Analytical study of the frequency response function of a nonlinear spring damper system

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Abstract— **A spring damper system with a nonlinear damping element is investigated using the Volterra series method to study the system frequency response function (FRF) characteristics. The relationship between the FRF and the characteristic parameters of the nonlinear damper is determined to produce an analytical description for the system FRF. Simulation studies are used to verify the theoretical analysis. These results provide an important basis for the FRF based analysis and design of nonlinear spring damper systems in the frequency domain.**

Keywords-FRF,GFRF,Volterra series, probing

I. Introduction

Frequency domain analysis of linear systems is well established and is used in almost every branch of science and engineering. The classical time-domain characterization of a linear system is given by:

$$y(t)=\int_{-\infty}^{\infty} d\tau h(\tau)u(t-\tau) \tag{1}$$

The corresponding frequency-domain description is:

$$Y(j\omega)=H(j\omega)U(j\omega) \tag{2}$$

In (1) and (2), *h(t)* is the impulse response function of the system, $H(j\omega)$ is the frequency response function (FRF), and *y(t)* and *u(t)* are the output and input of the system respectively. $Y(j\omega)$ and $U(j\omega)$ are the system output and input spectra which are the Fourier Transform of *y(t)* and *u(t)* respectively. Linear dynamic systems are completely characterized by the impulse response function *h(t)* or the FRF $H(j\omega)$.

For nonlinear systems, the situation is more complicated and the classical approach to the analysis of linear systems normally does not work. Volterra introduced the Volterra series, which is an extension of equation (1) to the nonlinear case. Bedrosian and Rice [2] developed a harmonic probing algorithm which can be used to determine the Generalized Frequency Response Functions which are the Fourier Transforms of the Volterra kernels: a frequency domain description for nonlinear systems. Volterra kernel transforms. Based on the Volterra series and the harmonic probing algorithm, many theories have been developed to analyze the frequency response of nonlinear systems. Wiener and Spina [3] derived expressions for the single sinusoid describing function in terms of the Volterra kernels. Peyton Jones and Billings [4] extended these results and derived algorithms for obtaining the general harmonic input describing function. Chua and Ng [5] extended the Volterra series from single input single output systems to multi-input single-output systems. Expressions for the output frequency characteristics of nonlinear systems have been derived for multitone and general inputs by Lang and Billings [6]. This result is a natural extension of the well known linear relationship (2) to the nonlinear case. Worden, Manson, and Tomlinson [7] extended the conventional harmonic probing algorithm to deal with the Volterra functional series model subject to a multi-tone input. Swain and Billings [8] derived a recursive algorithm to compute the generalized frequency response function matrix of multi-input multi-output (MIMO) nonlinear systems.

It is well known that the FRF is an important concept in the analysis of linear systems in the frequency domain and which is widely used in a variety of engineering areas. Although, rigorously speaking, the same definition of FRF does not hold for nonlinear systems, the FRF concept has been applied by researchers and engineers to perform approximate nonlinear system frequency domain analysis. However, as far as we are aware of, few results have been reported on the analysis of the effects of the characteristic parameters of nonlinear systems on the FRF although such results may be important for FRF based nonlinear system analysis and design. In order to partly solve this problem, in the present study, we focus on a single-degree-of-freedom (SDOF) spring damper system with a nonlinear damping characteristic, to derive an analytical relationship between the FRF and the damping characteristic parameters of the system. Volterra based nonlinear frequency domain methods are used to perform the analysis, and the results can be extended to more general cases. Simulation studies are conducted to verify the theoretical analysis. This work provides an important basis for the FRF based analytical studies including the design of nonlinear spring damper systems.

II. System Description

Consider the nonlinear spring damper system shown in Figure 1. A mass, m, supported on a nonlinear damper and a parallel spring, is subject to a harmonic disturbance of amplitude, F_d at the frequency Ω . The nonlinear damping characteristic is represented by a third order polynomial:

$$f(.) = a_1(.) + a_2(.)^2 + a_3(.)^3 \tag{3}$$

where a_1, a_2, a_3 are the parameters of the damper. The characteristic parameter of the spring is k_1, and the force transmitted to the support is $F_s(t)$. The system input and output equilibrium equations can be expressed as:

$$m\ddot{x}(t) + a_1\dot{x}(t) + a_2\dot{x}^2(t) + a_3\dot{x}^3(t) + kx(t) = F_d \sin(\Omega t) \tag{4}$$

$$F_s(t) = a_1\dot{x}(t) + a_2\dot{x}^2(t) + a_3\dot{x}^3(t) + kx(t) \tag{5}$$

For convenience of analysis, define

$$y_1(t) = x(t) \tag{6}$$

$$y_2(t) = F_s(t) \tag{7}$$

and $u_1 = F_d \sin(\Omega t)$ (8)

The system can then be described by a single input two output system as:

$$m\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_2\dot{y}_1{}^2(t) + a_3\dot{y}_1{}^3(t) + ky_1(t) = F_d \sin(\Omega t) \tag{9}$$

$$y_2(t) = a_1\dot{y}_1(t) + a_2\dot{y}_1^2(t) + a_3\dot{y}_1^3(t) + ky_1(t) \tag{10}$$

The objective of this study is to investigate how the system nonlinearity affects the FRF of the system. In order to achieve this, we need to derive the relationship between the system FRF and the nonlinear damping parameters a_2 and a_3.

