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# Comparing Axiomatizations of Free Pseudospaces

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## Abstract

Independently and pursuing different aims, Hrushovski and Srour [3] and Baudisch and Pillay [1] have introduced two free pseudospaces that generalize the well known concept of Lachlan's free pseudoplane. In this paper we investigate the relationship between these free pseudospaces, proving in particular, that the pseudospace of Baudisch and Pillay is a reduct of the pseudospace of Hrushovski and Srour.

**Key words:** free pseudoplane, free pseudospace, stable theories, equational theories

**MSC (2000):** 03C45

## 1 Introduction

Already back in 1974 Lachlan [8] introduced the free pseudoplane which is by now a well studied and well understood model-theoretic object. In particular, Hrushovski and Pillay [4] showed that 1-based or weakly normal theories do not contain a type-definable pseudoplane. Hence the free pseudoplane is the prototype of a stable and not 1-based theory. While the free pseudoplane is a 2-dimensional object in essence, two generalizations of the pseudoplane in form of 3-dimensional pseudospaces were independently introduced by Hrushovski and Srour [3] and Baudisch and Pillay [1]. The motivations for the construction of these pseudospaces differ, but the constructions itself share many common features.

The free pseudospace of Hrushovski and Srour is a coloured 3-dimensional pseudospace and was constructed as the first example of a stable and non-equational theory. Equational theories were introduced by Srour [10, 12–14] and further developed by Junker, Kraus, and Lascar [5–7]. A parameter-free formula  $\varphi(\bar{x}; \bar{y})$  with two sorts of variables  $\bar{x}$  and  $\bar{y}$  is called an equation, if every infinite conjunction  $\bigwedge_{i \in I} \varphi(\bar{x}; \bar{a}_i)$  of instances of  $\varphi$  is equivalent to a conjunction

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$\bigwedge_{i \in I_0} \varphi(\bar{x}, \bar{a}_i)$  with finite  $I_0 \subseteq I$ .<sup>1</sup> A theory is equational, if every formula is equivalent to a Boolean combination of equations.

By counting the number of types it is easy to see that equational theories are stable [10]. Thus Srouf posed the question whether the class of equational theories is a proper subclass of the class of stable theories. This question was answered affirmatively by Hrushovski and Srouf with the construction of their free pseudospace in the unfortunately unpublished manuscript [3]. The result from [4] mentioned above shows that Lachlan's pseudoplane is a typical example of a stable non-1-based theory. As equational theories provide a natural generalization of 1-based theories [10], this motivates the approach to search for a stable non-equational theory in form of a higher-dimensional version of the pseudoplane.

Independently of [3], Baudisch and Pillay [1] constructed another free pseudospace as an example of a non- $CM$ -trivial stable theory in which no infinite field is interpretable. This shows that the hierarchy of  $n$ -ample theories, developed by Pillay [9], is strict up to its second level. The first level of this hierarchy is again formed by non-1-based theories, whereas 2-ample theories correspond to non- $CM$ -trivial theories. It had already been conjectured in [1] that the pseudospace of Baudisch and Pillay is a reduct of the coloured pseudospace by Hrushovski and Srouf, but the actual verification turned out to be far from obvious.

This paper is organized as follows. In Sect. 2 we review the pseudoplane of Lachlan [8]. We also introduce a coloured version of this pseudoplane which will serve as an essential ingredient for the analysis of the coloured pseudospace of Hrushovski and Srouf.

In Sects. 3 and 4 we describe the free pseudospaces  $\Sigma$  of Baudisch and Pillay [1] and  $\Gamma$  of Hrushovski and Srouf [3]. Using the standard model of  $\Sigma$  from [1] we construct a standard model of  $\Gamma$ .

The main results follow in Sect. 5 where we investigate the relationship between the axiom systems  $\Sigma$  and  $\Gamma$ . We prove that  $\Sigma$  is a reduct of  $\Gamma$ . The main technical difficulty for this result lies in deriving from  $\Gamma$  the axioms of  $\Sigma$  which expresses the freeness conditions. We achieve this by analyzing paths and cycles in models of  $\Gamma$ . As a byproduct we obtain a simplification of the axiom system  $\Sigma$ .

In the final section we explain the original purpose of  $\Gamma$  as a stable non-equational theory. In particular, we include a proof for the non-equationality of  $\Gamma$  which is based on the proof sketch given in the draft [3].

## 2 The Free Pseudoplane

First we will review the free pseudoplane of Lachlan [8], because it is of fundamental importance for the higher dimensional pseudospaces that are the topic

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<sup>1</sup>This definition of equationality stems from [3]. Pillay and Srouf [10] give a different definition where they only consider equations in *one* free variable  $x$ . Whether the two definitions are equivalent is open. In [5] Junker provided a detailed comparison of these different formalisations of equationality.

of this paper. The language contains unary predicates  $B$  and  $C$  for lines and points, respectively, and a binary incidence relation  $I$  between lines and points. The free pseudoplane is axiomatized by the following axiom set  $\Delta$ :

- $\Delta_1$ : Every element is a point or a line, but not both.
- $\Delta_2$ :  $I \subseteq (B \times C) \cup (C \times B)$  is a symmetric relation between lines and points.
- $\Delta_3$ : Every point lies via  $I$  on infinitely many lines. Conversely, every line contains infinitely many points.
- $\Delta_4$ : There are no cycles, i.e., there do not exist mutually distinct elements  $x_0, \dots, x_n$ ,  $n \geq 2$ , with  $I(x_i, x_{i+1})$ ,  $0 \leq i \leq n-1$ , and  $I(x_n, x_0)$ .

The standard model  $N_0$  of  $\Delta$  has as its domain the set  $\omega^{<\omega}$  of finite sequences of natural numbers. The lines of  $N_0$  are the sequences of even length, whereas sequences of odd length are points. The incidence relation  $I(x, y)$  holds between elements  $x$  and  $y$ , if  $x$  is either a direct predecessor or a direct successor of  $y$ . Thus  $N_0$  is a countable model of  $\Delta$  which, moreover, is connected. It is well known that  $\Delta$  is a complete theory.

## The Coloured Pseudoplane

Next we will describe a coloured modification of the free pseudoplane where lines and points are equipped with colours. This modification is not of independent interest, but it will serve as an important building block in subsequent sections. The language is enriched by unary relations  $C_r, C_w, B_r$ , and  $B_w$  for red and white points and red and white lines, respectively. In addition to the axioms  $\Delta_1$  to  $\Delta_4$ , the axiom set  $\Delta'$  contains the following three axioms regarding the colours:

- $\Delta_5$ : Every line is either red or white, i.e., it fulfills exactly one of the predicates  $B_r$  or  $B_w$ . The analogous condition holds for points.
- $\Delta_6$ : Every point lies on infinitely many white and on infinitely many red lines.
- $\Delta_7$ : Every red (resp. white) line  $b$  contains exactly one white (resp. red) point, which is called the *exceptional point* of  $b$ .

Models of  $\Delta'$  are called *free coloured pseudoplanes*. The standard model  $N'_0$  of the coloured pseudoplane is derived from the standard model  $N_0$  of  $\Delta$  by colouring lines and points. Lines are coloured according to

$$\begin{aligned} B_r(N'_0) &= \{b \mid b \in B(N'_0), b_{\ell(b)} \text{ is even}\} \\ B_w(N'_0) &= \{b \mid b \in B(N'_0), b_{\ell(b)} \text{ is odd}\} , \end{aligned}$$

where  $\ell(b)$  denotes the length of the sequence  $b$ , and  $b_{\ell(b)}$  is its last element. By this construction every point lies on infinitely many red and white lines.

It remains to colour the points. If the predecessor point  $c$  of a line  $b$  in  $B(N'_0)$  has a different colour than  $b$ , then  $c$  is the exceptional point of  $b$ , and all successors of  $b$  are coloured with the colour of  $b$ . If, on the other hand,  $b$  and  $c$

are of the same colour, then we can choose the exceptional point freely among the successors of  $b$ . Therefore  $\Delta_7$  is fulfilled, and hence  $N'_0$  is a model of  $\Delta'$ .

It is not hard to directly construct an isomorphism between two countable connected free coloured pseudoplanes. Therefore  $\Delta'$  has only one countable connected model up to isomorphism.

