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Robust Identification of Strongly Stabilizable Systems

Pertti M. Mäkilä, *Member, IEEE,* and Jonathan R. Partington

Abstract—**For strongly stabilizable systems for which a strongly stabilizing controller is known approximately, we consider system identification in the graph, gap, and chordal metrics using robust H_∞ identification of the closed-loop transfer function in the framework proposed by Helmicki, Jacobson, and Nett. Error bounds are derived showing that robust convergence is guaranteed and that the identification can be satisfactorily combined with a model reduction step. Two notions of robust identification of stable systems are compared, and an alternative robust identification technique based on smoothing, which can be used to yield polynomial models directly, is developed.**

I. Introduction

ROBUST identification in H_∞ for stable linear systems (in either continuous time or discrete time) involves the measurement of a finite number of frequency response values of the system transfer function (which in general may be corrupted by noise). From these, one obtains an approximation to the original system which should converge in the H_∞ sense as the number of measurements and the noise level simultaneously converge to infinity and zero, respectively.

This notion of "worst-case" identification essentially originates with Helmicki, Jacobson, and Nett, [7]–[10], who gave a concept of robustly convergent identification, and provided the first algorithm achieving this: it was nonlinear. In addition, they produced linear algorithms tuned to *a priori* information about the unknown system. Gu and Khargonekar [4], [5] provided an untuned linear algorithm that, though not robustly convergent, performs satisfactorily in practice, and gave a large class of rapidly convergent nonlinear algorithms. In Partington [16] and [17] it was shown that there is no robustly convergent (untuned) linear algorithm: additionally some rapidly convergent nonlinear algorithms were given, and links with interpolation explored. Connections with approximation were also explored by Mäkilä [12] and Mäkilä and Partington [13].

For unstable infinite-dimensional systems it is also of interest to perform a robust identification procedure. For practical reasons, one normally works with a stable closed-loop transfer function (that is, one assumes the existence and possibly incomplete knowledge of a stabilizing controller), and makes measurements of the closed-loop frequency response, which again may be corrupted by noise. Given this, one wishes to obtain an estimate for the unstable plant, and the appropriate ways to do this are in the graph, gap, and chordal metrics; as a further application, one may then wish to design a more robust controller for the original unstable plant, and this is the correct framework in which to do this (see, e.g., [18]) As we shall demonstrate in Section III, if one relies on identifying one closed-loop transfer function, it is essential that the controller itself be stable, and so we shall make this assumption.

In Section II, we set up the mathematical background to the problem in detail and say more about the identification process. In particular, we give a discussion of notions of robust identification in $A(D)$ and H_∞ for stable systems. For stabilizable systems, three metrics are useful for comparing plants, and these are discussed.

Section III then contains some detailed estimates for the approximation errors resulting from identification of the closed-loop transfer function, which show that one can achieve robust convergence in the framework we consider. It is also shown that performing a model reduction step on the identified model is justifiable (and in general desirable), since robust convergence of the low-order model is still achievable.

Section IV discusses a smoothing technique which provides an alternative to the Helmicki-Jacobson-Nett method of obtaining stable transfer function models via a Nehari step. This has the advantage that it can be used to produce identified models which are polynomials (or lie in some other preassigned model set) in a direct fashion.

In Section V, a very simple closed-loop example is analyzed, and the effects of robust convergence are demonstrated.

II. Mathematical Background

We shall be considering infinite-dimensional linear time-invariant systems, and our results will be applicable to both discrete time and continuous time systems. In continuous time, the transfer function $G(s)$ of a stable system is a bounded analytic function in the right-half complex plane, i.e., an element of $H_\infty(\mathbb{C}_+)$ acting on $H_2(\mathbb{C}_+)$ (the space of analytic functions in the right-half plane with square integrable boundary values) by multiplication; in discrete time, a stable transfer function $g(z)$

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P. M. Mäkilä is with the Department of Engineering, Åbo Akademi, Swedish University of Åbo, SF-20500, Åbo, Finland.

J. R. Partington is with the School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom.

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(obtained by taking z-transforms) is analytic and bounded on the set of complex numbers of modulus greater than one. In either case, we can and will transform the situation in a norm-preserving manner so as to consider functions analytic and bounded on the unit disk: in the first case, by defining $f(z) = G(\mathscr{M}z)$ so that $G(s) = f(\mathscr{M}s)$ where $\mathscr{M}$ is the Möbius map $\mathscr{M}z = (1 - z)/(1 + z)$; and in the second case, by defining $f(z) = g(1/z)$. Our results will be stated in the scalar case for clarity but they extend to the multivariable case with minor changes—see the remarks at the end of Section III.

