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Robust Identification from Band-Limited Data

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Abstract—Consider the problem of identifying a scalar bounded-input/bounded-output stable transfer function from pointwise measurements at frequencies within a bandwidth. We propose an algorithm which consists of building a sequence of maps from data to models converging uniformly to the transfer function on the bandwidth when the number of measurements goes to infinity, the noise level to zero, and asymptotically meeting some gauge constraint outside. Error bounds are derived, and the procedure is illustrated by numerical experiments.

Index Terms—Approximate modeling, linear systems, Nehari extension, robust identification.

I. INTRODUCTION

This paper is concerned with the problem of harmonic identification, that is, of recovering a single-input/single-output (SISO) and bounded-input/bounded-output (BIBO)-stable transfer function from a family of experimental pointwise values on the imaginary axis. Such data are common in engineering practice as they may be obtained from asymptotic outputs associated to sine inputs or from numerical simulations of distributed parameter systems (see [6] and [17], for example). In [9], a setting to approach this issue was proposed in which the error in measurements is handled in a deterministic fashion, and the identification procedure consists of a map from finite sets of data to (stable) transfer functions that converge uniformly to the “true” transfer function when the noise goes uniformly to zero and the number of data goes to infinity.

In the present work, we shall consider the (realistic) case where the experiments are only available in some range of frequencies corresponding to the bandwidth of the system. In this case, none of the algorithms that were proposed [8], [9], [11], [12] converges, and we shall see that the setting itself has to be modified. We shall adapt to the new situation by requiring the map from data to models to converge uniformly in the bandwidth while meeting some norm constraints at remaining frequencies.

Our working space will be the unit disc rather than the half-plane, the two frameworks being equivalent by means of a Möbius transform. Since the transfer function of a BIBO-stable system is continuous on the imaginary axis, including at infinity, a model for us has to be found in the disc algebra.

Let $H_{\infty}$ be the familiar Hardy space of bounded analytic functions in the disc and $A(D)$ (the disc algebra) be the subspace of such functions that are continuous on the closed disc. On a couple of occasions in this section, we shall also use the symbol $H_{\infty}$ to mean the Hardy space of the right half-plane $\Pi_+: = \{s \in \mathbb{C}; \text{Re}s > 0\}$, but the context will always keep the meaning clear. The algebra $A(\Pi_+)$ of the right half-plane will then consist of those functions in $H_{\infty}$ of this half-plane that extend continuously to the imaginary axis, including at infinity. The symbol $C(X)$ stands for the space of complex continuous functions on $X$ endowed with the sup norm.

Spaces $X$ used in this paper will be arcs on the unit circle or intervals on the imaginary axis.

In the problem of robust $H_{\infty}$ identification of functions in the disc algebra as stated in the above-mentioned references, one is given experimental data as complex numbers $(\alpha_k)_{k=0}^{\infty} = (f(z_k) + z_k e_k)_{k=0}^{\infty}$, where $f$ is an unknown function in the disc algebra $A(D)$, and $z_N, \ldots, z_N$ are points on the unit circle $\mathbb{T}$, while $(\eta_{k}, \ldots, \eta_{k})$ is some unknown but bounded noise sequence which can be due to nonlinear effects or measurement errors, for example. From the $(\alpha_k)$, one wishes to construct an approximation $f_N$ such that in the limit, as the noise level tends to zero and the number of observations tends to infinity, one has convergence in the $H_{\infty}$ norm, that is

$$\lim_{N \to \infty} \sup_{\|\varphi\|_{\infty} \leq 1} \|f_N - f\|_{\infty} = 0,$$

for all $f \in A(D)$. (1)

This convergence requirement corresponds to a continuity property of the model $f_N$ with respect to the number of measurements and the noise level, as explained in Remark 1 below. To approach this problem, a two-stage algorithm has been found useful [8], [9], [11], [12]. To proceed, one first computes a trigonometric polynomial $p_N$, which interpolates the given data (but is not in $A(D)$), and one applies then the (nonlinear) Nehari extension [19] to obtain the best approximation to $p_N$ by a function $f_N$ bounded and analytic in the disc (it will in fact be rational).

When the points $z_k$ are equally spaced on the circle, $p_N$ can be obtained using the classical Jackson or de la Vallée-Poussin trigonometric polynomials [11], [20]. When the points are not equally spaced, the problem becomes computationally harder, but one can design a transformation from the given points into equally spaced ones and proceed as before (see, e.g., [13]) or else rely on a more general principle of linear programming [14].

In the last reference, the overall error of the identification procedure can be expressed as a sum of two terms, one corresponding to the noise and the other to the maximum gap between the interpolation points. One such theoretical bound is $4\epsilon + 3\text{dist}(f, P_p)$, where $\epsilon \geq \|\varphi\|_{\infty}$ and $P_p$ is the space of polynomials of degree $p$ and the maximum gap is less than $1/p$. Thus, the error goes to zero as $\epsilon \to 0$, provided the maximum gap between the measurement points $(z_k)$ goes to zero.

However, in practical applications, one may not be able to measure $f$ at all points on the circle. For example, in the identification of continuous-time, linear, time-invariant, and BIBO-stable control systems by frequency response measurements, which can be reduced to the above problem by means of the Möbius transformation $s = (1 + s)/(1 - z)$ and $G(s) = f(z)$ where $G$ is the transfer-function, one is not able to measure $G(\omega)$ for arbitrarily high values of $\omega$. Moreover, one is not normally concerned about modeling $G$ arbitrarily well at high frequencies. In some cases, one may even prefer to have a linear model valid for a restricted set of frequencies, since the linearity assumption would hold only locally with respect to the frequency. In these circumstances, no algorithm can guarantee uniform convergence over the whole imaginary axis without further a priori knowledge on $G$ [14]. It is nevertheless natural to ask whether the unknown function $G$ can be recovered in a robust fashion at least in the range of frequencies where measurements are available, through a model which is still under control at the remaining frequencies.

