Article:

Baratchart, L., Leblond, J., Partington, J.R. et al. (1 more author) (1997) Robust identification from band-limited data. IEEE Transactions on Automatic Control, 42 (9). pp. 1318-1325. ISSN 0018-9286

https://doi.org/10.1109/9.623101

Robust Identification from Band-Limited Data

L. Baratchart, J. Leblond, J. R. Partington, and N. Torkhani

***Abstract*— Consider the problem of identifying a scalar bounded-input/bounded-output stable transfer function from pointwise measurements at frequencies within a bandwidth. We propose an algorithm which consists of building a sequence of maps from data to models converging uniformly to the transfer function on the bandwidth when the number of measurements goes to infinity, the noise level to zero, and asymptotically meeting some gauge constraint outside. Error bounds are derived, and the procedure is illustrated by numerical experiments.**

***Index Terms*—Approximate modeling, linear systems, Nehari extension, robust identification.**

I. Introduction

This paper is concerned with the problem of harmonic identification, that is, of recovering a single-input/single-output (SISO) and bounded-input/bounded-output (BIBO)-stable transfer function from a family of experimental pointwise values on the imaginary axis. Such data are common in engineering practice as they may be obtained from asymptotic outputs associated to sine inputs or from numerical simulations of distributed parameter systems (see [6] and [17], for example). In [9], a setting to approach this issue was proposed in which the error in measurements is handled in a deterministic fashion, and the identification procedure consists of a map from finite sets of data to (stable) transfer functions that converge uniformly to the "true" transfer function when the noise goes uniformly to zero and the number of data goes to infinity.

In the present work, we shall consider the (realistic) case where the experiments are only available in some range of frequencies corresponding to the bandwidth of the system. In this case, none of the algorithms that were proposed [8], [9], [11], [12] converges, and we shall see that the setting itself has to be modified. We shall adapt to the new situation by requiring the map from data to models to converge uniformly in the bandwidth while meeting some norm constraints at remaining frequencies.

Our working space will be the unit disc rather than the half-plane, the two frameworks being equivalent by means of a Möbius transform. Since the transfer function of a BIBO-stable system is continuous on the imaginary axis, including at infinity, a model for us has to be found in the disc algebra.

Let H_∞ be the familiar Hardy space of bounded analytic functions in the disc and $A(\mathbb{D})$ (the disc algebra) be the subspace of such functions that are continuous on the closed disc. On a couple of occasions in this section, we shall also use the symbol H_∞ to mean the Hardy space of the right half-plane $\Pi_+ = \{s \in \mathbb{C}; \mathrm{Re}\, s > 0\}$, but the context will always keep the meaning clear. The algebra $A(\Pi_+)$ of the right half-plane will then consist of those functions in H_∞ of this half-plane that extend continuously to the imaginary axis, including at infinity. The symbol $C(X)$ stands for the space of complex continuous functions on X endowed with the sup norm. Spaces X used in this paper will be arcs on the unit circle or intervals on the imaginary axis.

Manuscript received February 25, 1995; revised July 25, 1996.

L. Baratchart, J. Leblond, and N. Torkhani are with INRIA, BP 93, 06902 Sophia-Antipolis Cedex, France (e-mail: leblond@sophia.inria.fr).

J. R. Partington is with the School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K.

Publisher Item Identifier S 0018-9286(97)05053-8.

In the problem of robust H_∞ identification of functions in the disc algebra as stated in the above-mentioned references, one is given experimental data as complex numbers $(a_k)_{k=-N}^N = (f(z_k) + \eta_k)_{k=-N}^N$, where f is an unknown function in the disc algebra $A(\mathbb{D})$, and $z_{-N}, \cdots, z_N$ are points on the unit circle $\mathbb{T}$, while $(\eta_{-N}, \cdots, \eta_N)$ is some unknown but bounded noise sequence which can be due to nonlinear effects or measurement errors, for example. From the (a_k), one wishes to construct an approximation f_N such that in the limit, as the noise level tends to zero and the number of observations tends to infinity, one has convergence in the H_∞ norm, that is

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \le \epsilon} \|f_N - f\|_\infty = 0, \qquad \text{for all } f \in A(\mathbb{D}). \tag{1}$$

This convergence requirement corresponds to a continuity property of the model f_N with respect to the number of measurements and the noise level, as explained in Remark 1 below. To approach this problem, a two-stage algorithm has been found useful [8], [9], [11], [12]. To proceed, one first computes a trigonometric polynomial p_N which interpolates the given data (but is not in $A(\mathbb{D})$), and one applies then the (nonlinear) Nehari extension [19] to obtain the best approximation to p_N by a function f_N bounded and analytic in the disc (it will in fact be rational).

When the points (z_k) are equally spaced on the circle, p_N can be obtained using the classical Jackson or de la Vallée–Poussin trigonometric polynomials [11], [20]. When the points are not equally spaced, the problem becomes computationally harder, but one can design a transformation from the given points into equally spaced ones and proceed as before (see, e.g., [13]) or else rely on a more general principle of linear programming [14].

In the last reference, the overall error of the identification procedure can be expressed as a sum of two terms, one corresponding to the noise and the other to the maximum gap between the interpolation points. One such theoretical bound is $4\epsilon + 5\mathrm{dist}\,(f, P_p)$, where $\epsilon \ge \|\eta\|_\infty$ and P_p is the space of polynomials of degree p and the maximum gap is less than $1/p$. Thus, the error goes to zero as $\epsilon \to 0$, provided the maximum gap between the measurement points (z_k) goes to zero.

However, in practical applications, one may not be able to measure f at all points on the circle. For example, in the identification of continuous-time, linear, time-invariant, and BIBO-stable control systems by frequency response measurements, which can be reduced to the above problem by means of the Möbius transformation $s = (1+z)/(1-z)$ and $G(s) = f(z)$ where G is the transfer-function, one is not able to measure $G(i\omega)$ for arbitrarily high values of ω. Moreover, one is not normally concerned about modeling G arbitrarily well at high frequencies. In some cases, one may even prefer to have a linear model valid for a restricted set of frequencies, since the linearity assumption would hold only locally with respect to the frequency. In these circumstances, no algorithm can guarantee uniform convergence over the whole imaginary axis without further *a priori* knowledge on G [14]. It is nevertheless natural to ask whether the unknown function G can be recovered in a robust fashion at least in the range of frequencies where measurements are available, through a model which is still under control at the remaining frequencies.

Let us stay with the half-plane for a while and discuss a bit further the situation where measurements are only available in the bandwidth, say Ω. In this connection, some work on band-limited identification has been published by Bai and Raman [1] in which they essentially approximate separately the real and imaginary parts of the transfer function by polynomials over the frequency interval Ω, plugging in some arbitrary polynomial weight of sufficiently high degree to become the denominator off the approximant so as to end up with a stable and proper model. In doing so, they are not concerned about controlling the behavior of the set Ω and, since their scheme is (real) linear, it is a routine matter to check, by the same arguments as in [12], that their sequence of estimates is unbounded outside Ω for almost every noise in l^∞ (i.e., for every noise sequence in a set of second category in the sense of Baire). In fact, we claim that *any* H^∞ band-limited identification scheme *must* incorporate some constraints that impinge on the behavior of the transfer-function outside Ω. This can be inferred from two facts.