### 3 The Free Pseudospace of Baudisch and Pillay

In this section we describe a 3-dimensional analogue of the pseudoplane as developed by Baudisch and Pillay [1]. In addition to points and lines the pseudospace contains also planes. The language  $L$  of this pseudospace consists of unary predicates  $A, B, C$  for planes, lines and points, respectively, and binary predicates  $I$  and  $J$  for the incidence relations between planes and lines as well as between lines and points.

Before we describe the axioms of the pseudospace we need to introduce some terminology. By  $A, B$ , and  $C$  we also denote the set of planes, lines, and points, respectively. We will usually use letters  $a, a', a_i \dots$  for planes,  $b, b', b_i \dots$  for lines, and  $c, c', c_i \dots$  for points, and we will often refrain from indicating explicitly the type of an element denoted in this way. For a plane  $a$  we define the sets  $B(a) = \{b \in B \mid J(a, b)\}$  and  $C(a) = \{c \in C \mid c \in a\}$ . For a point  $c$  the sets  $A(c)$  and  $B(c)$  are defined analogously.

Elements  $d_0, \dots, d_n$  form a *walk* if consecutive elements are incident to each other. If all elements are pairwise distinct except possibly for  $d_0$  and  $d_n$ , the walk is called a *path*. If in addition  $d_0 = d_n$ , the path is also called a *cycle*. The length of a path is the number of distinct elements in it. If all elements are planes or lines, then we speak of an *AB-path*. *BC-paths* are defined analogously.

The free pseudospace of [1] is axiomatized by the following axioms:

$\Sigma_0$ : Every element fulfills exactly one of the relations  $A, B$ , or  $C$ . The relations  $J \subseteq (A \times B) \cup (B \times A)$  and  $I \subseteq (B \times C) \cup (C \times B)$  are symmetric.

$\Sigma_1$ :  $(A, B, J)$  is a free pseudoplane.

$\Sigma_2$ :  $(B(a), C(a), I)$  is a free pseudoplane for every plane  $a$ .

$\Sigma_3$ : The intersection of two planes is either empty, or a point, or a line.

$\Sigma_4$ : Let  $a$  be a plane and  $X = (a, b, \dots, b', a)$  be a cycle of length  $n$ . Then there exists a *BC-path* between  $b$  and  $b'$  of length at most  $n - 1$ , which only contains points from  $X$  and lines from  $a$ .

For the axioms  $\Sigma_1, \dots, \Sigma_4$  we also consider their dual versions  $\Sigma_1^*, \dots, \Sigma_4^*$  which are formed by interchanging planes and points (e.g.,  $\Sigma_2^*$  reads: for every point  $c$ ,  $(A(c), B(c), J)$  is a free pseudoplane). The axiom set  $\Sigma$  of the free pseudospace of Baudisch and Pillay [1] comprises of the axioms  $\Sigma_0, \dots, \Sigma_4$  together with their dual versions  $\Sigma_1^*, \dots, \Sigma_4^*$ .

In [1] Baudisch and Pillay construct a particular countable connected model  $M_0$  of  $\Sigma$  which is called the standard model. Further, it is shown that the theory  $\Sigma$  is complete,  $\omega$ -stable, and not *CM*-trivial.

## A Simplified Axiomatization

It is apparent from the axioms  $\Sigma$  that points and planes are completely dual to each other. Many arguments can therefore be simplified by establishing some property only for points and lines, which immediately implies this property for planes and lines as well. In this subsection we will simplify  $\Sigma$  and point out that it is in fact not necessary to include the point-plane duality in the axiomatization. It already follows from the first half  $\Sigma_1, \dots, \Sigma_4$  of the axioms of  $\Sigma$ .

First, let us introduce a convention which will help to ease the notation. From the axioms of  $\Sigma$  it is clear that planes and lines are uniquely determined by the set of its points. This allows us to relax our notation and occasionally identify planes and lines with the set of its points, i.e.,  $a = C(a)$  and  $b = \{c \mid I(b, c)\}$ . Hence we can use expressions like  $c \in b$ ,  $b \subset a$  or  $a \cap b$ , which are considered as abbreviations for the respective formulas involving the incidence relations  $I$  and  $J$ .

The next two lemmas are the first steps towards proving the point-plane duality in  $\Sigma$ .

**Lemma 3.1** *Every model of  $\Sigma_0$ ,  $\Sigma_1$ , and  $\Sigma_2$  fulfills  $\Sigma_2^*$ .*

*Proof.* Let  $c$  be a point. We have to show that  $(A(c), B(c), J)$  is a free pseudoplane, i.e., we have to check the axioms  $\Delta_1$  to  $\Delta_4$ . Axioms  $\Delta_1$  and  $\Delta_2$  follow immediately from  $\Sigma_0$ .

For  $\Delta_3$  let  $a \in A(c)$ . By  $\Sigma_2$ ,  $(B(a), C(a), I)$  is a free pseudoplane. Because  $c \in C(a)$  there exist infinitely many lines in  $a$  that contain  $c$ . Therefore every plane in  $(A(c), B(c), J)$  contains infinitely many lines. That every line  $b$  lies in infinitely many planes follows from  $\Sigma_1$ , because every plane  $a \supset b$  also contains  $c$ . Finally,  $(A(c), B(c), J)$  does not contain cycles as this is already true for  $(A, B, J)$  by  $\Sigma_1$ . Hence also  $\Delta_4$  is fulfilled.  $\square$

**Lemma 3.2** *Every model of  $\Sigma_0$ ,  $\Sigma_1^*$ , and  $\Sigma_3$  satisfies  $\Sigma_3^*$ .*

*Proof.* Let  $c$  and  $c'$  be two distinct points. If there is none or exactly one plane containing  $c$  and  $c'$ , then  $\Sigma_3^*$  is already fulfilled for  $c$  and  $c'$ .

Assume therefore that  $a$  and  $a'$  are two distinct planes that both contain  $c$  and  $c'$ . Then  $\{c, c'\} \subseteq a \cap a'$ , and by  $\Sigma_3$  there exists a line  $b$  such that  $c, c' \in b$  and  $a \cap a' = b$ . By  $\Sigma_1^*$  this line  $b$  is uniquely determined by  $c$  and  $c'$ . Hence the planes containing  $c$  and  $c'$  are exactly the planes that contain  $b$ .  $\square$

In the next lemma we notice that also  $\Sigma_3$  follows from the other axioms.

**Lemma 3.3** *Every model of  $\Sigma_0$ ,  $\Sigma_1$ ,  $\Sigma_1^*$ ,  $\Sigma_2$ , and  $\Sigma_4$  satisfies  $\Sigma_3$ .*

*Proof.* Assume that  $a$  and  $a'$  are two distinct planes which both contain two distinct points  $c$  and  $c'$ . We will first show that both  $c$  and  $c'$  lie on a common line  $b$ . By  $\Sigma_2$  we can choose appropriate lines such that  $X = (a, b_1, c, b_2, a', b_3, c', b_4, a)$  forms a cycle. Now, by axiom  $\Sigma_4$  there exists a  $BC$ -path  $Y$  connecting  $b_1$  and

$b_4$  which contains only points from  $X$  and lines from  $a$ . There are three possibilities for such a path: either  $Y = (b_1, c, b_4)$ , yielding the desired line  $b = b_4$ , or  $Y = (b_1, c', b_4)$ , yielding  $b = b_1$ , or, finally,  $Y = (a, b_1, c, b, c', b_4, a)$  with some line  $b$  fulfilling the claim. Thus we have obtained a line  $b \subset a$  with  $c, c' \in b$ .

As the role of  $a$  and  $a'$  can be interchanged, we also get a line  $b' \subset a'$  with  $c, c' \in b'$ . By  $\Sigma_1^*$  this line is already uniquely determined by  $c$  and  $c'$ , i.e.,  $b = b'$  is a common line of  $a$  and  $a'$ . By  $\Sigma_1$  the intersection of  $a$  and  $a'$  cannot contain more than one line, and hence  $\Sigma_3$  is fulfilled.  $\square$

In the following proposition we provide a simplified axiomatization for  $\Sigma$ . As it seems more difficult to verify that also  $\Sigma_4^*$  is derivable from the remaining axioms, we have to postpone the proof until Sect. 5.