Let us stay with the half-plane for a while and discuss a bit further the situation where measurements are only available in the bandwidth, say $\Omega$. In this connection, some work on band-limited identification has been published by Bai and Raman [1] in which they essentially approximate separately the real and imaginary parts of the transfer function by polynomials over the frequency interval $\Omega$, plugging
in some arbitrary polynomial weight of sufficiently high degree to become the denominator off the approximant so as to end up with a stable and proper model. In doing so, they are not concerned about controlling the behavior of the set \( \Omega \) and, since their scheme is (real) linear, it is a routine matter to check, by the same arguments as in [12], that their sequence of estimates is unbounded outside \( \Omega \) for almost every noise in \( L^\infty \) (i.e., for every noise sequence in a set of second category in the sense of Baire). In fact, we claim that any \( H^\infty \) band-limited identification scheme must incorporate some constraints that impose on the behavior of the transfer-function outside \( \Omega \). This can be inferred from two facts.

1) In the space \( C(\Omega) \), the subspace \( A(\Pi^+) |_{g} \) obtained by restricting \( A(\Pi^+) \) to \( \Omega \) is dense.

2) If \( G \in C(\Omega) \) does not belong to \( A(\Pi^+) |_{g} \), any sequence of functions in \( A(\Pi^+) \) (or even in \( H_\infty \) that converges to \( G \) on \( \Omega \) is unbounded in \( H_\infty \).

Fact 1) is an easy consequence of Runge’s theorem, while Fact 2) follows from the weak-* compactness of balls in \( H_\infty \), and we refer the reader to [2] for a proof which is phrased on the disc rather than the half-plane (and also works in \( L^p(\Omega) \) for \( 1 \leq p \leq \infty \)).

Altogether, 1) and 2) indicate that no matter the data, we can always construct an excellent model on \( \Omega \) at the cost of nearly destabilizing it at the remaining frequencies, a problem which is familiar to identification practitioners. At this point, it is perhaps interesting to draw a parallel with the seemingly different process of stochastic parametric identification; there, the constraints on the model are often imposed in terms of bounded rational degree, and the analog of the above-mentioned phenomenon would be that allowing the degree to grow too large destabilizes the model because it starts fitting the noise.

It might be argued that all one needs to do is to prescribe plausible values for \( G \) outside the bandwidth and to use standard robust identification techniques. However, this approach would prevent us from recovering \( G \) asymptotically on \( \Omega \). Indeed, Fact 1) is not applicable to the whole axis, and we should incur an irreducible error at each frequency.

In this paper, we propose to constrain the behavior of the model to lie within some tolerance of a prescribed function at nonmeasured frequencies. Thus, back to the disc, we propose the following modified setup. We suppose that \( 0 < \theta < \pi \) and consider \( I = \{ \phi^\theta, \phi \leq \theta \leq 2\pi - \alpha \} \), which is a proper closed symmetric subarc of the unit circle. We define \( J \) to be the closure of the complement of \( I \), i.e., \( J = \{ \phi^\theta, \alpha < \theta \leq \pi \} \). Also, we define the norm

\[
\|g\|_{\ell_\infty} = \text{ess sup} \{\|g(\phi^\theta)\|: \phi^\theta \in I \}
\]

for \( g \in L^\infty(I) \) and similarly for \( J \).

We provide ourselves with measurements \( a_k = f(z_k) + \eta_k \), with \( k = -N, \ldots, N \), where the \( z_k \) all lie within \( I \) with \( z_{-N} = e^{-i\alpha} \) and \( z_N = e^{i\alpha} \). We shall assume that the function \( f \) satisfies an \textit{a priori} estimate of the form

\[
|f(z) - h(z)| \leq r(z), \quad \text{for all } z \in J
\]

for some functions \( h \) and \( r \) belonging to \( C(J) \), with \( r \) a nonnegative gauge function that vanishes at the endpoints of \( J \).

This may seem absurd since \( f \) cannot be known exactly and therefore \( h \) cannot be determined to within a precision less than \( \epsilon \). However, there is actually no contradiction since, in the algorithm, the values of \( h \), just like those of \( f \), are assumed to be available only up to an uncertainty of \( \epsilon \).

Our aim is to find an approximate model \( f_N \) of \( f \) on \( I \) converging robustly on \( I \), namely

\[
\lim_{N \to \infty} \sup_{\|\phi\|_{\infty} \leq \epsilon} \|f_N - f\|_{\ell_\infty} = 0.
\]

Moreover, we also require that this approximation procedure asymptotically meets the gauge constraint on \( J \)

\[
\lim_{N \to \infty} \sup_{\|\phi\|_{\infty} \leq \epsilon} \left\{ \sup_{z \in J} |f_N(z) - h(z)| - r(z) \right\} \leq 0.
\]

However, from our incomplete set of data, we cannot constrain the model \( f_N \) to converge robustly to \( f \) on the whole circle; on \( J \), we will only get that \( f_N \) converges weakly-* to \( f \)

\[
\lim_{N \to \infty} \int_J f_N u \, d\theta = \int_J f u \, d\theta, \quad \text{for all } u \in L^1(J).
\]

Note also that this scheme is not untuned in the terminology of [9], and this is natural since we emphasized the necessity of constraining the model on \( I \) in one way or another. Here, we need a pointwise bound of the form (3) on \( J \).

A few comments on the role of \( r \) are perhaps in order. On the one hand, it seems more secure to choose \( r \) to be large on \( J \) so that (3) will be satisfied for a large class of functions \( h \). On the other hand, if one wants to get accurate modeling at infinity, it is necessary to have a good guess for the behavior of \( f \) outside the bandwidth, that is, to be able to make \( r \) small. Indeed, the approximation \( f_N \) to \( f \) that we are about to construct is such that \( |f_N - h| \to r \) uniformly on \( J \) as \( N \to \infty \) and \( \epsilon \to 0 \). Thus, if \( |f - h| \) is significantly smaller than \( r \), the values of \( f \) and \( f_N \) will not be close to each other on \( J \) and the weak-* convergence of \( f_N \) to \( f \) will cause \( f_N \) to oscillate on \( J \) with an amplitude which depends on the size of \( r \). Still, the model \( f_N \) asymptotically meets the gauge constraint (3) which is the main feature of our approach and warrants applications where one is not so much concerned with the behavior at high frequencies except for its boundedness.