1) In the space $C(\Omega)$, the subspace $A(\Pi_+)|_\Omega$ obtained by restricting $A(\Pi_+)$ to Ω is dense.
2) If $G \in C(\Omega)$ does not belong to $A(\Pi_+)|_\Omega$, any sequence of functions in $A(\Pi_+)$ (or even in H_∞) that converges to G on Ω is unbounded in H_∞.

Fact 1) is an easy consequence of Runge's theorem, while Fact 2) follows from the weak–* compactness of balls in H_∞, and we refer the reader to [2] for a proof which is phrased on the disc rather than the half-plane (and also works in $L^p(\Omega)$ for $1 \le p \le \infty$). Altogether, 1) and 2) indicate that no matter the data, we can always construct an excellent model on Ω at the cost of nearly destabilizing it at the remaining frequencies, a problem which is familiar to identification practitioners. At this point, it is perhaps interesting to draw a parallel with the seemingly different process of stochastic parametric identification; there, the constraints on the model are often imposed in terms of bounded rational degree, and the analog of the above-mentioned phenomenon would be that allowing the degree to grow too large destabilizes the model because it starts fitting the noise.

It might be argued that all one needs to do is to prescribe plausible values for G outside the bandwidth and to use standard robust identification techniques. However, this approach would prevent us from recovering G asymptotically on Ω. Indeed, Fact 1) is not applicable to the whole axis, and we should incur an irreducible error at each frequency.

In this paper, we choose to constrain the behavior of the model to lie within some tolerance of a prescribed function at nonmeasured frequencies. Thus, back to the disc, we propose the following modified setup. We suppose that $0 < a < \pi$ and consider $I = \{e^{i\theta}: a \le \theta \le 2\pi - a\}$, which is a proper closed symmetric subarc of the unit circle. We define J to be the closure of the complement of I, i.e., $J = \{e^{i\theta}: -a \le \theta \le a\}$. Also, we define the norm

$$\|g\|_{I,\infty} = \text{esssup } \{|g(e^{i\theta})|: e^{i\theta} \in I\} \tag{2}$$

for g in $L^\infty(I)$ and similarly for J.

We provide ourselves with measurements $a_k = f(z_k) + \eta_k$, with $k = -N, \cdots, N$, where the z_k all lie within I with $z_{-N} = e^{-ia}$ and $z_N = e^{ia}$. We shall assume that the function f satisfies an *a priori* estimate of the form

$$|f(z) - h(z)| \le r(z), \qquad \text{for all } z \in J \tag{3}$$

for some functions h and r belonging to $C(J)$, with r a nonnegative gauge function that vanishes at the endpoints of J.

This may seem absurd since f cannot be known exactly and therefore h cannot be determined to within a precision less than ϵ. However, there is actually no contradiction since, in the algorithm, *the values of h, just like those of f, are assumed to be available only up to an uncertainty of* ϵ.

Our aim is to find an approximate model f_N of f on I converging robustly on I, namely

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \le \epsilon} \|f_N - f\|_{I,\infty} = 0.$$

Moreover, we also require that this approximation procedure asymptotically meets the gauge constraint on J

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \sup_{\|\eta\|_\infty \le \epsilon} \left\{ \sup_{z\in J} |f_N(z) - h(z)| - r(z) \right\} \le 0.$$

However, from our incomplete set of data, we cannot constrain the model f_N to converge robustly to f on the whole circle; on J, we will only get that f_N converges weakly–* to f

$$\lim_{N\to\infty} \int_J f_N u \, d\theta = \int_J f u \, d\theta, \qquad \text{for all } u \in L^1(J).$$

Note also that this scheme is not untuned in the terminology of [9], and this is natural since we emphasized the necessity of constraining the model on J in one way or another. Here, we need a pointwise bound of the form (3) on J.

A few comments on the role of r are perhaps in order. On the one hand, it seems more secure to choose r to be large on J so that (3) will be satisfied for a large class of functions h. On the other hand, if one wants to get accurate modeling at infinity, it is necessary to have a good guess for the behavior of f outside the bandwidth, that is, to be able to make r small. Indeed, the approximation f_N to f that we are about to construct is such that $|f_N - h| \to r$ uniformly on J as $N \to \infty$ and $\epsilon \to 0$. Thus, if $|f - h|$ is significantly smaller than r, the values of f and f_N will not be close to each other on J and the weak–$*$ convergence of f_N to f will cause f_N to oscillate on J with an amplitude which depends on the size of r. Still, the model f_N asymptotically meets the gauge constraint (3) which is the main feature of our approach and warrants applications where one is not so much concerned with the behavior at high frequencies except for its boundedness.

In this paper, we describe an identification procedure meeting the above requirements and derive error bounds in the case of equally spaced points with a suitable choice of h (Section II); the procedure rests on an extension of results demonstrated in [2]. We then report on a numerical experiment from real data measured on a hyperfrequency filter by the French National Center for Spacial Research (CNES); see Section III.

We shall make the standing assumption, required for system-theoretical reasons though not for mathematical ones, that the unknown function f and the analytic model f_N we are seeking are real symmetric, namely that $f(\bar{z}) = \overline{f(z)}$ and the same for f_N. Thus we need only take measurements in $a \le \theta \le \pi$ and obtain the others by complex conjugation. The reference function h is also assumed to verify this hypothesis on the (symmetric) arc J.

II. An Algorithm for Approximate Modeling

Suppose, for some unknown function $f \in A(\mathbb{D})$, that we are given the values $(a_k) = (f(z_k) + \eta_k)_{k=-N}^N$, where z_k belongs to I and (η_k) is a noise sequence, assumed to be ϵ-small in the l^∞ norm. We also assume that $z_0 = -1$ and that $z_{-k} = \bar{z}_k$, $a_{-k} = \bar{a}_k$, and $\eta_{-k} = \bar{\eta}_k$ for $1 \le k \le N$, which is the real-symmetric assumption made above.

Although we are seeking models in $A(\mathbb{D})$ only, we shall need to make excursions into H_∞. If $g \in H_\infty$ and $\sup_{z\in\mathbb{D}} |g(z)| = \|g\|_\infty$, recall (see, e.g., [10, ch. 3]) that the radial limit $\lim_{r\to 1} g(re^{i\theta})$ exists for almost every θ (even nontangential limits exist), and this serves as a definition for $g(e^{i\theta})$. In this way, $g(e^{i\theta})$ becomes a member of $L \vee \infty(\mathbb{T})$, with norm $\|g\|_\infty$, whose Fourier coefficients of negative index do vanish and whose restriction to any subset of positive measure on $\mathbb{T}$ is nonzero if g is nonzero.

Given functions $\alpha \in L^\infty(I)$, $\beta \in L^\infty(J)$ we denote by $\alpha \vee \beta$ the $L^\infty(\mathbb{T})$ function which is equal to α on I and to β on the interior

$\mathring{J}$ of J; when $\inf \alpha > 0$ and $\inf \beta > 0$, we also denote by $w_{\alpha, \beta}$ the outer function

$$w_{\alpha, \beta} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(\alpha \vee \beta) \, d\theta \right\}. \tag{4}$$

This function is characterized by the following properties (see, e.g., [10, ch. 5]): $w_{\alpha, \beta}(0) > 0$, $w_{\alpha, \beta}$ and $w_{\alpha, \beta}^{-1}$ are both in H_∞, and $|w_{\alpha, \beta}| = \alpha \vee \beta$, that is

$$|w_{\alpha, \beta}(z)| = \begin{cases} \alpha(z), & \text{a.e. on } I, \\ \beta(z), & \text{a.e. on } \mathring{J}. \end{cases} \tag{5}$$

Moreover, observe that $w_{\alpha, \beta} = w_{\alpha, 1} w_{1, \beta}$ so that $w_{\alpha, \beta}^{-1} = w_{1/\alpha, 1/\beta}$.