III. THE FREQUENCY RESPONSE FUNCTION

For a multi-input multi-output (MIMO) nonlinear system, under certain conditions, the output of the j_1 th subsystem can be described by a Volterra functional polynomial as (Worden, Manson and Tomlinson [7], Swain and Billings [8]):

$$y_{j_1}(t) = \sum_{n=1}^{N} y_{j_1}^{(n)}(t) \tag{11}$$

where $y_{j_1}^{(n)}(t)$ is nth order component of $y_{j_1}(t)$, N is the maximum order of the system nonlinearity, and

$$y_{j_1}^{(n)}(t) = \sum_{\beta_1=1}^{r}\sum_{\beta_2=\beta_1}^{r}\cdots\sum_{\beta_n=\beta_{n-1}}^{r}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n^{(j_1;\beta_1,\cdots\beta_n)}(\tau_1,\cdots,\tau_n)u_{\beta_1}(t-\tau_1)\cdots u_{\beta_n}(t-\tau_n)d\tau_1\cdots d\tau_n \tag{12}$$

where r is the number of inputs and $h_n^{(j_1;\beta_1,\cdots\beta_n)}(\tau_1,\cdots,\tau_n)$ is the nth order Volterra kernel of the j_1 th subsystem.

The output of a single input two output nonlinear system is therefore given by:

$$y_{j_1}(t) = \sum_{n=1}^{N} y_{j_1}^{(n)}(t) \qquad j_1 = 1, 2 \tag{13}$$

$$y_{j_1}^{(n)}(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n^{j_1:\overbrace{1,\cdots 1}^{n}}(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u_1(t-\tau_i)d\tau_i \quad j_1=1,2 \tag{14}$$

When system (13) (14) is subjected to a multitone input:

$$u(t) = \sum_{i=1}^{K}|A_i|\cos(w_i t + \angle A_i) = \sum_{i=1}^{K}\left(\frac{A_i}{2}e^{jw_i t} + \frac{A_i^*}{2}e^{-jw_i t}\right) = \sum_{i=-K, i\neq 0}^{K}\frac{A_i}{2}e^{jw_i t} \tag{15}$$

where $A_{-i} = A_i^*, w_{-i} = -w_i$, according to the output spectrum of nonlinear systems derived by Lang and Billings [6], the frequency domain input-output relationship of the system can be:

$$Y_{j_1}(jw) = \sum_{n=1}^{N}\frac{1}{2^n}\sum_{w_{k_1}+\cdots w_{k_n}=w} H_n^{j_1:\overbrace{1,\cdots 1}^{n}}(jw_{k_1},\cdots,jw_{k_n})A(w_{k_1})\cdots A(w_{k_n}) \tag{16}$$

$j_1 = 1, 2$

where $Y_{j_1}(jw)$, $j_1=1, 2$, are the output spectra of the j_1 th output. $H_n^{j_1:\overbrace{1,\cdots 1}^{n}}(jw_{k_1},\cdots,jw_{k_n})$ is the multi-dimensional Fourier Transform of the nth order impulse response of the j_1 th subsystem called the nth order generalized frequency response function (GFRF)

$$H_n^{j_1:\overbrace{1,\cdots 1}^{n}}(jw_{k_1},\cdots,jw_{k_n}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n^{j_1:\overbrace{1,\cdots 1}^{n}}(\tau_1,\cdots,\tau_n)e^{-j(w_1\tau_1+\cdots+w_n\tau_n)}d\tau_1\cdots d\tau_n \tag{17}$$

$$k_l \in \{-1,+1\}, l = 1,\cdots,n$$

$$A(w) = \begin{cases}|A_k|e^{j\angle A_k} \; if w \in \{w_k, k=\pm 1\}, \; where |A_{\pm 1}| = F_d, \; w_{\pm 1} = \pm\Omega, \; and \angle A_{\pm 1} = \mp\pi/2 \\ 0, \; otherwise\end{cases} \tag{18}$$

When the spring damper system (9) and (10) is equivalently described by the Volterra model (13) (14) and is subjected to the multitone input (15), the output spectra of the system is given by (16). Consider $j_1 = 2$ in (16), and write out the terms in the expression for $Y_2(jw)$ up to 5th order of nonlinearity to yield

$$\begin{aligned} Y_2(jw) &= \frac{1}{2}H_1^{2:1}(jw)A(w) + \frac{1}{2^2}\sum_{w_{k_1}+w_{k_2}=w} H_2^{2:11}(jw_{k_1}, jw_{k_2})A(w_{k_1})A(w_{k_2}) \\ &+ \frac{1}{2^3}\sum_{w_{k_1}+w_{k_2}+w_{k_3}=w} H_3^{2:111}(jw_{k_1}, jw_{k_2}, jw_{k_3})A(w_{k_1})A(w_{k_2})A(w_{k_3}) \\ &+ \frac{1}{2^4}\sum_{w_{k_1}+\ldots+w_{k_4}=w} H_4^{2:1111}(jw_{k_1},\ldots, jw_{k_4})A(w_{k_1})\ldots A(w_{k_4}) \\ &+ \frac{1}{2^5}\sum_{w_{k_1}+\ldots+w_{k_5}=w} H_5^{2:11111}(jw_{k_1},\ldots, jw_{k_5})A(w_{k_1})\ldots A(w_{k_5}) + \cdots \end{aligned} \tag{19}$$

