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Input–Output Stabilization of Linear Systems on $\mathbb{Z}$

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***Abstract*—A formal framework is set up for the discussion of generalized autoregressive with external input models of the form $Ay = Bu$, where A and B are linear operators, with the main emphasis being on signal spaces consisting of bounded sequences parametrized by the integers. Different notions of stability are explored, and topological notions such as the idea of a closed system are linked with questions of stabilizability in this very general context. Various problems inherent in using $\mathbb{Z}$ as the time axis are analyzed in this operatorial framework.**

***Index Terms*—Input–output (I/O) stabilization, linear systems, operator closures, stabilizability.**

I. Introduction

In this paper, we are concerned with stabilization of linear systems specified by an input–output (I/O) relationship $Ay = Bu$, and our primary interest is in signal spaces which are defined on the whole set of integers $\mathbb{Z}$. This is known to lead to technical complications that are not present for signal spaces defined on the nonnegative integers $\mathbb{N}$ (see [7], [9], [11], [12], [17], [19], [18], and [21]). We will present a general framework allowing signal spaces that contain signals which persist in time. In fact, many of our results hold for linear I/O systems defined on abstract linear (normed) spaces. Special cases of linear models of the type $Ay = Bu$ are the bread and butter in many branches of technology: they are used in hundreds of millions of mobile phones and in hundreds of millions of control systems [e.g., in proportional–integral derivative (PID) controllers] [15], [1]. The main objectives are as follows: to establish the intrinsic limitations of the basic one-operator model $y = Pu$ on $\mathbb{Z}$, when P is an unbounded operator, and to show that the two-operator model $Ay = Bu$ provides a natural remedy within the I/O formalism.

The generality of the approach taken here is motivated by the fact that there are many ways to define interesting linear spaces of persistent signals that lead to, say, H_∞ or L_1/ℓ_1 optimal control [20], [22], [23]. (Excellent general references on H_∞ and L_1/ℓ_1 optimal control are [30] and [3], respectively.) In addition, in several important fields of study such as system identification, stochastic control, and telecommunication systems, it is customary to consider signals that are not bounded (and may not even have finite variance). The general framework presented here covers also I/O stabilization problems for such signal space settings. However, the signal space that we use here mostly as an example of a persistent signal space, is the space $\ell_\infty(\mathbb{Z})$ of bounded sequences (and its vector generalizations). The $\ell_\infty(\mathbb{Z})$ space contains the full time axis versions of the nonlinear bounded power space and the nonlinear quasi-stationary signal space studied in [30] and in [16], respectively. (These spaces of signals are nonlinear, that is, not linear, as they are not closed under addition.)

The basic I/O plant model $y = Pu$, where we are writing $P = B$, is a special case of the general I/O model $Ay = Bu$ with $A = I$, the identity operator. Unfortunately, strictly unstable, linear convolution operators are, in general, not stabilizable in an $\ell_\infty(\mathbb{Z})$ signal space setting [19], and so the standard plant model $y = Pu$ is not convenient for the present study. However, a study of the causal, single-input–single-output (SISO) case in [21] indicates that the more general I/O model definition $Ay = Bu$ is convenient if A and B are chosen as bounded linear operators on $\ell_\infty(\mathbb{Z})$. Here we study a similar situation for general linear, possibly noncausal, multiple-input–multiple-output (MIMO) systems defined over general signal spaces. Note that the general linear I/O model $Ay = Bu$ is often used in time series analysis, in system identification, and in control studies to mention a few examples.

Due to the popularity of the basic plant model $y = Pu$ in many engineering courses on control and in applications, it is also of great interest to study whether some extended definition of the plant P could serve as a convenient plant description so that $y = P_E u$, where P_E is some linear extension of P, i.e., $P_E u = Pu$ on $D(P; \ell_\infty(\mathbb{Z})^m)$. Jacob [9]–[11] has shown that in I/O stabilization studies on the $\ell_2(\mathbb{Z})$ signal space of square summable signals, it suffices to use the operator closure of P (in $\ell_2(\mathbb{Z})$) as P_E to recover the standard stabilization results for the half time axis $\ell_2(\mathbb{N})$ setting [25], [8], [30]. Such an approach to stabilization is here called the closure approach.

The operator closure of a strictly unstable, linear, time-invariant, finite-dimensional convolution operator on $\ell_2(\mathbb{Z})$, or on $\ell_\infty(\mathbb{Z})$, is not causal [7], [17], [12], [19], [9], and this gives one motivation for studying I/O stabilization for both causal and noncausal systems. In the present work we show that the closure approach fails even for some finite dimensional linear convolution systems defined over persistent signal spaces on $\mathbb{Z}$. It would be interesting to study the interpretation of such results within the elegant behavioral approach to dynamical systems [28], [29], [26].

An additional motivation for studying noncausal systems is that many problems in image processing and signal processing, and increasingly in control and systems, involve noncausal operators (see, for example, [28] and [14]). Furthermore, the standard technical definitions of the concepts of causality and time-invariance have been introduced having in mind especially the signal space $\ell_2(\mathbb{N})$. It is easy to find examples of signal spaces

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in which the usual technical definitions lead to difficulties [23], [2], and so this gives a good motivation to work out the general linear system results in an independent manner.

Although various full-time axis control problems, and closed-loop identification problems, have been studied for a long time, it appears that [7] is the first to explicitly demonstrate some of the difficulties of using unstable linear convolution operators in full time axis stabilization studies (a brief discussion appears in [8]). In fact, the literature contains numerous erroneous treatments of such problems as discussed in [18]. Reference [21] seems to be the first to demonstrate that the general linear I/O model $Ay = Bu$, by allowing one to describe open-loop unstable behavior without the need to introduce unbounded convolution operators, avoids many of the limitations of the basic I/O model $y = Pu$ in full time axis stabilization studies. We shall also provide a new type of argument concerning the limitations of unbounded convolution operators for linear normed spaces of equivalence classes of signals obtained from interesting linear seminormed spaces of persistent signals on $\mathbb{Z}$.

The rest of the paper is organized as follows. Mathematical background material and notation are introduced in Section II. In Section III, a formal framework is established for considering generalized autoregressive with external input (ARX) systems of the form $Ay = Bu$, and the links between time-invariant systems and convolutions made more precise. Section IV treats stability, of which we consider here three distinct definitions. The feedback system is introduced in Section V, and the link between closed systems and stabilizability established in this very general context. An intrinsic difficulty due to unstable (unbounded) convolution operators is also analyzed in this section. The simpler case when the plant and controller operators are bounded is analyzed in Section VI. "The closure approach," highlighting some problems of stabilizability on the signal space $\ell_\infty(\mathbb{Z})$ and on $\ell_q(\mathbb{Z}_-)$ for $1 \le q \le \infty$, is treated in Section VII. Some conclusions are drawn in Section VIII.

II. Mathematical Preliminaries and Notation

We use the standard notation $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, and $\mathbb{Z}_-$ for the complex numbers (or the complex plane), the reals, the integers, the nonnegative integers, the positive integers, and the negative integers, respectively. Furthermore, $\mathbb{R}^n$ denotes the linear space of all real n-tuples $v = [v_1, \ldots, v_n]'$, $v_i \in \mathbb{R}$, $i = 1, \ldots, n$, equipped with the norm

$$|v| \equiv \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2}.$$

(The superscript $'$ denotes vector transpose.)

Let $\ell_q(Q)^n$, $n \in \mathbb{Z}_+$ and $q \ge 1$, denote the linear normed space of all sequences $x = \{x(t) \in \mathbb{R}^n\}_{t \in Q}$ such that

$$\|x\|_q \equiv \left(\sum_{t \in Q} |x(t)|^q \right)^{1/q} < \infty. \tag{1}$$

Here, Q denotes $\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{Z}_-$. For $q = \infty$, the space $\ell_\infty(Q)^n$ is defined analogously using the norm $\|x\|_\infty$ defined as

$$\|x\|_\infty \equiv \sup_{t \in Q} |x(t)|. \tag{2}$$

Note that in the $\ell_\infty(Q)$ case the vector norm $|v|$, $v \in \mathbb{R}^n$, is sometimes replaced with the equivalent norm $\sup_{i=1,\ldots,n} |v_i(t)|$.

Let $c_+(\mathbb{Z})^n$ and $c_-(\mathbb{Z})^n$ denote the spaces of all real bounded sequences $x = \{x(t) \in \mathbb{R}^n\}_{t \in \mathbb{Z}}$ such that the limit $\lim_{t \to +\infty} x(t)$, respectively, the limit $\lim_{t \to -\infty} x(t)$, exists. These spaces are subspaces of $\ell_\infty(\mathbb{Z})^n$. Furthermore, let $c_0(\mathbb{Z})^n$ denote the linear subspace of $\ell_\infty(\mathbb{Z})^n$ such that $x \in \ell_\infty(\mathbb{Z})^n$ is in $c_0(\mathbb{Z})^n$ if and only if $\lim_{t \to \pm\infty} x(t) = 0$. The spaces $c_{+0}(\mathbb{Z})^n$ and $c_{-0}(\mathbb{Z})^n$ denote the subspaces of $\ell_\infty(\mathbb{Z})^n$ of elements x satisfying $\lim_{t \to +\infty} x(t) = 0$ and $\lim_{t \to -\infty} x(t) = 0$, respectively.

The spaces $\ell_\infty(\mathbb{Z})^n$, $c_\pm(\mathbb{Z})^n$, $c_{\pm 0}(\mathbb{Z})^n$, and $c_0(\mathbb{Z})^n$ are Banach spaces, i.e., complete linear normed spaces, equipped with the norm (2).

Let $\mathbb{T}$ denote the unit circle of the complex plane, i.e., $\mathbb{T} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \subset \mathbb{C}$, and let $L_2(\mathbb{T})$ and $L_\infty(\mathbb{T})$ denote the spaces of square integrable and essentially bounded complex-valued functions on $\mathbb{T}$, respectively.