Given a complex number c we let $e_c(e^{i\theta})$ be the function defined on J by

$$e_c(e^{i\theta}) = \frac{1}{2a}(c(\theta + a) - \bar{c}(\theta - a)) \tag{6}$$

thus, e_c is linear in θ and satisfies $e_c(e^{ia}) = c$ and $e_c(e^{-ia}) = \bar{c}$. All we shall need beyond the values a_k to make our procedure effective is to specify numerically r and approximate values b_k of h at points z'_k on J. When nothing is known on the shape of f except being proper and stable, a particularly simple choice is $h = e_{f(e^{ia})}$ and $b_k = e_{a_N}(z'_k)$ on J; since ϵ is a bound for $|e_{f(e^{ia})} - e_{a_N}|$ on J, this allows h to be assigned numerically up to some precision less than ϵ. There is nothing so special about the function e_c defined in (6) except that $e_c(e^{ia}) = c$, $e_c(e^{-ia}) = \bar{c}$, and e_c goes uniformly to zero on J with c; any function with the same properties could be used in its place, and this choice was mainly for simplicity and definiteness. If one wants a strictly proper model, one may use quadratic interpolants rather than linear ones for h to interpolate the value zero at one. We then need to choose r large enough so that (3) is satisfied. Of course, there is no way to ensure beforehand that it is the case, and this is revealed *a posteriori* only if the convergence gets ruined, which means that r is too small somewhere on J.

We begin with a result asserting that robust band-limited identification, as defined in the introduction, is possible at least when r satisfies a Lipschitz condition. The arguments in the proof will turn out to be constructive, providing us with an algorithm to solve the problem. Although, in practice, we use only a finite number of measurement points, it is convenient to state the convergence result in terms of an infinite sequence.

Theorem 2.1 (Convergence Result): Assume the sequence (z_k) is dense in I, and let (z'_k) be a sequence dense in J. Let r be a nonnegative Lipschitz-continuous function on $\mathbb{T}$ of exponent μ, $0 < \mu \le 1$, which is zero on I. For every $N, M \in \mathbb{N}$, there exists a mapping $T_{r, N, M}: \mathbb{C}^{N+1} \times \mathbb{C}^M \to A(\mathbb{D})$ such that writing

$$\mathcal{E}(N, M, f, h, a, b) = \sup_{z \in \mathbb{T}} [|T_{r, N, M}(a_0, \cdots, a_N, b_1, \cdots, b_M)(z) - f \vee h(z)| - r(z)]$$

for $f \in L^\infty(I)$ and $h \in L^\infty(J)$, we have

$$\mathcal{E}(N, M, f, h, a, b) \to 0 \text{ as } N, M \to \infty \text{ and } \epsilon \to 0 \tag{7}$$

where $\epsilon = \max\{|a_k - f(z_k)|, |b_k - h(z'_k)|\}$, provided that $f \vee h \in C(\mathbb{T})$ and $|f - h| \le r$ on J.

Remark 1: The robustness property (7) is to be interpreted practically as a continuity property of $\mathcal{E}$ with respect to N, M, and ϵ. More precisely, it means that for every $\eta_0 > 0$, there exist $N_0, M_0 > 0$, and $\epsilon_0 > 0$ such that if $N > N_0$, $M > M_0$, and $\epsilon < \epsilon_0$, then $\mathcal{E}(N, M, f, h, a, b) < \eta_0$. In particular, since r is zero on I, (7) implies that, for ϵ small enough and N, M large enough, $T_{r,N,M}(a_0, \cdots, a_N, b_1, \cdots, b_M)$ is near to f in $L^\infty(I)$ while $T_{r, N, M}(a_0, \cdots, a_N, b_1, \cdots, b_M) - h$ is approximately bounded by r on J.

In the case where measurements are equally spaced, we get the following more precise bounds for $\mathcal{E}$. We write ω_f for the modulus of continuity of f, that is

$$\omega_f(\alpha) = \sup_{|\theta - \phi| \le \alpha} |f(e^{i\theta}) - f(e^{i\phi})| \tag{8}$$

and let P_s denote the space of trigonometric polynomials of degree at most s.

Theorem 2.2 (Error Estimates): Suppose that we are given κ points $(z_k)_{k=-N}^{N}$ and $(z'_k)_{k=-M}^{M}$ that are equally spaced on $\mathbb{T}$ and $s \le \frac{1}{4}(\kappa - 1)$. Then, there is a choice of $h \in C(J)$ such that with $\hat{f} = f \vee h$, $\mathcal{E}(N, M, f, h, a, b)$ satisfies

$$\mathcal{E}(N, M, f, h, a, b) \le 4(2 + 1/s)(\mathrm{dist}(\hat{f}, P_s) + \epsilon) \tag{9}$$

where

$$\mathrm{dist}(\hat{f}, P_s) \le \frac{3}{2} \max_{0 \le \lambda \le 1} \left[\omega_f\left(\frac{\lambda\pi}{s+1}\right) + (1 - \lambda) \frac{\pi}{s+1} \frac{\|f\|_{I, \infty}}{a} \right]. \tag{10}$$

Remark 2: Observe that the bounds given by (9) and (10) are explicit and satisfy (7) of Theorem 2.1 (where $T_{r, N, M}(a_0, \cdots, a_N, b_1 \cdots, b_M)$ is taken to be f_N). It is of perhaps more interest to have a bound for $|f - f_N|$ on J, and this follows immediately from the triangle inequality as well, giving on J

$$|f - f_N| \le \mathcal{E}(N, M, f, h, a, b) + r.$$

Before proving Theorem 2.1, we need to establish a few facts concerning a bounded (dual) extremal problem, which plays here the same role as the Nehari extension does in robust identification over the whole circle. These results will extend some of those established in [2].

For every pair of functions $\rho, \tau \in C(J)$ with $\rho > 0$, we define

$$\mathcal{B}_{\rho, \tau} = \{\gamma \in H_\infty, |\tau - \gamma| \le \rho \text{ a.e. on } J\}.$$

Proposition 1: Let ψ be in $L^\infty(I)$, h and ρ be in $C(J)$ with $\rho > 0$, and consider the following minimization problem:

$$\|\psi - g_0\|_{I, \infty} = \min_{g \in \mathcal{B}_{\rho, h}} \|\psi - g\|_{I, \infty} = \beta_\infty. \tag{11}$$

1) Problem (11) admits a solution $g_0 \in \mathcal{B}_{\rho,h}$; when $\psi \vee h \in H_\infty + C(\mathbb{T})$, the solution g_0 is unique. We assume now that ψ is not already the trace on I of a function in $\mathcal{B}_{\rho,h}$ so that $\beta_\infty > 0$.
2) When $\psi \vee h \in H_\infty + C(\mathbb{T})$, we have that

$$\begin{cases} |\psi - g_0| = \beta_\infty, & \text{a.e. on } I, \\ |h - g_0| = \rho, & \text{a.e. on } J. \end{cases}$$

3) The function g_0 is a solution to problem (11) if and only if

$$v_0 = g_0 \, w_{1/\beta_\infty, 1/\rho} \tag{12}$$

is a solution to the implicit Nehari problem

$$\min_{v \in H_\infty} \|(\psi \vee h) \, w_{1/\beta_\infty, 1/\rho} - v\|_\infty = \|(\psi \vee h) \, w_{1/\beta_\infty, 1/\rho} - v_0\|_\infty = 1. \tag{13}$$

Proof: The case where ρ is constant on J is contained in [2, Ths. 3 and 4]. What we need here is to consider an arbitrary positive function $\rho \in C(J)$.