In this paper, we describe an identification procedure meeting the above requirements and derive error bounds in the case of equally spaced points with a suitable choice of \( h \) (Section II); the procedure rests on an extension of results demonstrated in [2]. We then report on a numerical experiment from real data measured on a hyperfrequency filter by the French National Center for Spacial Research (CINES); see Section III.

We shall make the standing assumption, required for system-theoretical reasons though not for mathematical ones, that the unknown function \( f \) and the analytic model \( f_N \) we are seeking are real symmetric, namely that \( f(z) = f(\overline{z}) \) and the same for \( f_N \). Thus we need only take measurements in \( \alpha \leq \theta \leq \pi \) and obtain the others by complex conjugation. The reference function \( h \) is also assumed to verify this hypothesis on the (symmetric) arc \( J \).

II. AN ALGORITHM FOR APPROXIMATE MODELING

Suppose, for some unknown function \( f \in A(\mathcal{D}) \), that we are given the values \( (a_k) = (f(z_k) + \eta_k)_{k=-N}^N \), where \( z_k \) belongs to \( I \) and \( \eta_k \) is a noise sequence, assumed to be \( \epsilon \)-small in the \( L^\infty \) norm. We also assume that \( z_0 = -1 \) and that \( z_{-k} = z_k, a_{-k} = a_k, \) and \( \eta_{-k} = \eta_k \) for \( 1 \leq k \leq N \), which is the real-symmetric assumption made above.

Although we are seeking models in \( A(\mathcal{D}) \) only, we shall need to make excursions into \( H_\infty \). If \( g \in H_\infty \) and \( \sup_{\phi \in \mathcal{D}} |g(\phi^{\epsilon})| \geq ||g||_\infty \), recall (see, e.g., [10, ch. 3]) that the radial limit \( \lim_{\epsilon \to 0} g(\phi^{\epsilon}) \) exists for almost every \( \theta \) (even nontangential limits exist), and this serves as a definition for \( g(\phi^{\epsilon}) \). In this way, \( g(\phi^{\epsilon}) \) becomes a member of \( \ell^\infty(\mathcal{T}) \), with norm \( ||g||_\ell \), whose Fourier coefficients of negative index do vanish and whose restriction to any subset of positive measure on \( \mathcal{T} \) is nonzero if \( g \) is nonzero.

Given functions \( \alpha \in \ell^\infty(I) \), \( \beta \in \ell^\infty(J) \) we denote by \( \alpha \vee \beta \) the \( \ell^\infty(\mathcal{T}) \) function which is equal to \( \alpha \) on \( I \) and to \( \beta \) on the interior
the outer function

\[ w_{\alpha, \beta} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + \epsilon}{e^{i\theta} - \epsilon} \log (\alpha \vee \beta) d\theta \right\} \]  

(4)

This function is characterized by the following properties (see, e.g., [10, ch. 5]): \( w_{\alpha, \beta}(0) > 0 \), \( w_{\alpha, \beta} \) and \( w_{\alpha, \beta}^{-1} \) are both in \( H_\infty \), and \( |w_{\alpha, \beta}| = \alpha \vee \beta \), that is

\[ |w_{\alpha, \beta}(\zeta)| = \begin{cases} \alpha(\zeta), & \text{a.e. on } I, \\ \beta(\zeta), & \text{a.e. on } J. \end{cases} \]  

(5)

Moreover, observe that \( w_{\alpha, \beta} = w_{\alpha, 1} w_{1, \beta} \) so that \( w_{\alpha, \beta}^{-1} = w_{1/\alpha, 1/\beta} \).

Given a complex number \( c \) we let \( e_c(e^{i\theta}) \) be the function defined on \( J \) by

\[ e_c(e^{i\theta}) = \frac{1}{2\pi} (c(\theta + a) - c(\theta - a)) \]  

(6)

thus, \( e_c \) is linear in \( \theta \) and satisfies \( e_c(e^{i\alpha}) = c \) and \( e_c(e^{-i\alpha}) = \overline{c} \). All we shall need beyond the values \( a_c \) to make the procedure effective is to specify numerically \( r \) and approximate values \( b_k \) of \( b_k \) at points \( z_k \) on \( J \). When nothing is known on the shape of \( f \) except being proper and stable, a particularly simple choice is \( h = e_f(e_{z_k}) \) and \( b_k = e_{a_k}(z_k) \) on \( J \); since \( c \) is a bound for \( |e_f(e_{z_k}) - e_{a_k}| \) on \( J \), this allows \( h \) to be assigned numerically up to some precision less than \( \epsilon \).

There is nothing so special about the function \( e_c \) defined in (6) except that \( e_c(e^{i\alpha}) = e_c(e^{-i\alpha}) = \overline{c} \), and \( e_c \) goes uniformly to zero on \( J \) with \( c \); any function with the same properties could be used in its place, and this choice was mainly for simplicity and definiteness. If one wants a strictly proper model, one may use quadratic interpolants rather than linear ones for \( h \) to interpolate the value zero at one. We then need to choose \( r \) large enough so that (3) is satisfied. Of course, there is no way to ensure beforehand that it is the case, and this is revealed \textit{a posteriori} only if the convergence gets ruined, which means that \( r \) is too small somewhere on \( J \).

We begin with a result asserting that robust band-limited identification, as defined in the introduction, is possible at least when \( r \) satisfies a Lipschitz condition. The arguments in the proof will turn out to be constructive, providing us with an algorithm to solve the problem. Although, in practice, we use only a finite number of measurement points, it is convenient to state the convergence result in terms of an infinite sequence.