Let X and Y be real linear spaces; then for an operator A mapping from a subspace of X into Y, it will be convenient to write $D(A; X)$ for its domain of definition. Next, $R(A; X)$ denotes the range (i.e., image) of the linear operator $A : X \to Y$, i.e.,

$$R(A; Y) = \{y = Ax \mid x \in D(A; X)\} \subset Y.$$

The null space (i.e., kernel) $N(A; X)$ of A is defined as

$$N(A; X) = \{x \in D(A; X) \mid Ax = 0\}.$$

Let $V \subset D(A; X)$. We use the standard notation $AV = \{Av \mid v \in V\} \subset Y$. Denote by $\|\cdot\|_X$, the norm in a linear normed space X. Let X and Y be linear normed spaces. We say that the operator $A : X \to Y$ is bounded if there exists a nonnegative number K such that

$$\|Ax\|_Y \le K \|x\|_X, \qquad x \in X.$$

Note that here $D(A; X) = X$. (Observe that in the literature, the above condition is sometimes relaxed so that it holds for $x \in D(A; X) \subseteq X$.) For a bounded operator $A : X \to Y$, the quantity

$$\|A\|_{\langle X, Y \rangle} = \sup_{x \ne 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

is called the norm induced by the domain and range space norms in X and Y, respectively, or simply the induced norm of A and it is often denoted as $\|A\|$.

Let $A : X \to X$ and $B : Y \to X$ be bounded operators. It is said that A and B are *left-coprime* if there exist bounded operators $C : X \to X$ and $D : X \to Y$ such that

$$(AC + BD)x = x, \qquad x \in X.$$

(This is slightly more general than the usual notion, allowing for the fact that A and B need not act on the same spaces.) Furthermore, it is said that the bounded operators C and D are right-coprime. Reference [25] provides a rather comprehensive treatment of coprimeness notions in control from an algebraic perspective.

Let $A : D(A;X) \to Y$ be a linear operator, where X and Y are linear normed spaces. The operator A is closed if the graph $\{(x, Ax)' \mid x \in D(A;X)\}$ is a closed set in the Cartesian product space $X \times Y$. A linear operator $A_E : D(A_E;X) \to Y$ is called an *extension* of A if $D(A;X) \subset D(A_E;X)$ and $A_E x = Ax$ for all $x \in D(A;X)$. An operator $\overline{A}$ which extends A and is closed, and has the property that any closed extension A_E of A extends $\overline{A}$, is called the *closure* of A. (Note that the closure of an operator need not always exist.) More detailed information about operator closures can be found in [6].

III. Discrete Linear Systems on $\mathbb{Z}$

In this section, we discuss discrete linear systems on $\mathbb{Z}$ in an abstract I/O context. In fact, many results of this paper can be interpreted as results on linear operator equations defined on abstract linear spaces without any dynamical systems context.

A. Basic Properties

Let $s(\mathbb{Z})^n$, $n \in \mathbb{Z}_+$, denote the linear space of (all) double-sided real sequences $x = \{x(t) \in \mathbb{R}^n\}_{t\in\mathbb{Z}}$.

Let S denote the *right shift* on $s(\mathbb{Z})^n$, that is for any $x \in s(\mathbb{Z})^n$

$$(Sx)(t) = x(t-1), \qquad t \in \mathbb{Z}.$$

Similarly, let L denote the *left shift* on $s(\mathbb{Z})^n$, so that for any $x \in s(\mathbb{Z})^n$

$$(Lx)(t) = x(t+1), \qquad t \in \mathbb{Z}.$$

Introduce the truncation operator $P_k : s(\mathbb{Z})^n \to s(\mathbb{Z})^n$, $k \in \mathbb{Z}$, by

$$(P_k x)(t) = \begin{cases} x(t), & t \le k \\ 0, & t > k \end{cases}.$$

This paper deals with linear operator equations on $\mathbb{Z}$ and so the following definition of a discrete linear I/O system will be employed.

Definition 3.1: The quadruple (A, B, Y^p, X^m), where

$$\begin{aligned} A &: D(A;Y^p) \to Y^p \\ B &: D(B;X^m) \to Y^p \end{aligned}$$

are linear operators, is called a (discrete) linear system, with m inputs (u) and p outputs (y), consisting of the set of trajectories

$$T(A,B,Y^p,X^m) \equiv \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in D(B;X^m) \times D(A;Y^p) \mid Ay = Bu \right\}.$$

Here $Y^p \subseteq s(\mathbb{Z})^p$ and $X^m \subseteq s(\mathbb{Z})^m$ are linear spaces.

It would be possible to present a more general definition of a linear system by considering the operator equation $Ay = Bu$ for the quintuple (A, B, Y^p, X^m, V^n), where $A : D(A;Y^p) \to V^n$ and $B : D(B;X^m) \to V^n$ are linear operators, and Y^p, X^m and V^n are linear spaces. This would allow a fully equivalent role for u and y, which would be beneficial in problems where there is no clear distinction between an input and an output. Here we are, however, only concerned with problems in which it is natural to decompose the system variables into an input u and an output y. (Typically y denotes then measured variables and u variables that can be manipulated and through which the information about y can be fed back into the plant to achieve some performance specifications for the resulting feedback system.)

Finally, it is possible to define an abstract linear system (A, B, Y, X) (or (A, B, Y, X, V)) via a linear operator equation $Ay = Bu$ in abstract linear spaces for continuous-time systems (some of the measure theoretic complications of this case are studied in [21]) and for systems which need not have any dynamical (systems) interpretation. In fact, many results of this paper hold in this very general context. Note also that m and p are both positive integers as this is the case of interest in this paper.

Remark 3.1: Note that if $(A, B, s(\mathbb{Z})^p, s(\mathbb{Z})^m)$ is a linear system, then (A, B, Y^p, X^m) is a linear system for any linear subspaces X^m and Y^p of $s(\mathbb{Z})^m$ and $s(\mathbb{Z})^p$, respectively. This is so because

$$\begin{aligned} D(A;Y^p) &= D(A;s(\mathbb{Z})^p) \cap \{y \in Y^p : Ay \in Y^p\} \\ D(B;X^m) &= D(B;s(\mathbb{Z})^m) \cap \{x \in X^m : Bx \in Y^p\} \end{aligned}$$

and, thus, $D(A;Y^p)$ and $D(B;X^m)$ are linear spaces.

It is interesting to note that (A, B, Y^p, X^m) being a linear system, in general, does not imply that (A, B, W^p, V^m) is a linear system, where $Y^p \subset W^p \subset s(\mathbb{Z})^p$ and $X^m \subset V^m \subset s(\mathbb{Z})^m$.

Remark 3.2: Let the quadruple (A, B, Y^p, X^m) be a linear system. Then, it is easy to verify that $T(A, B, Y^p, X^m)$ is a linear subspace of $X^m \times Y^p$.

Note that trajectories are the basis of the behavioral approach to dynamical systems [28], [29], [26]. The behavioral approach provides a very general setting to study dynamical systems. Here we are interested in the connections between linear operator theory and widely used I/O models.

We would now like to define discrete linear time-invariant (LTI) systems on $\mathbb{Z}$. This is somewhat subtle because the right shift S and the left shift L can have rather nontrivial properties depending on how the signal spaces Y^p and X^m are chosen.

We denote by $S_X : D(S_X;X) \to X$ and $L_X : D(L_X;X) \to X$, the right shift and the left shift, respectively, on the linear space X. Here, X denotes either X^m or Y^p. Suppose

$$S_X L_X x = L_X S_X x = x, \qquad x \in X \tag{3}$$

i.e., S_X is the inverse of L_X and *vice versa*. The relationships in (3) imply that

$$\begin{aligned} D(S_X;X) &= R(S_X;X) = X \\ D(L_X;X) &= R(L_X;X) = X. \end{aligned}$$

Note that (3) holds trivially when $X = s(\mathbb{Z})^n$. These relationships are valid on the spaces $\ell_q(\mathbb{Z})^n$, $q \ge 1$, $c_0(\mathbb{Z})^n$, and on many other signal spaces. However, there are other natural signal spaces where they do not hold (for example, signal spaces on $\mathbb{Z}_+$).

Definition 3.2: The (discrete) linear system (A, B, Y^p, X^m) is called an LTI system, with m inputs (u) and p outputs (y), if (3) holds, and if

$$\begin{aligned} S_{Y^p} D(A; Y^p) &\subseteq D(A; Y^p) \\ L_{Y^p} D(A; Y^p) &\subseteq D(A; Y^p) \\ S_{X^m} D(B; X^m) &\subseteq D(B; X^m) \\ L_{X^m} D(B; X^m) &\subseteq D(B; X^m) \end{aligned} \tag{4}$$

and

$$\begin{aligned} AS_{Y^p} &= S_{Y^p} A \qquad AL_{Y^p} = L_{Y^p} A \\ BS_{X^m} &= S_{Y^p} B \qquad BL_{X^m} = L_{Y^p} B. \end{aligned}$$

This means that A and B should commute with both the right shift and the left shift. (The aforementioned commutation conditions should be interpreted so that they hold on $D(A; Y^p)$ and on $D(B; X^m)$, respectively.) As the relationships in (3) are not valid on all interesting signal spaces, we use the previous definition of an LTI system rather sparingly. (Most of our results are valid for general linear systems.)

Proposition 3.1: Let the assumptions (4) of definition 3.2 hold. Then

$$\begin{aligned} S_{Y^p} D(A; Y^p) &= L_{Y^p} D(A; Y^p) = D(A; Y^p) \\ S_{X^m} D(B; X^m) &= L_{X^m} D(B; X^m) = D(B; X^m). \end{aligned}$$

Proof: We have $L_{Y^p} D(A; Y^p) \subseteq D(A; Y^p)$. Applying L_{Y^p} to both sides of the relation $S_{Y^p} D(A; Y^p) \subseteq D(A; Y^p)$, we have $D(A; Y^p) = L_{Y^p} S_{Y^p} D(A; Y^p) \subseteq L_{Y^p} D(A; Y^p)$, and hence $L_{Y^p} D(A; Y^p) = D(A; Y^p)$. The remaining identities are proven similarly. ■

In the sequel, we will leave out the subscript X in S_X and L_X, and write simply S and L, as this should not cause any confusion.