The first step is to make sure that $\mathcal{B}_{\rho,h}$ is nonempty. For this, put $m = \min_J \rho > 0$. Since any $g \in H_\infty$ such that $\|g - h\|_{J,\infty} \le m$ belongs to $\mathcal{B}_{\rho,h}$, the conclusion follows from the density of $A(\mathbb{D})_{|J}$ in $C(J)$ already pointed out (but for the half-plane) as Fact 1) in Section I. Next, setting

$$\gamma = g\, w_{1,1/\rho} \tag{14}$$

and taking (5) into account, we get

$$\begin{aligned}\min_{g\in\mathcal{B}_{\rho,h}} \|\psi - g\|_{I,\infty} &= \min_{\gamma\in\mathcal{B}_{1,hw_{1,1/\rho}}} \|\psi - \gamma\, w_{1,1/\rho}^{-1}\|_{I,\infty}\\ &= \min_{\gamma\in\mathcal{B}_{1,hw_{1,1/\rho}}} \|\psi\, w_{1,1/\rho} - \gamma\|_{I,\infty} = \beta_\infty.\end{aligned} \tag{15}$$

We are now back to the case of a constant bound on J so that the cited results of [2] apply. This yields γ_0 realizing the infimum above, hence $g_0 = \gamma_0\, w_{1,1/\rho}^{-1}$ as asserted in 1). If $\psi \vee h \in H_\infty + C(\mathbb{T})$, so does $(\psi \vee h)\, w_{1,1/\rho}$ since $H_\infty + C(\mathbb{T})$ is an algebra (see, e.g., [7, IX, Th. 2.2]; again from [2], we get uniqueness of γ_0, hence of g_0, thereby proving 1).

We turn to the proof of 2) and we observe, since $\beta_\infty > 0$ by assumption, that [2, Th. 4] implies $|\psi\, w_{1,1/\rho} - \gamma_0| = \beta_\infty$ a.e. on I and $|h\, w_{1,1/\rho} - \gamma_0| = 1$ a.e. on J. Now, 2) follows at once from (5) and (14).

With regard to 3), we get from [2, Th. 3] that $\gamma_0 w_{1/\beta_\infty,1}$ is the solution to (13) and from Section IV of the cited paper that the value of this problem is indeed one. Now, (12) follows immediately from (14). ∎

Notice that β_∞ is *defined* by (11) so that the weight $w_{1/\beta_\infty,1/\rho}$ depends on ρ, ψ, and h through β_∞. Hence, (13) is an implicit equation, and the right value for β_∞ is the one that makes the infimum equal to one. That such a value is unique will follow from Lemma 2.2 below.

We are now in a position to establish our main result.

Proof of Theorem 2.1: The first step is to construct a trigonometric polynomial p_N, say of degree d, depending on $a_0, \cdots, a_N, b_1, \cdots, b_M$ and interpolation points $z_0, \cdots, z_N, z'_1, \cdots, z'_M$. Here, we can use standard robustly convergent interpolation procedures as in [11], [14], and [18] (in reality, we also use conjugate values at conjugate interpolation points).

However, p_N cannot serve as a model because it does not belong to $A(\mathbb{D})$ in general. If $p_N \in A(\mathbb{D})$ for some N and some a_k's, we simply set $T_{r,N,M}(a_0, \cdots, a_N, b_1, \cdots, b_M) = p_N$, which meets all our requirements. We now assume throughout the proof that $p_N \notin A(\mathbb{D})$, and we notice in this case that p_N cannot be the trace of any H_∞ function on I. If g were such a function, $z^d(p_N - g) \in H_\infty$ would vanish on I, hence should vanish identically, yielding $p_N = g$ so that p_N would be in $A(\mathbb{D})$.

Let

$$\sigma_N(z) = r(z) + \varepsilon_N, \qquad \forall z \in J \tag{16}$$

for a sequence (ε_N) of positive numbers to be determined later. This defines a μ-Lipschitz-continuous positive function σ_N on J.

The next stage is to get a function $f_N \in \mathcal{B}_{\sigma_N,p_N}$ solution to the following bounded extremal problem:

$$\begin{aligned}\min\{\|p_N - g\|_{I,\infty},\ g \in \mathcal{B}_{\sigma_N,p_N}\}\\ = \|p_N - f_N\|_{I,\infty} = \beta_\infty(N).\end{aligned} \tag{17}$$

For simplicity, we will write in the sequel $\beta_\infty = \beta_\infty(N)$. It follows from Proposition 1 that $\beta_\infty > 0$ and that f_N does exist, is unique, and satisfies

$$|p_N - f_N| = \begin{cases}\beta_\infty, & \text{a.e. on } I\\ \sigma_N, & \text{a.e. on } J.\end{cases} \tag{18}$$

Again from Proposition 1, it follows that (17) is equivalent to finding v_N which solves the Nehari problem

$$\begin{aligned}&\min_{v\in H_\infty} \|p_N w_{1/\beta_\infty,1/\sigma_N} - v\|_\infty\\ &\quad = \|p_N w_{1/\beta_\infty,1/\sigma_N} - v_N\|_\infty = 1\end{aligned} \tag{19}$$

where f_N and v_N are related by

$$v_N = f_N\, w_{1/\beta_\infty,1/\sigma_N}.$$

This provides us with $f_N \in H_\infty$, and the problem is now, for each N, to choose ε_N, ensuring that $f_N \in A(\mathbb{D})$. Observe, indeed, that for arbitrary values of ε_N, the outer function $w_{1/\beta_\infty,1/\sigma_N}$ is discontinuous at $e^{\pm ia}$ and that neither v_N nor *a fortiori* f_N needs to be continuous on $\mathbb{T}$. The following lemma will allow us to obtain this continuity from an appropriate choice of ε_N.

Lemma 2.1: Under the hypotheses of Theorem 2.1, and still assuming $p_N \notin H_\infty$, the following assertions hold.

1) For every fixed N, the quantity β_∞ defined by (17) and (16) is continuous and decreasing with respect to ε_N, and the implicit equation
$$\varepsilon_N = \beta_\infty \tag{20}$$
admits a solution.
2) For every N and the choice (20) of ε_N, the outer function $w_{1/\beta_\infty,1/\sigma_N}$ is Lipschitz-continuous on $\mathbb{T}$ of exponent μ.
3) If $f \vee h \in C(\mathbb{T})$ and $|f - h| \le r$ on J, and if for every N we choose ε_N as given by (20), then
$$\lim_{\substack{N\to\infty\\ \epsilon\to 0}} \beta_\infty = 0. \tag{21}$$

Proof:

1) Observe from the convexity of the set $\mathcal{B}_{\sigma_N,p_N}$ and of the norm function $\|\ \|_{I,\infty}$ that β_∞ is a decreasing convex function of ε_N and hence is continuous.

   Now, $p_{N|I} \in C(I)$ which is contained in the $L^\infty(I)$ closure of $H_{\infty|I}$ (see [2]), so (16) and (17) imply that $\beta_\infty \to 0$ as $\varepsilon_N \to \infty$. Thus, for ε_N large enough, $\beta_\infty < \varepsilon_N$.

