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Magnetohydrodynamic drift equations: from Langmuir circulations to magnetohydrodynamic dynamo?

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We derive the closed system of averaged magnetohydrodynamic (MHD) equations for general oscillating flows. The used small parameter of our asymptotic theory is the dimensionless inverse frequency, and the leading term for a velocity field is chosen to be purely oscillating. The employed mathematical approach combines the two-timing method and the notion of a distinguished limit. The properties of commutators are used to simplify calculations. The derived averaged equations are similar to the original MHD equations, but surprisingly (instead of the commonly expected Reynolds stresses) a drift velocity plays a part of an additional advection velocity. In the special case of a vanishing magnetic field $\boldsymbol{h} \equiv 0$, the averaged equations produce the Craik–Leibovich equations for Langmuir circulations (which can be called 'vortex dynamo'). We suggest that, since the mathematical structure of the full averaged equations for $\boldsymbol{h} \neq 0$ is similar to those for $\boldsymbol{h} \equiv 0$, these full equations could lead to a possible mechanism of MHD dynamo, such as the generation of the magnetic field of the Earth.

Key words: dynamo theory, general fluid mechanics, magnetohydrodynamics

1. Introduction

The magnetohydrodynamic (MHD) dynamo theory represents a flourishing research area, based on mean-field electrodynamics and/or on multi-scale (or homogenization) theory (see Moffatt 1978, 1983; Frisch 1985; Roberts & Soward 1992; Childress & Gilbert 1995; Zheligovsky 2009; Hughes & Proctor 2010). These theories are aimed at deriving the averaged governing equations and obtaining their solutions that describe the growth of a magnetic field. In this paper we also derive the averaged MHD equations; however, the considered class of flows that oscillate in time (oscillatory flows) has been almost overlooked in the previous studies. There is no unique definition of an oscillatory flow. We suggest that such a flow contains velocity oscillations with frequency higher than the inverse characteristic times of all other coexisting motions. It can be expressed as $\sigma \gg 1$, where σ is the dimensionless frequency of oscillation. To derive the averaged equations, we employ the two-timing method (see e.g. Nayfeh 1973; Kevorkian & Cole 1996). We present it as an elementary, systematic and justifiable procedure that follows

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Vladimirov (2005, 2008, 2010) and Yudovich (2006). This procedure is complemented by novel material on the distinguished limit, which allows one to find and justify a proper slow time scale. The derived equations are similar to the original MHD equations, but surprisingly (instead of the commonly expected Reynolds stresses) a *drift velocity* (or just *a drift*) plays a part of an additional advection velocity. In the special case of a vanishing magnetic field, the averaged equations produce the well-known Craik–Leibovich equations for Langmuir circulations (see Craik & Leibovich 1976; Leibovich 1983; Craik 1985; Thorpe 2004). The aim of our short presentation of this special case is to attract attention to the close resemblance between the averaged equations with non-zero magnetic field and the Craik–Leibovich equations. The results we obtained partially overlap with those by Vladimirov (2010, 2011) and Herreman & Lesaffre (2011). The additional incentive for our research is to make all the calculations and derivations elementary and free of any physical and mathematical assumptions (except for the most common ones, such as the existence of differentiable solutions). We use only the Eulerian description and Eulerian averaging operation, since they are the most transparent and allow us to avoid additional steps and difficulties; for example, the Eulerian description does not have any difficulties in the cases of chaotic trajectories. Such a deliberately chosen simple framework could allow one to build further physical models based on highly reliable foundations. The first step in this direction has already been done: Herreman & Lesaffre (2011) have shown the existence of the *kinematic* Stokes drift dynamo.