In this paper, we shall be concerned with identifying unstable plants, that is, in our formulation, ones which are merely meromorphic inside the unit circle. We assume, however, that a strongly stabilizing controller exists and is known at least approximately, that is, that there is a function $C(z) \in H_\infty$ such that $G(z) = P(z)/(1 + C(z)P(z)) \in H_\infty$. (For a further discussion of this, see [12].) In fact, we shall make the stronger assumption that C and G are in the *disk algebra* $A(D) = H_\infty \cap C(\mathbb{T})$, the space of functions analytic in the open disk $D = \{|z| < 1\}$ and continuous in the closed disk $\{|z| \le 1\}$; this is desirable since we shall be interested in rational approximation and identification by means of boundary values on the unit circle. The disk algebra clearly contains all stable finite-dimensional (rational) systems, after the transformations described above.

For H_∞ identification of a function $G(z) \in A(D)$ in the framework of Helmicki, Jacobson, and Nett [7], [10] one is given a set of possibly corrupted values of the function G on the unit circle, say, $(g_1, \cdots, g_n) \in \mathbb{C}^n$, where $g_k = G(z_k) + \eta(z_k)$ for some interpolation points $z_1, \cdots, z_n$ (usually the nth roots of unity) and some noise function $\eta \in l_\infty(\mathbb{T})$, the space of bounded functions on the circle $\mathbb{T}$. From these, one obtains a function $G' = T_n(g_1, \cdots, g_n) = \tilde{T}_n(G, \eta) \in A(D)$ which approximates G.

For robust approximation of G one requires the following condition:

$$\lim_{n \to \infty, \epsilon = \|\eta\|_\infty \to 0} \|G' - G\|_\infty = 0 \tag{2.1}$$

where G' is an approximation to G produced from the corrupted measurements on the unit circle. This condition allows one to discuss robustness with a minimum of *a priori* information about the unknown system, and ties in well with the notions of robust stability in control theory. We shall see a further benefit of this property in Theorem 2.1 (in that this form of robustness guarantees convergence in the Helmicki, Jacobson, and Nett setting); further motivation is provided by (2.3) below, where we consider an uncertainty set-up in which the unknown system lies within some ball of systems. Yet another application is that one can consider *a priori* information about the unknown system of a more general nature.

Robust identification of G in the framework of Helmicki, Jacobson, and Nett is defined as follows: for $M \ge 0$ and $\rho > 1$ let $H_\infty(\rho, M)$ be the set of all functions G analytic and bounded in the open disk of radius ρ and such that

$$\|G\|_{\rho,\infty} = \sup\{|G(z)| : |z| < \rho\} \le M < \infty.$$

Then the Helmicki, Jacobson, and Nett [7], [10] condition is as follows:

$$\lim_{n \to \infty, \epsilon \to 0} \sup_{\|\eta\|_\infty \le \epsilon} \sup_{G \in S} \|G' - G\|_\infty = 0 \tag{2.2}$$

where $S = H_\infty(\rho, M)$. In other words, one demands uniform approximation over certain subsets of $A(D)$ which are *relatively compact*, that is such that any sequence of functions in the set has a norm-convergent subsequence. In fact $A(D)$-robustness as in (2.1) guarantees this for *any* relatively compact subset S.

Theorem 2.1: An untuned identification algorithm yielding G' as an approximation to G that satisfies the robustness condition (2.1) also satisfies (2.2) for any relatively compact S, in particular for $S = H_\infty(\rho, M)$, $\rho > 1$, $M \ge 0$.

Proof: Suppose that we have a robust identification algorithm satisfying (2.1). If condition (2.2) is breached, then there are a set S, a number $\epsilon > 0$, and a sequence of functions $(F_k) \in S$ and approximants F_k' constructed from k measurements and a noise level of at most $1/k$ such that $\|F_k - F_k'\| \ge \epsilon$ for all k. By relative compactness, there is a subsequence of the F_k which converges in the $A(D)$ norm to some function $F \in A(D)$. But then $\|F - F_k'\| > \epsilon/2$ for arbitrarily large values of k, although F_k' are approximants to F constructed with k points and a noise level in the measurements of F of at most $1/k + \|F - F_k\|$, which tends to zero as $k \to \infty$ in this subsequence. However, this contradicts (2.1). □

It seems unlikely that (2.1) and (2.2) are equivalent (even if (2.2) holds for all ρ and M), but algorithms satisfying (2.2) generally seem to satisfy (2.1) also.

For a relatively compact subset $S \subset A(D)$, the nature of one's *a priori* information may lead one to consider worst-case identification over the set

$$S_\delta = \{s + t : s \in S, t \in A(D), \|t\| \le \delta\}$$

for some $\delta > 0$. It follows easily, be treating t as an extra component to the noise, that the following condition is also satisfied for an identification procedure satisfying (2.1).

$$\lim_{n \to \infty, \epsilon \to 0, \delta \to 0} \sup_{\|\eta\|_\infty \le \epsilon} \sup_{G \in S_\delta} \|G' - G\|_\infty = 0. \tag{2.3}$$

Identification in $A(D)$ is commonly implemented as follows. Let $E_n : A(D) \times l_\infty(\mathbb{T}) \to \mathbb{C}^n$ be the experiment operator defined by $E_n(G, \eta) = (G(z_k) + \eta(z_k))_{k=1}^n$, and let $R_n : \mathbb{C}^n \to C(\mathbb{T})$ be a linear operator, the interpolation operator. Here we regard $\mathbb{C}^n$ as l_∞^n, that is, we give it the supremum norm.