   Then let $\varepsilon_N \to 0$. Assume that $\beta_\infty < \varepsilon_N$ so that in particular $\beta_\infty \to 0$. In view of (17), and since f_N remains bounded on J, this implies $p_N \in H_\infty$; see [2, Proposition 3], which is a contradiction. Hence, $\beta_\infty \ge \varepsilon_N$ eventually, which proves 1) by the intermediate value theorem.
2) Since the gauge function r is assumed to be μ-Lipschitz on J, so is σ_N from its definition (16) and also $1/\sigma_N$ as $\sigma_N \ge \varepsilon_N > 0$. Hence, writing $w_N = w_{1/\beta_\infty,1/\sigma_N}$ for simplicity
$$|w_N| = \begin{cases}1/\beta_\infty = 1/\varepsilon_N = 1/\sigma_N(e^{\pm}ia), & \text{on } I,\\ 1/\sigma_N, & \text{on } J\end{cases}$$
is μ-Lipschitz on $\mathbb{T}$, and it remains for us to show that w_N is also μ-Lipschitz. By standard arguments, this reduces to the analogous result on conjugate functions, see [7, III, Th. 1.3]. This achieves the proof of 2).
3) By the construction of p_N
$$\lim_{\substack{N\to\infty\\ \epsilon\to 0}} \sup_{\|\eta\|_\infty\le\epsilon} \|p_N - f \vee h\|_\infty = 0. \tag{22}$$
Choose $\varepsilon_N = \beta_\infty$ for each N, as this is possible by 1), and assume that (21) is false. Then, (22) implies that for N large enough and ϵ small enough we will get $\|p_N - h\|_{J,\infty} \le \varepsilon_N$,

and, since $|f-h| \le r$ on J by hypothesis, it turns out that $|f-p_N| \le \sigma_N$ on J. Hence, for such N and ϵ, $f \in \mathcal{B}_{\sigma_N, p_N}$ and necessarily $\beta_\infty \le \|p_N - f\|_{I,\infty}$, which, still from (22), tends to zero as $N \to \infty, \epsilon \to 0$, a contradiction. This proves 3) and the lemma. ■

To complete the proof of Theorem 2.1, choose $\varepsilon_N = \beta_\infty$. It follows from 2) of Lemma 2.1 that $p_N w_{1/\beta_\infty, 1/\sigma_N}$ is μ-Lipschitz hence *a fortiori* Dini-continuous on $\mathbb{T}$, and the Carleson–Jacobs theorem [7, IV, Th. 2.1] implies that the solution v_N to (19) belongs to $A(\mathbb{D})$. Again from 2) of Lemma 2.1

$$w_{\beta_\infty, \sigma_N} = (w_{1/\beta_\infty, 1/\sigma_N})^{-1}$$

is continuous (since it is μ-Lipschitz) so that

$$f_N = v_N w_{\beta_\infty, \sigma_N}$$

lies in $A(\mathbb{D})$.

We finally verify that $T_{r,N,M}(a_0, \cdots, a_N, b_1, \cdots, b_M) = f_N$ does the job. Indeed, on I we have the inequality $|f - f_N| \le |f - p_N| + |p_N - f_N|$, and the last term is equal to β_∞ by (18); thus, (22) and (21) give the desired behavior on I.

Moreover, on J, we get $|h - f_N| \le |h - p_N| + |p_N - f_N|$ and, since $\sigma_N = r + \beta_\infty$, the result for J follows from (18), (21), and (22). This establishes (7) and ends the proof of Theorem 2.1. ■

Remark 3: If the data are obtained by the Möbius transform of measurements in continuous time, the question arises as to whether the inverse transform of $f_N = T_{r,N,M}(a_0, \cdots, a_N, b_1, \cdots, b_M)$ is the transfer function of a BIBO-stable system. The answer is yes. Indeed, it follows from [15] that the solution ν_N of the Nehari problem associated to the μ-Lipschitz function $p_N w_{1/\beta_\infty, 1/\sigma_N}$ is itself μ-Lipschitz and hence has H^1 derivative. Hence, so does f_N. Transforming back to the half-plane yields a function G_N whose derivative is in $H^1(\Pi_+)$. Mimicking the classical proof of Hardy's inequality [7, II, ex. 8], one obtains that G_N is the Laplace transform of some impulse response belonging to $L^1(0, \infty)$ plus the constant $f_N(1)$ which is bounded in modulus by $|h(1)| + r(1)$.

Having established Theorem 2.1, we must tie one loose end to make the proof constructive, namely how does one find in practice β_∞ in order to solve the Nehari problem (19) and to select ε_N according to (20). This can be done by a dichotomy procedure which rests on Lemma 2.2 below.

For every $\varepsilon > 0$, define the map Δ_ε

$$\begin{aligned}\Delta_\varepsilon:]\,0, \infty[&\longrightarrow]0, \infty[\\ \delta &\longmapsto \min_{v \in H_\infty} \|p_N w_{1/\delta, 1/(\varepsilon+r)} - v\|_\infty.\end{aligned}$$

Lemma 2.2: If $p_N \notin H_\infty$, then for every $\varepsilon > 0$, the map Δ_ε is defined from $(0, \infty)$ onto $(0, \infty)$, is continuous, and is monotonically decreasing.

Proof of Lemma 2.2: Let $\varepsilon > 0$. Then for every real $\delta > 0$, the function $p_N w_{1/\delta, 1/(\varepsilon+r)} \in H_\infty + C(\mathbb{T})$. Hence, by [7, IV, Th. 1.3, Th. 1.7], there is a unique function $v_\delta \in H_\infty$ such that

$$\Delta_\varepsilon(\delta) = \|p_N w_{1/\delta, 1/(\varepsilon+r)} - v_\delta\|_\infty. \tag{23}$$

Let $\delta_1, \delta_2 > 0$, $\delta_1 \ne \delta_2$. Then, from the definition of Δ_ε, we get

$$\begin{aligned}\Delta_\varepsilon(\delta_1) &< \|p_N w_{1/\delta_1, 1/(\varepsilon+r)} - w_{\delta_2/\delta_1, 1}\, v_{\delta_2}\|_\infty\\ &= \|w_{\delta_2/\delta_1, 1}\,(p_N w_{1/\delta_2, 1/(\varepsilon+r)} - v_{\delta_2})\|_\infty.\end{aligned}$$

That the inequality above is strict follows from the uniqueness of v_δ and the fact that $w_{\delta_2/\delta_1, 1}\, v_{\delta_2} \ne v_{\delta_1}$. Indeed, $|p_N w_{1/\delta_1, 1/(\varepsilon+r)} - v_{\delta_1}| = \Delta_\varepsilon(\delta_1)$ and $|p_N w_{1/\delta_2, 1/(\varepsilon+r)} - v_{\delta_2}| = \Delta_\varepsilon(\delta_2)$ are constant a. e. on $\mathbb{T}$, while

$$|p_N w_{1/\delta_1, 1/(\varepsilon+r)} - w_{\delta_2/\delta_1, 1}\, v_{\delta_2}| = \Delta_\varepsilon(\delta_2)|w_{\delta_2/\delta_1, 1}|$$

assumes different values on I and J. Therefore

$$\begin{aligned}\Delta_\varepsilon(\delta_1) &< \max\left(\frac{\delta_2}{\delta_1}\|p_N w_{1/\delta_2, 1/(\varepsilon+r)} - v_{\delta_2}\|_{I,\infty},\right.\\ &\qquad\left.\|p_N w_{1/\delta_2, 1/(\varepsilon+r)} - v_{\delta_2}\|_{J,\infty}\right)\\ &= \Delta_\varepsilon(\delta_2)\,\max\left(\frac{\delta_2}{\delta_1}, 1\right).\end{aligned}$$

Taking $\delta_2 < \delta_1$ shows that Δ_ε is decreasing, and then $\delta_1 < \delta_2$ shows that it is continuous.