**Proposition 3.4** *Every model of  $\Sigma_0, \Sigma_1, \Sigma_1^*, \Sigma_2$ , and  $\Sigma_4$  fulfills all axioms of  $\Sigma$ .*

## 4 The Coloured Pseudospace of Hrushovski and Srouf

This section is devoted to another free pseudospace, introduced by Hrushovski and Srouf [3]. Although this pseudospace is very similar to the pseudospace of Baudisch and Pillay [1], it also contains a number of additional features. Before giving the full axiomatization we will provide an informal description.

As in the pseudospace of [1] models consist of points, lines, and planes. As before there are incidence relations  $I$  between points and lines and  $J$  between lines and planes, but additionally there are two direct incidence relations  $I_r$  and  $I_w$  between points and planes. Lines are either red or white, indicated by unary relations  $B_r$  and  $B_w$ . Points have on a given plane also a colour red or white, indicated by the relations  $I_r$  and  $I_w$ . The colour of one point can be different on different planes. Via  $I_r$  and  $I_w$  planes split into a red and a white section. A red line  $b$  of a plane  $a$  contains only points from the red section of  $a$ , except for one white point, the exceptional point of  $b$  in  $a$ . The same holds for white lines. Lines and points of a plane therefore form a free coloured pseudoplane. Finally, there are axioms stating that models are maximally free of cycles.

The language  $L'$  consists of unary relation symbols  $A, B, B_r, B_w$ , and  $C$  for planes, lines (red and white) and points, and binary relation symbols  $I, J, I_r$ , and  $I_w$  for the incidence relations. Therefore  $L'$  extends the language  $L$  from the previous section. The axiom set  $\Gamma$  from [3] contains the following axioms:

- $\Gamma_0$ : Every element fulfills exactly one of the relations  $A, B$ , or  $C$ .  
 $J \subset (A \times B) \cup (B \times A)$  is a symmetric incidence between planes and lines.  
 $I \subset (B \times C) \cup (C \times B)$  is a symmetric incidence between lines and points.
- $\Gamma_1$ : The intersection of two lines is either empty or a single point. Every line contains infinitely many points. The set of lines is nonempty.
- $\Gamma_2$ : For every plane  $a$  and every point  $c \in a$  there exist infinitely many lines in  $a$  that contain  $c$ .
- $\Gamma_3$ : Every line lies in infinitely many planes.

- $\Gamma_4$ : If  $b_1, \dots, b_n$ ,  $n \geq 2$ , are pairwise different lines with  $b_i \cap b_{i+1} \neq \emptyset$ ,  $1 \leq i \leq n-1$ , then  $b_1 \cap b_n = \emptyset$ , or there exists a point  $c$  with  $c \in b_i$  for all  $i = 1, \dots, n$ .
- $\Gamma_5$ : Planes are nonempty. The intersection of two planes is either empty or a point or a line.
- $\Gamma_6$ : If  $a_1, \dots, a_n$ ,  $n \geq 2$ , are pairwise distinct planes such that  $a_i \cap a_{i+1}$  is a line for  $i = 1, \dots, n-1$ , then  $a_1 \cap a_n = \emptyset$ , or  $a_1 \cap a_n$  is a point, or  $a_1, \dots, a_n$  contain a common line.
- $\Gamma_7$ : If  $a_1, a_2, a_3$  are three distinct planes such that  $a_i \cap a_j \neq \emptyset$ ,  $1 \leq i, j \leq 3$ , then  $a_1, a_2, a_3$  contain a common point.
- $\Gamma_8$ : If  $a_1, \dots, a_n$ ,  $n \geq 3$ , are pairwise distinct planes such that  $a_i \cap a_{i+1} \neq \emptyset$ ,  $1 \leq i \leq n-1$ , and  $a_i \cap a_{i+2} = \emptyset$ ,  $1 \leq i \leq n-2$ , then  $a_1 \cap a_n = \emptyset$ .
- $\Gamma_9$ : Lines are either red or white, i.e., every line fulfills exactly one of the relations  $B_r$  or  $B_w$ .
- $\Gamma_{10}$ :  $I_r, I_w \subset (A \times C) \cup (C \times A)$  are the symmetric incidence relations between planes and their red and white points.  $I_r \cap I_w = \emptyset$ . The red and white sections of a plane  $a$  are defined as  $a_r = \{x \mid I_r(x, a)\}$  and  $a_w = \{x \mid I_w(x, a)\}$ , respectively.
- $\Gamma_{11}$ :  $J(a, b) \leftrightarrow \forall x(I(x, b) \rightarrow I_r(x, a) \vee I_w(x, a))$  holds for all  $a \in A, b \in B$ .
- $\Gamma_{12}$ : For every plane  $a$  and every point  $c \in a$  there are infinitely many red and infinitely many white lines in  $a$  containing  $c$ .
- $\Gamma_{13}$ : For every red (resp. white) line  $b$  in a plane  $a$  there exists exactly one *exceptional point*  $c \in a_w \cap b$  (resp.  $c \in a_r \cap b$ ).
- $\Gamma_{14}$ : For every line  $b$  and every point  $c \in b$  there exist infinitely many planes  $a$  with  $b \subset a$ , such that  $c$  is the exceptional point of  $b$  in  $a$ .

To obtain consistent notation we slightly modified the description of  $\Gamma$  from [3] (where other symbols than  $A, B, B_r, B_w, C$  are used and incidence relations are not symmetric). We also grouped the axioms in such a way that the first axioms  $\Gamma_0$  to  $\Gamma_8$  do not refer to the colour relations  $B_r, B_w$  and  $I_r, I_w$ . We denote the axioms  $\Gamma_0$  to  $\Gamma_8$  by  $\Gamma'$ . Together with the colour axioms  $\Gamma_9$  to  $\Gamma_{14}$  we obtain the full set  $\Gamma$ . Clearly, every model of  $\Gamma$  is also a model of  $\Gamma'$ .

The notions of walks, paths, and cycles are easily adapted to the language  $L'$ . Because in  $\Gamma$  we also have direct point-plane incidence relations, walks in models of  $\Gamma$  are not necessarily walks in the sense of  $\Sigma$ .

In contrast to the pseudospace of Baudisch and Pillay the duality between points and planes is not so apparent from the axioms of  $\Gamma$ . Because of the colours (points are red or white, and planes do not have colours) full duality is not even possible. We will, however, show in the next section that the role of points and planes can be interchanged if colours are omitted.



First we will show the consistency of  $\Gamma$  by constructing a coloured version  $M'_0$  of the standard model  $M_0$  of  $\Sigma$  from [1]. The planes and lines form the free pseudoplane  $\omega^{<\omega}$ , where the set of planes is  $\{\eta \in \omega^{<\omega} \mid \ell(\eta) \text{ is even}\}$  and lines correspond to  $\{\eta \in \omega^{<\omega} \mid \ell(\eta) \text{ is odd}\}$ . The incidence  $J(\eta, \tau)$  holds, if  $\eta$  is a direct predecessor or successor of  $\tau$ . In analogy to Sect. 2, lines are coloured according to  $B_r(M'_0) = \{b \in B(M'_0) \mid b_{\ell(b)} \text{ is even}\}$  and  $B_w(M'_0) = \{b \in B(M'_0) \mid b_{\ell(b)} \text{ is odd}\}$ . Hence every plane contains infinitely many red and infinitely many white lines. Planes and lines therefore form a free coloured pseudoplane, where the colour of planes is neglected.

Now we inductively add points for the planes and define the relations  $I$ ,  $I_r$ , and  $I_w$ . The induction is carried out on the length of a plane  $a$  as an element of  $\omega^{<\omega}$ . The induction hypothesis consists of the following two assertions:

1. For all planes  $a$  of length  $\leq 2n$ , points  $C(a)$  have been added such that

$$P(a) = (B(a), C(a), I_a, B_{ra}, B_{wa}, I_{ra}, I_{wa})$$

is a connected free coloured pseudoplane. Here  $I_a, B_{ra}, B_{wa}$  are the restrictions of  $I, B_r, B_w$  to lines and points from  $a$ , and  $I_{ra}, I_{wa}$  are the unary relations  $I_r(\cdot, a)$  and  $I_w(\cdot, a)$ , respectively, which indicate the colour of a point in  $a$ .