An example of such an operator R_n (taken from [17]) is as follows. Let $z = e^{i\theta}$ with $0 \le \theta \le 2\pi$ denote a point of the unit circle, and define the Dirichlet kernel by

$$D_k(\theta) = \frac{\sin(k + 1/2)\theta}{2 \sin \theta/2} = (1/2) \sum_{r=-k}^{k} z^r$$

the Fejér kernel by

$$K_p(\theta) = (1/(p+1)) \sum_{k=0}^{p} D_k(\theta)$$

the Jackson trigonometric polynomials ($0 \le p < n$) by

$$J_{p,n}(a,\theta) = (2/n) \sum_{k=1}^{n} a_k K_p(\theta - t_k)$$

where $t_k = 2\pi k/n$ and $a_k = G(z_k) + \eta(z_k)$ for $1 \le k \le n$, and the discrete De La Vallée Poussin trigonometric polynomials ($n \ge 4p+1$) by

$$V_{p,n}(\theta) = \big((2p+1)J_{2p,n} - (p+1)J_{p,n}\big)/p.$$

We set $R_n(a_1, \cdots, a_n) = V_{p,n}$, a trigonometric polynomial of degree $2p$.

Then the following result shows that any such R_n can be used as the basis of a robustly convergent nonlinear algorithm by combining it with a Nehari approximation step—whereby a function of $C(\mathbb{T})$ is replaced by its best approximation from $A(D)$, as in [7], [10], [4], [5], [16] and [17].

Proposition 2.2: Let R_n be a uniformly bounded sequence of interpolation operators as above such that $\|R_n E_n(G,0) - G\|_\infty \to 0$ for all G in some dense subset Q of $A(D)$. Then the two-step identification algorithm T_n, consisting of R_n followed by a Nehari best approximation step, is robustly convergent in the sense of (2.1) and hence (2.2).

Proof: Let K be such that $\|R_n\| \le K$ for all n. Let $G \in A(D)$ and $\eta \in l_\infty(\mathbb{T})$. Then, using the linearity of E_n and of R_n, and writing $\tilde{T}_n(G,\eta)$ for the identified model, we have that

$$\|\tilde{T}_n(G,\eta) - R_n E_n(G,\eta)\|_\infty \le \|G - R_n E_n(G,\eta)\|_\infty$$

and hence

$$\|G - \tilde{T}_n(G,\eta)\|_\infty \le 2\|G - R_n E_n(G,0)\| + 2\|R_n E_n(0,\eta)\|_\infty.$$

Let $\delta > 0$ be given. Then there is a function $q \in Q$ such that $\|G - q\|_\infty \le \delta$. Thus, since

$$\|G - R_n E_n(G,0)\|_\infty \le \|q - R_n E_n(q,0)\|_\infty + (1+K)\delta$$

we see that

$$\|G - \tilde{T}_n(G,\eta)\|_\infty \le 2\|q - R_n E_n(q,0)\|_\infty + 2(1+K)\delta + 2K\|\eta\|_\infty.$$

This can be made as small as desired by first choosing δ and q, and then by demanding that n be sufficiently large and $\|\eta\|_\infty$ be sufficiently small. Hence, (2.1) follows, with $G' = \tilde{T}_n(G,\eta)$. □

When dealing with stabilizable systems P such that $G = P/(1+CP) \in A(D)$ for some $C \in A(D)$, robust approximation in a metric μ such as the graph, gap, or chordal metrics (defined below) requires that

$$\lim_{n\to\infty,\epsilon=\|\eta\|_\infty\to 0,\|C-C'\|\to 0} \mu(P,P') = 0 \tag{2.4}$$

where $P' = G'/(1 - C'G')$, with C' an $A(D)$ estimate of C and with G' an approximation to G produced as in (2.1). Again we may ask for uniformity of approximation over various relatively compact sets of meromorphic functions, but we do not do so here.

Three natural metrics can be defined to measure the closeness of one meromorphic function to another.

The first two assume that P_1 and P_2 have normalized coprime factorizations $P_k = N_k/D_k (k = 1, 2)$ with $N_k^* N_k + D_k^* D_k = 1$ on $\mathbb{T}$, and with X_k, Y_k in H_∞ such that $X_k N_k + Y_k D_k = 1$. See [19] for details. We write $\hat{G}_k = \begin{bmatrix} N_k \\ D_k \end{bmatrix}$, for $k = 1, 2$.

The *graph* metric $d(P_1, P_2)$ is defined by

$$d(P_1,P_2) = \max\left\{\inf_{Q\in H_\infty,\|Q\|\le 1} \|\hat{G}_1 - \hat{G}_2 Q\|_\infty, \inf_{Q\in H_\infty,\|Q\|\le 1} \|\hat{G}_2 - \hat{G}_1 Q\|_\infty\right\} \tag{2.5}$$

(see [3] and [19]).