\textit{Theorem 2.1 (Convergence Result):} Assume the sequence \( (z_k) \) is dense in \( I \), and let \( (z_k^*) \) be a sequence dense in \( J \). Let \( r \) be a nonnegative Lipschitz-continuous function on \( T \) of exponent \( \mu \), \( 0 < \mu \leq 1 \), which is zero on \( I \). For every \( N, M \in \mathbb{N} \), there exists a mapping \( T_{r, N, M} : \mathbb{C}^{N+1} \times \mathbb{C}^M \rightarrow \mathcal{A}(\mathcal{D}) \) such that

\[ \mathcal{E}(N, M, f, h, \alpha, \beta) = \sup_{z \in \mathcal{I}} \left( |T_{r, N, M}(a_0, \ldots, a_N, b_1, \ldots, b_M)(z)| - |f \vee h(z) - r(z)| \right) \]  

for \( f \in L_\infty(I) \) and \( h \in L_\infty(J) \), we have

\[ \mathcal{E}(N, M, f, h, \alpha, \beta) \rightarrow 0 \text{ as } N, M \rightarrow \infty \text{ and } \epsilon \rightarrow 0 \]  

(7)

where \( \epsilon = \max \left\{ \|a_k - f(z_k)\|, \|b_k - h(z_k^*)\| \right\} \), provided that \( f \vee h \in \mathcal{C}(\mathcal{T}) \) and \( |f - h| \leq r \) on \( J \).

\textit{Remark 1:} The robustness property (7) is to be interpreted practically as a continuity property of \( \mathcal{E} \) with respect to \( N, M, \) and \( \epsilon \). More precisely, it means that for every \( \eta_0 > 0 \), there exist \( N_0, M_0 > 0 \) and \( \epsilon_0 > 0 \) such that if \( N > N_0 \) and \( M > M_0 \), and \( \epsilon < \epsilon_0 \), then \( \mathcal{E}(N, M, f, h, \alpha, \beta) < \eta_0 \). In particular, since \( r \) is zero on \( I \), (7) implies that, for \( \epsilon \) small enough and \( N, M \) large enough, \( T_{r, N, M}(a_0, \ldots, a_N, b_1, \ldots, b_M) \) is near \( f \) in \( L_\infty(I) \) while \( T_{r, N, M}(a_0, \ldots, a_N, b_1, \ldots, b_M) - h \) is approximately bounded by \( r \) on \( J \).

In the case where measurements are equally spaced, we get the following more precise bounds for \( \mathcal{E} \). We write \( \omega_f \) for the modulus of continuity of \( f \), that is

\[ \omega_f(\epsilon) = \sup_{|\theta - \alpha| \leq \epsilon} |f(e^{i\theta}) - f(e^{i\alpha})| \]  

(8)

and let \( P_n \) denote the space of trigonometric polynomials of degree at most \( s \).

\textit{Theorem 2.2 (Error Estimates):} Suppose that we are given \( \kappa \) points \( (z_k)_{k=1}^{\infty} \) and \( (z_k^*)_{k=1}^{M} \) that are equally spaced on \( T \) and \( s \leq (\kappa - 1) \). Then, there is a choice of \( h \in \mathcal{C}(J) \) such that with \( \tilde{f} = f \vee h \), \( \mathcal{E}(N, M, f, h, \alpha, \beta) \leq 4(2 + 1/\kappa)(\delta s + \epsilon) \) satisfies

\[ \mathcal{E}(N, M, f, h, \alpha, \beta) \leq 4(2 + 1/\kappa)(\delta s + \epsilon) \]  

(9)

\[ \mathcal{E}(N, M, f, h, \alpha, \beta) \leq 4(2 + 1/\kappa)(\delta s + \epsilon) \]  

Before proving Theorem 2.1, we need to establish a few facts concerning a bounded (dual) extremal problem, which plays here the same role as the Nehari extension does in robust identification over the whole circle. These results will extend some of those established in [2].

For every pair of functions \( \rho, \tau \in \mathcal{C}(J) \) with \( \rho > 0 \), we define

\[ \mathcal{B}_{\rho, \tau} = \{ \gamma \in \mathcal{H}_\infty \mid |\gamma - \tau| \leq \rho \text{ a.e. on } J \}. \]  

(10)

\textit{Proposition 1:} Let \( \psi \) be in \( L_\infty(I) \), \( h \) and \( \rho \) be in \( \mathcal{C}(J) \) with \( \rho > 0 \), and consider the following minimization problem:

\[ \min_{\psi \in \mathcal{B}_{\rho, \tau}} \|\psi - g_0\|_{L_\infty} = \min_{\psi \in \mathcal{B}_{\rho, \tau}} \|\psi - g_0\|_{L_\infty} = \beta_{\infty}. \]  

(11)

1) Problem (11) admits a solution \( g_0 \in \mathcal{B}_{\rho, \tau} \); when \( \psi \in h \in \mathcal{H}_\infty + C(T) \), the solution \( g_0 \) is unique. We assume now that \( \psi \) is not already the trace on \( I \) of a function in \( \mathcal{B}_{\rho, \tau} \) so that \( \beta_{\infty} > 0 \).

2) When \( \psi \in h \in \mathcal{H}_\infty + C(T) \), we have that

\[ \min_{\psi \in \mathcal{B}_{\rho, \tau}} \|\psi - g_0\|_{L_\infty} = \beta_{\infty}, \]  

\[ \min_{\psi \in \mathcal{B}_{\rho, \tau}} \|\psi - g_0\|_{L_\infty} = \beta_{\infty}. \]  

(12)

3) The function \( g_0 \) is a solution to problem (11) if and only if

\[ r_0 \equiv g_0 w_1/|\beta_{\infty}|/\rho \]  

(13)

is a solution to the implicit Nehari problem

\[ \min_{r \in \mathcal{F}_{\infty}} \|\psi \vee h \|_{L_\infty}/\rho - v_{\infty} = 1. \]  

(14)
Proof: The case where $\rho$ is constant on $J$ is contained in [2, Ths. 3 and 4]. What we need here is to consider an arbitrary positive function $\rho \in C(J)$.