2. Axiom  $\Gamma_{14}$  is fulfilled for all lines of length at most  $2n - 1$ , i.e., for every line  $b \in \omega^{<2n}$  and every point  $c \in b$  there exist infinitely many planes  $a \in \omega^{\leq 2n}$  with  $b \subset a$ , such that  $c$  is the exceptional point of  $b$  in  $a$ .

At the end of this construction the set of all points is formed by the union  $\bigcup \{C(a) \mid \ell(a) \text{ even}\}$ . Let us already remark here that in the construction we have many choices concerning the distribution of the colours. All these choices, however, will provide elementarily equivalent models.

In the initial step of the construction we choose  $C(\langle \rangle)$  as a countable set of points. Colours of  $B(\langle \rangle)$  are already determined by the colouring of  $(A, B, J)$  in such a way that  $B(\langle \rangle)$  contains infinitely many red and white lines. On  $B(\langle \rangle) \cup C(\langle \rangle)$  we define the relations  $I_{\langle \rangle}, I_{r\langle \rangle}, I_{w\langle \rangle}$  such that  $P(\langle \rangle)$  is a countable connected free coloured pseudoplane. In the initial step, the second part of the induction claim does not apply. Colours of  $C(\langle \rangle)$  in planes of length two are chosen in the next step of the construction.

For the induction step, assume that the induction hypothesis holds for all planes of length at most  $2n$ . Let  $b$  be a line of length  $2n + 1$ . We will simultaneously define  $P(a)$  for all planes  $a$  of length  $2n + 2$  which are incident to  $b$ . Let  $a$  be one of these planes, i.e.,  $b = a_{2n+1}$  is the predecessor line of  $a$ . Let further  $C^0$  be the set of points of the line  $b$  and let  $C^1$  be a countable set of new elements. We define the points of  $a$  as  $C(a) = C^0 \cup C^1$ . As in the initial step of the induction, the colours of  $C^0$  in planes of length  $2n + 2$  have not been determined yet. This will be done below, observing the second part of the induction hypothesis regarding axiom  $\Gamma_{14}$ .

Now we define  $I_a, I_{ra}, I_{wa}$  on  $B(a) \cup C(a)$  such that  $P(a)$  becomes a connected countable free coloured pseudoplane. We do not introduce any new

points on the line  $b$ , i.e.,  $I_a(b, c)$  holds if and only if  $c \in C^0$ . Colours of points can be chosen independently in each plane  $a$  of length  $2n + 2$ . Additionally, the exceptional point of  $b$  is chosen such that for each  $c \in C^0$  there are infinitely many planes  $a$  of length  $2n + 2$  such that  $a_{2n+1} = b$  and  $c$  is the exceptional point of  $b$  on  $a$ . This is possible because  $C^0$  is countable and also  $b$  contains countably many successor planes  $a$  on which the exceptional point can be chosen arbitrarily. Hence also the second part of the induction claim is fulfilled.

Finally, the set of all points of  $M'_0$  is the union of all sets  $C(a)$ , and the relations  $I, I_r, I_w$  are the unions of the respective relations  $I_a, I_{ra}, I_{wa}$  for all planes  $a$ .

The next theorem is our main result in this section.

**Theorem 4.1**  $M'_0$  is a model of  $\Gamma$ , and hence  $\Gamma$  is consistent.

The theorem is proven by the following remarks and lemmas.

Let  $M_0$  be the  $L$ -reduct of  $M'_0$ . Then  $M_0$  is exactly the standard model of  $\Sigma$  constructed in [1]. Therefore  $\Sigma$  is valid in  $M'_0$ . It remains to show that also  $\Gamma$  is fulfilled in  $M'_0$ . To show this we will first derive all axioms of  $\Gamma'$  from  $\Sigma$ . As  $M'_0$  is a model of  $\Sigma$  this implies the validity of  $\Gamma'$  in  $M'_0$ .

**Lemma 4.2** Every model of  $\Sigma$  satisfies  $\Gamma_0$  to  $\Gamma_6$ .

*Proof.* The axioms  $\Gamma_0$  and  $\Sigma_0$  are identical. Axiom  $\Sigma_1^*$  implies  $\Gamma_1$  and  $\Gamma_4$ . Axiom  $\Gamma_2$  follows from  $\Sigma_2$ , and  $\Gamma_3$  follows from  $\Sigma_1$ . Finally,  $\Gamma_5$  and  $\Gamma_6$  follow from  $\Sigma_1$  and  $\Sigma_3$ .  $\square$

Deriving  $\Gamma_7$  and  $\Gamma_8$  from  $\Sigma$  requires some extra arguments.

**Lemma 4.3** Every model of  $\Sigma$  satisfies  $\Gamma_7$ .

*Proof.* Let  $a_1, a_2, a_3$  be distinct planes and let  $c_1, c_2, c_3$  be points such that  $c_1 \in a_2 \cap a_3$ ,  $c_2 \in a_1 \cap a_3$ ,  $c_3 \in a_1 \cap a_2$ . We have to show the existence of a point  $c \in a_1 \cap a_2 \cap a_3$ . If  $c_1 \in a_1$ , then  $c = c_1$  is such a point. Likewise, if  $c_2 \in a_2$  or  $c_3 \in a_3$ . Assume now that

$$c_i \notin a_i \text{ for } i = 1, \dots, 3. \quad (1)$$

We will derive a contradiction. By assumption  $c_1, c_2$ , and  $c_3$  are pairwise distinct. Hence there exists a cycle

$$(a_1, b_3, c_3, b'_3, a_2, b_1, c_1, b'_1, a_3, b_2, c_2, b'_2, a_1)$$

with pairwise distinct lines  $b_1, b_2, b_3, b'_1, b'_2, b'_3$ . Choosing such lines is possible by (1) and because  $(B(a_i), C(a_i), I)$  is a free pseudoplane. By  $\Sigma_4$  there exists a  $BC$ -path  $X$  between  $b_3$  and  $b'_2$ , containing only lines from  $a_1$  and points from  $\{c_1, c_2, c_3\}$ . The point  $c_1$  cannot occur in  $X$  because  $c_1 \notin a_1$ . Hence we have either

- (a)  $X = (b_3, c_3, b'_2)$  or
- (b)  $X = (b_3, c_2, b'_2)$  or
- (c)  $X = (b_3, c_3, b', c_2, b'_2)$  with  $b' \subset a_1$ .

In every case there exists a line  $b''_1 \subset a_1$  containing both  $c_2$  and  $c_3$ , namely in (a)  $b''_1 = b'_2$ , in (b)  $b''_1 = b_3$  and in (c)  $b''_1 = b'$ . Analogously, using  $c_2 \notin a_2$  and  $c_3 \notin a_3$  we get lines  $b''_2 \subset a_2$  and  $b''_3 \subset a_3$  with  $c_1, c_3 \in b''_2$  and  $c_1, c_2 \in b''_3$ . By (1) the lines  $b''_1, b''_2$ , and  $b''_3$  are pairwise distinct. Hence there exists a cycle of lines and points

$$(b''_1, c_3, b''_2, c_1, b''_3, c_2, b''_1) ,$$

contradicting  $\Sigma_1^*$ . □

**Lemma 4.4** *Every model of  $\Sigma$  satisfies  $\Gamma_8$ .*

*Proof.* Let  $a_1, \dots, a_n$ ,  $n \geq 3$ , be distinct planes with  $c_i \in a_i \cap a_{i+1}$ ,  $1 \leq i < n$ , and  $a_i \cap a_{i+2} = \emptyset$ ,  $1 \leq i < n - 1$ . We have to prove  $a_1 \cap a_n = \emptyset$ . We will show this by induction on  $n$ . The base case  $n = 3$  is clear. Let  $n > 3$  and assume that  $\Gamma_8$  is valid for all  $3 \leq k < n$ . By hypothesis we have

$$a_i \cap a_j = \emptyset \text{ for } 1 \leq i < j \leq n \text{ with } i + 1 \neq j \text{ and } (i, j) \neq (1, n). \quad (2)$$