Likewise, the *gap* metric $\delta(P_1, P_2)$ can be defined by

$$\delta(P_1,P_2) = \max\left\{\inf_{Q\in H_\infty} \|\hat{G}_1 - \hat{G}_2 Q\|_\infty, \inf_{Q\in H_\infty} \|\hat{G}_2 - \hat{G}_1 Q\|_\infty\right\} \tag{2.6}$$

(see [3]). Since $\delta(P_1, P_2) \le d(P_1, P_2) \le 2\delta(P_1, P_2)$ these define the same topology, which is the natural topology for considering robust stabilization of systems.

A third metric, the *chordal* metric $\mathscr{K}(P_1, P_2)$, may be defined as follows. For two complex numbers w_1 and w_2 the chordal distance between them is

$$\mathscr{K}(w_1,w_2) = \frac{|w_1 - w_2|}{\sqrt{(1+|w_1|^2)(1+|w_2|^2)}} \tag{2.7}$$

with $\mathscr{K}(w,\infty) = 1/\sqrt{1+|w|^2}$. For two meromorphic functions P_1, P_2 in the disk we write

$$\mathscr{K}(P_1,P_2) = \sup\{\mathscr{K}(P_1(z),P_2(z)) : |z| < 1\}. \tag{2.8}$$

More about this metric can be found in [6]. Reference [2] shows that the chordal metric coincides with the gap metric when restricted to systems with no unstable zeros or poles. Also, [15] shows that the chordal metric gives the same topology as the other two metrics, at least if we restrict to the set of functions with coprime factorizations in terms of functions analytic in a disk of radius ρ for some $\rho > 1$.

III. Estimates for Identification Errors

In this section, we consider the accuracy with which an unknown unstable SISO open-loop transfer function can be approximated by an identified model, obtained from a set of frequency-response measurements of a stable closed-loop.

Let P be the given system, and C a stabilizing controller, such that C and $G = P/(1 + CP)$ are both in H_∞. By taking measurements of G on the unit circle, we obtain an identified model G'. However, C itself may not be known exactly, and we may only have an approximation C' to it. These combine to give an approximation P' to P.

The following two examples show that the convergence of a set of approximate measurements of C and $G = P/(1 + CP)$ need not yield a convergent set of approximants to $P = G/(1 - CG)$ in the case when C is itself unstable. For convenience, we shall work directly with continuous-time systems and denote the variable by s.

Example: i) Let $P(s) = 1$, $C(s) = 1/s$, $G(s) = s/(s + 1)$. A close approximation to C in the gap topology is $C' = 1/(s + a)$ for a close to zero. If we assume that $G' = G$, then $P' = s(s + a)/(s^2 + as + a)$, which does not converge to P in the gap topology as $a \to 0$ (consider the value at the origin.)

Example: ii) Let $P(s) = 1$, $C(s) = 1/s$, $G(s) = s/(s + 1)$ again, and suppose that $C' = C$, i.e., C is known exactly, and that $G' = (s + a)/(s + 1)$, which is close to G in H_∞ for a close to zero. Now $P' = s(s + a)/(s^2 - a)$, which again does not converge to P in the gap topology as $a \to 0$.

Our first result estimates the distance of P' from P in the graph, gap, and chordal metrics under the necessary assumption that C is itself stable. As an alternative to using $G = P/(I + CP)$ one can use any one of the four stable transfer functions in the matrix

$$\begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} [C \quad I]$$

and similar results hold with slightly different error estimates.

Theorem 3.1: Let P be a possibly unstable open-loop transfer function and $C \in H_\infty$ a stable controller such that the closed-loop transfer function $G = P/(1 + CP) \in H_\infty$ is stable; let G' be an approximation to the closed-loop transfer function G, with $\|G - G'\| = \alpha$, C' an approximation to the controller C, with $\|C - C'\| = \beta$, and $P' = G'/(1 - C'G')$ the corresponding approximation to the open-loop transfer function P. Then $\|GC - G'C'\| \le \gamma = \|G\| \beta + \alpha \|C\| + \alpha\beta$, and the following error bounds hold for the graph metric $d(P, P')$, the gap metric $\delta(P, P')$ and the chordal metric $\mathscr{K}(P, P')$.

i) $d(P, P') \le 2\eta/(1 - \eta)$, where $\eta = \sqrt{\alpha^2 + \gamma^2}(1 + \|C\|)$;

ii) $\delta(P, P') \le 2\eta/(1 - \eta)$, where η is as in i);

iii) $\mathscr{K}(P, P') \le (1 + \|C\|)^2 \left(\dfrac{\alpha}{1 - \max(\alpha, \gamma)(1 + \|C\|)} + \beta \max\left(1, \dfrac{\|G\|(\|G\| + \alpha)}{1 - \gamma(1 + \|C\|)}\right)\right)$.