When X^m and Y^p are linear normed spaces it is natural to regard the space $X^m \times Y^p$ as a linear normed space equipped with an appropriate product space norm. In this paper, we will put

$$\left\| \begin{pmatrix} u \\ y \end{pmatrix} \right\|_{X^m \times Y^p} = \max\{\|u\|_{X^m}, \|y\|_{Y^p}\}.$$

This is a convenient choice here as our many general results are illustrated or made more specific mostly on the $\ell_\infty(\mathbb{Z})^n$ signal space (and on some of its subspaces). We can then regard the space $X^m \times Y^p$ as a topological space, where the topology is induced by the product space norm. Other norms, such as $\left(\|u\|_{X^m}^2 + \|y\|_{Y^p}^2\right)^{1/2}$, give the same topology.

Definition 3.3: Let the quadruple (A, B, Y^p, X^m) be a linear system. The system is said to be closed, if the set of trajectories $T(A, B, Y^p, X^m)$ is a closed subspace of $X^m \times Y^p$.

Remark 3.3: Note that if (A, B, Y^p, X^m) is a linear system, with Y^p, X^m linear normed spaces and $A : Y^p \to Y^p$, $B : X^m \to Y^p$ bounded operators ($D(A; Y^p) = Y^p$ and $D(B; X^m) = X^m$), then the system (A, B, Y^p, X^m) is closed.

Closedness of an LTI system is an important property in the stability analysis to be presented later.

Example 1: Consider the LTI system $(A, B, \ell_\infty(\mathbb{Z}), \ell_\infty(\mathbb{Z}))$ with $A = I - S$ and $B = I$, so that the input u and the output y satisfy

$$y(t) - y(t-1) = u(t), \qquad t \in \mathbb{Z}.$$

What is the set of trajectories $T(A, B, \ell_\infty(\mathbb{Z}), \ell_\infty(\mathbb{Z}))$ for this LTI system? By the previous theorem, this set is closed in the product topology of $\ell_\infty(\mathbb{Z}) \times \ell_\infty(\mathbb{Z})$.

Clearly any $(u, y)' \in \ell_\infty(\mathbb{Z})^2$, such that

$$y(t) = C_1 + \sum_{k \le 0} u(t+k), \qquad t \in \mathbb{Z}$$

where the sum defines an $\ell_\infty(\mathbb{Z})$ sequence, belongs to $T(A, B, \ell_\infty(\mathbb{Z}), \ell_\infty(\mathbb{Z}))$. Here, C_1 is any real number. Note that such trajectories must satisfy $y \in c_-(\mathbb{Z})$ and

$$\lim_{t \to -\infty} u(t) = 0.$$

Similarly, any $(u, y)' \in \ell_\infty(\mathbb{Z})^2$, such that

$$y(t) = C_2 + \sum_{k \ge 1} u(t+k), \qquad t \in \mathbb{Z}$$

where the sum defines an $\ell_\infty(\mathbb{Z})$ sequence, belongs to $T(A, B, \ell_\infty(\mathbb{Z}), \ell_\infty(\mathbb{Z}))$. Here C_2 is any real number. Note that such trajectories must satisfy $y \in c_+(\mathbb{Z})$ and

$$\lim_{t \to +\infty} u(t) = 0.$$

However, the aforementioned trajectories do not exhaust $T(A, B, \ell_\infty(\mathbb{Z}), \ell_\infty(\mathbb{Z}))$. In fact, most trajectories do not converge to any limit when $t \to \pm\infty$: Take any sequence $y \in \ell_\infty(\mathbb{Z})$ and put $u = (I - S)y$. Then $u \in \ell_\infty$. Thus, any $(u, y)'$ defined in this manner belongs to $T(A, B, \ell_\infty(\mathbb{Z}), \ell_\infty(\mathbb{Z}))$. The following remark, which will have an application later in the theory of stability, makes this precise.

Remark 3.4: A vector $w \in \ell_\infty(\mathbb{Z})$ lies in the range of $(I - S)$ if and only if it satisfies the condition that, for some $M > 0$

$$\left| \sum_{t=k+1}^{m} w(t) \right| \le M, \qquad \text{for all} \quad m, k \in \mathbb{Z}, \quad \text{with} \quad m > k. \tag{5}$$

For if $w = (I - S)v$, then

$$\begin{aligned} \left| \sum_{t=k+1}^{m} w(t) \right| &= \\ |(v(m) - v(m-1)) + \cdots + (v(k+1) - v(k))| &= \\ |v(m) - v(k)| &\le 2\|v\|_\infty \end{aligned}$$

and if (5) holds, then the choice

$$v(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{j=1}^{n} w(j), & \text{if } n > 0 \\ -\sum_{j=n+1}^{0} w(j), & \text{if } n < 0 \end{cases}$$

provides a $v \in \ell_\infty(\mathbb{Z})$ with $w = (I - S)v$.

Note that the range of $(I - S)$ is not a closed subspace of $\ell_\infty(\mathbb{Z})$, for example it contains the vectors $(\ldots, 0, 0, 1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$ for each n but their norm limit $(\ldots,0,0,1,1/2,1/3,\ldots)$ is not in the range of $(I-S)$. Even when we restrict to $(I-S): c_0(\mathbb{Z}) \to c_0(\mathbb{Z})$ the range fails to be closed, as may be seen by considering the vectors $(\ldots,0,0,1,1/2,1/3,\ldots,1/n,-s_n/n,\ldots,-s_n/n,0,0,\ldots)$, where $s_n = 1 + 1/2 + \cdots + 1/n$ and the term s_n/n occurs n times.

A fuller discussion of the range of a shift-invariant A is included in Section III-B. We stress that the set of trajectories can be a closed set, even when the range of the operator is not itself closed.

This simple example illustrates the fact that the set of trajectories $T(A,B,Y^p,X^m)$, in general, does not define the output y as a function of u. That is, an input u need not result in a unique output y. This indicates in a very clear manner that the usual operator setting $y = Pu$, where $P: D(P;X^m) \to Y^p$ is a linear operator, cannot accommodate the more general linear systems setup studied here.

A popular technical definition of a causal operator is as follows. The linear operator $P: D(P;X^m) \to Y^p$ is said to be *causal* if

$$P_k(Y^p)P = P_k(Y^p)PP_k(X^m), \qquad \text{for } k \in Z$$

where $P_k(Y^p)$ and $P_k(X^m)$ denote the truncation operators on Y^p and X^m, respectively. (We could have used the notation P_k for both of these operators, as these are defined in complete analogy to the truncation operator P_k on $s(\mathbb{Z})^n$.) However, this definition of causality runs into technical problems on certain interesting signal spaces [23], and so to obtain the greatest degree of generality, most of our results do not use the notion of causality. The aforementioned definition is valid, however, on $\ell_q(\mathbb{Z})^n$, $q \geq 1$, on $c_0(\mathbb{Z})^n$, and on many other signal spaces.

An important subclass of linear systems can be defined as follows.

Definition 3.4: Let $P: D(P;s(\mathbb{Z})^q) \to s(\mathbb{Z})^r$ denote the linear operator

$$\begin{aligned}(Px)(t) &\equiv \sum_{k\in\mathbb{Z}} H(t,k)x(t-k)\\ &= \lim_{K,L\to\infty} \sum_{-K\leq k\leq L} H(t,k)x(t-k)\end{aligned}$$

where the $H(t,k) \in \mathbb{R}^{r\times q}$, $t,k \in \mathbb{Z}$, are real matrices. The operator P is called a linear convolution operator.

The convolution operator P is causal if $H(t,k) = 0$ for all $k < 0$. The operator P is called anticausal if $H(t,k) = 0$ for all $k > 0$.

If $H(t,k) = G(k)$ for all $t \in \mathbb{Z}$, $k \in \mathbb{Z}$, for some sequence of real matrices $\{G(k)\}_{k\in\mathbb{Z}}$, then the linear convolution operator

$$\begin{aligned}(Px)(t) &\equiv \sum_{k\in\mathbb{Z}} G(k)x(t-k)\\ &= \lim_{K,L\to\infty} \sum_{-K\leq k\leq L} G(k)x(t-k)\end{aligned} \tag{6}$$

is called an LTI convolution operator, as $(I,P,s(\mathbb{Z})^r,s(\mathbb{Z})^q)$ is an LTI system. Here, I denotes the identity operator on $s(\mathbb{Z})^q$.

In this paper, the aforementioned (old) conventions on causality and time-invariance of convolution operators are used also for convolution operators defined on suitable linear subspaces of $s(\mathbb{Z})^n$.

For completeness, we state the following result.

Theorem 3.1: Let $A: D(A;s(\mathbb{Z})^p) \to s(\mathbb{Z})^p$ and $B: D(B;s(\mathbb{Z})^m) \to s(\mathbb{Z})^p$ be LTI convolution operators. Then, the quadruple $(A,B,s(\mathbb{Z})^p,s(\mathbb{Z})^m)$ is an LTI system.

The proof follows readily from Definition 3.2 and (6) and is, therefore, omitted. Note that, by Remark 3.1, it is clear that $(A,B,s(\mathbb{Z})^p,\ell_\infty(\mathbb{Z})^m)$ and $(A,B,\ell_\infty(\mathbb{Z})^p,\ell_\infty(\mathbb{Z})^m)$ are LTI systems, too.

B. Closed Ranges and Existence of Inverses

For this section, we restrict to the case $m = p = 1$, which is enough to indicate some of the issues involved. It is simplest to begin with a discussion of the $\ell_2(\mathbb{Z})$ case. Let A be a bounded *shift-invariant* operator on $\ell_2(\mathbb{Z})$, that is, an operator such that $AS = SA$, where S, as usual, denotes the right shift. Then it is well-known that by means of Fourier series A is unitarily equivalent to M_h, the operator of multiplication on $L_2(\mathbb{T})$ by a bounded function $h \in L_\infty(\mathbb{T})$.

Proposition 3.2: Let $M_h: L_2(\mathbb{T}) \to L_2(\mathbb{T})$ be as before. Then, the following possibilities can occur.

1) h vanishes on a subset $E \subseteq \mathbb{T}$ of positive measure. Then, A is not injective, so no left inverse exists (in this case, A cannot be causal).
2) h does not vanish on a subset of positive measure, but $h^{-1} \notin L_\infty(\mathbb{T})$. Then, $R(A;\ell_2(\mathbb{Z}))$ is not closed, and A has an unbounded left inverse.
3) $h^{-1} \in L_\infty(\mathbb{T})$. Then, $R(A;\ell_2(\mathbb{Z})) = \ell_2(\mathbb{Z})$ and A is invertible.