Denote

$$Q_1(jw) = \frac{1}{2}H_1^{2:1}(jw)A(w) \tag{20}$$

$$Q_2(jw) = \frac{1}{2^2}\sum_{w_{k_1}+w_{k_2}=w} H_2^{2:11}(jw_{k_1}, jw_{k_2})A(w_{k_1})A(w_{k_2}) \tag{21}$$

$$Q_3(jw) = \frac{1}{2^3}\sum_{w_{k_1}+w_{k_2}+w_{k_3}=w} H_3^{2:111}(jw_{k_1}, jw_{k_2}, jw_{k_3})A(w_{k_1})A(w_{k_2})A(w_{k_3}) \tag{22}$$

$$Q_4(jw) = \frac{1}{2^4}\sum_{w_{k_1}+\ldots+w_{k_4}=w} H_4^{2:1111}(jw_{k_1},\ldots, jw_{k_4})A(w_{k_1})\ldots A(w_{k_4}) \tag{23}$$

$$Q_5(jw) = \frac{1}{2^5}\sum_{w_{k_1}+\ldots+w_{k_5}=w} H_5^{2:11111}(jw_{k_1},\ldots, jw_{k_5})A(w_{k_1})\ldots A(w_{k_5}) \tag{24}$$

And consider the case where the system is subjected to the harmonic input (8). After substituting Ω into (20)-(24) for w and some manipulations then

$$Y_2(j\Omega) = Q_1(j\Omega) + Q_2(j\Omega) + Q_3(j\Omega) + Q_4(j\Omega) + Q_5(j\Omega) + \cdots \tag{25}$$

$$Q_1(j\Omega) = \frac{1}{2}H_1^{2:1}(j\Omega)A(\Omega) \tag{26}$$

$$Q_2(j\Omega) = 0 \tag{27}$$

$$Q_3(j\Omega)=\frac{|A(\Omega)|^2 A(\Omega)}{2^3}\begin{bmatrix} H_3^{2:111}(-j\Omega, j\Omega, j\Omega)+ \\ H_3^{2:111}(j\Omega, -j\Omega, j\Omega)+ \\ H_3^{2:111}(j\Omega, j\Omega, -j\Omega) \end{bmatrix} \tag{28}$$

$$Q_4(j\Omega)=0 \tag{29}$$

$$Q_5(j\Omega)=\frac{1}{2^5}|A(\Omega)|^4 A(\Omega)\sum H_5^{2:11111}(\cdot) \tag{30}$$

$$\sum H_5^{2:11111}(\cdot)=\begin{bmatrix} H_5^{2:11111}(-j\Omega, j\Omega, j\Omega, -j\Omega, j\Omega)+H_5^{2:11111}(-j\Omega, j\Omega, j\Omega, j\Omega, -j\Omega)+ \\ H_5^{2:11111}(-j\Omega, -j\Omega, j\Omega, j\Omega, j\Omega)+H_5^{2:11111}(-j\Omega, j\Omega, -j\Omega, j\Omega, j\Omega)+ \\ H_5^{2:11111}(j\Omega, j\Omega, -j\Omega, j\Omega, -j\Omega)+H_5^{2:11111}(j\Omega, j\Omega, j\Omega, -j\Omega, -j\Omega)+ \\ H_5^{2:11111}(j\Omega, -j\Omega, j\Omega, -j\Omega, j\Omega)+H_5^{2:11111}(j\Omega, -j\Omega, -j\Omega, j\Omega, j\Omega)+ \\ H_5^{2:11111}(j\Omega, j\Omega, -j\Omega, -j\Omega, j\Omega)+H_5^{2:11111}(j\Omega, -j\Omega, j\Omega, j\Omega, -j\Omega) \end{bmatrix} \tag{31}$$

Consequently, when subject to a harmonic input, the FRF of system (9) (10) between the input and the second output can be expressed as

$$H(j\Omega)=\frac{Y_2(j\Omega)}{A(\Omega)}$$

$$=\frac{1}{2}H_1^{21}(j\Omega)+\frac{|A(\Omega)|^2}{2^3}\begin{bmatrix} H_3^{2:111}(-j\Omega, j\Omega, j\Omega)+ \\ H_3^{2:111}(j\Omega, -j\Omega, j\Omega)+ \\ H_3^{2:111}(j\Omega, j\Omega, -j\Omega) \end{bmatrix}+\frac{1}{2^5}|A(\Omega)|^4\sum H_5^{2:11111}(\cdot)+\cdots \tag{32}$$

Equation (32) shows that the effect of a_2 and a_3 on the generalized frequency response functions $H_n^m(\cdot)$ of the system have to be determined first.