In § 2 the notation used is introduced and the list of the main definitions (the averaging operation, etc.) is presented. Section 3 is devoted to the general formulation of the problem, its two-timing dimensionless version, and the distinguished limit arguments. The chosen small parameter represents the dimensionless inverse frequency $\varepsilon \equiv 1/\sigma$. In the chosen class of oscillatory flows, the zero-order terms for both mean velocity and mean magnetic field vanish, $\overline{\boldsymbol{u}}_0 \equiv 0$, $\overline{\boldsymbol{h}}_0 \equiv 0$. The consideration of the distinguished limit leads to a slow time variable s that is connected to physical time t as $s = t/\varepsilon$. In § 4 we study the equations for the first four successive approximations, which lead to the *MHD drift equations*. This study shows that (in the considered class of flows) the first two terms (of zero and first order) of the oscillating part of the velocity must be potential. The MHD drift equations describe the evolution of the first-order mean vorticity (and the related mean velocity) and the first-order mean magnetic field; in order to derive these equations, we consider the zero-, first-, second- and third-order equations. We also show that the drift velocity that appears in the MHD drift equations coincides with the Stokes drift. In § 5 we consider the special case of pure hydrodynamics (with zero magnetic field) and show that the MHD drift equations give the Craik–Leibovich equations. The well-known isomorphism between rotating flows and stratified flows (for translationally invariant motions) leads us to the equivalence between the instability that causes Langmuir circulations and Taylor's instability of an inversely stratified equilibrium. The aim of this example is to attract attention to its possible generalizations for MHD flows. Section 6 contains an extensive discussion of our studies as well as their connection to the known results.

2. Functions and notation

The variables $\boldsymbol{x} = (x_1, x_2, x_3)$, t, s and τ serve as dimensionless Cartesian coordinates, physical time, slow time and fast time, respectively. The used definitions, notation and properties are listed below.

(i) A dimensionless function $f = f(\boldsymbol{x}, s, \tau)$ belongs to the class $\mathbb{O}(1)$ if $f = O(1)$ and all partial $\boldsymbol{x}$, s and τ derivatives of f (required for our consideration) are also $O(1)$. In this paper, all small parameters appear as explicit multipliers, while all functions always belong to the $\mathbb{O}(1)$ class.

(ii) The class $\mathbb{H}$ of *hat functions* (or oscillating functions with non-zero mean) is defined as

$$\widehat{f} \in \mathbb{H} : \widehat{f}(\boldsymbol{x}, s, \tau) = \widehat{f}(\boldsymbol{x}, s, \tau + 2\pi), \tag{2.1}$$

where the τ dependence is always 2π-periodic; the dependences on $\boldsymbol{x}$ and s are not specified.

(iii) The subscripts t, τ and s denote the related partial derivatives.

(iv) For an arbitrary $\widehat{f} \in \mathbb{H}$ the *averaging operation* is

$$\langle \widehat{f}\, \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0+2\pi} \widehat{f}(\boldsymbol{x}, s, \tau)\, \mathrm{d}\tau, \quad \forall\ \tau_0, \tag{2.2}$$

where during the τ integration we keep $s = \text{const.}$ and $\langle \widehat{f}\, \rangle$ does not depend on τ_0.

(v) The class $\mathbb{T}$ of *tilde functions* (or purely oscillating functions, or fluctuations) is such that

$$\widetilde{f} \in \mathbb{T} : \widetilde{f}(\boldsymbol{x}, s, \tau) = \widetilde{f}(\boldsymbol{x}, s, \tau + 2\pi), \quad \text{with } \langle \widetilde{f}\, \rangle = 0, \tag{2.3}$$

and they represent a special case of hat functions with zero average.

(vi) The class $\mathbb{B}$ of *bar functions* (or mean functions) is defined as

$$\overline{f} \in \mathbb{B} : \overline{f}_\tau \equiv 0, \quad \overline{f}(\boldsymbol{x}, s) = \langle \overline{f}(\boldsymbol{x}, s) \rangle. \tag{2.4}$$

(vii) We introduce the *tilde integral* (or the fluctuating part of an integral of a fluctuating function) $\widetilde{f}^\tau$ as

$$\widetilde{f}^\tau \equiv \int_0^\tau \widetilde{f}(\boldsymbol{x}, s, \eta)\, \mathrm{d}\eta - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^\mu \widetilde{f}(\boldsymbol{x}, s, \eta)\, \mathrm{d}\eta \right) \mathrm{d}\mu. \tag{2.5}$$

The tilde integration is inverse to the τ differentiation $(\widetilde{f}^\tau)_\tau = (\widetilde{f}_\tau)^\tau = \widetilde{f}$; the proof is omitted. The τ derivative of a tilde function always represents a tilde function. However, the τ integration of a tilde function can produce a hat function. (For example, let us take $\widetilde{\phi} = \overline{\phi}_0 \sin \tau$, where $\overline{\phi}_0$ is an arbitrary bar function. One can see that $\langle \widetilde{\phi}\, \rangle \equiv 0$. However, $\langle \int_0^\tau \widetilde{\phi}(\boldsymbol{x}, s, \rho)\, \mathrm{d}\rho \rangle = \overline{\phi}_0 \neq 0$, unless $\overline{\phi}_0 \equiv 0$.) Formula (2.5) keeps the result of integration inside the $\mathbb{T}$ class.