Assume now, that there exists a point  $c_n \in a_1 \cap a_n$ . We will construct a contradiction, similarly as in the previous lemma. By (2) and the assumption there exists a cycle

$$(a_1, b, c_1, b', a_2, \dots, a_n, b'', c_n, b''', a_1) .$$

Applying  $\Sigma_4$  yields a path  $X$  between  $b$  and  $b'''$ , containing only lines from  $a_1$  and points from  $\{c_1, \dots, c_n\}$ . By (2), the points  $c_2, \dots, c_{n-1}$  cannot appear in  $X$ . Hence we have  $X = (b, c_1, b''')$ , or  $X = (b, c_n, b''')$ , or  $X = (b, c_1, b_1, c_n, b''')$  with  $b_1 \subset a_1$ . In each case there exists a line  $b_1$ , containing the points  $c_1$  and  $c_n$ . Analogously, (2) yields lines  $b_2, \dots, b_n$  with  $c_{i-1} \in b_i$  and  $c_i \in b_i$ ,  $2 \leq i \leq n$ . By (2) all lines  $b_1, \dots, b_n$  are distinct. Hence there is a cycle

$$(b_1, c_1, b_2, c_2, \dots, b_n, c_n, b_1) ,$$

contradicting axiom  $\Sigma_1^*$ . □

Finally, the colour axioms in  $\Gamma$  follow directly from the construction of  $M'_0$ :

**Lemma 4.5**  *$M'_0$  satisfies the axioms  $\Gamma_9$  to  $\Gamma_{14}$ .*

## 5 The Relationship between the Two Pseudospaces

The aim of this section is to prove Theorem 5.5:  $\Sigma$  and  $\Gamma'$  are equivalent. Already in the last section we have shown that the axioms  $\Gamma'$  are derivable from  $\Sigma$ . Now we will prove that the axioms of  $\Gamma'$  also imply all axioms from  $\Sigma$ . We will first derive  $\Sigma_4$  from  $\Gamma'$ , which requires the following lemma.

**Lemma 5.1** *Let  $M \models \Gamma'$  and let  $X = (a, b, c_0, a_1, c_1, \dots, c_{n-1}, a_n, c_n, b', a)$  be a cycle in  $M$  consisting of planes  $a = a_0, a_1, \dots, a_n$ , lines  $b, b'$ , and points  $c_0, \dots, c_n$ . Then there exists a BC-path  $Y = (b, c'_0, b'_1, c'_1, \dots, c'_{m-1}, b'_m, c'_m, b')$  such that  $\{c'_0, \dots, c'_m\} \subseteq \{c_0, \dots, c_n\}$  and  $c'_i \in a$ ,  $0 \leq i \leq m$ . Additionally, we have  $b'_i \subset a$ ,  $1 \leq i \leq m$ .*

*Proof.* The last sentence follows from the first part of the lemma. Namely, if  $a'_i \neq a$  is a plane with  $b'_i \subset a'_i$ , then  $c'_{i-1}, c'_i \in a'_i \cap a$ , and hence  $b'_i = a'_i \cap a$ . The first part of the lemma is shown by induction on  $n$ .

*Base cases.* For  $n = 0$  we have  $X = (a, b, c_0, b', a)$ , and the claim is true.

For  $n = 1$  we have  $X = (a, b, c_0, a_1, c_1, b', a)$ , i.e.,  $c_0, c_1 \in a \cap a_1$ . Then there exists a line  $b'' = a \cap a_1$ , and hence there is the walk  $(b, c_0, b'', c_1, b')$ . If  $b = b''$  or  $b'' = b'$ , then the walk can be shortened. In the following we will not explicitly mention, if a walk can be shortened in such a way.

For  $n = 2$  we have  $X = (a, b, c_0, a_1, c_1, a_2, c_2, b', a)$ . By  $\Gamma_7$  there exists a point  $c \in a \cap a_1 \cap a_2$ . We will distinguish four cases.

*Case 1.*  $c = c_0$ , i.e., in particular  $c \neq c_2$ . Then there exists  $b'' = a \cap a_2$  such that  $c_0, c_2 \in b''$ . Therefore the desired path  $Y$  is obtained from the walk  $(b, c_0, b'', c_2, b')$ .

*Case 2.*  $c = c_1$ , hence  $c \neq c_0$  and  $c \neq c_2$ . Then there exist lines  $b'' = a \cap a_1$  and  $b''' = a \cap a_2$  such that  $c_0, c_1 \in b''$  and  $c_1, c_2 \in b'''$ . Therefore we have the walk  $(b, c_0, b'', c_1, b''', c_2, b')$ .

*Case 3.*  $c = c_2$ . Like case 1.

*Case 4.*  $c \neq c_0$ ,  $c \neq c_1$ , and  $c \neq c_2$ . Then we have lines  $b_0 = a \cap a_1$ ,  $b_1 = a_1 \cap a_2$ , and  $b_2 = a_2 \cap a$ , i.e., there exists the walk  $(a, b_0, a_1, b_1, a_2, b_2, a)$ . Because  $a, a_1, a_2$  are pairwise distinct, they contain a common line  $b''$  by  $\Gamma_6$ , which is identical with  $b_0, b_1$ , and  $b_2$  by  $\Gamma_5$ , i.e.,  $b'' = b_0 = b_1 = b_2$ . Therefore we get the walk  $(b, c_0, b_0, c_2, b')$ .

*Induction step.* Let the claim be true for  $k < n$ ,  $n \geq 3$ . If there exists an index  $i$ ,  $1 \leq i \leq n-1$  such that  $c_i \in a$ , then we choose  $b'' \subset a$  with  $c_i \in b''$ . Hence we have the paths  $(a, b, c_0, \dots, a_i, c_i, b'', a)$  and  $(a, b'', c_i, a_{i+1}, \dots, a_n, c_n, b', a)$ . By induction hypothesis there exist  $BC$ -paths connecting  $b$  and  $b''$  as well as  $b''$  and  $b'$ , that only use points from  $\{c_0, \dots, c_n\}$  lying in  $a$ . We obtain the desired  $BC$ -path between  $b$  and  $b'$  by concatenation. We can therefore make the following

**Assumption 1**  $c_i \notin a$  for  $1 \leq i \leq n-1$ .

If there exist  $i$  and  $j$  such that  $0 \leq i \leq n$ ,  $1 \leq j \leq n$ ,  $i \neq j, j-1$  and  $c_i \in a_j$ , then we can shorten the path  $X$  to  $X' = (a, b, c_0, \dots, a_i, c_i, a_j, c_j, \dots, b', a)$  if  $i < j-1$ , and to  $X' = (a, b, c_0, \dots, c_{j-1}, a_j, c_i, a_{i+1}, \dots, b', a)$  if  $i > j$ . In this case the induction hypothesis for  $X'$  yields the claim. In addition to Assumption 1 we therefore make

**Assumption 2**  $c_i \notin a_j$  for  $0 \leq i \leq n$ ,  $1 \leq j \leq n$ ,  $i \neq j, j-1$ .

Finally, if  $c_0$  and  $c_n$  are on a common line  $b''$ , then we directly get the path  $Y$  from  $(b, c_0, b'', c_n, b')$ . We therefore also assume

**Assumption 3**  $c_0$  and  $c_n$  do not lie on a common line.

From Assumptions 1 to 3 we will derive a contradiction, hence for any given  $X$  at least one of these assumptions does not hold, and thus the claim is proved. By  $\Gamma_8$  there exists some  $j$ ,  $0 \leq j \leq n-2$  such that  $a_j \cap a_{j+2} \neq \emptyset$ . Let  $c \in a_j \cap a_{j+2}$ . For this situation we will prove the following claim.

**Claim 1** *If there exists a point  $c \in a_j \cap a_{j+2}$  with  $0 \leq j \leq n-2$ , then there exists a  $BC$ -path  $Y'$  which connects  $b$  and  $b'$  and does not use any points except  $c_0$ ,  $c_n$ , and  $c$ . Further,  $c$  appears in  $Y'$ , and we have  $c \neq c_0$ ,  $c \neq c_n$ , and  $c \in a$ .*

*Proof of Claim 1.* To prove the first sentence we will distinguish two cases.

*Case 1.*  $j = 0$ , i.e.,  $c \in a \cap a_2$ . Then there exists  $b'' \subset a$  such that  $c \in b''$ . Hence we get cycles  $(a, b, c_0, a_1, c_1, a_2, c, b'', a)$  and  $(a, b'', c, a_2, \dots, a_n, c_n, b', a)$ . Applying the induction hypothesis twice yields  $BC$ -paths between  $b$  and  $b''$  as well as between  $b''$  and  $b'$ . By Assumption 1 these paths only contain points from  $\{c_0, c_n, c\}$ . By concatenation we get a  $BC$ -path connecting  $b$  and  $b'$ .