IV. Generalized Frequency Response Functions

Given a parametric model of a nonlinear system, an effective method for determining the system GFRFs is the harmonic probing method introduced by Bedrosian and Rice [2]. The basic idea of this method is to apply an input *u(t)* which is a combination of exponentials such that

$$u(t)=\sum_{i=1}^{R} e^{jw_i t} \qquad 1\le R\le N \tag{33}$$

to excite the system under study, and then to substitute the exponential input and the output response of the Volterra series model into the system parametric model. The Rth order GFRF of the system can then be obtained by extracting the coefficient of $e^{j(w_1+\cdots w_R)}$ from the resulting expression.

To use the harmonic probing method to determine the GFRFs of system (9) (10), the multi-frequency excitation (33) is applied as input to excite the system to yield the response of from the system Volterra series model (13) (14) as

$$\begin{aligned} y_{j_1}(t) &= \sum_{n=1,n\ne R}^{N}\sum_{m(n)} G_{m_1(n)\cdots m_R(n)}^{j_1:\underbrace{1,\cdots,1}_{n}}(jw_1,\cdots,jw_R)e^{j[m_1(n)w_1+\cdots+m_R(n)w_R]t} \\ &+ R!H_R^{j_1:\underbrace{1,\cdots,1}_{R}}(jw_1,\cdots,jw_R)e^{j(w_1+\cdots+w_R)t} \qquad j_1=1,2 \end{aligned} \tag{34}$$

$$G_{m_1(n)\cdots m_R(n)}(jw_1,\cdots,jw_R)=\frac{n!}{m_1(n)!\cdots m_R(n)!}H_n(\underbrace{jw_1,\cdots,jw_1}_{m_1(n)},\cdots,\underbrace{jw_R,\cdots,jw_R}_{m_R(n)}) \tag{35}$$

Substitute (33) and (34) into the system parametric model (9) (10) for *u(t)* and $y_{j_1}(t)$, $j_1=1,2$ respectively and extract the coefficient from the resulting expressions to obtain the equations from which the GFRFs of the nonlinear spring damper system

$\left[H_n^{1:\underbrace{1,\cdots 1}_{n}}(jw_1,\cdots,jw_n), H_n^{2:\underbrace{1,\cdots 1}_{n}}(jw_1,\cdots,jw_n)\right]$ n=1, 2, 3 can be derived.

Following this procedure, the GFRFs of system (9) (10) up to the 5th order nonlinearity have been determined for the analysis of the FRF of the system.

For the first order GFRFs, the harmonic input

$$u_1(t)=e^{jw_1 t} \tag{36}$$

is applied to excite the system. Because R=1, (34) can be written as

$$\begin{cases} y_1(t)=H_1^{1:1}(jw_1)e^{jw_1 t}+\cdots \\ y_2(t)=H_1^{2:1}(jw_1)e^{jw_1 t}+\cdots \end{cases} \tag{37}$$

Substituting (36) and (37) into (9) and (10), and extracting the coefficient of $e^{jw_1 t}$, gives the first order GFRFs:

$$\begin{cases} H_1^{1:1}(jw_1)=\dfrac{1}{m(jw_1)^2+a_1(jw_1)+k} \\ H_1^{2:1}(jw_1)=\dfrac{a_1(jw_1)+k}{m(jw_1)^2+a_1(jw_1)+k} \end{cases} \tag{38}$$

For the 2nd order GFRFs, the harmonic input

$$u_1(t)=e^{jw_1 t}+e^{jw_2 t} \tag{39}$$

is applied to excite the system. Now, R=2, and (34) is given by:

$$\begin{cases} y_1(t)=H_1^{1:1}(jw_1)e^{jw_1 t}+H_1^{1:1}(jw_2)e^{jw_2 t}+2H_2^{1:11}(jw_1,jw_2)e^{j(w_1+w_2)t}+\cdots \\ y_2(t)=H_1^{2:1}(jw_1)e^{jw_1 t}+H_1^{2:1}(jw_2)e^{jw_2 t}+2H_2^{2:11}(jw_1,jw_2)e^{j(w_1+w_2)t}+\cdots \end{cases} \tag{40}$$

Substitute (39) and (40) into (9) and (10), and extracting the coefficient of $e^{j(w_1+w_2)t}$, yields the 2nd order GFRFs:

$$\begin{cases} H_2^{1:11}(jw_1,jw_2)=-a_2F_0(jw_1,jw_2) \\ H_2^{2:11}(jw_1,jw_2)=ma_2(jw_1+jw_2)^2F_0(jw_1,jw_2) \end{cases} \tag{41}$$

$$F_0(jw_1,jw_2)=\frac{H_1^{1:1}(jw_1)H_1^{1:1}(jw_2)(jw_1)(jw_2)}{m(jw_1+jw_2)^2+a_1j(w_1+w_2)+k} \tag{42}$$