(viii) The unique solution of a partial differential equation (PDE) inside the tilde class,

$$\widetilde{f}_\tau \equiv \partial \widetilde{f} / \partial \tau = 0 \quad \Rightarrow \quad \widetilde{f} \equiv 0, \tag{2.6}$$

follows from (2.5). One can also see that $\widehat{f}_\tau = \widetilde{f}_\tau$ and $\langle \widehat{f}_\tau \rangle = \langle \widetilde{f}_\tau \rangle = 0$.

(ix) The commutator of two vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$ is

$$[\boldsymbol{f}, \boldsymbol{g}] \equiv (\boldsymbol{g} \cdot \nabla)\boldsymbol{f} - (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} = \nabla \times (\boldsymbol{f} \times \boldsymbol{g}), \tag{2.7}$$

where the last part of the equality is valid only for solenoidal fields $\boldsymbol{f}$ and $\boldsymbol{g}$. The commutator is antisymmetric and satisfies Jacobi's identity for any vector fields $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$:

$$[\boldsymbol{f},\boldsymbol{g}]=-[\boldsymbol{g},\boldsymbol{f}], \quad [\boldsymbol{f},[\boldsymbol{g},\boldsymbol{h}]]+[\boldsymbol{h},[\boldsymbol{f},\boldsymbol{g}]]+[\boldsymbol{g},[\boldsymbol{h},\boldsymbol{f}]]=0. \tag{2.8}$$

As the average operation (2.2) is proportional to integration over τ, integration by parts yields

$$\langle[\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{g}}_\tau]\rangle=-\langle[\widetilde{\boldsymbol{f}}_\tau,\widetilde{\boldsymbol{g}}]\rangle=-\langle[\widetilde{\boldsymbol{f}}_\tau,\widetilde{\boldsymbol{g}}]\rangle, \quad \langle[\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{g}}^\tau]\rangle=-\langle[\widetilde{\boldsymbol{f}}^\tau,\widetilde{\boldsymbol{g}}]\rangle=-\langle[\widetilde{\boldsymbol{f}}^\tau,\widetilde{\boldsymbol{g}}]\rangle. \tag{2.9}$$

(x) For any tilde function $\widetilde{\boldsymbol{f}}$ and bar function $\overline{\boldsymbol{g}}$, conditions (2.8) and (2.9) give

$$\langle[\widetilde{\boldsymbol{f}},[\overline{\boldsymbol{g}},\widetilde{\boldsymbol{f}}^\tau]]\rangle=[\overline{\boldsymbol{g}},\overline{\boldsymbol{V}}], \quad \text{where } \overline{\boldsymbol{V}}\equiv\langle[\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{f}}^\tau]\rangle/2. \tag{2.10}$$

3. Two-timing problem and distinguished limits

The governing equation for MHD dynamics of a homogeneous inviscid incompressible fluid with velocity field $\boldsymbol{u}^*$, magnetic field $\boldsymbol{h}^*$, vorticity $\boldsymbol{\omega}^*\equiv\operatorname{curl}^*\boldsymbol{u}^*$ and current $\boldsymbol{j}^*\equiv\operatorname{curl}^*\boldsymbol{h}^*$ is taken in the vorticity form as

$$\partial\boldsymbol{\omega}^*/\partial t^*+[\boldsymbol{\omega}^*,\boldsymbol{u}^*]^*-[\boldsymbol{j}^*,\boldsymbol{h}^*]^*=0, \quad \text{in } \mathscr{D}^*, \tag{3.1a}$$

$$\partial\boldsymbol{h}^*/\partial t^*+[\boldsymbol{h}^*,\boldsymbol{u}^*]^*=0, \quad \operatorname{div}^*\boldsymbol{u}^*=0, \quad \operatorname{div}^*\boldsymbol{h}^*=0, \tag{3.1b}$$

where asterisks mark dimensional variables and operations, t^* is time, $\boldsymbol{x}^*=(x_1^*,x_2^*,x_3^*)$ are Cartesian coordinates, and $[\cdot,\cdot]^*$ stands for the dimensional commutator (2.7). In this paper we deal with the transformations of equations, and hence the form of the flow domain $\mathscr{D}^*$ and particular boundary conditions can be specified at later stages.