The same inequalities hold if the roles of C, G, and P and their primed equivalents are reversed.

Proof: First, the estimate to $GC - G'C'$ follows on writing

$$GC - G'C' = G(C - C') + (G - G')C'.$$

i) Note that G and $1 - CG$ are coprime, and hence $(G/(1 - CG))$ is a coprime factorization of P.

We now adopt an approach very similar to that of [18, theorem 4.1] using [19, lemma 7.3.2]. Observe first that if $\|G\| \ge \|1 - CG\|$, then since $\|C\| \, \|G\| + \|1 - CG\| \ge 1$, we have that $\|G\| \ge 1/(1 + \|C\|)$. Similarly, if $\|1 - CG\| \ge \|G\|$, then $\|1 - CG\| \ge 1/(1 + \|C\|)$. Either way,

$$|G^*G + (1 - CG)^*(1 - CG)| \ge \theta^2$$

on the imaginary axis, where $\theta = 1/(1 + \|C\|)$.

We also have that $\|[G - G', \; (1 - CG) - (1 - C'G')]\|_\infty \ge \sqrt{\alpha^2 + \gamma^2}$, and hence if we form a normalized coprime factorization

$$\frac{(G/R)}{(1 - CG)/R}$$

of P, then $\|[G/R - G'/R, \; (1 - CG)/R - (1 - C'G')/R]\|_\infty \le \eta = \sqrt{\alpha^2 + \gamma^2}/\theta$. The result now follows from [19, lemma 7.3.2], as required.

ii) This follows immediately from i), since the gap metric is less than or equal to the graph metric ([3]).

iii) An easy estimate of the chordal metric $\mathscr{K}(P, P')$ (at a point s) follows from the fact that

$$\mathscr{K}(P, P') \le \min(|P - P'|, |1/P - 1/P'|).$$

(See, e.g., [6].)

In this case, we have

$$P - P' = \frac{(G - G') + GG'(C - C')}{(1 - CG)(1 - C'G')}$$

and

$$1/P - 1/P' = \frac{G - G'}{GG'} + (C - C').$$

As in i) we consider the cases when a) $|G|/|1 - CG| \le 1$ and $|1 - CG| \ge 1/(1 + \|C\|)$ and b) $|G|/|1 - CG| \ge 1$ and $|G| \ge 1/(1 + \|C\|)$.

In the first case

$$P - P' = \frac{(G - G')}{(1 - CG)(1 - C'G')} + \frac{(C - C')GG'}{(1 - CG)(1 - C'G')}.$$

Hence

$$|P - P'| \le \frac{\alpha}{(1/(1 + \|C\|))(1/(1 + \|C\|) - \gamma)} + \frac{\beta \|G\|(\|G\| + \alpha)}{(1/(1 + \|C\|))(1/(1 + \|C\|) - \gamma)}.$$

In the second case

$$1/P - 1/P' = C' - C + \frac{G' - G}{GG'}.$$

Hence

$$|1/P - 1/P'| \leq \beta + \frac{\alpha}{(1/(1 + \|C\|))(1/(1 + \|C\|) - \alpha)}.$$

Finally, taking the supremum over all points s, we obtain

$$\mathscr{K}(P, P') \leq (1 + \|C\|)^2 \left(\frac{\alpha}{1 - \max(\alpha, \gamma)(1 + \|C\|)} + \beta \max\left(1, \frac{\|G\|(\|G\| + \alpha)}{1 - \gamma(1 + \|C\|)}\right)\right).$$

□

Various algorithms for H_∞ identification of stable systems (corresponding to functions f in the disk algebra) have been given recently in [7]–[10], [4], [5], [16] and [17]. Some very rapidly convergent ones are to be found in [5] and [17]: these guarantee an error bound for identification with n frequency response measurements (each subject to noise not exceeding ϵ) of $O(E_n(f)) + O(\epsilon)$, where

$$E_n(f) = \inf\{\|f - p\| : p \text{ a trigonometric polynomial of degree } n\}.$$

In the case when $f \in H_\infty(\rho)$ (i.e., analytic and bounded in a disk of radius ρ) for some $\rho > 1$ it is easy to see that $E_n(f)$ decreases at an exponential rate with n. Combining these algorithms with the result above we obtain the following.

Corollary 3.2: Let P, C, and G be as in Theorem 3.1, let G' be an identified model as produced from n frequency response measurements of G using the algorithms of [5] and [17], let C' be an estimate of the controller C and let P' be the identified (possibly unstable) plant given by $P' = G'/(1 - C'G')$. Then the error between P and P' satisfies

$$\mu(P, P') \leq K_1 E_n(G) + K_2 \epsilon + K_3 \|C - C'\| \quad (3.1)$$

where μ is any of the graph, gap, or chordal metrics, and where K_1, K_2, and K_3 are constants which depend on G and C but not on n, $\|C - C'\|$ or ϵ.