*Case 2.*  $1 \leq j \leq n-2$ . Then we have a cycle  $(a, b, \dots, a_j, c, a_{j+2}, \dots, b', a)$ , and by induction hypothesis and Assumption 1 we get a  $BC$ -path that contains only the points  $c_0, c_n, c$ .

Thus the path  $Y'$  only uses the points  $c_0$ ,  $c_n$ , and  $c$ . The point  $c$  is included in  $Y'$  because all choices for  $Y'$  omitting  $c$ , i.e.,  $(b, c_0, b')$ ,  $(b, c_n, b')$ , and  $(b, c_0, b'', c_n, b')$  contradict Assumption 3. Thus the only possible configurations for  $Y'$  are  $(b, c, b')$ ,  $(b, c, b'', c_n, b')$ , and  $(b, c_0, b'', c, b'', c_n, b')$ , in which case  $c \neq c_0$  and  $c \neq c_n$  follow by Assumption 3 and because  $Y'$  is a path. Hence  $c$  appears in  $Y'$ , and therefore we get  $c \in a$  by induction hypothesis.  $\square$

As argued before, axiom  $\Gamma_8$  assures the existence of an index  $j$ ,  $0 \leq j \leq n-2$  such that  $a_j$  and  $a_{j+2}$  contain a common point  $c$ . Applying Claim 1 to this point we obtain a  $BC$ -path between  $b$  and  $b'$ , using the point  $c$  and possibly also  $c_0$  and  $c_n$ . We will call this path  $Y_1$ . In the next claim we prove that  $c$  is contained in all planes  $a_i$ .

**Claim 2** *For all  $1 \leq i \leq n$  we have  $c \in a_i$ .*

*Proof of Claim 2.* We will prove inductively the following claim: if  $c \in a_i$  and  $c \in a_j$ ,  $0 \leq i < j \leq n+1$  (with  $a_0 = a_{n+1} = a$ ), then  $c \in a_k$  for all  $i \leq k \leq j$ . The proof proceeds by induction on  $l = j - i$ .

*Base case.* For  $l = 1$  there is nothing to show. Let  $l = 2$ , i.e.,  $c \in a_i \cap a_{i+2}$ . By  $\Gamma_7$  we have  $a_i \cap a_{i+1} \cap a_{i+2} \neq \emptyset$ . Let  $c' \in a_i \cap a_{i+1} \cap a_{i+2}$ . Claim 1 for  $c'$  yields a  $BC$ -path  $Y_2$  between  $b$  and  $b'$ , that contains  $c'$  and possibly also  $c_0$  and  $c_n$ . As  $(B, C, I)$  is a free pseudoplane,  $BC$ -paths are unique, and therefore  $Y_1 = Y_2$  and in particular  $c = c'$ . Hence  $c \in a_{i+1}$ .

*Induction step.* Let  $l \geq 3$  and let the claim be true for all  $k < l$ . Then we have the situation  $(a_i, c_i, \dots, c_{i+l-1}, a_{i+l}, c, a_i)$ . By  $\Gamma_8$  there exists an index  $m$  such that  $i \leq m \leq i+l-2$  and  $a_m \cap a_{m+2} \neq \emptyset$ . Let  $c' \in a_m \cap a_{m+2}$ . As before, applying Claim 1 to  $c'$  we get a  $BC$ -path  $Y_2$  between  $b$  and  $b'$ , using only the points  $c_0, c_n$ , and  $c'$ . Then we have  $Y_1 = Y_2$  and hence  $c = c'$ . Therefore  $c \in a_i \cap a_{m+1} \cap a_{i+l}$  and the induction hypothesis yields  $c \in a_{i+k}$ ,  $0 \leq k \leq l$ .  $\square$

By Claim 1 we have  $c \in a$  and by Assumption 1  $c \neq c_i$ ,  $1 \leq i \leq n-1$ . Together with Claim 1 we get  $c \neq c_i$ ,  $0 \leq i \leq n$ . By Claim 2 this means  $c, c_i \in a_i \cap a_{i+1}$ ,  $0 \leq i \leq n$ . Thus by  $\Gamma_5$  there exist lines  $b_i = a_i \cap a_{i+1}$  such that  $c, c_i \in b_i$ ,  $0 \leq i \leq n$ . The lines  $b_0, \dots, b_{n-1}$  are pairwise distinct, because if  $b_i = b_j$ ,  $0 \leq i < j \leq n-1$ , then  $c_i \in b_i = b_j = a_j \cap a_{j+1}$  in contradiction to Assumption 2. By the same argument the lines  $b_1, \dots, b_n$  are pairwise distinct.

Additionally, Assumption 3 yields  $b_0 \neq b_n$ , hence all of  $b_0, \dots, b_n$  are pairwise distinct. Therefore we get an  $AB$ -cycle  $(a, b_0, a_1, b_1, \dots, b_{n-1}, a_n, b_n, a)$  in contradiction to  $\Gamma_6$ . Hence Assumptions 1 to 3 cannot hold simultaneously, and the proof is complete.  $\square$

This enables us to prove the validity of  $\Sigma_4$  in  $\Gamma$ .

**Lemma 5.2** *Every model of  $\Gamma'$  satisfies  $\Sigma_4$ .*

*Proof.* Let  $X = (a, b, \dots, b', a)$  be an  $ABC$ -cycle. We have to construct a  $BC$ -path  $Y$  connecting  $b$  and  $b'$  and consisting only of points from  $X$  which are in  $a$ . To apply the previous lemma we transform  $X$  to a cycle  $X'$  that contains no lines except  $b$  and  $b'$ . To achieve this we apply the following steps a) to c) to the inner part  $b, \dots, b'$  of  $X$ :

- a) Every walk of the form  $a_1, b_1, c_1$  is replaced by  $a_1, c_1$ . Similarly, every walk  $c_1, b_1, a_1$  is shortened to  $c_1, a_1$ .
- b) Every walk  $c_1, b_1, c_2$  is substituted by  $c_1, a_1, c_2$ , where the plane  $a_1$  contains the line  $b_1$  and does not occur in  $X$ .
- c) Finally, every walk of the form  $a_1, b_1, a_2$  is changed to  $a_1, c_1, a_2$  with an arbitrary point  $c_1$  from  $b_1$  that does not occur in  $X$ .

After these steps have been performed on  $X$  we apply the following rule:

- d) If the cycle  $X$  obtained after the steps a) to c) starts with  $a, b, a_1$ , then we choose some point  $c_0$  from  $b$ , not contained in  $X$ , and replace  $a, b, a_1$  by  $a, b, c_0, a_1$ . Similarly, if  $X$  ends with  $a_n, b', a$ , then we insert a new point  $c_n \in b'$ , obtaining  $a_n, c_n, b', a$ .

The cycle  $X'$  thus obtained has the form  $X' = (a, b, c_0, a_1, c_1, \dots, a_n, c_n, b', a)$  and contains only planes, the lines  $b$  and  $b'$ , and all points from  $X$  as well as new points inserted by the rules c) and d). Applying Lemma 5.1 to  $X'$  yields a  $BC$ -path  $Y$  between  $b$  and  $b'$  with points from  $X'$  and lines from  $a$ . The new points introduced by the rules c) and d) can be chosen arbitrarily from infinitely many possibilities. Therefore, as  $BC$ -paths are unique, these new points cannot appear in  $Y$ . Hence the path  $Y$  is in accordance with the requirements from axiom  $\Sigma_4$ .  $\square$

It remains to derive axiom  $\Sigma_4^*$  from  $\Gamma$ . This requires a similar result as Lemma 5.1, but with a considerably simpler proof.