For the 3rd order GFRFs, the harmonic input applied is

$$u_1(t)=e^{jw_1 t}+e^{jw_2 t}+e^{jw_3 t} \tag{43}$$

Since R=3, from (34)

$$\begin{cases} y_1(t)=H_1^{1:1}(jw_1)e^{jw_1 t}+H_1^{1:1}(jw_2)e^{jw_2 t}+H_1^{1:1}(jw_3)e^{jw_3 t}+ \\ 2H_2^{1:11}(jw_1,jw_2)e^{j(w_1+w_2)t}+2H_2^{1:11}(jw_1,jw_3)e^{j(w_1+w_3)t} \\ +2H_2^{1:11}(jw_2,jw_3)e^{j(w_2+w_3)t}+6H_3^{1:111}(jw_1,jw_2,jw_3)e^{j(w_1+w_2+w_3)t}+\cdots \\ \\ y_2(t)=H_1^{2:1}(jw_1)e^{jw_1 t}+H_1^{2:1}(jw_2)e^{jw_2 t}+H_1^{2:1}(jw_3)e^{jw_3 t}+ \\ 2H_2^{2:11}(jw_1,jw_2)e^{j(w_1+w_2)t}+2H_2^{2:11}(jw_1,jw_3)e^{j(w_1+w_3)t} \\ +2H_2^{2:11}(jw_2,jw_3)e^{j(w_2+w_3)t}+6H_3^{2:111}(jw_1,jw_2,jw_3)e^{j(w_1+w_2+w_3)t}+\cdots \end{cases} \tag{44}$$

Substituting (43) and (44) into (9) and (10), and extracting the coefficient of $e^{j(w_1+w_2+w_3)t}$, yields the 3rd GFRFs

$$\begin{cases} H_3^{1:111}(jw_1,jw_2,jw_3)=\dfrac{a_2^2F_1(jw_1,jw_2,jw_3)-a_3F_2(jw_1,jw_2,jw_3)}{\beta(jw_1+jw_2+jw_3)} \\ H_3^{2:111}(jw_1,jw_2,jw_3)=\dfrac{-m(jw_1+jw_2+jw_3)^2}{\beta(jw_1+jw_2+jw_3)}\left[a_2^2F_1(jw_1,jw_2,jw_3)-a_3F_2(jw_1,jw_2,jw_3)\right] \end{cases} \tag{45}$$

$$\beta(jw_1+jw_2+jw_3)=m(jw_1+jw_2+jw_3)^2+a_1(jw_1+jw_2+jw_3)+k \tag{46}$$

$$F_1(jw_1, jw_2, jw_3) = \frac{2}{3} \times \begin{bmatrix} H_1^{1:1}(jw_1)F_0(jw_2, jw_3)(jw_1)(jw_2 + jw_3) \\ +H_1^{1:1}(jw_2)F_0(jw_1, jw_3)(jw_2)(jw_1 + jw_3) \\ +H_1^{1:1}(jw_3)F_0(jw_1, jw_2)(jw_3)(jw_1 + jw_2) \end{bmatrix} \quad (47)$$

$$F_2(jw_1, jw_2, jw_3) = H_1^{1:1}(jw_1)H_1^{1:1}(jw_2)H_1^{1:1}(jw_3)(jw_1)(jw_2)(jw_3) \quad (48)$$

For GFRFs of higher orders, the same procedure yields expressions which are not presented due to page limitations. The 1st, 3rd and 5th order GFRFs will be used in the next section to determine an analytical expression for the system FRF $H(j\Omega)$.

V. Analytical Description of the Frequency Response Function

The FRF of the system (9) (10) of interest in the present study has been given in (32). Truncating the expression at 5th order yields

$$H(j\Omega) = \frac{Y_2(j\Omega)}{A(\Omega)} = \frac{1}{2}H_1^{2:1}(j\Omega) + \frac{|A(\Omega)|^2}{2^3}\begin{bmatrix} H_3^{2:111}(-j\Omega, j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, -j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, j\Omega, -j\Omega) \end{bmatrix} + \frac{1}{2^5}|A(\Omega)|^4 \sum H_5^{2:11111}(\cdot) \quad (49)$$

which is valid under the assumption that the system nonlinear effects beyond 5th order are negligible.

To derive an analytical expression for the FRF $H(j\Omega)$ in terms of the system damping characteristic parameters a_2 and a_3, substitute $H_3^{2:111}(\cdot)$ and $H_5^{2:11111}(\cdot)$ evaluated at the relevant frequency points into (49) for $\left[H_3^{2:111}(-j\Omega, j\Omega, j\Omega) + H_3^{2:111}(j\Omega, -j\Omega, j\Omega) + H_3^{2:111}(j\Omega, j\Omega, -j\Omega)\right]$ and $\sum H_5^{2:11111}(\cdot)$ respectively. In order to illustrate the approach, consider the derivation of the analytical relationship between $H(j\Omega)$ and a_2, a_3 in three relatively simple cases: (i) $a_2 \neq 0$, $a_3 \neq 0$ and ignoring the nonlinear effects higher than 3rd order; (ii) $a_2 = 0$, $a_3 \neq 0$ and considering nonlinear effects up to 5th order; (iii) $a_2 \neq 0$, $a_3 = 0$ and considering nonlinear effects up to 5th order.