The first step is to make sure that $B_{x,h}$ is nonempty. For this, put $m = \min_{I} \rho > 0$. Since any $g \in H_\infty$ such that $\|g - h\|_{L_\infty} \leq m$ belongs to $B_{x,h}$, the conclusion follows from the density of $A(\mathbb{D})|I|$ in $C(J)$ already pointed out (but for the half-plane) as Fact 1) in Section I. Next, setting
\[
\gamma = g w_{1,1}/\rho
\]
and taking (5) into account, we get
\[
\min_{x \in \mathbb{R}} \|x - g\|_{L_\infty} = \min_{\gamma \in \mathbb{R}} \|\gamma + w_{1,1}^{-1}/\rho\|_{L_\infty} = \min_{\gamma \in \mathbb{R}} \|\gamma w_{1,1}/\rho - \gamma\|_{L_\infty} = \beta_\infty.
\]
(14)

We are now back to the case of a constant bound on $J$ so that the cited results of [2] apply. This yields $g_0$ realizing the infimum above, hence $g_0 = \gamma_0 w_{1,1}/\rho$ as asserted in 1). If $\psi \vee h \in H_\infty + C(T)$, so does $(\psi \vee h) w_{1,1}/\rho$ since $H_\infty + C(T)$ is an algebra (see, e.g., [7, IX, Th. 2.2]); again from [2], we get uniqueness of $\gamma_0$, hence of $g_0$, thereby proving 1).

We turn to the proof of 2) and we observe, since $\beta_\infty > 0$ by assumption, that [2, Th. 4] implies $|\psi w_{1,1}/\rho - \gamma_0| = \beta_\infty$ a.e. on $I$ and $|h w_{1,1}/\rho - \gamma_0| = 1$ a.e. on $J$. Now, 2) follows at once from (5) and (14).

With regard to 3), we get from [2, Th. 3] that $\gamma_0 w_{1,1}/\beta_\infty$ is the solution to (13) and from Section IV of the cited paper that the value of this problem is indeed one. Now, (12) follows immediately from
\[
(14).
\]

Notice that $\beta_\infty$ is defined by (11) so that the weight $w_{1,1}/\beta_\infty$ depends on $\rho$, $\psi$, and $h$ through $\beta_\infty$. Hence, (13) is an implicit equation, and the right value for $\beta_\infty$ is the one that makes the infeum equal to one. That such a value is unique will follow from Lemma 2.2 below.

We are now in a position to establish our main result.

Proof of Theorem 2.1: The first step is to construct a trigonometric polynomial $p_N$, say of degree $d$, depending on $a_0, \ldots, a_n, b_1, \ldots, b_M$ and interpolation points $z_0, \ldots, z_N, z_1, \ldots, z_M$. Here, we can use standard robustly convergent interpolation procedures as in [11], [14], and [18] (in reality, we also use conjugate values at conjugate interpolation points).

However, $p_N$ cannot serve as a model because it does not belong to $A(\mathbb{D})$ in general. If $p_N \in A(\mathbb{D})$ for some $N$ and some $a_k$'s, we simply set $T_{x,N,M}(a_0, \ldots, a_N, b_1, \ldots, b_M) = p_N$, which meets all our requirements. We now assume throughout the proof that $p_N \notin A(\mathbb{D})$, and we notice in this case that $p_N$ cannot be the trace of any $H_\infty$ function on $I$. If $g$ were such a function, $w_d(p_N - g) \in H_\infty$ would vanish on $I$, hence should vanish identically, yielding $p_N = g$ so that $p_N$ would be in $A(\mathbb{D})$.

Let
\[
\sigma_N(z) = r(z) + \varepsilon_N \quad \forall z \in J
\]
for a sequence $(\varepsilon_N)$ of positive numbers to be determined later. This defines a $\mu$-Lipschitz-continuous positive function $\sigma_N$ on $J$.

The next stage is to get a function $f_N \in B_{\sigma_N, \rho}$ solution to the following bounded extremal problem:
\[
\min \{ \|p_N - g\|_{L_\infty} : g \in B_{\sigma_N, \rho} \} = \|p_N - f_N\|_{L_\infty} = \beta_\infty(N).
\]
(17)

For simplicity, we will write in the sequel $\beta_\infty = \beta_\infty(N)$. It follows from Proposition 1 that $\beta_\infty > 0$ and that $f_N$ does exist, is unique, and satisfies
\[
\|p_N - f_N\|_{L_\infty} = \begin{cases} \beta_\infty, & \text{a.e. on } I \\ 0, & \text{a.e. on } J. \end{cases}
\]
(18)

Again from Proposition 1, it follows that (17) is equivalent to finding $v_N$ which solves the Nehari problem
\[
\min_{v \in \mathbb{D} \cap I} \|p_N w_{1,1}/\beta_\infty \varepsilon_N - v\|_{L_\infty} = \|p_N w_{1,1}/\beta_\infty \varepsilon_N - v\|_{L_\infty} = 1
\]
(19)

where $f_N$ and $v_N$ are related by
\[
v_N = f_N w_{1,1}/\beta_\infty \varepsilon_N
\]
This provides us with $f_N \notin H_\infty$, and the problem is now, for each $N$, to choose $\varepsilon_N$, ensuring that $f_N \in A(\mathbb{D})$. Observe, indeed, that for arbitrary values of $\varepsilon_N$, the outer function $w_{1,1}/\beta_\infty \varepsilon_N$ is discontinuous at $e^{j\pi} \mu$ and that neither $v_N$ nor $a_{\text{fortiori}} f_N$ needs to be continuous on $T$. The following lemma will allow us to obtain this continuity from an appropriate choice of $\varepsilon_N$.

Lemma 2.1: Under the hypotheses of Theorem 2.1, and still assuming $p_N \notin H_\infty$, the following assertions hold.

1) For every fixed $N$, the quantity $\beta_\infty$ defined by (17) and (16) is continuous and decreasing with respect to $\varepsilon_N$, and the implicit equation
\[
\varepsilon_N = \beta_\infty
\]
(20)

admits a solution.