As a continuous and positive decreasing map, Δ_ε has a limit at ∞. Given $\xi > 0$, there exists a function $g \in H_\infty$ such that $\|p_N w_{1, 1/(\varepsilon+r)} - g\|_{J,\infty} < \xi$ because $p_N w_{1, 1/(\varepsilon+r)} \in H_\infty + C(\mathbb{T})$ and $H_{\infty|_J}$ is dense in $C(J)$ (this follows at once from Fact 1) in Section I). For every $n > 0$, we have

$$\Delta_\varepsilon(n) \le \|p_N w_{1/n, 1/(\varepsilon+r)} - g w_{1/n, 1}\|_\infty$$

which implies that for n large enough

$$\begin{aligned}\Delta_\varepsilon(n) \le \max&\left(\frac{1}{n}\|p_N w_{1, 1/(\varepsilon+r)} - g\|_{I,\infty},\right.\\ &\left.\|p_N w_{1, 1/(\varepsilon+r)} - g\|_{J,\infty}\right) < \xi.\end{aligned}$$

As ξ is arbitrarily small, we necessarily get $\lim_{\delta \to \infty} \Delta_\varepsilon(\delta) = 0$. To analyze the behavior of Δ_ε when $\delta \to 0$, we write

$$\begin{aligned}\Delta_\varepsilon(\delta) = \max&\left(\frac{1}{\delta}\|p_N w_{1, 1/(\varepsilon+r)} - v_\delta\, w_{\delta, 1}\|_{I,\infty},\right.\\ &\left.\|p_N w_{1, 1/(\varepsilon+r)} - v_\delta\, w_{\delta, 1}\|_{J,\infty}\right).\end{aligned}$$

We claim that if the first argument of the $\max$ remains bounded as $\delta \to 0$, then the second does not. Indeed, $v_\delta w_{\delta, 1}$ would otherwise be a family of H_∞ functions converging to $p_N w_{1/\delta, 1/(\varepsilon+r)}$ in $L^\infty(I)$ as δ tends to zero but remaining bounded on J; in view of $p_N \notin H_\infty$, this would contradict [2, Proposition 3] (Fact 2) of Section I rephrased on the disc). Thus, we get $\lim_{\delta \to 0} \Delta_\varepsilon(\delta) = \infty$. This shows that Δ_ε is onto $(0, \infty)$. ■

By Lemma 2.2, we can associate to every $\varepsilon > 0$ a unique $\beta_\infty(\varepsilon) > 0$ such that $\Delta_\varepsilon(\beta_\infty(\varepsilon)) = 1$, and $\beta_\infty(\varepsilon)$ may be computed by a dichotomy procedure in view of the monotonicity of Δ_ε.

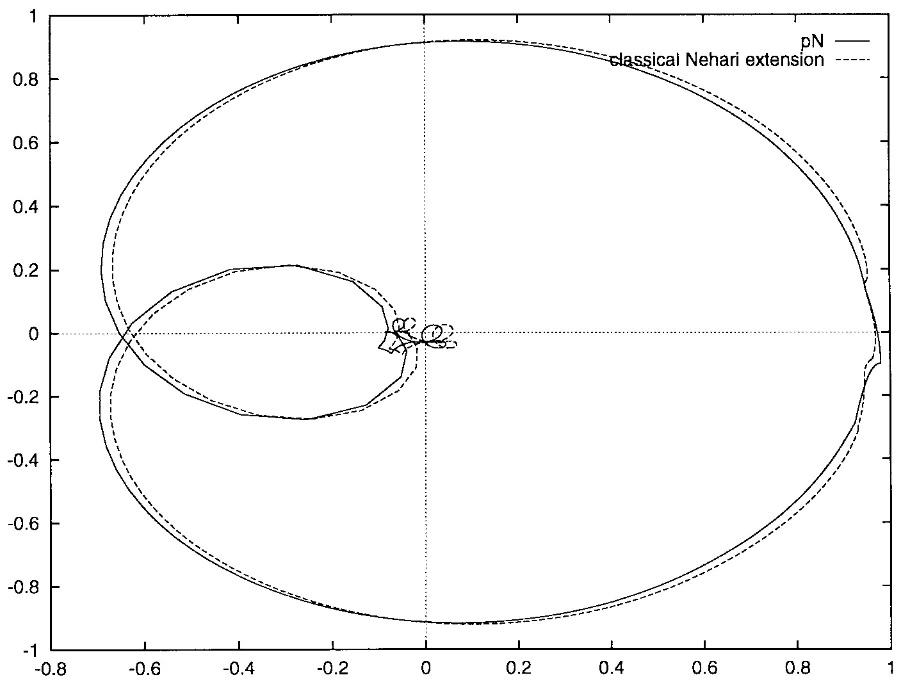
Given p_N, which in turn defines Δ_ε, what we want to find now is the unique value $\varepsilon = \varepsilon_N$ for which $\beta_\infty(\varepsilon) = \varepsilon$ so that both (19) and (20) are satisfied. In view of the monotonicity asserted in 1) of Lemma 2.1, this can again be solved by dichotomy.

This process, which is somehow similar in spirit to the γ-iteration used in H_∞-control, settles our constructive approach to Theorem 2.1. However, it requires solving a series of Nehari problems, the solution of which can be numerically estimated only when the function to be approximated is continuous. Indeed, in this case, it can be represented arbitrarily well in $L^\infty(\mathbb{T})$ by a rational function (using for instance the Jackson polynomials previously introduced to compute p_N) whose Hankel operator has finite rank and thus possesses a finite singular-value decomposition allowing one to solve the associated Nehari problem in various fashions (see, e.g., [4] and [5]).

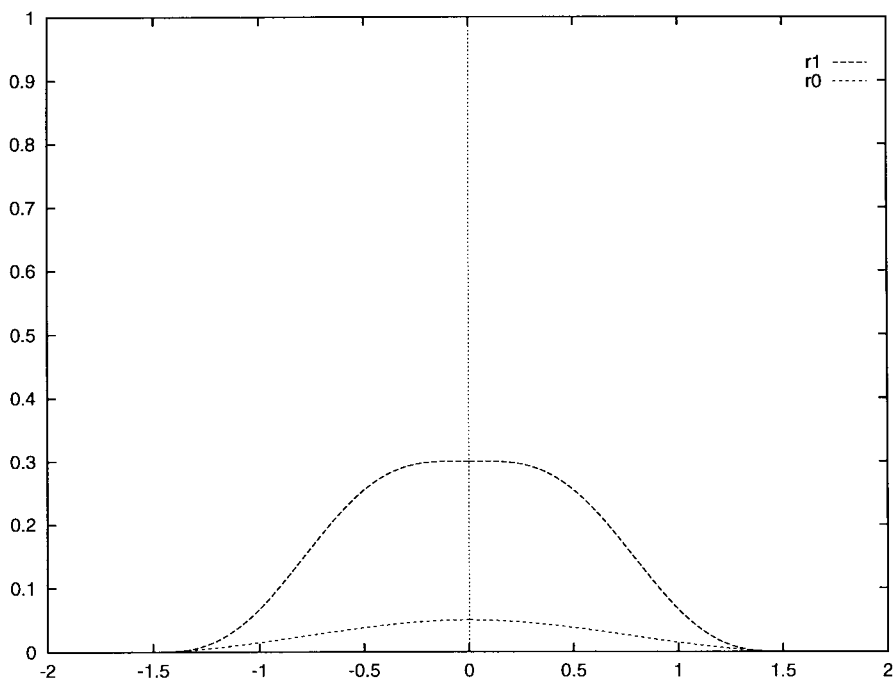
Now, the typical Nehari problem we must solve here is associated to a function of the form

$$p_N w_{1/\delta, 1/(\varepsilon+r)} \tag{24}$$

for some positive numbers ε and δ, and such a function is clearly discontinuous at $e^{\pm ia}$ in general. However, (24) is continuous at any other point on $\mathbb{T}$, because it is even μ-Lipschitz there; indeed, an outer