**Lemma 5.3** *In a model of  $\Gamma'$  let  $X = (c, b, a_0, c_0, \dots, c_{n-1}, a_n, b', c)$  be a cycle consisting of planes  $a_0, \dots, a_n$ , lines  $b, b'$ , and points  $c_0, \dots, c_{n-1}$ . Then there exists an  $AB$ -path  $Y = (b, a'_0, b'_0, \dots, b'_{m-1}, a'_m, b')$  with  $\{a'_0, \dots, a'_m\} \subseteq \{a_0, \dots, a_n\}$  and  $c \in a'_i$ ,  $0 \leq i \leq m$ . Additionally, we have  $c \in b'_i$ ,  $0 \leq i \leq m-1$ .*

*Proof.* The last assertion  $c \in b'_i$  follows from  $c \in a'_i \cap a'_{i+1} = b'_i$ . We will show the first part of the claim by induction on  $n$ .

*Base case.* For  $n = 0$  we have  $X = (c, b, a_0, b', c)$ , and the claim holds.

For  $n = 1$  we have  $X = (c, b, a_0, c_0, a_1, b', c)$ . Because  $c, c_0 \in a_0 \cap a_1$  there is a line  $b_0 = a_0 \cap a_1$ , and hence we get the walk  $(b, a_0, b_0, a_1, b')$ .

For  $n = 2$  we have  $X = (c, b, a_0, c_0, a_1, c_1, a_2, b', c)$ . By  $\Gamma_7$  there exists a point  $c' \in a_0 \cap a_1 \cap a_2$ . We will distinguish four cases.

*Case 1.*  $c' = c_0$ . Then  $c_0 \in a_2$ , and there exists the cycle  $(c, b, a_0, c_0, a_2, b', c)$ . We can therefore continue as in the case  $n = 1$ .

*Case 2.* The case  $c' = c_1$  is analogous to case 1.

*Case 3.*  $c' = c$ , and therefore in particular  $c' \neq c_0$  and  $c' \neq c_1$ . Then there exist lines  $b_0 = a_0 \cap a_1$  and  $b_1 = a_1 \cap a_2$ , yielding the path  $(b, a_0, b_0, a_1, b_1, a_2, b')$ .

*Case 4.*  $c' \neq c$ ,  $c' \neq c_0$ , and  $c' \neq c_1$ . Then there exist lines  $b_0 = a_0 \cap a_1$ ,  $b_1 = a_1 \cap a_2$ , and  $b_2 = a_2 \cap a_0$ , i.e., we have the walk  $(a_0, b_0, a_1, b_1, a_2, b_2, a_0)$ . From this we infer  $b_0 = b_1 = b_2$  (cf. the resp. part of the proof of Lemma 5.1), and thus we obtain the path  $(b, a_0, b_0, a_2, b')$ .

*Induction step.* Let the claim be true for  $k < n$ ,  $n \geq 3$ . By  $\Gamma_8$  there exists an index  $i$ ,  $0 \leq i \leq n - 2$  such that  $a_i \cap a_{i+2} \neq \emptyset$ . Let  $c' \in a_i \cap a_{i+2}$ . Applying the induction hypothesis to  $(c, b, a_0, \dots, c_{i-1}, a_i, c', a_{i+2}, c_{i+2}, \dots, a_n, b', c)$  yields the desired  $AB$ -path  $Y$ .  $\square$

**Lemma 5.4** *Every model of  $\Gamma'$  satisfies  $\Sigma_4^*$ .*

*Proof.* Let  $X = (c, b, \dots, b', c)$  be an  $ABC$ -cycle. We search for an  $AB$ -path between  $b$  and  $b'$  with planes from  $X$  that contain  $c$ . We transform  $X$  to a circle  $X'$  that does not contain lines others than  $b, b'$  by applying the dual procedure of the proof of Lemma 5.2. Applying Lemma 5.3 to  $X'$  yields an  $AB$ -path  $Y$  between  $b$  and  $b'$  with planes from  $X'$  and lines containing  $c$ . The planes inserted in the construction of  $X'$  cannot appear in  $Y$ , because  $AB$ -paths are unique, and for the new planes there exist infinitely many different choices. Therefore  $Y$  meets the requirements of  $\Sigma_4^*$ .  $\square$

These preparations enable us to characterize the relationship between  $\Sigma$  and  $\Gamma$  as follows:

**Theorem 5.5**  *$\Sigma$  and  $\Gamma'$  are equivalent.*

*Proof.* In the last section we have already shown  $\Sigma \models \Gamma'$ .

Concerning the converse  $\Gamma' \models \Sigma$ , axiom  $\Sigma_0$  is identical to  $\Gamma_0$ . To derive  $\Sigma_1$ ,  $\Sigma_1^*$ , and  $\Sigma_2$  from  $\Gamma'$  we have to check the axioms  $\Delta$  for the respective pseudoplanes. Axioms  $\Delta_1$  and  $\Delta_2$  defining the incidence relations are apparently fulfilled.  $\Delta_3$  is easily checked to follow from  $\Gamma_1$ ,  $\Gamma_2$ , and  $\Gamma_3$ . For  $\Delta_4$  we have to verify the absence of cycles.  $AB$ -cycles do not exist by  $\Gamma_5$  and  $\Gamma_6$ . By  $\Gamma_1$  and  $\Gamma_4$  there are also no  $BC$ -cycles. Clearly, then there are also no cycles in the pseudoplanes mentioned in  $\Sigma_2$ .

$\Sigma_3$  is equivalent to  $\Gamma_5$ . By Lemmas 3.1 and 3.2 the axioms  $\Sigma_2^*$  and  $\Sigma_3^*$  hold in  $\Gamma'$ . Finally,  $\Sigma_4$  and  $\Sigma_4^*$  were proved in Lemmas 5.2 and 5.4.  $\square$

**Corollary 5.6**  *$\Sigma$  is the  $L$ -reduct of  $\Gamma$ , i.e., if  $M$  is an  $L'$ -structure such that  $M \models \Gamma$ , then  $M|L$  is a model of  $\Sigma$ .*

This corollary also clarifies the duality between points and lines in models of  $\Gamma$ , namely, if the colours are removed, then points and planes can be interchanged. In fact, this duality is a very natural concept, that does not even have to be required axiomatically. We have already formulated this observation before as Proposition 3.4, but now we can give the full proof.

*Proof of Proposition 3.4.* Let  $M$  be an  $L$ -structure satisfying  $\Sigma_0, \Sigma_1, \Sigma_1^*, \Sigma_2$ , and  $\Sigma_4$ . By Lemmas 3.1, 3.2, and 3.3,  $M$  also satisfies  $\Sigma_2^*, \Sigma_3$ , and  $\Sigma_3^*$ . In the proof of  $\Sigma \models \Gamma'$  we only used the above mentioned axioms of  $\Sigma$ . In particular, the proofs of Lemmas 4.3 and 4.4 do not involve  $\Sigma_4^*$ . Therefore  $M \models \Gamma'$ , and with Theorem 5.5 we get  $M \models \Sigma$ .  $\square$

## 6 On the Non-Equationality of $\Gamma$

Baudisch and Pillay proved in [1] that the pseudospace  $\Sigma$  is a complete and  $\omega$ -stable theory. Once we know that  $\Sigma$  is a reduct of  $\Gamma$ , the same line of arguments can be also used to show the completeness and  $\omega$ -stability of  $\Gamma$ . This involves in particular exploring the fine structure of sufficiently saturated models of  $\Gamma$  and a detailed type analysis together with the computation of Morley ranks. In comparison to [1], however, the details are somewhat more tedious due to the richer language of  $\Gamma$  (a complete analysis was carried out in [2]). We will omit this altogether and proceed to explain the original purpose of  $\Gamma$  as an example of a stable and non-equational theory.

Computing Morley ranks in  $\Gamma$  it turns out that, as in  $\Sigma$ , the Morley rank of a plane  $a$  is  $\omega$ . However, in contrast to  $\Sigma$ , where we have  $MD(a) = 1$ , the Morley degree of  $a$  increases to 2 in  $\Gamma$ , owing to the fact that  $a$  splits into a white and a red section. For these we get  $MR(a_r) = MR(a_w) = \omega$  and  $MD(a_r) = MD(a_w) = 1$ . We collect all these facts in the following list. For the proof we refer to [3] and [2]. For the rest of this section we work in a big saturated model of  $\Gamma$ .

### Fact 6.1

1.  $\Gamma$  is complete and  $\omega$ -stable.
2. The set of points of a line  $b$  is indiscernible over  $b$ . Similarly, for every plane  $a$ , every two points from the red section  $a_r$  of  $a$  are conjugate over  $a$ .
3. For each plane  $a$ ,  $MR(a_r) = \omega$  and  $MD(a_r) = 1$ .