Proof: In the first case, $h\chi_E = 0$, where χ_E is the characteristic function (indicator function) of E, and thus A annihilates any $\ell_2(\mathbb{Z})$ sequence that is the sequence of Fourier coefficients of a function vanishing on E. Note that A cannot be causal, as this would correspond to a function $h \in H^\infty$, and such functions cannot vanish on sets of positive measure unless they are identically zero (see, for example, [5, p. 17]).

Otherwise, we cannot have $hu = 0$ unless u is zero almost everywhere, and so A is injective. However, if the range of A is closed, then A has a bounded left inverse by the Banach open mapping theorem [24]. Furthermore, if $(Af,g) = 0$ for all f, then $A^*g = 0$, so $g = 0$ (note that A^* is equivalent to $M_h^* = M_{\overline{h}}$, which is injective). Thus, the range of A is necessarily the whole space, and the inverse is a two-sided inverse. Consequently $\|h^{-1}\|_\infty = \|A^{-1}\| < \infty$. ■

We have seen already the example $I-S$ as a bounded injective operator on $\ell_2(\mathbb{Z})$ without closed range.

Remark 3.5: There is a difference in the $\ell_2(\mathbb{N})$ case, where A corresponds to multiplication on the Hardy space H_2. Here, the range can be closed and not the whole space: For example, S (the unilateral shift), has a one-sided inverse.

To understand the $\ell_\infty(\mathbb{Z})$ case better, we need to find a way of representing shift-invariant operators as convolutions.

Theorem 3.2: If A is a bounded shift-invariant operator on $c_0(\mathbb{Z})$, then it is given by convolution with a (not necessarily causal) impulse response, i.e.,

$$Au(t) = \sum_{k=-\infty}^{\infty} a_k u(t-k) \tag{7}$$

and, furthermore, $\sum_{k=-\infty}^{\infty} |a_k| < \infty$. The same is true on $c_{-0}(\mathbb{Z})$ if A is causal.

Proof: Let $X = c_0(\mathbb{Z})$ or $c_{-0}(\mathbb{Z})$, and let $a \in X$ denote Ae_0, where $e_0(t) = 1$ for $t = 0$, and 0 otherwise. Then, the shift invariance easily implies that (7) holds for any u with finite support. The fact that A is bounded also implies that the sequence $(a(t))$ is in ℓ_1, since otherwise $Au(0) = \sum_{-N}^{N} a_k u(-k)$ could be made arbitrarily large by choosing $u(k) = -\operatorname{sign}(a_k)$ for $k = -N, \ldots, N$, even though $\|u\|_\infty = 1$, contradicting the boundedness of A.

In the case where $X = c_0(\mathbb{Z})$, there is nothing more to prove, as the sequences of finite support are dense in X. In the case where, $X = c_{-0}(\mathbb{Z})$, when A is causal, we know that $Au(t) = Au_t(t)$, where $u_t \in c_0(\mathbb{Z})$ is defined by $u_t(k) = u(k)$ for $k \leq t$, otherwise 0. Then

$$Au(t) = Au_t(t) = \sum_{k=0}^{\infty} a_k u_t(t-k) = \sum_{k=0}^{\infty} a_k u(t-k)$$

as required. ■

Remark 3.6: Consider the example $Au(t) = \mathrm{Blim}_{k \to -\infty} u(k)$ for all $t \in \mathbb{Z}$, where Blim is a generalized (Banach) limit [24, p. 82] (i.e., a continuous linear functional defined on the space of all bounded sequences, which coincides with the limit of the sequence whenever this exists). This shows that the result of Theorem 3.2 no longer holds on $c_{-}(\mathbb{Z})$ or $\ell_\infty(\mathbb{Z})$.

Remark 3.7: A bounded shift-invariant operator A as in Theorem 3.2 has a (necessarily bounded) inverse if and only if $\sum_{k=-\infty}^{\infty} a_k z^k \neq 0$ for all z on the unit circle $\mathbb{T}$. If in addition A is causal, then it has a causal inverse if and only if $\sum_{k=0}^{\infty} a_k z^k \neq 0$ for all z in the closed unit disc $\overline{\mathbb{D}}$. This follows easily from Wiener's lemma, which can be proven using the Gelfand theory of commutative Banach algebras (see [24, p. 266]).

IV. I/O Stability

Three I/O stability notions will be discussed in this section. These I/O stability notions will be used later in the closed-loop stability analysis.

Definition 4.1: Let (A, B, Y^p, X^m) be a linear system, where X^m and Y^p are linear spaces. The system is said to be weakly (X^m, Y^p) stable (solvable) if

$$(\{u\} \times Y^p) \cap T(A, B, Y^p, X^m) \neq \emptyset \qquad \text{for all} \quad u \in X^m.$$

(Here, the notation $\emptyset$ denotes the empty set.) This stability notion states simply that any input in X^m should produce some outputs in Y^p. Weak stability is a solvability notion for the generalized linear equation $Ay = Bu$, where y is regarded as the unknown variable. Observe that it always holds that

$$(\{0\} \times Y^p) \cap T(A, B, Y^p, X^m) \neq \emptyset$$

as

$$(\{0\} \times \{0\}) \in T(A, B, Y^p, X^m)$$

due to the fact that X^m and Y^p are linear spaces (and so contain at least a zero element by definition, implying that $D(A; Y^p)$ and $D(B; X^m)$ contain at least a zero element as A and B are linear operators).

Note that the previous definition allows the possible multivaluedness of the relationship between the input u and the output y. Let $Y(u) \subset Y^p$ denote the set of all y such that $(u, y)' \in T(A, B, Y^p, X^m)$. If $Y(u)$ is empty for some $u \in X^m$, then the system is not weakly stable.

Definition 4.2: Let (A, B, Y^p, X^m) be a linear system, where X^m and Y^p are linear normed spaces. The system is said to be bounded-input–bounded-output (BIBO) (X^m, Y^p) stable if $Y(u)$ is a singleton for each $u \in X^m$.

Observe that we require here that X^m and Y^p are linear normed spaces, as this is customary with BIBO stability for signal spaces defined on $\mathbb{N}$. Clearly, BIBO stability implies weak stability. For $Y^p = \ell_\infty(\mathbb{Z})^p$ and $X^m = \ell_\infty(\mathbb{Z})^m$, this defines the usual BIBO stability notion. Note that we are using here the convention that $Y(u)$ should be a singleton for each u as this simplifies the statement of some our later results.

It should be emphasized that the type of argument used in [7] and in [17], [19] to discuss the impossibility to stabilize unstable, causal, finite-dimensional, convolution operators on the full time axis does not apply to the above two types of stability notions, but rather to the type of gain stability notion to be introduced next.

Definition 4.3: Let (A, B, Y^p, X^m) be a linear system, where Y^p and X^m are linear normed spaces. The system is said to be (X^m, Y^p) gain stable if there exists a nonnegative constant K such that

$$\|y\|_{Y^p} \leq K \|u\|_{X^m}$$

holds for all $(u, y)' \in T(A, B, Y^p, X^m)$ and $(\{u\} \times Y^p) \cap T(A, B, Y^p, X^m) \neq \emptyset$ for all $u \in X^m$.

That is, (X^m, Y^p) gain stability implies BIBO (X^m, Y^p) stability, but the reverse implication need not hold.

In the case that $X^m = \ell_\infty(\mathbb{Z})^m$ and $Y^p = \ell_\infty(\mathbb{Z})^p$, we call this form of stability simply ℓ_∞ gain stability.

The main reason why we have introduced the notion of weak (I/O) stability is to allow us to study the intrinsic difficulties of the basic I/O model $y = Pu$ under as mild assumptions as possible. Clearly this notion is not useful for a more detailed I/O performance assessment as weak stability does not imply (when X^m and Y^p are linear normed spaces) even the existence of an inequality of the form

$$\|y\|_{Y^p} \leq K_1 \|u\|_{X^m} + K_2$$

for some nonnegative constants K_1 and K_2 independent of $[u', y']'$ in $T(A, B, Y^p, X^m)$.

Proposition 4.1: Let the linear system (A, B, Y^p, X^m) be weakly (X^m, Y^p) stable. Then

$$D(B, X^m) = X^m \qquad \text{and} \qquad R(B; Y^p) \subseteq R(A; Y^p).$$

Proof: Assume that there exists $u \in X^m$ such that $u \notin D(B; X^m)$. Then

$$(\{u\} \times Y^p) \cap T(A, B, Y^p, X^m) = \emptyset$$

in contradiction with the weak stability of (A, B, Y^p, X^m). Hence, $D(B; X^m) = X^m$ as claimed. Similarly, assume that there exists $u \in D(B; X^m)$ such that $Bu \notin R(A; Y^p)$. Then

$$(\{u\} \times Y^p) \cap T(A, B, Y^p, X^m) = \emptyset$$

as $Ay \neq Bu$ for all $y \in D(A; Y^p)$, in contradiction with the weak stability of (A, B, Y^p, X^m). Hence, $R(B; Y^p) \subseteq R(A; Y^p)$ must hold. This completes the proof. ■

Proposition 4.2: Let the linear system (A, B, Y^p, X^m) be (X^m, Y^p) gain stable, where X^m and Y^p are linear normed spaces. Then

$$N(A; Y^p) = \{y \in D(A; Y^p) \mid Ay = 0\} = \{0\}.$$

Proof: Assume that $y \in N(A; Y^p)$ but $y \neq 0$. Then $(0, y)' \in T(A, B, Y^p, X^m)$, in contradiction with the gain stability of (A, B, Y^p, X^m) as $\|y\|_{Y^p} > 0$. This completes the proof. ■

The next result states closedness of a linear system as a necessary condition for gain stability of the system when X^m and Y^p are linear normed spaces.

Theorem 4.1: Let the linear system (A, B, Y^p, X^m) be (X^m, Y^p) gain stable, where X^m and Y^p are linear normed spaces. Then, the system (A, B, Y^p, X^m) is closed.