Proof: This result follows from Theorem 3.1 on observing that $\alpha = O(E_n(G)) + O(\epsilon)$ and that $\gamma = O(\alpha) + O(\|C - C'\|)$. □

These estimates are realistic, since in the special case $C = 0$, $G = P$, the three metrics are locally equivalent to the H_∞ norm, and in general the dependence of the estimates in this case on n and ϵ are realistic. The special case $G = 0$ may also be considered: this shows that the dependence of the error estimates on C is also realistic.

To obtain low-order models for G (and hence P) a model reduction step is appropriate, and theoretical results for stable systems were given in [17] and [13]. No suitable results are yet available for truncated balanced realizations, but for an optimal Hankel-norm approximant $\hat{G}$ of degree k to an identified model G' for G, the error bound in [17] is as follows:

$$\|G - \hat{G}\| \leq (8k + 1)a_k(G) + k(C_1 E_n(G) + C_2\|\eta\|_\infty) \quad (3.2)$$

for absolute constants C_1 and C_2; here

$$a_k(G) = \inf\{\|G - r\|_\infty : r \in A(D), \text{ rational}, \deg(r) \leq k\}.$$

Combining this with the estimates above we obtain the following.

Corollary 3.3: Let P, P', C, C', G, and G' be as in Corollary 3.2, let G' be a degree-k optimal Hankel-norm approximation to G', let $\hat{C}$ be a rational approximation to C and let $\hat{P} = \hat{G}/(1 - \hat{C}\hat{G})$ be the resulting reduced-order identified model for P. Then the degree of $\hat{P}$ is at most deg $(\hat{G})$ + deg $(\hat{C})$ and the error between P and $\hat{P}$ satisfies

$$\mu(P, P') \leq K_0 k a_k(G) + K_1 k E_n(G) + K_2 k \epsilon + K_3 \|C - \hat{C}\| \quad (3.3)$$

where μ is any of the graph, gap, or chordal metrics, and where K_0, K_1, K_2, and K_3 are constants which depend on G and C but not on k, n, $\|C - \hat{C}\|$ or ϵ.

Proof: This follows from Theorem 3.1, exactly as Corollary 3.2 did, but using the estimate (3.2). □

In the multivariable case, similar results hold. The closed-loop system $G(z) = P(z)(I + C(z)P(z))^{-1}$ is now required to be analytic and matrix valued on the disk with continuous boundary values, and can again be approximately identified by means of matrix-valued frequency response measurements. Error estimates similar to those in this section can be obtained for the approximation error in P in the gap and gap metrics (the chordal metric is more difficult to define in this context) by repeating the above calculations.

IV. An Identification Algorithm Using Smoothing

In this section, we shall be reconsidering the question of identification of a *stable* transfer function G from a set of corrupted values $g_k = G(z_k) + \eta(z_k)$ measured at the nth roots of unity, as in Section II. An extension to stabilizable systems may be derived as in Section III: to illustrate this, we shall give an example in Section V using both of these techniques for closed-loop identification.

Virtually all identification algorithms in the literature proceed by first obtaining a not necessarily stable rational model from the given data (usually a trigonometric polynomial) and then using a Nehari approximation step to approximate this by a stable model. In this section, we propose an alternative technique based on smoothing.

To motivate this new technique, note that the effectiveness of the two-stage process is limited by the effectiveness with which one can approximate the unknown system by trigonometric polynomials (that is, polynomials in z and $1/z$). It turns out that, if we consider the sets $H_\infty(\rho, M)$, approximation by rational functions, in particular by trigonometric polynomials, performs no better in the worst case than approximation by standard polynomials.

To make these comments precise, let P_n be the set of polynomials of degree at most n, T_n the set of trigonometric polynomials of degree at most n and R_n the set of rational L_∞ functions of degree at most n. Then the following result is true.

Theorem 4.1:

$$\begin{aligned}&\sup_{G\in H_\infty(\rho,M)} \inf_{p_n\in P_n} \|G-p_n\|_\infty\\ &\quad= \sup_{G\in H_\infty(\rho,M)} \inf_{t_n\in T_n} \|G-t_n\|_\infty\\ &\quad= \sup_{G\in H_\infty(\rho,M)} \inf_{r_n\in R_n} \|G-r_n\|_\infty = M/\rho^{n+1}.\end{aligned}$$

Proof: Clearly, since $P_n \subset T_n \subset R_n$, of the first three expressions above the first is greater than or equal to the second, and the second is greater than or equal to the third. From a result of Babenko (see [11], page 126]) the extremal function for the P_n problem is $G(z) = Mz^{n+1}/\rho^{n+1}$, giving

$$\sup_{G\in H_\infty(\rho,M)} \inf_{p_n\in P_n} \|G-p_n\|_\infty = M/\rho^{n+1}.$$