In case (i), substitute the results of $H_3^{2:111}(jw_1, jw_2, jw_3)$ given by (45) and evaluate at $(-\Omega, \Omega, \Omega)$, $(\Omega, -\Omega, \Omega)$ and $(\Omega, \Omega, -\Omega)$ respectively into (49), yields

$$\begin{aligned} H(j\Omega) &= \frac{1}{2}H_1^{2:1}(j\Omega) + \frac{|A(\Omega)|^2}{2^3}\begin{bmatrix} H_3^{2:111}(-j\Omega, j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, -j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, j\Omega, -j\Omega) \end{bmatrix} \\ &= p_1(j\Omega) - p_2(j\Omega)a_2^2 + p_3(j\Omega)a_3 \end{aligned} \quad (50)$$

$$p_1(j\Omega) = \frac{1}{2}H_1^{2:1}(j\Omega) \quad (51)$$

$$p_2(j\Omega) = \frac{m\Omega^6 \left|H_1^{1:1}(j\Omega)\right|^2 H_1^{1:1}(j\Omega)\left|A(\Omega)\right|^2}{2\beta(j\Omega)\beta(j2\Omega)} \quad (52)$$

$$p_3(j\Omega) = -\frac{3j\Omega^5 m \left|H_1^{1:1}(j\Omega)\right|^2 H_1^{1:1}(j\Omega)\left|A(\Omega)\right|^2}{2^3\beta(j\Omega)} \quad (53)$$

In case (ii), substituting $H_3^{2:111}(jw_1, jw_2, jw_3)$ evaluated at $(-\Omega, \Omega, \Omega)$, $(\Omega, -\Omega, \Omega)$ and $(\Omega, \Omega, -\Omega)$ and $H_5^{2:11111}(jw_1, \cdots, jw_5)$ evaluated at the ten 5-dimensional frequency space points in the definition of $\sum H_5^{2:11111}(\cdot)$ in (31) into (49) yields

$$\begin{aligned} H(j\Omega) &= \frac{1}{2}H_1^{2:1}(j\Omega) + \frac{|A(\Omega)|^2}{2^3}\begin{bmatrix} H_3^{2:111}(-j\Omega, j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, -j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, j\Omega, -j\Omega) \end{bmatrix} \\ &+ \frac{1}{2^5}|A(\Omega)|^4 \sum H_5^{2:11111}(\cdot) = p_1(j\Omega) + p_3(j\Omega)a_3 + p_4(j\Omega)a_3^2 \end{aligned} \quad (54)$$

where p_1 and p_3 are as defined above and

$$p_4(j\Omega) = -\left(\frac{1}{2^5\beta^4(\Omega)\beta^2(-\Omega)}\right)|A(\Omega)|^4\Omega^8 m\left[\frac{18}{\beta(\Omega)} - \frac{9}{\beta(-\Omega)} + \frac{9}{\beta(3\Omega)}\right] \quad (55)$$

In case (iii), the same computation as in case (ii) yields

$$\begin{aligned} H(j\Omega) &= \frac{1}{2}H_1^{2:1}(j\Omega) + \frac{|A(\Omega)|^2}{2^3}\begin{bmatrix} H_3^{2:111}(-j\Omega, j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, -j\Omega, j\Omega) + \\ H_3^{2:111}(j\Omega, j\Omega, -j\Omega) \end{bmatrix} \\ &+ \frac{1}{2^5}|A(\Omega)|^4 \sum H_5^{2:11111}(\cdot) = p_1(j\Omega) - p_2(j\Omega)a_2^2 + p_5(j\Omega)a_2^4 \end{aligned} \quad (56)$$

where p_1 and p_2 are the same as defined above and

$$p_5(j\Omega) = \frac{m\Omega^{10}\left|\mathrm{A}(\Omega)\right|^4}{2^5\beta^4(\Omega)\beta^2(-\Omega)\beta(2\Omega)} \times \begin{bmatrix} \frac{32}{\beta(\Omega)\beta(2\Omega)} - \frac{48}{\beta(3\Omega)\beta(-2\Omega)} + \\ \frac{48}{\beta(2\Omega)\beta(3\Omega)} + \frac{16}{\beta(-\Omega)\beta(-2\Omega)} \end{bmatrix} \quad (57)$$

More general results which consider both a_2 and a_3 and take higher order system nonlinearities into account can be determined following the same approach. These results explicitly reveal how the FRF depends on the system nonlinear damping characteristic parameters, and are therefore of considerable significance for the analysis of the effects of the nonlinear parameters on the system behavior and for the design of this system in the frequency domain.

VI. Verification of the Analytical Result

In order to verify the analytically determined FRF, consider the nonlinear spring damper system of (9) (10) excited by the harmonic input (8) with the parameters *m* and *k* given by *k*=16000 N/m; *m*=240 kg.

Simulation studies were conducted to determine the Nyquist plots of the system FRF under the three different cases of (i) (ii) (iii) in section 5 and to compare the simulation results with the FRFs analytically determined using (50) (54) and (56) respectively.

For case (i), where $a_2 \neq 0$, $a_3 \neq 0$, simulation studies were conducted for $a_1 = 296$, $F_d = 100$ to compare the system FRFs with the analytically determined results for the following five choices of a_2 and a_3: (1) $a_2 = 200$, $a_3 = 200$; (2) $a_2 = 500$, $a_3 = 500$; (3) $a_2 = 1200$, $a_3 = 1200$; (4) $a_2 = 100$, $a_3 = 1000$; (5) $a_2 = 1000$, $a_3 = 500$.