Proof: Note that by the stability assumption $D(B; X^m) = X^m$. Now let $\{(u^{(n)}, y^{(n)})' \in T(A, B, Y^p, X^m)\}_{n \geq 1}$ be a convergent sequence. Denote the limit point by $(u^*, y^*)' \in X^m \times Y^p$. As the linear system is stable, it follows that there is an element $(u^*, y(u^*))' \in T(A, B, Y^p, X^m)$ such that $Ay(u^*) = Bu^*$. Now

$$A(y^{(n)} - y(u^*)) = B(u^{(n)} - u^*)$$

as the set of trajectories $T(A, B, Y^p, X^m)$ is a linear space by Remark 3.2. However, by the gain stability assumption, there exists a number $K \geq 0$, independent of n, such that

$$\|y^{(n)} - y(u^*)\|_{Y^p} \leq K \|u^{(n)} - u^*\|_{X^m}.$$

The right-hand side of this inequality tends to zero when $n \to \infty$. Hence $y^* = y(u^*)$, as Y^p is a linear normed space, and so $(u^*, y^*)' \in T(A, B, Y^p, X^m)$ and, hence, the LTI system is closed. This completes the proof. ■

The following result provides a converse to Propositions 4.1 and 4.2, and Theorem 4.1.

Theorem 4.2: Suppose that (A, B, Y^p, X^m) is a closed linear system, where X^m and Y^p are Banach spaces, and that we have $D(B, X^m) = X^m$, $R(B; Y^p) \subseteq R(A; Y^p)$ and $N(A, Y^p) = \{0\}$. Then, (A, B, Y^p, X^m) is gain stable.

Proof: The given conditions imply that for each $u \in X^m$ there is a unique $y \in Y^p$ such that $Ay = Bu$. Thus, there is a linear operator $G : X^m \to Y^p$ whose graph $\{(u, Gu) : u \in X^m\}$ is equal to $T(A, B, Y^p, X^m)$. Hence, by the closed graph theorem [24, p. 50], G is bounded, which implies that the system is (X^m, Y^p) gain stable. ■

The literature of automatic continuity theory (a collection of results asserting that operators satisfying certain algebraic conditions are automatically bounded) also contains results implying that under some circumstances BIBO stability is equivalent to gain stability. We shall apply the following result, which is a special case of [4, Cor. 5.3.45] (a result expressed in the language of locally compact topological groups).

Theorem 4.3: Let X, Y be shift-invariant Banach spaces contained in $s(\mathbb{Z})$. Then every shift-invariant linear mapping from X into Y is automatically continuous *if and only if* there is no eigenvalue λ of the shift S on Y such that $(\lambda I - S)X$ has infinite codimension in X.

Corollary 4.1: Every BIBO stable linear shift-invariant operator between signal spaces $c_0(\mathbb{Z})^n$, $c_{-0}(\mathbb{Z})^n$, or $c_{+0}(\mathbb{Z})^n$ is gain stable.

Proof: It is easy to see that we may reduce the problem to the SISO case $m = p = 1$.

If λ is an eigenvalue of S, then there is a nonzero vector u such that $Su = \lambda u$ and, hence, $u(t) = a\lambda^{-t}$ for all $t \in \mathbb{Z}$. This is impossible. ■

The above result also applies to the case of $\ell_2(\mathbb{Z})^n$, a result that is well-known and goes back at least as far as [13].

Remark 4.1: The case of $\ell_\infty(\mathbb{Z})^n$ is different, since $\lambda = 1$ is an eigenvalue of the shift on ℓ_∞, and the range of $I - S$ (characterized in Remark 3.4) has infinite codimension; for we may easily find a sequence (u_j) of vectors in ℓ_∞ such that no finite nonzero linear combination lies in the range of $I - S$. One such example is obtained by taking $u_j = (\ldots, 0, 0, 1, 1/2^{1/j}, 1/3^{1/j}, \ldots)$ for $j = 1, 2, \ldots$. Thus BIBO stability is not the same as gain stability in this context.

Some related discussions can be found in [22].

V. Feedback System

We are now ready to discuss feedback systems, that is, systems constructed by an interconnection of a plant and a controller, together with the various associated notions of closed-loop stability. We shall see, for example, that closed-loop stability implies that both plant and controller are closed systems.

A. Feedback System and Stability

Thus, consider the interconnected system

$$\begin{aligned} Hu &= Fy + \Lambda_1 w \\ Ay &= Bu + \Lambda_2 w \end{aligned} \tag{8}$$

where (H, F, X^m, Y^p) and (A, B, Y^p, X^m) are linear systems, $w \in V^q \subset s(\mathbb{Z})^q$ are external signals, where V^q is a linear space, and $\Lambda_1 : D(\Lambda_1; V^q) \to X^m$ and $\Lambda_2 : D(\Lambda_2; V^q) \to Y^p$ are linear operators.

Here, (A, B, Y^p, X^m) and (H, F, X^m, Y^p) are called the plant and the controller, respectively. We shall write the interconnected system as

$$\Gamma x = \Lambda w \tag{9}$$

where

$$\Gamma = \begin{pmatrix} H & -F \\ -B & A \end{pmatrix} \quad \Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$$

and $x = (u, y)'$. Note that $(\Gamma, \Lambda, X^m \times Y^p, V^q)$ is a linear system. This linear system is called *the feedback system* associated with the plant and the controller.

Theorem 5.1: Let the linear feedback system $(\Gamma, \Lambda, X^m \times Y^p, V^q)$ be $(V^q, X^m \times Y^p)$ gain stable, where X^m, Y^p and V^q are linear normed spaces. Then, the feedback system $(\Gamma, \Lambda, X^m \times Y^p, V^q)$ is a closed linear system. Furthermore, then $D(\Lambda; V^q) = V^q$ and

$$R(\Lambda; X^m \times Y^p) \subseteq R(\Gamma; X^m \times Y^p).$$

If in addition Λ is surjective (i.e., onto $X^m \times Y^p$, or in other words $R(\Lambda; X^m \times Y^p) = X^m \times Y^p$), and both $A : Y^p \to Y^p$ and $H : X^m \to X^m$ are bounded operators, then the plant (A, B, Y^p, X^m) and the controller (H, F, X^m, Y^p) are closed linear systems.

Proof: That the gain stability of the feedback system implies closedness of the feedback system follows by a direct application of Theorem 4.1. The domain and range relationships of the theorem follow at once from Proposition 4.1.

The plant closedness result is proved as follows. Suppose that $(v_n, z_n) \in T(A, B, Y^p, X^m)$, so $Az_n = Bv_n$, and one has $v_n \to \hat{v}$ and $z_n \to \hat{z}$. Since Λ is surjective, we may choose $\Lambda_1 w_n = Hv_n$ and $\Lambda_2 w_n = -Bv_n = -Az_n$, so that the feedback (8) have the solution $u = v_n$ and $y = 0$.

Now, pass to the limit, using the fact that the mapping from w to $\begin{pmatrix} u \\ y \end{pmatrix}$ is bounded; thus taking $\Lambda_1 w = H\hat{v}$, $\Lambda_2 w = -A\hat{z}$ we have the solution $u = \hat{v}$ and $y = 0$. This implies that $Ay = Bu + \Lambda_2 w$, or $B\hat{v} = A\hat{z}$, thus $(\hat{v}, \hat{z})$ lies in $T(A, B, Y^p, X^m)$, which is therefore closed.

A similar proof shows that the controller is a closed system. ■

An important special case of the feedback system (9) is obtained when $q = m + p$, $V^q = X^m \times Y^p$, and $\Lambda = I$ is the identity operator on $X^m \times Y^p$. Then, (9) simplifies to

$$\Gamma x = w. \tag{10}$$

This is a linear equation system defined in a linear normed space setting. We shall in the sequel use this simplified form of the feedback system.

Let us now consider stability conditions for the feedback system, i.e., closed-loop stability conditions.

Theorem 5.2: Let the linear plant (A, B, Y^p, X^m) and the linear controller (H, F, X^m, Y^p) be given, where X^m and Y^p are linear spaces. The associated feedback system $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ is weakly stable if and only if (iff)

$$R(\Gamma; X^m \times Y^p) = X^m \times Y^p.$$

Proof: This follows directly from Proposition 4.1 using the definitions of the feedback system and its weak stability. ■

Example 2: Consider the LTI feedback system

$$\begin{aligned} y(t) &= \sum_{i \geq 0} u(t - i - 1) + w_2(t) \\ u(t) &= -k_1 y(t) - k_2 \sum_{j \geq 0} y(t - j - 1) + w_1(t) \end{aligned}$$

where $k_2 \neq 0$.

This corresponds to PI control of a marginally stable first-order system. Let $X^m = Y^p = \ell_\infty(\mathbb{Z})$. (Here, $m = p = 1$.) It follows by the definition of a convolution sum that the following conditions must hold:

$$\begin{aligned} \lim_{t \to -\infty} u(t) &= 0 \\ \lim_{t \to -\infty} y(t) &= 0. \end{aligned}$$

Furthermore, clearly also $y(t) - w_2(t)$ and $u(t) + k_1 y(t) - w_1(t)$ have to tend to zero when $t \to -\infty$. However, with the two earlier limit conditions, these conditions give that $w_1(t)$ and $w_2(t)$ have to tend to zero when $t \to -\infty$. Therefore

$$R(\Gamma, \ell_\infty(\mathbb{Z}) \times \ell_\infty(\mathbb{Z})) \subseteq c_{-0}(\mathbb{Z}) \times c_{-0}(\mathbb{Z})$$

and so $R(\Gamma, \ell_\infty(\mathbb{Z}) \times \ell_\infty(\mathbb{Z}))$ is a proper subspace of $\ell_\infty(\mathbb{Z}) \times \ell_\infty(\mathbb{Z})$ only. Hence, no PI controller, of the convolution summation form, with nonzero integral effect, can even weakly stabilize the first-order marginally stable convolution system. This is in stark contrast to the situation for the signal space $\ell_\infty(\mathbb{N})$ defined on the standard singly-infinite time axis $\mathbb{N}$.

Let $\Gamma_L^{-1} : R(\Gamma; X^m \times Y^p) \to D(\Gamma; X^m \times Y^p)$ denote the linear operator, when it exists, such that

$$\Gamma_L^{-1} \Gamma x = x, \qquad \text{for any } x \in D(\Gamma; X^m \times Y^p).$$

Here, X^m and Y^p are linear spaces.