2) For every $N$ and the choice (20) of $\varepsilon_N$, the outer function $w_{1,1}/\beta_\infty \varepsilon_N$ is Lipschitz-continuous on $T$ of exponent $\mu$.

3) If $f \vee h \in C(T)$ and $|f - h| \leq \varepsilon$ on $J$, and for every $N$ we choose $\varepsilon_N$ as given by (20), then
\[
\lim_{N \to \infty} \beta_\infty = 0.
\]
(21)

Proof:

1) Observe from the convexity of the set $B_{\sigma_N, \rho}$ and of the norm function $\|\cdot\|_{L_\infty}$ that $\beta_\infty$ is a decreasing convex function of $\varepsilon_N$ and hence is continuously.

Now, $p_N[I] \in C(I)$ which is contained in the $L_\infty(I)$ closure of $H_\infty[I]$ (see [2]), so (16) and (17) imply that $\beta_\infty \to 0$ as $\varepsilon_N \to \infty$. Thus, for $\varepsilon_N$ large enough, $\beta_\infty < \varepsilon_N$.

Then let $\varepsilon_N \to 0$. Assume that $\beta_\infty < \varepsilon_N$ so that in particular $\beta_\infty \to 0$. In view of (17), and since $f_N$ remains bounded on $J$, this implies $p_N \notin H_\infty$; see [2, Proposition 3], which is a contradiction. Hence, $\beta_\infty \geq \varepsilon_N$ eventually, which proves 1) by the intermediate value theorem.

2) Since the gauge function $r$ is assumed to be $\mu$-Lipschitz on $J$, so is $\sigma_N$ from its definition (16) and also $1/\sigma_N$ as $\sigma_N \geq \varepsilon_N > 0$. Hence, writing $w_N = w_{1,1}/\beta_\infty \varepsilon_N$ for simplicity
\[
w_N = \begin{cases} 1/\beta_\infty = 1/\varepsilon_N = 1/\sigma_N(e^{j\pi} \mu), & \text{on } I \\ 1/\sigma_N, & \text{on } J \end{cases}
\]
is $\mu$-Lipschitz on $T$, and it remains for us to show that $w_N$ is also $\mu$-Lipschitz. By standard arguments, this reduces to the analogous result on conjugate functions, see [7, III, Th. 1.3]. This achieves the proof of 2).

3) By the construction of $p_N$
\[
\lim_{N \to \infty} \max_{|h| \leq \varepsilon_N} \|p_N - f \vee h\|_{L_\infty} = 0.
\]
(22)

Choose $\varepsilon_N = \beta_\infty$ for each $N$, as this is possible by 1), and assume that (21) is false. Then, (22) implies that for $N$ large enough and $\varepsilon$ small enough we will get $\|p_N - h\|_{L_\infty} \leq \varepsilon_N$, which is the contradiction we were after.
and, since \([f - h] \leq r\) on \(J\) by hypothesis, it turns out that
\[
[f - p_N] \leq \sigma_N
\]
on \(J\). Hence, for such \(N\) and \(f \in B_{\delta, N, p_N}\) and
necessarily \(\beta_\infty \leq \|p_N - f\|_{\infty}\), which, still from \((22)\),
tends to zero as \(N \rightarrow \infty, \epsilon \rightarrow 0\), a contradiction. This proves 3) and the lemma.

To complete the proof of Theorem 2.1, choose \(\varepsilon_N = \beta_\infty\). It follows from 2) of Lemma 2.1 that \(p_N w_1/1/\beta_\infty, 1/\epsilon + r\) is \(\mu\)-Lipschitz hence a forteriori
Dini-continuous on \(T\), and the Carleson-Jacobs theorem [7, IV, Th. 2.1] implies that the solution \(u_{N, \cdot}\) to \((A(D))\). Again from 2) of Lemma 2.1
\[
w_0 \circ \sigma_N^{-1} = (w_1/1/\beta_\infty, 1/\sigma_N)^{-1}
\]
is continuous (since it is \(\mu\)-Lipschitz) so that
\[
f_N = v_N w_0 \circ \sigma_N^{-1}
\]
lies in \(A(D)\).

We finally verify that \(T_{\epsilon, N, M}(a_0, \ldots, a_N, b_1, \ldots, b_M) = f_N\) does the job. Indeed, on \(I\) we have the inequality \([f - f_N] \leq [f - p_N] + [p_N - f_N]\), and the last term is equal to \(\beta_\infty\) by \((18)\); thus, \((22)\) and \((21)\) give the desired behavior on \(J\).

Moreover, on \(J\), we get \([h - f_N] \leq [h - p_N] + [p_N - f_N]\) and, since \(\sigma_N = r + \beta_\infty\), the result for \(J\) follows from \((18)\), \((21)\), and \((22)\). This establishes \((7)\) and ends the proof of Theorem 2.1.

Remark 3: If the data are obtained by the Möbius transform of measurements in continuous time, the question arises as to whether the inverse transform of \(f_N = T_{\epsilon, N, M}(a_0, \ldots, a_N, b_1, \ldots, b_M)\) is the transfer function of a BIBO-stable system. The answer is yes. Indeed, it follows from \((16)\) that the solution \(u_N\) of the Nehari problem associated to the \(\mu\)-Lipschitz function \(p_N w_1/1/\beta_\infty, 1/\epsilon + r\) is itself \(\mu\)-Lipschitz and hence has \(H^1\) derivative. Hence, so does \(f_N\).

Transforming back to the half-plane yields a function \(G_N\) whose derivative is in \(H^1(\Pi_+\)). Mimicking the classical proof of Hardy's inequality [7, II, ex. 8], one obtains that \(G_N\) is the Laplace transform of some impulse response belonging to \(L^1(0, \infty)\) plus the constant \(f_N(1)\) which is bounded in modulus by \([h(1)] + r(1)\).

Having established Theorem 2.1, we must tie one loose end to make the proof constructive, namely how does one find in practice \(\beta_\infty\) in order to solve the Nehari problem \((19)\) and to select \(\varepsilon_N\) according to \((20)\). This can be done by a dichotomy procedure which rests on Lemma 2.2 below.