The solid lines in Figures 2-6 represent the simulation results of the FRF obtained by a FFT analysis. The dashed lines show the FRF analytically determined using (50) when nonlinear terms up to 3rd order are considered. The dotted lines represent the FRF which only takes the system linear effect into account. Figures 2, 3 and 4 clearly indicate how the theoretical analysis works for the three choices of a_2 and a_3. From Figure 2, it can be observed that the analytical result is very similar to the simulation results when $a_2 = a_3 = 200$. Figure 3 shows that as the value of a_2 a_3 increases to 500, which means an increase of the nonlinear effects of the system, the analytical results can still represent the simulation results reasonably well. A detailed analysis of Figure 4 indicates that when $a_2 = a_3$=1200, the analytically determined FRF can represent the simulation results over all frequencies apart from those between 7.7 and 8.7, which is near to the resonant frequency of 8.16 rads/s. The further the frequency is away from the resonant frequency, the more accurately the analytical expressions can represent the FRF of the system. Further simulation studies show that when $a_2 = a_3$ increase beyond 1200, basically the analytically determined FRF can not be used to well represent the real system FRF. This implies that the system nonlinearity higher than 3rd order has to be taken into account in these cases to achieve a better analytical representation of the system FRF. Figures 5 and 6 show a comparison of the analytical and simulation results for two other choices of a_2 and a_3, and demonstrates that the conclusions for the case of $a_2 = a_3$ also holds for these more complicated situations.

For case (ii), where $a_2 = 0$, $a_3 \neq 0$, simulations studies were conducted to compare the system FRF with the analytical results for the following choices of a_1 and F_d: (1) $a_1 = 29.6$, $F_d = 1$, $a_3 = 3000$; (2) $a_1 = 29.6$, $F_d = 1$, $a_3 = 5000$; (3) $a_1 = 29.6$, $F_d = 1$, $a_3 = 10000$; (4) $a_1 = 296$, $F_d = 100$, $a_3 = 300$; (5) $a_1 = 296$, $F_d = 100$, $a_3 = 500$; (6) $a_1 = 296$, $F_d = 100$, $a_3 = 1000$

In Figures 7-12, solid and dotted lines again represent the simulated FRF and the FRF which only takes linear effects into account. The dash-dotted lines show the analytical FRF determined by (54) which considers system nonlinear effects up to 5th order.

The results in Figures 10-12 reflect the same phenomena as demonstrated by Figures 2-6 but for the cases $a_2 = 0$, $a_3 = 300$; $a_2 = 0$, $a_3 = 500$; and $a_2 = 0$, $a_3 = 1000$. The results in Figures 7-9 show similar phenomena but under a smaller $a_1 = 29.6$, a smaller input amplitude $F_d = 1$, and a greater range of change of a_3 from $a_3 = 3000$ to $a_3 = 10000$. Notice that a smaller value of a_1 and a greater value of F_d both correspond to severe nonlinear effects. Because of this, for the two scenarios of smaller $a_1 = 29.6$, smaller $F_d = 1$ and greater $a_1 = 296$, greater $F_d = 100$, similar comparison results between the simulated and analytical FRFs can be observed.

For case (iii), where $a_2 \neq 0$, $a_3 = 0$, simulation studies were conducted when $F_d = 100$ to compare the system FRF with the analytical results for the following choices of a_1 and a_2: (1) $a_1 = 1960$, $a_2 = 20000$; (2) $a_1 = 1960$, $a_2 = 30000$; (3) $a_1 = 2960$, $a_2 = 40000$; (4) $a_1 = 2960$, $a_2 = 50000$

The comparison between the system FRF and the analytically determined results is shown in Figures 13-16 for the four different choices of a_1 and a_2 respectively. Conclusions similar to those for cases (i) and (ii) can be reached regarding these results.

Simulation studies for the above three different cases all indicate that the analytically obtained FRF can match the real results over a considerable range of parameter values and frequencies although only up to 3rd or 5th order system nonlinear effects have been considered. If higher order nonlinearities were taken into account in the analytical expressions (50), (54) and (56) for the system FRF, it should be possible to get significant improvements in Figures 4,5,9,12,14 and 16 where the match between the analytical and real FRF result is not very satisfactory close to the system resonance frequency 8.16 rad/s.

Given a specific harmonic input, and the parameters k, a_1, m in the linear part of the system (9) (10), it is shown via theoretical analysis in section 5 and verified by simulation studies above that the FRF of the nonlinear spring damper system can be expressed as an explicit analytical function of the system damping characteristic parameters a_2 and a_3. This result can be used directly in the FRF based analysis and design of the system (9) (10). This approach can be extended to address the frequency domain analysis and design issues of more general nonlinear systems.

VII. CONCLUSIONS

The FRF of a nonlinear spring damper system has been studied. For the first time, an analytical relationship between the FRF and the nonlinear damping characteristic parameters has been derived for this system using Volterra series method. Simulation studies have been used to verify the theoretical analysis. It has been observed from the simulation results that the orders of nonlinearity considered in the analytical FRF description have significant impact on the accuracy of the derived analytical expressions when severe nonlinear effects are involved. This work provides a significant basis for the FRF based analytical study and design of nonlinear spring damper systems in the frequency domain.