This operator is the left inverse of $\Gamma : D(\Gamma; X^m \times Y^p) \to R(\Gamma; X^m \times Y^p)$. It is clear that Γ_L^{-1} exists iff $N(\Gamma; X^m \times Y^p) = \{0\}$. Now, (10) gives that

$$x = \Gamma_L^{-1} w$$

for any $w \in R(\Gamma; X^m \times Y^p)$. However, $R(\Gamma; X^m \times Y^p)$ could be a proper subspace of $X^m \times Y^p$, it is seen that the existence of Γ_L^{-1} need not even imply weak stability of the feedback system. However, the above discussion gives directly the following stability condition.

Proposition 5.1: Let $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ be a linear feedback system, where X^m and Y^p are linear normed spaces. The feedback system is BIBO $(X^m \times Y^p, X^m \times Y^p)$ stable iff

$$R(\Gamma; X^m \times Y^p) = X^m \times Y^p \quad \text{and} \quad N(\Gamma; X^m \times Y^p) = \{0\}.$$

Note that this result does not require that

$$D(\Gamma; X^m \times Y^p) = X^m \times Y^p.$$

In particular, it does not require that Γ is a bounded operator for closed-loop BIBO $(X^m \times Y^p, X^m \times Y^p)$ stability to be possible.

This gives directly the next result.

Proposition 5.2: Let $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ be a linear feedback system, where X^m and Y^p are linear normed spaces. The feedback system is $(X^m \times Y^p, X^m \times Y^p)$ gain stable iff it is BIBO $(X^m \times Y^p, X^m \times Y^p)$ stable and the left inverse $\Gamma_L^{-1} : X^m \times Y^p \to D(\Gamma; X^m \times Y^p)$ of Γ is bounded on $X^m \times Y^p$, so that there exists a nonnegative real number K such that

$$\|\Gamma_L^{-1} x\|_{X^m \times Y^p} \leq K \|x\|_{X^m \times Y^p}, \qquad x \in X^m \times Y^p.$$

Recall that the left inverse Γ_L^{-1} satisfies

$$\Gamma_L^{-1} \Gamma x = x, \qquad x \in D(\Gamma; X^m \times Y^p).$$

However, as here $R(\Gamma; X^m \times Y^p) = X^m \times Y^p$ due to stability, it is seen that

$$\Gamma \Gamma_L^{-1} z = z, \qquad z \in X^m \times Y^p.$$

Theorem 5.3: Let $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ be a closed linear feedback system, where X^m and Y^p are Banach spaces. Let the feedback system be BIBO $(X^m \times Y^p, X^m \times Y^p)$ stable and let $D(\Gamma; X^m \times Y^p)$ be a complete subspace (i.e., a Banach space) of $X^m \times Y^p$. Then, the feedback system is $(X^m \times Y^p, X^m \times Y^p)$ gain stable.

Proof: Note that by assumption Γ is a closed operator from the Banach space $D(\Gamma; X^m \times Y^p)$ into the Banach space $X^m \times Y^p$. Hence, by the closed-graph theorem, Γ is a bounded operator. By BIBO stability of the feedback system, it is seen that $R(\Gamma; X^m \times Y^p) = X^m \times Y^p$ and that $N(\Gamma; X^m \times Y^p) = \{0\}$. That is, Γ is a one-to-one mapping from a Banach space onto a Banach space. Hence, the left inverse $\Gamma_L^{-1} : X^m \times Y^p \to D(\Gamma; X^m \times Y^p)$ of Γ exists. It follows by an application of the open mapping theorem [6, pp. 141–143] that Γ_L^{-1} is a bounded operator on the Banach space $X^m \times Y^p$, and so the feedback system is $(X^m \times Y^p, X^m \times Y^p)$ gain stable. This completes the proof. ■

Again we would like to emphasize that this result does not require that $D(\Gamma; X^m \times Y^p) = X^m \times Y^p$. The next example demonstrates among other things that the conditions given in the above result need not be necessary for gain stability.

Example 3: Consider the linear plant $(A, B, \ell_2(\mathbb{Z}), \ell_2(\mathbb{Z}))$ controlled by the linear controller $(H, F, \ell_2(\mathbb{Z}), \ell_2(\mathbb{Z}))$, where $A = I$

$$B = S \sum_{k \geq 0} S^k$$

$H = I$, $F = f$, and f is a real number.

So, here a first-order, marginally stable (using the standard control engineering terminology), convolution system is controlled by a proportional controller. Let us study the feedback system $\Gamma : \ell_2(\mathbb{Z})^2 \to \ell_2(\mathbb{Z})^2$. Note that $D(\Gamma; \ell_2(\mathbb{Z})^2)$ is a proper subspace of $\ell_2(\mathbb{Z})^2$ for any f. Furthermore, the feedback system is closed and $D(\Gamma; \ell_2(\mathbb{Z})^2)$ is not a complete space.

Take first $f = 0$. Then it is easy to check that $R(\Gamma; \ell_2(\mathbb{Z})^2)$ is a proper subspace of $\ell_2(\mathbb{Z})^2$. Hence, the feedback system is not weakly stable for $f = 0$. (So, this agrees with the fact that the plant $(A, B, \ell_2(\mathbb{Z}), \ell_2(\mathbb{Z}))$ is not weakly stable.)

Take now $f \neq 0$, $f \neq -2$. Direct computation gives then that

$$R(\Gamma; \ell_2(\mathbb{Z})^2) = \ell_2(\mathbb{Z})^2 \quad N(\Gamma, \ell_2(\mathbb{Z})^2) = \{0\}$$

and so the feedback system is BIBO $(\ell_2(\mathbb{Z})^2, \ell_2(\mathbb{Z})^2)$ stable by proposition 5.1. This implies that the left inverse $\Gamma_L^{-1} : \ell_2(\mathbb{Z})^2 \to D(\Gamma; \ell_2(\mathbb{Z})^2)$ of Γ exists. In fact, we compute that

$$\Gamma_L^{-1} = (I - (1+f)S)^{-1} \begin{pmatrix} I - S & (I - S)f \\ S & I - S \end{pmatrix}.$$

This is a bounded operator. Hence, the feedback system is $(\ell_2(\mathbb{Z})^2, \ell_2(\mathbb{Z})^2)$ gain stable for $f \neq 0$, $f \neq -2$, by Proposition 5.2. (Note that Γ_L^{-1} is causal for $-2 < f < 0$.)

Let us now introduce three notions of stabilizability.

Let (A, B, Y^p, X^m) be the plant in the linear feedback system $(\Gamma, I, X^m \times, Y^p, X^m \times Y^p)$, where X^m and Y^p are linear spaces. The plant is said to be $X^m \times Y^p$ *weakly stabilizable* if there exists a linear controller (H, F, X^m, Y^p), which makes the feedback system weakly $(X^m \times Y^p, X^m \times Y^p)$ stable.

Let now X^m and Y^p be linear normed spaces. The plant (A, B, Y^p, X^m) is said to be $X^m \times Y^p$ *gain stabilizable*, respectively, $X^m \times Y^p$ *BIBO stabilizable*, if there exists a linear controller, which makes the linear feedback system $(X^m \times Y^p, X^m \times Y^p)$ gain stable, respectively, BIBO $(X^m \times Y^p, X^m \times Y^p)$ stable.

Clearly, gain stabilizability implies weak (and BIBO) stabilizability, but the reverse implication does not hold in general.

The following result gives a necessary and sufficient condition for weak stabilizability. (It follows directly from Theorem 5.2.)

Proposition 5.3: Let the linear plant (A, B, Y^p, X^m) be given in the linear feedback system $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$, where X^m and Y^p are linear spaces. The plant (A, B, Y^p, X^m) is weakly stabilizable if and only if

$$R(\Gamma; X^m \times Y^p) = X^m \times Y^p$$

holds for some linear controller (H, F, X^m, Y^p).

By Theorem 5.1, a necessary condition for gain stabilizability is that the plant and the controller are closed systems.

By the results in [7], [17], and [19], it is impossible to gain stabilize the LTI plant (A, B, Y^p, X^m) when either A or B is a strictly unstable, finite-dimensional LTI convolution operator and X^m and Y^p are $\ell_q(\mathbb{Z})$ spaces, $1 \leq q \leq \infty$.

B. Seminorms and Equivalence Classes

It is often natural to start with a signal space which is only a linear seminormed space rather than a linear normed space. A standard procedure is then to consider equivalence classes of signals such that an equivalence class corresponds to all signals whose differences are of zero seminorm. This allows one to obtain a linear normed space of signals. We will illustrate this procedure as it reveals some fundamental problems associated with the use of unbounded operators.

Consider the linear space, X_{ls}^n, of bounded signals $x \in \ell_\infty(\mathbb{Z})^n$ equipped with the seminorm

$$\|x\|_{ls} \equiv \limsup_{M,N \to \infty} \left(\frac{1}{M + N + 1} \sum_{t=-M}^{N} |x(t)|_2^2 \right)^{1/2}$$

where $|\cdot|_2$ denotes the usual Euclidean length of a vector in $\mathbb{R}^n$. This space allows one to generalize H_∞ optimization into a setup that contains, as a nonlinear subspace, those bounded signals that allow Wiener's generalized harmonic analysis (GHA) [27], [22], [23]. (Such bounded GHA signals are popular in studies of robust control and system identification [30], [16].)

Let $E(x)^n$ denote the equivalence class associated with $x \in X_{ls}^n$, i.e., the set all signals z in X_{ls}^n such that $\|z-x\|_{ls}=0$. Let X_E^n denote the linear normed space of equivalence classes $E(x)^n$ of signals x in X_{ls}^n. Let the quadruple (A,B,X_{ls}^p,X_{ls}^m) be a linear system. If for any $x \in X_{ls}^p$ and any $v \in X_{ls}^m$, it holds that $AE(x)^p \subset E(z)^p$ and $BE(v)^m \subset E(w)^p$ for some $z \in X_{ls}^p$ and $w \in X_{ls}^p$, then we can interpret A and B as linear operators on X_E^p and on X_E^m, respectively. (Clearly, then $AE(0)^p \subset E(0)^p$ and $BE(0)^m \subset E(0)^p$.)

We shall say that the (causal) convolution operator $P: D(P;X_{ls}^n) \to X_{ls}^q$

$$(Px)(t) = \sum_{k\geq 0} G(k)x(t-k),\ t \in \mathbb{Z}$$

such that

$$\limsup_{k\to\infty} \max_{i,j} |G_{ij}(k)| > 0$$

has a unit impulse response bounded away from zero. Here, $G(k)$ is a real $q \times n$ matrix for $k \geq 0$. Note that P is a linear operator.