For every \(\varepsilon > 0\), define the map \(\Delta_\varepsilon\)
\[
\Delta_\varepsilon[\cdot] : [0, \infty] \rightarrow [0, \infty],
\]
\[
\delta \mapsto \min_{v \in H^\infty} \|p_N w_1/1/\epsilon + r - v\|_{\infty},
\]

\textbf{Lemma 2.2:} If \(p_N \not\in H^\infty\), then for every \(\varepsilon > 0\), the map \(\Delta_\varepsilon\) is bounded from (0, \(\infty\)) onto (0, \(\infty\)), is continuous, and is monotonically decreasing.

\textbf{Proof of Lemma 2.2:} Let \(\varepsilon > 0\). Then for every real \(\delta > 0\), the function \(p_N w_1/1/\epsilon + r \in H^\infty + C(T)\). Hence, by [7, IV, Th. 1.3, Th. 1.7], there is a unique function \(v_\delta \in H^\infty\) such that
\[
\Delta_\varepsilon(\delta) = \|p_N w_1/1/\epsilon + r - v_\delta\|_{\infty}. \tag{23}
\]

Let \(\delta_1, \delta_2 > 0, \delta_1 \neq \delta_2\). Then, from the definition of \(\Delta_\varepsilon\), we get
\[
\Delta_\varepsilon(\delta_1) < \|p_N w_1/1/\epsilon + r - v_{\delta_2}\|_{\infty} = \|v_{\delta_2}\|_{\infty}.
\]

That the inequality above is strict follows from the uniqueness of the \(v_\delta\) and the fact that \(v_{\delta_2} \neq v_{\delta_1}\). Indeed, \([p_N w_1/1/\epsilon + r - v_{\delta_1}] = \Delta_\varepsilon(\delta_1)\) and \([p_N w_1/1/\epsilon + r - v_{\delta_2}] = \Delta_\varepsilon(\delta_2)\) are constant a. e. on \(T\), while
\[
\|p_N w_1/1/\epsilon + r - v_{\delta_2}\|_{\infty} = \Delta_\varepsilon(\delta_2) = \|v_{\delta_2}\|_{\infty}.
\]

By Lemma 2.2, \(\Delta_\varepsilon\) may be computed by a dichotomy procedure in view of the monotonicity of \(\Delta_\varepsilon\).

Given \(p_N\), which in turn defines \(\Delta_\varepsilon\), what we want to find now is the unique value \(\varepsilon = \varepsilon_N\) for which \(\beta_\varepsilon(\varepsilon_N) = 0\), if any. To analyze the behavior of \(\Delta_\varepsilon\) when \(\delta \rightarrow 0\), we write
\[
\Delta_\varepsilon(\delta) = \max\left\{\frac{1}{\delta}, \frac{\|p_N w_1/1/\epsilon + r - v_\delta w_\delta\|_{\infty}}{\|w_\delta\|_{\infty}}\right\}. \tag{24}
\]

We claim that if the first argument of the \(\max\) remains bounded as \(\delta \rightarrow 0\), then the second does not. Indeed, \(v_\delta w_\delta\) would otherwise be a family of \(H^\infty\) functions converging to \(p_N w_1/1/\epsilon + r\) in \(L^\infty(I)\) as \(\delta \rightarrow 0\) but remaining bounded on \(J\); in view of \(p_N \not\in H^\infty\), this would contradict [2, Proposition 3] (Fact 2) of Section I (rephrased on the disc). Thus, we get \(\lim_{\delta \rightarrow 0} \Delta_\varepsilon(\delta) = 0\). This shows that \(\Delta_\varepsilon\) is bounded from \(0, \infty\).

By Lemma 2.2, we can associate to every \(\varepsilon > 0\) a unique \(\beta_\varepsilon(\varepsilon) > 0\) such that \(\Delta_\varepsilon(\beta_\varepsilon(\varepsilon)) = 1\), and \(\beta_\varepsilon(\varepsilon)\) may be computed by a dichotomy procedure in view of the monotonicity of \(\Delta_\varepsilon\).

Given \(p_N\), which in turn defines \(\Delta_\varepsilon\), what we want to find now is the unique value \(\varepsilon = \varepsilon_N\) for which \(\beta_\varepsilon(\varepsilon_N) = 0\) so that both \((19)\) and \((20)\) are satisfied. In view of the monotonicity asserted in 1) of Lemma 2.1, this can again be solved by dichotomy.

This process, which is somehow similar in spirit to the \(\gamma\)-iteration used in \(H^\infty\)-control, settles our constructive approach to Theorem 2.1. However, it requires solving a series of Nehari problems, the solution of which can be numerically estimated only when the function to be approximated is continuous. Indeed, in this case, it can be represented arbitrarily well in \(L^\infty(T)\) by a rational function (using for instance the Jackson polynomials previously introduced to compute \(p_N\)) whose Hankel operator has finite rank and thus possesses a finite singular-value decomposition allowing one to solve the associated Nehari problem in various fashions (see, e.g., [4] and [5]).

Now, the typical Nehari problem we must solve here is associated to a function of the form
\[
p_N w_1/1/\epsilon + r \tag{24}
\]
for some positive numbers \(\varepsilon\) and \(\delta\), and such a function is clearly discontinuous at \(\varepsilon \pm \in\) in general. However, \((24)\) is continuous at any other point on \(T\), because it is even \(\mu\)-Lipschitz there; indeed, an outer
function whose log-modulus is \( \mu \)-Lipschitz in the neighborhood of some point is itself \( \mu \)-Lipschitz at this point. This is the local version of 2) of Lemma 2.1, and it is proved in the same manner except that we must appeal, this time, to a local version of the regularity theorem for conjugate functions (see the proof of [7, III, Corollary 1.4]). To circumvent the discontinuity problem at \( e^{\pm i\alpha} \), we introduce another Nehari problem, equivalent to (19). Let \( p \) be the first-order trigonometric polynomial which coincides with \( p_N \) at \( e^{\pm i\alpha} \) so that \( (p_N - p)w_{1/s,1/(e+i\gamma)} \) is continuous on \( \mathbb{T} \). The Nehari problem

\[
\min_{g \in C(\mathbb{T}, \infty)} \left\| (p_N - p)w_{1/s,1/(e+i\gamma)} - g \right\|_{\infty}
\]

(25)
is clearly equivalent to (19) under the transformation \( v = g + pw_{1/s,1/(e+i\gamma)} \) and consequently assumes the same value. The dichotomy procedures described before may now be performed numerically by solving (25) iteratively, and this was done in the example presented in Section III.