(a)



(b)

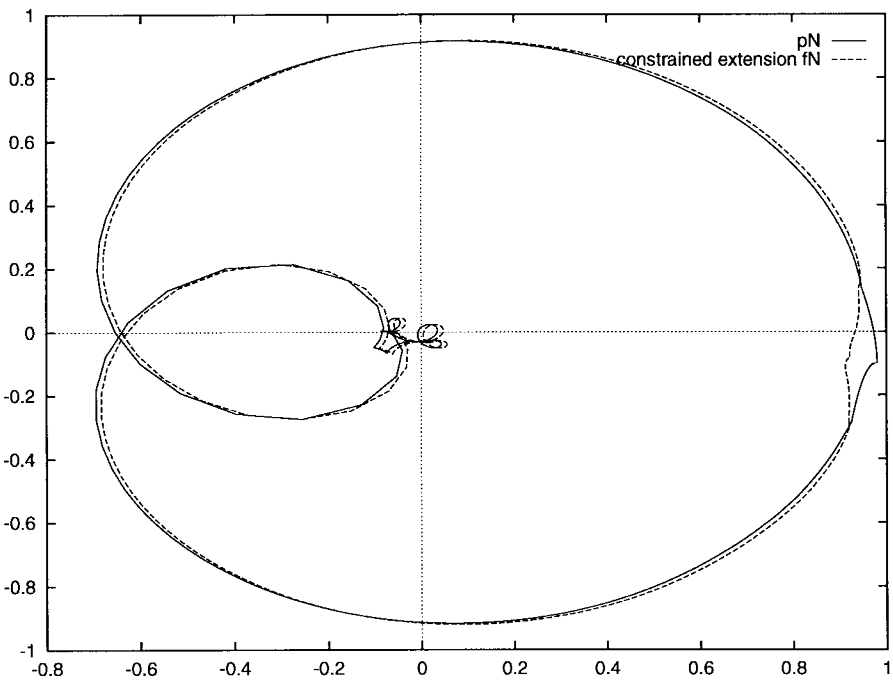
Fig. 1. (a) $\mathrm{dist}(p_N, H^\infty) = 0.0236$ and (b) the gauge functions r_0 and r_1 on J.

function whose $\log$-modulus is μ-Lipschitz in the neighborhood of some point is itself μ-Lipschitz at this point. This is the local version of 2) of Lemma 2.1, and it is proved in the same manner except that we must appeal, this time, to a local version of the regularity theorem for conjugate functions (see the proof of [7, III, Corollary 1.4]). To circumvent the discontinuity problem at $e^{\pm ia}$, we introduce another Nehari problem, equivalent to (19). Let p be the first-order trigonometric polynomial which coincides with p_N at $e^{\pm ia}$ so that $(p_N - p)w_{1/\delta,1/(\varepsilon+r)}$ is continuous on $\mathbb{T}$. The Nehari problem

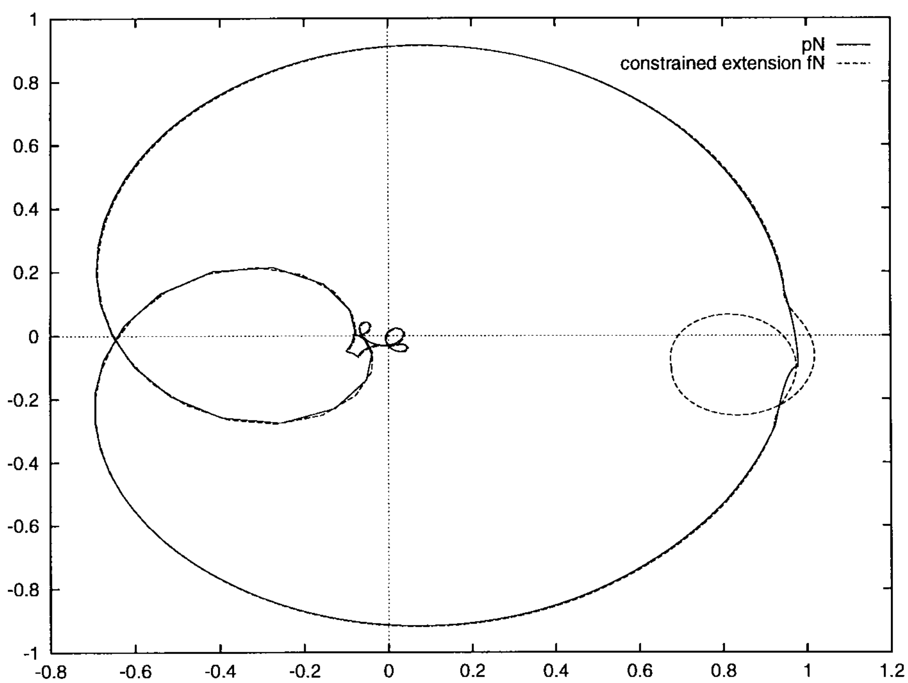
$$\min_{g\in H_\infty} \|(p_N - p)w_{1/\delta,1/(\varepsilon+r)} - g\|_\infty \tag{25}$$

is clearly equivalent to (19) under the transformation $v = g + pw_{1/\delta,1/(\varepsilon+r)}$ and consequently assumes the same value. The dichotomy procedures described before may now be performed numerically by solving (25) iteratively, and this was done in the example presented in Section III.

Proof of Theorem 2.2: We are given that the points (z_k) and (z'_k) are the κth roots of unity for some integer κ. We take here h to be the function $e_{f(e^{ia})}$ defined by (6). The approximate values b_k of h on J will be taken to be $e_{a_N}(z'_k)$. This choice is mainly for definiteness and is not essential, although it leads to simpler estimates.



(a)



(b)

Fig. 2. (a) gauge function $r_0, \beta_\infty = 0.0122$, and (b) gauge function $r_1, \beta_\infty = 0.003\,28$.