We accept that the considered class of (unknown) oscillatory solutions $\boldsymbol{u}^*,\boldsymbol{h}^*$ possesses characteristic scales of velocity U, magnetic field H, length L and high frequency σ^*,

$$U,H,L,\sigma^*\gg 1/T, \quad T\equiv L/U, \tag{3.2}$$

where T is a dependent time scale. In the chosen system of units, the dimensions of U and H coincide; we choose them to be of the same order $U=H$. The dimensionless variables and frequency are

$$\boldsymbol{x}\equiv\boldsymbol{x}^*/L, \quad t\equiv t/T, \quad \boldsymbol{u}\equiv\boldsymbol{u}^*/U, \quad \boldsymbol{h}\equiv\boldsymbol{h}^*/U, \quad \sigma\equiv\sigma^*T\gg 1. \tag{3.3}$$

We assume that the flow has its own intrinsic slow time scale T_{slow} (which can be different from T) and consider solutions of (3.1) in the form of hat functions (2.1):

$$\boldsymbol{u}^*=U\widehat{\boldsymbol{u}}(\boldsymbol{x},s,\tau), \quad \boldsymbol{h}^*=U\widehat{\boldsymbol{h}}(\boldsymbol{x},s,\tau), \quad \text{with } \tau\equiv\sigma t,\ s\equiv\Omega t,\ \Omega\equiv T/T_{slow}. \tag{3.4}$$

Then the use of the chain rule and transformation to dimensionless variables give

$$\left(\frac{\partial}{\partial\tau}+\frac{\Omega}{\sigma}\frac{\partial}{\partial s}\right)\widehat{\boldsymbol{\omega}}+\frac{1}{\sigma}[\widehat{\boldsymbol{\omega}},\widehat{\boldsymbol{u}}]-\frac{1}{\sigma}[\widehat{\boldsymbol{j}},\widehat{\boldsymbol{h}}]=0, \tag{3.5a}$$

$$\left(\frac{\partial}{\partial\tau}+\frac{\Omega}{\sigma}\frac{\partial}{\partial s}\right)\widehat{\boldsymbol{h}}+\frac{1}{\sigma}[\widehat{\boldsymbol{h}},\widehat{\boldsymbol{u}}]=0, \quad \operatorname{div}\widehat{\boldsymbol{h}}=0, \quad \operatorname{div}\widehat{\boldsymbol{u}}=0. \tag{3.5b}$$

In order to keep variable s 'slow' in comparison with τ, we have to accept that $\Omega/\sigma\ll 1$. Then (3.5) contains two independent small parameters:

$$\varepsilon\equiv\frac{1}{T\sigma^*}=\frac{1}{\sigma}, \quad \varepsilon_1\equiv\frac{1}{T_{slow}\sigma^*}\equiv\frac{\Omega}{\sigma}. \tag{3.6}$$

Here we must make an auxiliary (but technically essential) assumption: after the use of the chain rule (3.5), variables s and τ are considered to be *mutually independent*:

$$\tau, s \text{ are independent variables.} \tag{3.7}$$

(From the mathematical viewpoint, increasing the number of independent variables in a PDE represents a very radical step, which leads to an entirely new PDE. This step can be partially justified *a posteriori* by estimations of the error (or the residual) of the obtained solution (rewritten back to the original variable t) substituted into the original equation (3.1).) In a rigorous asymptotic procedure with $\sigma \to \infty$, one has to consider asymptotic paths on the $(\varepsilon, \varepsilon_1)$ plane such that $(\varepsilon, \varepsilon_1) \to (0, 0)$. Each of these paths can be prescribed by a particular function $\Omega(\sigma)$. One may expect that there are infinitely many different solutions of (3.5) corresponding to different $\Omega(\sigma)$. However, a unique path $\Omega = \Omega_d(\sigma)$ can be established; it is called a *distinguished limit*. Definitions of the *distinguished limit* vary in different books and papers (see Nayfeh 1973; Kevorkian & Cole 1996). We accept here the definition that uses a function $\Omega(\sigma) = \sigma^\alpha$ (with a constant $\alpha < 1$). The value of $\alpha = \alpha_d$ gives the distinguished limit when: (i) the solution for $\alpha = \alpha_d$ is given by a valid asymptotic procedure; (ii) all solutions for $\alpha_d < \alpha < 1$ contain terms secular in s; and (iii) for any $\alpha < \alpha_d$ the system of equations for successive approximations contains internal contradictions and it is unsolvable (unless a velocity field degenerates). (The nature of secular terms can be easily understood. For instance, if a true solution is proportional to $\sin \Omega_d t$, but one mistakenly takes $t = \Omega s$ with $\Omega > \Omega_d$, then the true solution is proportional to $\sin(\Omega_d s/\Omega)$ with a small parameter Ω_d/Ω; the decomposition with respect to this small parameter produces the required secular terms in s.) In the considered class of flows (defined in (3.11) below), it can be proven that the distinguished limit solution is given by $\alpha_d = -1$:

$$\Omega_d(\sigma) = 1/\sigma, \quad \tau = \sigma t, \quad s = t/\sigma. \tag{3.8}$$