However,

$$\sup_{G\in H_\infty(\rho,M)} \inf_{r_n\in R_n} \|G-r_n\|_\infty$$

is at least as large as the $(n+1)$st singular value of a Hankel matrix corresponding to $G(z) = Mz^{n+1}/\rho^{n+1}$, namely the $(n+1)$-by-$(n+1)$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & M/\rho^{n+1}\\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & M/\rho^{n+1} & 0\\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots\\ 0 & M/\rho^{n+1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0\\ M/\rho^{n+1} & 0 & \cdots & 0 & 0\end{pmatrix}.$$

(see e.g., [20, ch. 16]). This matrix has $(n+1)$ singular values all equal to M/ρ^{n+1} and so the result follows. □

It is therefore of interest to study an identification process that yields a polynomial model directly. For generality, we start with an approximation set (ϕ_k) in $A(D)$, i.e., a sequence of linearly independent elements of $A(D)$ whose linear span is dense in $A(D)$. One important example is the polynomial set $\{z^{k-1}\}_{k\geq 1}$, or more generally the simple rational set $\{[(z-a)/(1-az)]^{k-1}\}_{k\geq 1}$, for a fixed constant a with $-1 < a < 1$. We also make the assumption that each member of the approximation set is continuously differentiable on the unit circle.

Here, we give an identification algorithm based on polynomial models: more general algorithms can be defined similarly. Given data $(g_1,\cdots,g_n)$ and a fixed constant $\beta > 0$ define the identified model to be that polynomial p of degree at most $m-1$ which minimizes the quantity

$$\Lambda_n = \max\left\{\sup_{1\leq k\leq n} |g_k - p(z_k)|, \lambda_n\|p'\|_\infty\right\} \tag{4.1}$$

where $m = m(n)$ is any integer-valued function such that $\lim_{n\to\infty} m(n) = \infty$, and $\lambda_n = \beta^{-1}d_n$ with d_n the distance $|z_2 - z_1|$ between consecutive interpolation points.

The problem of minimizing expressions such as (4.1) has been discussed by Barrodale *et al.* [1]. They observe that an alternative form, where one chooses p to minimize the quantity

$$\Lambda_n^* = \max\left\{\sup_{1\leq k\leq n} |g_k - p(z_k)|_*,\ \lambda_n \sup_{0\leq\omega\leq 2\pi} |p'(e^{i\omega})|_*\right\} \tag{4.2}$$

where, for a complex number $z = x + iy$, $|z|_*$ denotes max $(|x|, |y|)$, can be solved by linear programming, since it is essentially a problem involving the solution of an overdetermined system of linear equations in an l_∞ sense. Since $|z|_* \leq |z| \leq \sqrt{2}|z|_*$, this alternative criterion is worth considering and similar error bounds can be derived to the ones we shall now present.

A further computational simplification, which avoids calculating any H_∞ norms, is to replace the expression $\|p'\|_\infty$ by the somewhat larger expression $\|p'\|_1 = \sum_{k=1}^{m-1} k|p_k|$, where $p(z) = \sum_{k=0}^{m-1} p_k z^k$.

Theorem 4.2: The algorithm using (4.1) above is robustly convergent over $A(D)$.

Proof: Given $G \in A(D)$ and $\delta > 0$ there exists a polynomial F of degree r, say, such that $\|F - G\| < \delta$. Hence

$$\Lambda_n \leq \max(\epsilon + \delta, \lambda_n\|F'\|_\infty)$$

for n large enough that $r \leq m(n) - 1$, where $\epsilon > 0$ is the noise level. Now if p is the polynomial minimizing (4.1), and ω_G is the modulus of continuity of G, then

$$\begin{aligned}\|G-p\|_\infty &\leq \Lambda_n + \omega_G(d_n) + \Lambda_n d_n/\lambda_n + \epsilon\\ &= (1+\beta)\Lambda_n + \omega_G(d_n) + \epsilon\\ &\leq (1+\beta)\max(\epsilon+\delta, \lambda_n\|F'\|_\infty)\\ &\qquad + \omega_G(d_n) + \epsilon \end{aligned}\tag{4.3}$$

by the triangle inequality, noting that $\|p'\| \leq \Lambda_n/\lambda_n$. This can be made arbitrarily small by choosing ϵ small and n large (δ and F having been chosen first), and hence (4.1) gives an algorithm which is robustly convergent over $A(D)$. □

Note that this technique provides us with a polynomial model directly; as an alternative, one may find that the rational model set is more appropriate given certain *a priori* information about the system. The calculations are very similar and will not be repeated here: as a result, one would now obtain an identified model which is a polynomial in $(z-a)/(1-az)$.

One important observation here is that the identification points do not need to be equidistant for the algorithm above to be used. Provided that as n tends to infinity the mesh d_n (the maximum distance between consecutive interpolation points) tends to zero, the algorithm is still robustly convergent and the same error bounds hold. Also, if we use $\|p'\|_1$ rather than $\|p'\|_\infty$ this does not affect the robust convergence and the only change in the proof of Theorem 4.2 is to replace $\|F'\|_\infty$ by $\|F'\|_1$.

V. Numerical Example

To illustrate the methods of Sections III and IV, in this section we include a numerical example of closed-loop identification, in which both the controller and the closed-loop transfer function are only given approximately.