To construct the trigonometric polynomial p_N we use the noisy values $(a_k)_{k=-N}^{N}$ of f on I together with the values (b_k) on J to produce the discrete de la Vallée–Poussin polynomial $V_{s,\kappa}$ with $\kappa \geq 4s+1$, as in [11]. Because $\epsilon = \max\{|a_k - f(z_k)|, |b_k - h(z_k')|\}$, this is equivalent to using measurements of $\hat{f} = f \vee h$ with an error of at most ϵ and hence (see [11, Th. 3.1]):

$$\|\hat{f} - p_N\|_\infty \leq (4 + 2/s)(\operatorname{dist}(\hat{f}, P_s) + \epsilon).$$

Now $\operatorname{dist}(\hat{f}, P_s) \leq \frac{3}{2}\omega_{\hat{f}}(\pi/s+1)$ by Jackson's theorem [16], which, given the definition of $\hat{f}$, implies that (10) holds. One could improve upon this bound by considering a smoother extension $\hat{f}$ to f. One way to do this would be to choose h to be a function cubic (in θ) which matches f and its derivatives at the points $z_{\pm N}$ (this, of course, assumes one is able to estimate these derivatives). To get the (b_k), one could use in this case a cubic polynomial matching the noisy values $a_{\pm(N-1)}$ and $a_{\pm N}$, as in [14].

Recall now that the final model f_N is the solution to the extremal problem (17). Moreover, by the proof of Lemma 2.1, we see that $\beta_\infty(N) \leq \|\hat{f} - p_N\|_\infty$. This gives us the following estimates for the error in the identified model f_N:

$$\|f - f_N\|_{I,\infty} \le \|f - p_N\|_{I,\infty} + \|p_N - f_N\|_{I,\infty} \le 2\|\hat{f} - p_N\|_\infty$$

and on J

$$\begin{aligned}|h - f_N| &\le |h - p_N| + |p_N - f_N| \\ &\le \|h - p_N\|_{J,\infty} + (r + \beta_\infty(N)) \le r + 2\|\hat{f} - p_N\|_\infty.\end{aligned}$$

■

III. Numerical Examples

Our main example consists of real data measured on a hyperfrequencies filter of the CNES. The bandwidth I is defined by $a = \pi/2$, and we are given 801 noisy pointwise values (a_k), so that $N = 400$. We first complete these data outside the bandwidth by rough estimates, and we construct the trigonometric polynomial p_N using discrete de la Vallée–Poussin polynomials. Fig. 1(a) shows the result of the classical Nehari extension to p_N, which gives rise to an error of value 0.0236 in $L^\infty(\mathbb{T})$. We then compute the solution f_N to the constrained approximation problem associated to p_N for different gauge functions r until an acceptable tradeoff is found between β_∞ and r; these gauge functions are plotted in Fig. 1(b). If no satisfactory compromise can be found, one can change the reference behavior on J, using the previous computations, in order to make a more accurate choice. The corresponding results are shown in Fig. 2(a) and (b).

We have also considered the function $f(z) = 3(z^2 + 1)/(z^2 + 2z + 5)$, already studied (using information on the whole circle) in [8], [11], and [12]. Full details can be found in [3].

IV. Conclusion

In this paper, we presented a framework for robust band-limited identification which extends the existing one for robust identification on the whole axis (or circle) that was introduced in [9]. We also developed a constructive algorithm to perform such a band-limited identification, which recovers the transfer-function on the bandwidth in a robust fashion while meeting gauge constraints at the remaining frequencies. The procedure is very similar in spirit to the two-stage algorithms proposed in [8], [9], [11], and [12] but appeals to a bounded extremal problem which may be seen as a generalization of the classical Nehari problem. We also derived error bounds in a standard case and presented examples on real data.

There are at least two further questions which, in our opinion, deserve further study. The first arises from the observation that the identification procedure can be applied to any sequence of data $a_0, a_1, \cdots, b_1, b_2, \cdots$; the question is: "What is the limit behavior of $T_{r,N,M}(a_0, \cdots, a_N, b_1, \cdots, b_M)$ if the data do not converge (pointwise in l_∞) to some interpolation sequence $f(a_0), f(a_1), \cdots, h(b_1), h(b_2), \cdots$ with $f \in A(\mathbb{D})$, $f \vee h \in C(\mathbb{T})$, and $|f - h| \le r$ on J?" The second question stems from the fact that our identification scheme converges uniformly to f on I but only weak–$*$ to f on J. This is enough to recover f uniformly on compact subsets of the half-plane (or of the disk) by the Poisson formula but not to recover f itself. Now, still assuming (3), what additional hypotheses would be needed on f in order to design an algorithm producing some stronger type of convergence? We think both questions are important in connection with the practical value of such schemes.

References

[1] E. W. Bai and S. Raman, "A linear interpolatory algorithm for robust system identification with corrupted measurement data," *IEEE Trans. Automat. Contr.,* vol. 38, pp. 1236–1241, 1993.

[2] L. Baratchart, J. Leblond, and J. R. Partington, "Hardy approximation to L^∞ functions on subsets of the circle," *Constructive Approximation,* vol. 12, no. 3, pp. 423–436, 1996.

[3] L. Baratchart, J. Leblond, J. R. Partington, and N. Torkhani, "Robust identification in the disc algebra from band-limited data," *INRIA Res. Rep.,* no. 2488, 1995.

[4] T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems, A Dynamic Game Approach.* Boston: Birkhäuser, 1991.

[5] B. Francis, *A Course In H^∞ Control Theory,* LNCIS. New York: Springer-Verlag, 1987.

[6] J. H. Friedman and P. P. Khargonekar, "Application of identification in H_∞ to lightly damped systems: Two case studies," *IEEE Trans. Contr. Syst. Tech.,* vol. 3, pp. 279–289, 1995.

[7] J. B. Garnett, *Bounded Analytic Functions.* New York: Academic, 1981.

[8] G. Gu and P. P. Khargonekar, "Linear and nonlinear algorithms for identification in H^∞ with error bounds," *IEEE Trans. Automat. Contr.,* vol. 37, pp. 953–963, 1992.

[9] A. J. Helmicki, C. A. Jacobson, and C. N. Nett, "Control-oriented system identification: A worst-case/deterministic approach in H_∞," *IEEE Trans. Automat. Contr.,* vol. 36, pp. 1163–1176, 1991.

[10] K. Hoffman, *Banach Spaces of Analytic Functions.* New York: Dover, 1988.

[11] J. R. Partington, "Robust identification and interpolation in H_∞," *Int. J. Contr.,* vol. 54, pp. 1281–1290, 1991.

[12] ——, "Robust identification in H_∞," *J. Math. Anal. Appl.,* vol. 166, pp. 428–441, 1992.

[13] ——, "Algorithms for identification in H_∞ with nonequally spaced function measurements," *Int. J. Contr.,* vol. 58, pp. 21–31, 1993.

[14] ——, "Interpolation in normed spaces from the values of linear functionals," *Bull. London Math. Soc.,* vol. 26, pp. 165–170, 1994.

[15] V. V. Peller, "Hankel operators and continuity properties of best approximation operators," *Algebra i Analiz.,* vol. 2, pp. 162–190, 1990.

[16] M. J. D. Powell, *Approximation Theory and Methods.* Cambridge: Cambridge Univ. Press, 1981.

[17] N. P. Rubin and D. J. N. Limebeer, "H_∞ identification for robust control," in *Proc. Amer. Contr. Conf.,* Baltimore, MD, 1994, pp. 2040–2044.

[18] N. Torkhani, "Robust interpolation and approximation for disc algebra functions on subsets of the circle," *J. Comput. Appl. Math.,* to be published.

[19] N. J. Young, *An Introduction to Hilbert Space.* Cambridge: Cambridge Univ. Press, 1988.

[20] A. Zygmund, *Trigonometric Series.* Cambridge: Cambridge Univ. Press, 1959.