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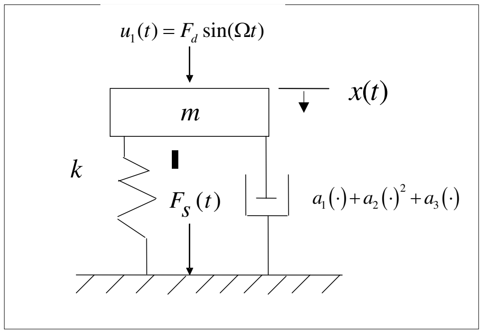
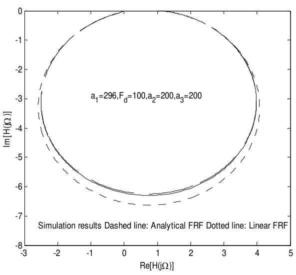


Figure 1. The single-degree-of-freedom (SDOF) spring damper system



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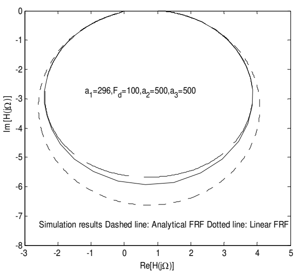

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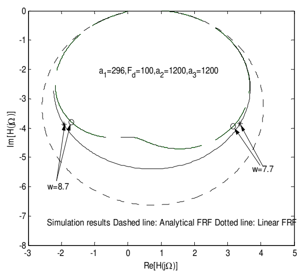

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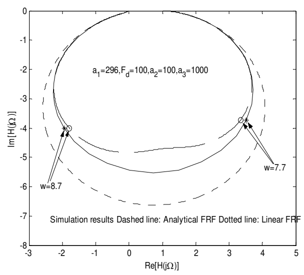

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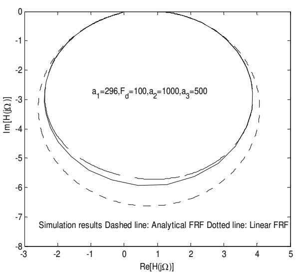

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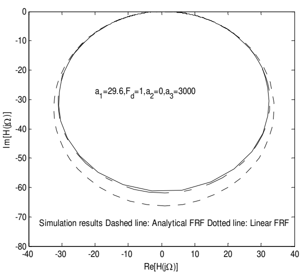

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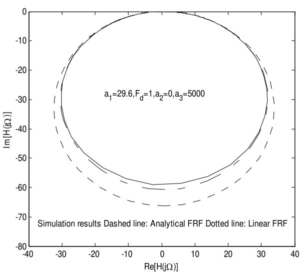

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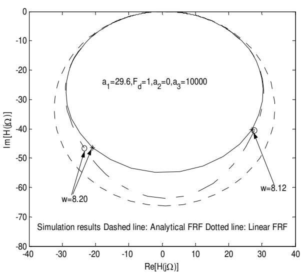


Figure 9

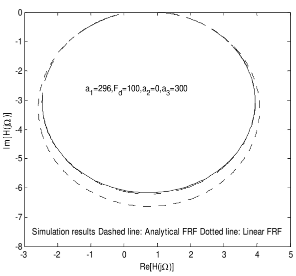
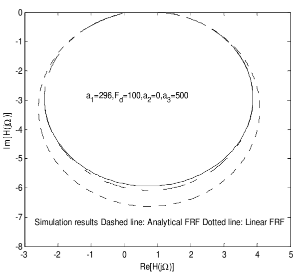


Figure 10



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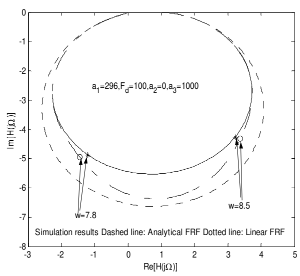

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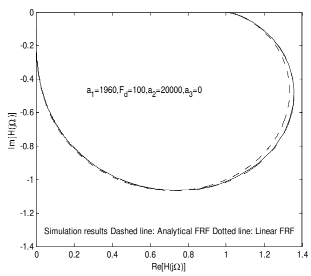

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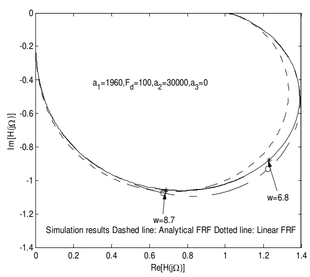

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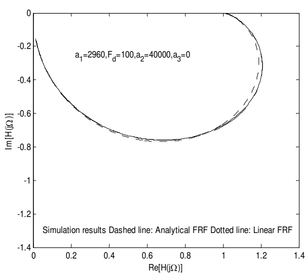

Figure 15

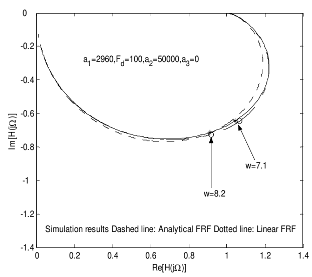

Figure 16