectrum at a desired frequency band in practices by using nonlinearities.

6 Conclusions

The super-harmonics and inter-modulations in the output frequencies of Volterra systems, especially of the nonlinear Volterra systems described by a NDE model, are demonstrated clearly, and some interesting properties of the system output frequencies are revealed explicitly for the first time in a uniform and analytical form. These properties provide several novel insights into the nonlinear behaviour in output spectrum of Volterra systems such as the periodicity and opposite property in the frequency domain, and reveal clearly the nonlinear effects on system output spectrum from different kind and degree of nonlinearities. These results can be used for the design of nonlinear systems or nonlinear filters to achieve a special output spectrum in a desired frequency band by taking advantage of nonlinearities, and provide an important and significant guidance to the analysis and design of nonlinear systems in the frequency domain by using the existing theory for Volterra systems. Further study will focus on the application issues.

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References

[1] Solomou, M. Evans, C. Rees, D. Chiras, N. Frequency domain analysis of nonlinear systems driven by multiharmonic signals, Proceedings of the 19th IEEE conference on Instrumentation and Measurement Technology, Vol 1, pp: 799- 804, 2002

[2] Taylor J. H., Describing Functions, an article in the Electrical Engineering Encyclopedia, John Wiley & Sons, Inc., New York, 1999

[3] W. J., Rugh. Nonlinear system theory --- the Volterra/Wiener approach. Baltimore and London: the Johns Hopkins University Press. 1981

[4] V. Volterra, Theory of Functionals and of Integral and Integro-differential Equations, Dover, New York, 1959.

[5] S. Boyd and L. O. Chua, “Fading memory and the problem of approximating nonlinear operators with Volterra series”, IEEE Trans. Circuits Syst., CAS-32 pp1150-1161, 1985
[6] W. Sandberg, “The mathematical foundations of associated expansions for mildly nonlinear systems”, IEEE Trans. Circuits Syst., CAS-30, pp441-455, 1983
[7] Rasool Dalirrooy Fard, Mehdi Karrari, and O. P. Malik, Synchronous Generator Model Identification for Control Application Using Volterra Series, IEEE Trans. Energy Conversion, Vol 20, No 4, pp 852- 858, 2005
[8] Doyle III, F. J., Pearson, R. K., & Ogunnaike, B. A.. Identification and control using Volterra models. Berlin: Springer, 2002
[9] S. French, Practical Nonlinear System Analysis by Wiener Kernel Estimation in the Frequency Domain, Biol. Cybernetics 24, 111-119, 1976
[10] N. Boutabba, L. Hassine, N. Loussaief, F. Kouki, H. Bouchriha, Volterra series analysis of the photocurrent in an Al/6T/ITO photovoltaic device, Organic Electronics 4, pp 1-8, 2003
[11] K. J. Friston, A. Mechelli, R. Turner, and C. J. Price, Nonlinear Responses in fMRI: The Balloon Model, Volterra Kernels, and Other Hemodynamics, NeuroImage 12, 466–477, 2000
[12] Junjie Yang and Sheldon X.-D. Tan, Nonlinear transient and distortion analysis via frequency domain Volterra series, Cirucits, Systems and Signal Processing, VOL. 25, NO. 3, PP. 295 – 314, 2006
[13] D.A.George, “Continuous nonlinear systems”, Technical Report 355, MIT Research Laboratory of Electronics, Cambridge, Mass. Jul. 24, 1959
[14] Chua L.O. and Ng C.Y., “Frequency domain analysis of nonlinear systems: general theory”, IEE Journal of Electronic Circuits and Systems, 3(4), pp. 165 – 185, 1979.
[15] X.J. Jing, Z.Q. Lang, S. A. Billings and G. R. Tomlinson, “The Parametric Characteristics of Frequency Response Functions for Nonlinear Systems”, International Journal of Control, Vol. 79, No. 12, December, pp 1552 – 1564, 2006
[16] Z. Q. Lang, and S. A. Billings. “Output frequency characteristics of nonlinear systems”. International Journal of Control, 64(6), pp1049-1067, 1996
[17] S.A. Billings and J.C. Peyton-Jones, “Mapping nonlinear integro-differential equation into the frequency domain”, International Journal of Control, Vol 54, 863-879, 1990
[18] G. M. Raz and B. D. Van Veen, “Baseband Volterra filters for implementing carrier based nonlinearities,” IEEE Trans. Signal Processing, vol.46, no. 1, pp. 103 – 114, Jan. 1998.
[19] Z. Q. Lang and S. A. Billings, Output frequencies of nonlinear systems, International Journal of Control, Vol 57, No 5, 713-730, 1997
[20] W. A. Frank, Sampling requirements for Volterra system identification, IEEE Signal Processing Letter, vol. 3, no. 9, pp. 266 – 268, Sep. 1996.
[21] Z. Q. Lang and S. A. Billings, Evaluation of Output Frequency Responses of Nonlinear Systems Under Multiple Inputs, IEEE Trans. Circuits and Systems—II: Analog and Digital Signal Processing, VOL. 47, NO. 1, pp 28-38, 2000
[22] Bedrosian, E., Rice, S. O.: The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputs. Proc. IEEE 59, 1688 (1971)

[23] Hua-Liang Wei, Zi-Qiang Lang, and Stephen A. Billings, An Algorithm for Determining the Output Frequency Range of Volterra Models With Multiple Inputs, IEEE Trans. Circuits and Systems—II: Express Brief, VOL. 54, NO. 6, pp 532-536, JUNE 2007
[24] S. A. Billings and Z. Q. Lang, Non-linear systems in the frequency domain: energy transfer filters, International Journal of Control, 75:14, 1066 – 1081, 2002
[25] Li Zhou and Eduardo A. Misawa, Low Frequency Vibration Suppression Shape Filter and High Frequency Vibration Suppression Shape Filter. American Control Conference, Portland, OR, USA, 4742-4747, June 8-10, 2005
[26] Nam S.W. and Powers E.J, Application of higher-order spectral analysis to cubically nonlinear-system identification, IEEE Transactions on Signal Processing, 42(7), pp. 1746 – 1765, 1994
[27] J. J. Bussgang, L. Ehrman, and J. W. Graham, Analysis of nonlinear systems with multiple inputs, Proc. IEEE, vol. 62, no. 8, pp. 1088-1119, Aug. 1974
[28] M. Schetzen, The Volterra and Wiener Theory of Nonlinear Systems, J. Wiley and Sons, 1980
[29] J. Schoukens, J. Nemeth, P. Crama, Y. Rolain, R. Pintelon, Fast approximate identification of nonlinear systems. *13th IFAC Symposium on System Identification*, Rotterdam, The Netherlands, 27-29 August, 2003, pp. 61-66