Theorem 5.4: Let the linear system (A,B,X_{ls}^p,X_{ls}^m) be given, where A and B are (causal) convolution operators. Let either A or B (or both) have a unit impulse response bounded away from zero. Then, the quadruple (A,B,X_E^p,X_E^m) is not a linear system.

Proof: It suffices to consider the case that the unit impulse response, $\{G(k) \in \mathbb{R}^{p\times m}\}_{k\geq 0}$, of B is bounded away from zero. It then follows that there exists a positive number $\delta > 0$ and a sequence of positive integers $\{k_q\}_{q\geq 1}$ such that $k_{q+1} > k_q$, $q \geq 1$

$$\lim_{q\to\infty} \frac{q}{k_q} = 0 \tag{11}$$

and

$$\inf_{q\geq 1} \max_{i,j} |G_{ij}(k_q)| \geq \delta. \tag{12}$$

Take $u(t)$ such that $u(t)=0$ for $t \geq 0$ and $u(-k)=0$ for any $k>0$ such that k is not equal to k_i for any $i \geq 1$. Then

$$y(0) = \sum_{q\geq 1} G(k_q)u(-k_q).$$

A necessary condition for the existence of $y(0)$ is that

$$\lim_{q\to\infty} G(k_q)u(-k_q) = 0. \tag{13}$$

However, by (12) there exists sequences of indexes $\{i_q \in \{1,\ldots,p\}\}_{q\geq 1}$ and $\{j_q \in \{1,\ldots,m\}\}_{q\geq 1}$ such that $|G_{i_q,j_q}(k_q)| \geq \delta$ for $q \geq 1$. Put $u_{j_q}(-k_q)=1$ and set other components of $u(-k_q)$ equal to zero for $q \geq 1$. Clearly, $u \in X_{ls}^m$ and the condition (13) does not hold. It is also easy to check that $\|u\|_{ls}=0$ by (11). Hence, $BE(0)^m$ is not a subset of $E(0)^p$. This completes the proof. ■

This means that when A or B is of the type of unbounded linear operator on X_{ls} as stated in this theorem, the set of trajectories $T(A,B,X_E^p,X_E^m)$ cannot be defined in any reasonable manner.

VI. Bounded Plant and Controller Operators

By our earlier discussions, everything simplifies significantly if one concentrates on bounded operators only. Hence, we will here be interested in the case that $\Gamma: X^m \times Y^p \to X^m \times Y^p$ is a bounded operator on $X^m \times Y^p$. (X^m and Y^p are of course linear normed spaces.) Thus

$$D(\Gamma; X^m \times Y^p) = X^m \times Y^p.$$

So, A, B, H, and F are all restricted to be bounded operators on their respective spaces. We now want to discuss the conditions these operators must satisfy for the linear feedback system $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ to be $(X^m \times Y^p, X^m \times Y^p)$ gain stable.

Proposition 6.1: Let $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ be a linear feedback system, where X^m and Y^p are linear normed spaces. Furthermore, let $\Gamma: X^m \times Y^p \to X^m \times Y^p$ be a bounded operator. If $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ is $X^m \times Y^p, X^m \times Y^p)$ gain stable, then $N(\Gamma; X^m \times Y^p) = \{0\}$, and in the linear plant (A,B,Y^p,X^m) and in the linear controller (H,F,X^m,Y^p), the operators A and B, respectively, the operators H and F, are left-coprime.

Proof: Gain stability implies here, by Proposition 5.2, that Γ must have an inverse Γ^{-1} such that

$$\Gamma^{-1}\Gamma x = \Gamma\Gamma^{-1}x = x, \qquad x \in X^m \times Y^p.$$

Hence, $N(\Gamma; X^m \times Y^p) = \{0\}$. Furthermore, Γ^{-1} must be bounded. Note that the plant operators A, B and the controller operators H, F, are all bounded operators. Let us decompose this inverse as

$$\Gamma^{-1} = \begin{pmatrix} X & -V \\ -Y & W \end{pmatrix}$$

where $X: X^m \to X^m$, $Y: X^m \to Y^p$, $V: Y^p \to X^m$, and $W: Y^p \to Y^p$ are bounded linear operators. Therefore

$$\begin{aligned} HX + FY &= I \\ BV + AW &= I. \end{aligned}$$

That is, A and B are left-coprime and also H and F are left-coprime. This completes the proof. ■

Example 4: The following example is based on [21]. We work with the signal space $\ell_\infty(\mathbb{Z})$, and consider the system

$$\begin{aligned} u &= ay + w_1 \\ Ay &= SAu + w_2 \end{aligned}$$

where S is the shift, $(Sx)(t) = x(t-1)$, and A is a stable convolution operator $(Ax)(t) = \sum_{k=0}^{\infty} a^k x(t-k)$, with $0 < |a| < 1$. Thus, $B = SA$, $H = I$, $F = -aI$, in the notation of (8). The operator Γ is given by

$$\Gamma = \begin{pmatrix} H & -F \\ -B & A \end{pmatrix} = \begin{pmatrix} I & -aI \\ -SA & A \end{pmatrix}$$

and is invertible with inverse

$$\Gamma^{-1} = \begin{pmatrix} A & aI \\ SA & I \end{pmatrix}.$$

Note that, perhaps surprisingly, the operators A and SA are left-coprime, indeed $A - aSA = I$.

A closely related version of this example was also considered in [21], namely

$$u = ay + w_1$$
$$y = Su + w_2.$$

Although, as explained in [21], this system has "spurious" solutions that are not in $\ell_\infty(\mathbb{Z})$, we now see that the operator Γ is invertible, indeed

$$\Gamma = \begin{pmatrix} I & -aI \\ -S & I \end{pmatrix}, \qquad \text{and}$$
$$\Gamma^{-1} = \begin{pmatrix} (I-aS)^{-1} & a(I-aS)^{-1} \\ S(I-aS)^{-1} & (I-aS)^{-1} \end{pmatrix}.$$

Theorem 6.1: Let (A, B, Y^p, X^m) be a linear system, where X^m and Y^p are normed linear spaces. Then, there is a linear controller (H, F, X^m, Y^p) where H, F and H^{-1} are bounded operators, gain stabilizing the feedback system $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$, with $\Gamma = \begin{pmatrix} H & -F \\ -B & A \end{pmatrix}$, if and only if there exist bounded operators $C : Y^p \to Y^p$, $D : Y^p \to X^m$ such that $C^{-1} : Y^p \to Y^p$ is bounded and

$$(AC + BD)x = x, \qquad x \in Y^p.$$

Proof: Let us start with the necessity part. Thus, let the feedback system be gain stabilized by some linear controller (H, F, X^m, Y^p) such that H, F and H^{-1} are bounded operators. Then it can be verified by direct computation that

$$\Gamma^{-1} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

where $\theta_{11} = H^{-1} + H^{-1}F(A - BH^{-1}F)^{-1}BH^{-1}$, $\theta_{12} = H^{-1}F(A - BH^{-1}F)^{-1}$, $\theta_{21} = (A - BH^{-1}F)^{-1}BH^{-1}$, and $\theta_{22} = (A - BH^{-1}F)^{-1}$.

Now, as Γ^{-1} is a bounded operator, it is seen that this is equivalent to $(A - BH^{-1}F)^{-1} : Y^p \to Y^p$ being a bounded operator. (As here by assumption A, B, H, F, and H^{-1} are all bounded operators.) However, then

$$(A(A - BH^{-1}F)^{-1} + B[-H^{-1}F(A - BH^{-1}F)^{-1}])x = x$$

for all $x \in Y^p$. That is, we can take

$$C = (A - BH^{-1}F)^{-1} \quad D = -H^{-1}F(A - BH^{-1}F)^{-1}$$

and $C^{-1} = A - BH^{-1}F$ is a bounded operator. This completes the proof of the necessity part.

Sufficiency is proved as follows. Take $H = I$ and $F = -DC^{-1}$. Then

$$A - BH^{-1}F = A + BDC^{-1} = (AC + BD)C^{-1} = C^{-1}.$$

Hence, the above formula for Γ^{-1} gives that

$$\Gamma^{-1} = \begin{pmatrix} I - DB & -D \\ CB & C \end{pmatrix}$$

so indeed the chosen controller gain stabilizes the feedback system. This completes the proof. ■

Note that this result deals with gain stabilization by stable controllers, as the controllers in the result are obviously gain stable, when we regard the plant output y and the external signals w as inputs to the controller.

It is also easy to verify that if the linear controller (H, F, X^m, Y^p) gain stabilizes the linear feedback system $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$, then so does the controller $(H + QB, F + QA, X^m, Y^p)$, where $Q : Y^p \to X^m$ is an arbitrary bounded operator. (This holds also when H^{-1} is not bounded.)

VII. Closure Approach

An important special case of the linear feedback system in (10) corresponds to

$$\Gamma = \begin{pmatrix} I & -F \\ -B & I \end{pmatrix}.$$

This is, in fact, the standard case in the I/O approach to robust control for signal spaces defined on $\mathbb{N}$.

However, it is known [7], [17], [19] that this standard feedback system description leads, for $\ell_q(\mathbb{Z})$, $1 \le q \le \infty$, signal spaces, to the conclusion that unstable, finite-dimensional, LTI convolution operator systems are not gain stabilizable, in stark contrast to the situation for the corresponding signal spaces defined on $\mathbb{N}$.

Furthermore, [7], [17], and [19] discuss replacing an unstable LTI convolution operator B by its operator closure, when the latter exists. Jacob [9], [11] studies this idea in a more systematic manner for the $\ell_2(\mathbb{Z})$ signal space (also the MIMO case).

Let $\overline{F} : D(\overline{F}; Y^p)$ and $\overline{B} : D(\overline{B}; X^m)$ denote the closure of F and B, respectively. Here, X^m and Y^p are linear normed spaces. We will denote the resulting linear feedback system as $(\overline{\Gamma}, I, X^m \times Y^p, X^m \times Y^p)$. This feedback system is called the closure of $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$. It involves the closure of the operator Γ.