We take the very simple unstable plant $P(s) = 1/s$ and a constant controller $C'(s) = 1$, giving $1/(s+1)$ as the 'ideal' stable closed-loop system. To reflect errors in the controller and the measured transfer function, let $C(s) = C'(s) + \eta_C(s)$, where η_C has modulus 0.05 and random complex argument, let $G(s) = P/(1 + CP)$ and let $G_0 = G + \eta_G$, where η_G also has modulus 0.05 and random argument.

Closed-loop $A(D)$-identification based on frequency response measurements of G_0 using the algorithm of [17] followed by model reduction produces an approximation G' to G, and hence an approximation $P' = G'/(1 - C'G')$ to P, as follows.

Starting with 21 points and noise level 0.05, the identified model for G had Hankel singular values equal to $0.467, 0.054, 0.050, 0.038, 0.030, 0.009, \cdots$, and hence a first-order model G' was appropriate: this was produced by means of a truncated balanced realization (which tends to produce best results for this particular procedure, to judge from our numerical experience: see e.g., [17].) In fact $\|G' - (1/(s+1))\|_\infty = 0.046$ and the final open-loop model was

$$P'(s) = \frac{1.014 + 0.048s}{0.021 + s}.$$

The simulation was repeated using 41 points and a noise level of 0.025. In this case $\|G' - (1/(s+1))\|_\infty = 0.017$ and the final open-loop model was

$$P'(s) = \frac{0.967 + 0.0008s}{0.005 + s}$$

indicating that convergence in the gap topology is taking place.

The methods of Section IV were also used to identify the closed-loop system G. The true transformed system is $0.5z + 0.5$, and taking $m = 2$ (polynomials of degree 1) naturally gives good results. Three values of β were tried.

For $\beta = 1$ the first example gave $G'(z) = 0.427z + 0.487$ with $\Lambda_{21} = 0.127$ and $P'(s) = (0.971 + 0.063s)/(0.092 + s)$, the second $G'(z) = 0.468z + 0.514$ with $\Lambda_{41} = 0.072$ and $P'(s) = (1.030 + 0.049s)/(0.019 + s)$.

With $\beta = 2$ these became $G'(z) = 0.492z + 0.478$ with $\Lambda_{21} = 0.076$ and $P'(s) = (0.975 - 0.014s)/(0.029 + s)$; and $G'(z) = 0.506z + 0.489$, with $\Lambda_{41} = 0.046$ and $P'(s) = (0.978 - 0.017s)/(0.005 + s)$, respectively. Taking $\beta = 5$ gave the same results as $\beta = 2$. Note that in general, high values of β will tend to amplify the effects of noise since they diminish the effects of large values of p'.

One could now use one of these identified models to design a robust controller for the original plant P, e.g., by using the techniques of [14]; a controller based on the identified model P' will in the limit (as the number of points tends to infinity and the noise levels tend to zero) tend to optimality for the unknown plant P also.

VI. Conclusions

The notion of robust identification and approximation given in (2.1) extends naturally to the problem of identification of unstable systems in the gap topology, using closed-loop transfer function measurements, and satisfactory low-order models can be produced by these means, as shown in Section III.

In addition, it is of interest to have a robust identification algorithm that produces a model in some specified set (e.g., a polynomial) and the methods of Section IV give a means of achieving this.

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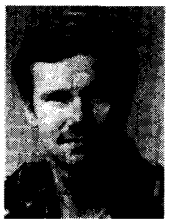
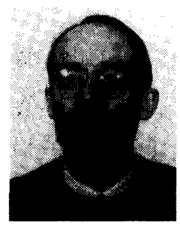


Pertti M. Mäkilä (SM'83) was born in Turku, Åbo, Finland, in 1954. He received the M.Sc. and Ph.D. degrees in chemical engineering, from Åbo Akademi University, Turku, Finland, in 1978 and 1983, respectively.

He has been a Senior Fulbright Scholar at the University of California, Berkeley, a Gendron Fellow at the Pulp and Paper Research Institute of Montreal and Vancouver, Canada, and a Manager of paper mill surveys at the R & D Department of Valmet Inc., Turku. From 1986 to 1989, he was a Senior Research Fellow of the Academy of Finland. During this time, he enjoyed a longer research visit of nearly 1 ½ years at McGill University, Montreal. He is presently in the Department of Chemical Engineering, Åbo Akademi University, Turku, Finland. His research interests include system identification and robust control.



Jonathan R. Partington was born in Norwich, U.K., in 1955. He obtained the Ph.D. degree in pure mathematics from the University of Cambridge, Cambridge, England, in 1980, and continued to work in abstract functional analysis until 1985, when he moved to the Cambridge University Engineering Department as a Senior Research Associate.

Since 1989, he has been a Lecturer in the School of Mathematics at the University of Leeds. He has written one book, an introduction to Hankel operators, and his current research interests include applications of functional analysis to systems theory, particularly approximation and identification.