**Proof of Theorem 2.2:** We are given that the points \( z_k \) and \( z'_k \) are the \( \kappa \)th roots of unity for some integer \( \kappa \). We take here \( h \) to be the function \( e_{f(s, \omega)} \) defined by (6). The approximate values \( b_k \) of \( h \) on \( J \) will be taken to be \( e_{\alpha_N}(z'_k) \). This choice is mainly for definiteness and is not essential, although it leads to simpler estimates.
To construct the trigonometric polynomial \( p_N \) we use the noisy values \( (a_k)_{k=-N}^{N} \) of \( f \) on \( I \) together with the values \( (b_k) \) on \( J \) to produce the discrete de la Vallée–Poussin polynomial \( V_{\infty N} \) with \( \epsilon \geq 4s+1 \), as in [11]. Because \( \epsilon = \max \{ |a_k - f(z_k)|, |b_k - h(z_k')| \} \), this is equivalent to using measurements of \( f = f \vee h \) with an error of at most \( \epsilon \) and hence (see [11, Th. 3.1]):

\[
\| \hat{f} - p_N \|_\infty \leq (4 + 2/s) \text{dist}(\hat{f}, P_s) + \epsilon.
\]

Now \( \text{dist}(\hat{f}, P_s) \leq \frac{2}{3} \omega_j(\pi/s+1) \) by Jackson's theorem [16], which, given the definition of \( \hat{f} \), implies that (10) holds. One could improve upon this bound by considering a smoother extension \( \hat{f} \) to \( f \). One way to do this would be to choose \( h \) to be a function cubic (in \( \theta \)) which matches \( f \) and its derivatives at the points \( z_{\pm N} \) (this, of course, assumes one is able to estimate these derivatives). To get the \( (b_k) \), one could use in this case a cubic polynomial matching the noisy values \( a_{\pm(N+1)} \) and \( a_{\pm N} \), as in [14].

Recall now that the final model \( f_N \) is the solution to the extremal problem (17). Moreover, by the proof of Lemma 2.1, we see that \( \beta_\infty(N) \leq \| \hat{f} - p_N \|_\infty \). This gives us the following estimates for the
error in the identified model $f_N$:
\[ \| f - f_N \|_{i, \infty} \leq \| f - p_N \|_{i, \infty} + \| p_N - f_N \|_{i, \infty} \leq 2 \| \hat{f} - p_N \|_{\infty} \]
and on $J$
\[ | h - f_N | \leq | h - p_N | + | p_N - f_N | \]
\[ \leq \| h - p_N \|_{i, \infty} + (r + \beta_\infty(N)) \leq r + 2 \| \hat{f} - p_N \|_{\infty} . \]

### III. Numerical Examples

Our main example consists of real data measured on a hyper-frequencies filter of the CNES. The bandwidth $I$ is defined by $a = \pi / 2$, and we are given 801 noisy pointwise values $(a_k)$, so that $N = 400$. We first complete these data outside the bandwidth by rough estimates, and we construct the trigonometric polynomial $p_N$ using discrete de la Vallée-Poussin polynomials. Fig. 1(a) shows the result of the classical Nehari extension to $p_N$, which gives rise to an error of value 0.0236 in $L^\infty (T)$. We then compute the solution $f_N$ to the constrained approximation problem associated to $p_N$ for different gauge functions $r$ until an acceptable tradeoff is found between $\beta_\infty$ and $r$; these gauge functions are plotted in Fig. 1(b). If no satisfactory compromise can be found, one can change the reference behavior on $J$, using the previous computations, in order to make a more accurate choice. The corresponding results are shown in Fig. 2(a) and (b).

We have also considered the function $f(z) = 3(z^2 + 1)/(z^2 + 2z + 5)$, already studied (using information on the whole circle) in [8], [11], and [12]. Full details can be found in [3].

### IV. Conclusion

In this paper, we presented a framework for robust band-limited identification which extends the existing one for robust identification on the whole axis (or circle) that was introduced in [9]. We also developed a constructive algorithm to perform such a band-limited identification, which recovers the transfer-function on the bandwidth in a robust fashion while meeting gauge constraints at the remaining frequencies. The procedure is very similar in spirit to the two-stage algorithms proposed in [8], [9], [11], and [12] but appeals to a bounded extremal problem which may be seen as a generalization of the classical Nehari problem. We also derived error bounds in a standard case and presented examples on real data.

There are at least two further questions which, in our opinion, deserve further study. The first arises from the observation that the identification procedure can be applied to any sequence of data $a_0, a_1, \cdots, b_1, b_2, \cdots$; the question is: “What is the limit behavior of $T_{r,N,M}(a_0, \cdots, a_N, b_1, \cdots, b_M)$ if the data do not converge (pointwise in $L^\infty$) to some interpolation sequence $f(a_0), f(a_1), \cdots, h(b_1), h(b_2), \cdots$ with $f \in A(D)$, $f \in C(\bar{T})$, and $|f - h| \leq r$ on $J^?''$? The second question stems from the fact that our identification scheme converges uniformly to $f$ on $I$ but only weak-$\ast$ to $f$ on $J$. This is enough to recover $f$ uniformly on compact subsets of the half-plane (or of the disk) by the Poisson formula but not to recover $f$ itself. Now, still assuming (3), what additional hypotheses would be needed on $f$ in order to design an algorithm producing some stronger type of convergence? We think both questions are important in connection with the practical value of such schemes.

### References