Remark 7.1: Let $(\Gamma, I, X^m \times Y^p, X^m \times Y^p)$ be a linear feedback system and let its closure exist. It is easy to see that if $\Gamma : D(B; X^m) \times D(F, Y^p) \to X^m \times Y^p$ is a bounded operator on its domain of definition, then its closure is also bounded on the domain of definition of the closure. Finally, if Γ is not bounded, then neither is $\overline{\Gamma}$.

It is known that for finite-dimensional LTI convolution operators with a bounded closure on $\ell_q(\mathbb{Z})$, $1 \le q \le \infty$, the domain of definition of the closure is the whole underlying linear normed space. The difficult case then corresponds to $\overline{\Gamma}$ not being bounded. This is for example the case when Γ is closed but not bounded (Γ is then its own unbounded closure).

Theorem 7.1: Let the linear convolution operator plant (I, B, Y^p, X^p) be given by

$$y(t) = (Bu)(t) = \sum_{k \ge 0} G^k u(t-k)$$

where X^p and Y^p are linear spaces, and G is a real $p \times p$ matrix with p independent eigenvectors, such that $|\lambda| \ge 1$ for all eigenvalues λ of G. Let X^p or Y^p contain $\ell_\infty(\mathbb{Z})^p$ as a linear subspace.

Then there does not exist a linear controller (H, F, X^p, Y^p) of the form

$$H = \sum_{k=-q}^{r} H_k S^k$$
$$F = \sum_{k=-q}^{r} F_k S^k$$

stabilizing the plant in the weak sense, where q and r are nonnegative integers, and $\{H_k\}_{k=-q}^{r}$ and $\{F_k\}_{k=-q}^{r}$ are sequences of real square matrices of size $p \times p$ such that both $\{H_k\}$ and $\{F_k\}$ contain a nonzero matrix.

Proof: The conditions on G imply that there is a nonsingular matrix M such that $G^n = M^{-1}\mathrm{diag}(\lambda_1^n, \ldots, \lambda_p^n)M$ for all $n \in \mathbb{Z}$, where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of G counted according to multiplicity. Thus, there is a constant $C > 0$ such that $|G^n x| \leq C|x|$ for all $n \leq 0$ and any $x \in \mathbb{R}^p$. (Recall that $|\cdot|$ denotes the usual Euclidean length of a vector.)

Assume that the linear controller (H, F, X^p, Y^p) of the aforementioned form weakly stabilizes the plant. Then, the feedback system satisfies

$$y(t) = \sum_{k \geq 0} G^k u(t-k) + w_2(t)$$
$$\sum_{k=-q}^{r} H_k u(t-k) = \sum_{k=-q}^{r} F_k y(t-k) + w_1(t)$$

and, hence, $G^k u(-k) \to 0$ as $k \to \infty$, which implies that $u(t) \to 0$ as $t \to -\infty$. Moreover, $y(t) - w_2(t) = G^t \sum_{r=-\infty}^{t} G^{-r} u(r) \to 0$ as $t \to -\infty$. Therefore

$$\lim_{t \to -\infty} \left[\left(\sum_{k=-q}^{r} F_k w_2(t-k) \right) + w_1(t) \right] = 0.$$

Hence, taking $w_2 = 0$, we see that

$$\lim_{t \to -\infty} w_1(t) = 0$$

which is not possible for an arbitrary signal w_1 in X^p if X^p contains $\ell_\infty(\mathbb{Z})^p$ as a linear subspace. Finally, if Y^p contains $\ell_\infty(\mathbb{Z})^p$ as a linear subspace, then it is clearly impossible to satisfy the aforementioned condition for an arbitrary w_2. Therefore, the range of the operator Γ in the feedback system is a proper subspace of $X^p \times Y^p$ only. This contradicts the assumption that (H, F, X^p, Y^p) stabilizes the plant in a weak sense, and completes the proof. ∎

A similar result holds for anti-causal systems $y(t) = \sum_{k \geq 0} G^k u(t+k)$, with the eigenvalues of G lying in the closed unit disc. The proof is similar, and we omit it.

Example 5: A simple modification of this is the following LTI feedback system in an $\ell_\infty(\mathbb{Z})$ signal setting:

$$y(t) = \sum_{k \geq 0} u(t-k-1) + w_2(t) \tag{14}$$
$$\sum_{k=-q}^{r} h_k u(t+k) = \sum_{k=-q}^{r} f_k y(t+k) + w_1(t) \tag{15}$$

where q, r are nonnegative integers and not all h_k, respectively, f_k, are zero. Similar arguments show that $R(\Gamma; \ell_\infty(\mathbb{Z})^2)$ is a proper subspace of $\ell_\infty(\mathbb{Z})^2$, and the feedback system $(\Gamma, I, \ell_\infty(\mathbb{Z})^2, \ell_\infty(\mathbb{Z})^2)$ cannot be made (weakly) stable by any finite-dimensional controller of the form in (15). This is in stark contrast to the situation for the signal space $\ell_\infty(\mathbb{N})$.

Note that as the plant is here a closed system [19], this conclusion does not change by trying to replace the plant with its closure! We see also that it does not matter whether the controller is causal or not, nor whether the feedback system is causal or not. However, it is quite easy to handle the associated feedback system (in an $\ell_\infty(\mathbb{Z})$ setting)

$$y(t) - y(t-1) = u(t-1) + w_2(t)$$
$$\sum_{k=-q}^{r} h_k u(t+k) = \sum_{k=-q}^{r} f_k y(t+k) + w_1(t).$$

There are no unbounded convolution operators here and for example the proportional controller $u(t) = -y(t)$ not only weakly stabilizes the plant, but also $\ell_\infty(\mathbb{Z}) \times \ell_\infty(\mathbb{Z})$ gain stabilizes the feedback system (and the left inverse Γ_L^{-1} of Γ is then a causal operator).

It is interesting to note that the usefulness of the closure approach in the $\ell_2(\mathbb{Z})$ setting, depends crucially on the properties of the right shift on $\ell_2(\mathbb{Z})$. If one is interested in a control experiment which has taken place in the past, then it is natural to use $\mathbb{Z}_-$ as the time axis.

Theorem 7.2: Let $1 \leq q \leq \infty$ and let $B : D(B; \ell_q(\mathbb{Z}_-)^p) \to \ell_q(\mathbb{Z}_-)^p$ be the convolution operator defined by

$$(Bu)(t) = \sum_{k \geq 0} G^k u(t-k), \qquad t \in \mathbb{Z}_-$$

where G is a real symmetric matrix of size $p \times p$ with at least one eigenvalue λ satisfying $|\lambda| > 1$. Then, $B : D(B; \ell_q(\mathbb{Z}_-)^p) \to \ell_q(\mathbb{Z}_-)^p$ does not have an operator closure, i.e., B is not a closable operator.

Proof: It follows by the assumptions on G that there exists a nonzero real vector $x \in \mathbb{R}^p$ such that

$$Gx = \lambda x.$$

We utilize the fact that [6, p. 145] B is closable if and only if $u_i \in D(B; \ell_q(\mathbb{Z}_-)^p)$, $u_i \to 0$, $Bu_i \to y \in \ell_q(\mathbb{Z}_-)^p$, imply that $y = 0$. So we put $u_i(-i) = \lambda^{-i} x$ and $u_i(t) = 0$ for $t \neq -i \leq -1$. (Here, $i \geq 1$.) Hence $u_i \in \ell_q(\mathbb{Z}_-)^p$. Denote $y_i = Bu_i$, $i \geq 1$. Now

$$y_i(t) = \lambda^t x, \qquad -i \leq t \leq -1$$

and $y_i(t) = 0$ for $t < -i$. Clearly, $y_i \in \ell_q(\mathbb{Z}_-)^p$ and so $u_i \in D(B; \ell_q(\mathbb{Z}_-)^p)$ for any $i \geq 1$. But $y_i \to y \equiv \{\lambda^t x\}_{t \leq -1} \in \ell_q(\mathbb{Z}_-)^p$ as $|\lambda| > 1$. So, $u_i \to 0$ while $y_i = Bu_i \to y \neq 0$ and hence B does not have a closure. ∎

This means that we cannot extend the operator B in any natural way to an operator B_E such that at least the so extended linear system $(I, B_E, \ell_q(\mathbb{Z}_-)^p, \ell_q(\mathbb{Z}_-)^p)$ would be $\ell_q(\mathbb{Z}_-)^p \times \ell_q(\mathbb{Z}_-)^p$ gain stabilizable. This result clearly generalizes to more general strictly unstable, finite-dimensional convolution operators, see also [19]. Note that it follows from the previous result, by a simple time reversal argument, that the analogous (anticausal) convolution operator

$$(Bu)(t) = \sum_{k \geq 0} G^k u(t+k)$$

is not closable on $\ell_q(\mathbb{Z}_+)^p$ (nor on $\ell_q(\mathbb{N})^p$), when the matrix G satisfies the same assumptions as in the previous result. Thus, there are simple, time-invariant, linear convolution systems defined on $\mathbb{N}$, which are neither stabilizable nor can be made stabilizable by considering operator closures.

VIII. Conclusion

We have studied I/O stabilization of linear systems of the form $Ay = Bu$ defined over rather general signal spaces. In fact, most of our linear system results can be interpreted as general results in linear operator theory, and they may, therefore, have applications beyond control and systems. An advantage of such a general setting is that, it has allowed us to show that the familiar I/O model $y = Pu$ even when generalized via operator closures, does not provide a satisfactory starting point to problems of I/O stabilization over persistent signal spaces defined on $\mathbb{Z}$. Furthermore, our arguments show that this fact has nothing to do with the causality notion, but rather it is due to the definition of linear convolution operators.

It is an important fact that the linear models $Ay = Bu$ and $y = Pu$ have drastically different degrees of usefulness in problems of I/O stabilization. It is clear that the presented general setup can be applied to various specific persistent signal spaces and this is a promising line of research. Note that we have avoided rather systematically the use of transform domain methods as these have been used (during the past 50 years or so) rather carelessly to study problems of I/O stabilization on the full time axis [18]. The first rigorous application of transform domain methods to problems of I/O stabilization, in the $\ell_2(\mathbb{Z})$ setting, seems to be that of Jacob [9]. It is hoped that the present work contributes toward clarifying an important problem area in control and systems theory.

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