Building on this analysis, the next result from [3] is the key lemma for showing the non-equationality of  $\Gamma$ . In fact, it is the only place in the whole argument where equations come into play.

**Lemma 6.2 (Hrushovski, Srour [3])** *Let  $\varphi(x, \bar{y})$  be an equation and let  $\bar{d}$  be parameters corresponding to the variables  $\bar{y}$ . Let further  $D$  be the realization set of the instance  $\varphi(x, \bar{d})$ . Then for every line  $b$  and every plane  $a$  the following holds:*



1. If  $b$  is almost in  $D$ , i.e., all points of  $b$  except finitely many are in  $D$ , then already all points of  $b$  are in  $D$ .
2. If  $MR(a_r \setminus D) < \omega$ , then  $a \subseteq D$ .

*Proof.* For the first item let us assume that there exists a point  $c \in b \setminus D$ , and let  $c'$  be an arbitrary point from  $b$ . By Fact 6.1 points are indiscernible over lines, i.e., there exists an automorphism  $f$  mapping  $c$  to  $c'$  and fixing  $b$ . We will denote  $f(D)$  by  $D_{c'}$ . Because  $c \notin D$  we also have  $c' \notin D_{c'}$ . As  $b$  is almost in  $D$  and is fixed by  $f$ , the line  $b$  is also almost in  $D_{c'}$ . Varying the point  $c'$  we get  $\bigcap_{c' \in b} D_{c'} = \emptyset$ , because  $c' \notin D_{c'}$ . The sets  $D_{c'}$  are all defined by instances of the equation  $\varphi$ , hence there exist points  $c_1, \dots, c_n$  from  $b$  such that

$$\bigcap_{i=1}^n D_{c_i} = \bigcap_{c' \in b} D_{c'} = \emptyset .$$

But by assumption  $b$  is almost in  $D_{c_i}$  for  $1 \leq i \leq n$  and therefore also almost in  $\bigcap_{i=1}^n D_{c_i}$ , which gives a contradiction.

For part 2 we first prove  $a_r \subseteq D$  by a similar argument as in part 1. Assume that there exists a point  $c \in a_r \setminus D$ , and let  $c' \in a_r$  be arbitrary. By Fact 6.1 there exists an automorphism  $f$  such that  $f(c) = c'$  and  $f(a) = a$ . Let again  $D_{c'}$  denote  $f(D)$ . By  $c \notin D$  we get  $c' \notin D_{c'}$ . As  $f$  also fixes the red section  $a_r$  we have  $f(a_r \setminus D) = a_r \setminus D_{c'}$ . Morley ranks are preserved by automorphisms, hence  $MR(a_r \setminus D_{c'})$  is finite. As all the sets  $D_{c'}$  are defined by instances of the equation  $\varphi$ , there exist points  $c_1, \dots, c_n$  such that  $\bigcap_{i=1}^n D_{c_i} = \bigcap_{c' \in a_r} D_{c'} = \emptyset$ . Therefore  $\bigcup_{i=1}^n a_r \setminus D_{c_i} = a_r$ . But  $MR(a_r \setminus D_{c_i}) < \omega$ , contradicting  $MR(a_r) = \omega$ . This shows  $a_r \subseteq D$ .

It remains to show  $a_w \subseteq D$ . For this let  $c \in a_w$ . There exists a red line  $b$  in  $a$  that contains  $c$ , i.e.,  $c$  is the exceptional point of  $b$  in  $a$ . Then  $b$  is contained almost in  $a_r$ , hence also almost in  $D$ . By part 1 we conclude that the whole line  $b$  lies in  $D$ , hence in particular  $c \in D$ .  $\square$

This lemma enables us to give the proof of the main theorem of [3] stating the non-equationality of the pseudospace  $\Gamma$ . More concretely, the theorem also exhibits a non-equational formula:  $I_r(x; y)$  defining the red section of the plane specified by the parameter  $y$ . Before giving the precise argument, let us provide a more intuitive explanation why  $I_r(x; y)$  is no equation. Let  $a_i$ ,  $i \in \omega$ , be pairwise distinct planes which intersect in a common line  $b$ . Moreover, assume that  $b$  is red, but the exceptional point  $c_i$  of  $b$  is different on all planes  $a_i$ . Then  $\bigcap_{i \in \omega} (a_i)_r = b \setminus \{c_i \mid i \in \omega\}$ . It is clear that the intersection  $\bigcap_{i \in \omega} (a_i)_r$  is not equal to a finite sub-intersection. Hence  $I_r(x; y)$  is no equation.

Of course, to prove non-equationality of  $\Gamma$  we need to show that  $I_r(x; y)$  is not even equivalent to a Boolean combination of equations. This is done in the proof of the following theorem.

**Theorem 6.3 (Hrushovski, Srour [3])**  $\Gamma$  is not equational. More precisely, the formula  $I_r(x; y)$ , defining the red section of a plane, is not equivalent to a Boolean combination of equations.

*Proof.* We fix a big saturated model  $M$  of  $\Gamma$  and denote by  $\varphi^M$  the realization set of a formula  $\varphi$  in  $M$ . Let now  $a$  be a plane and assume that its red section  $a_r$  can be defined by a Boolean combination of equations

$$a_r = \left( \bigvee_{i=1}^n \left( \bigwedge_{j=1}^{n_i} \psi_{ij} \wedge \bigwedge_{j=1}^{n'_i} \neg \psi'_{ij} \right) \right)^M,$$

where  $\psi_{ij}$  and  $\psi'_{ij}$  are instances of equations  $\varphi_{ij}$  and  $\varphi'_{ij}$ . Finite conjunctions and finite disjunctions of equations are again equations (cf. [5]). Using the abbreviations  $\varphi_i = \bigwedge_{j=1}^{n_i} \varphi_{ij}$  and  $\varphi'_i = \bigvee_{j=1}^{n'_i} \varphi'_{ij}$  we can therefore write  $a_r$  as

$$a_r = \left( \bigvee_{i=1}^n \psi_i \wedge \neg \psi'_i \right)^M$$

where  $\psi_i$  and  $\psi'_i$  are instances of the equations  $\varphi_i$  and  $\varphi'_i$ .

Because  $MR(a_r) = \omega$ , there exists an index  $j$ ,  $1 \leq j \leq n$ , such that  $MR((\psi_j \wedge \neg \psi'_j)^M) = \omega$ . Let  $Y = (\psi_j \wedge \neg \psi'_j)^M$ . From  $MD(a_r) = 1$  and  $MR(a_r) = MR(Y) = \omega$  we conclude  $MR(a_r \setminus Y) < \omega$ .

Because  $a_r \setminus \psi_j^M \subseteq a_r \setminus Y$ , we get  $MR(a_r \setminus \psi_j^M) \leq MR(a_r \setminus Y)$ , hence in particular  $MR(a_r \setminus \psi_j^M)$  is finite. Part 2 of Lemma 6.2 then yields  $a \subseteq \psi_j^M$ . As  $Y \subseteq a_r$ , this implies  $a_w \subseteq \psi_j^M$ . As in the proof of part 2 of Lemma 6.2 this extends to  $a \subseteq \psi_j^M$ . Namely, if  $c \in a_r$ , then there exists a white line  $b$  in  $a$  such that  $c$  is the exceptional point of  $b$  in  $a$ . As  $b$  is almost in  $\psi_j^M$  we get by part 1 of Lemma 6.2  $b \subseteq \psi_j^M$  and hence  $c \in \psi_j^M$ . Now we have  $Y \subseteq a_r$  and  $a \subseteq \psi_j^M$  which implies  $Y = \emptyset$ . But this means  $MR(a_r) = MR(a_r \setminus Y) < \omega$  in contradiction to  $MR(a_r) = \omega$ .  $\square$

The free pseudospace  $\Gamma$  was constructed as a first example of a stable and non-equational theory. Recently, Sela [11] has shown that also non-abelian free groups are stable non-equational. Already Hrushovski and Srouf remark in [3] that, although  $\Gamma$  is not equational, it is almost equational, a weakening of equationality where the forking relation is controlled by equations (cf. [6]). It remains as an open problem to construct a theory that is simple (or even stable) but not almost equational.

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