In this paper, we will show that the first three successive approximations (with s from (3.8)) do produce a valid asymptotic solution, and hence the above requirement (i) is fulfilled. The proof of (ii) and (iii) is similar to that considered in Vladimirov (2010, 2011); the proof is beyond the style and scope of the present paper and is omitted.

Hence the governing equations are

$$\widehat{\boldsymbol{\omega}}_\tau + \varepsilon[\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{u}}] - \varepsilon[\widehat{\boldsymbol{j}}, \widehat{\boldsymbol{h}}] + \varepsilon^2 \widehat{\boldsymbol{\omega}}_s = 0, \tag{3.9a}$$

$$\widehat{\boldsymbol{h}}_\tau + \varepsilon[\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{u}}] + \varepsilon^2 \widehat{\boldsymbol{h}}_s = 0, \quad \operatorname{div} \widehat{\boldsymbol{u}} = 0, \quad \operatorname{div} \widehat{\boldsymbol{h}} = 0, \tag{3.9b}$$

where $\varepsilon \equiv 1/\sigma \to 0$. Let us look for the solutions of (3.9) in the form of regular series

$$(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{u}}) = \sum_{k=0}^{\infty} \varepsilon^k (\widehat{\boldsymbol{h}}_k, \widehat{\boldsymbol{u}}_k), \quad \widehat{\boldsymbol{h}}_k, \widehat{\boldsymbol{u}}_k \in \mathbb{H} \cap \mathbb{O}(1), \quad k = 0, 1, 2, \ldots. \tag{3.10}$$

In this paper we study a class of flows with

$$\overline{\boldsymbol{u}}_0 \equiv 0, \quad \overline{\boldsymbol{h}}_0 \equiv 0, \tag{3.11}$$

which is natural from a physical viewpoint if one considers, say, how the secondary vorticity develops on the background of wave motion.

4. Successive approximations and MHD drift equation

The substitution of (3.10) and (3.11) into (3.9) produces the equations of successive approximations. The zero-order equations are $\widehat{\boldsymbol{\omega}}_{0\tau} = \widetilde{\boldsymbol{\omega}}_{0\tau} = 0$ and $\widehat{\boldsymbol{h}}_{0\tau} = \widetilde{\boldsymbol{h}}_{0\tau} = 0$. Their unique solution (2.6) is $\widetilde{\boldsymbol{\omega}}_0 \equiv 0$ and $\widetilde{\boldsymbol{h}}_0 \equiv 0$. Taking into account (3.11) we can write

$$\widehat{\boldsymbol{\omega}}_0 \equiv 0, \quad \widehat{\boldsymbol{h}}_0 \equiv 0, \tag{4.1}$$

which means that the zero-order flow is potential, purely oscillating, and the zero-order magnetic field vanishes. This leads to the similar first-order approximation of (3.9)–(4.1), $\widehat{\boldsymbol{\omega}}_{1\tau} = 0$, $\widehat{\boldsymbol{h}}_{1\tau} = 0$, which have the unique solution

$$\widetilde{\boldsymbol{\omega}}_1 \equiv 0, \quad \widetilde{\boldsymbol{h}}_1 \equiv 0, \quad \overline{\boldsymbol{\omega}}_1 = \boxed{?}, \quad \overline{\boldsymbol{h}}_1 = \boxed{?}, \tag{4.2}$$

where mean functions remain undetermined. The second-order equations that take into account (3.11), (4.1) and (4.2) are $\widetilde{\boldsymbol{\omega}}_{2\tau} + [\overline{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_0] = 0$ and $\widetilde{\boldsymbol{h}}_{2\tau} + [\overline{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0] = 0$, which after tilde integration (2.5) yield

$$\widetilde{\boldsymbol{\omega}}_2 = [\widetilde{\boldsymbol{u}}_0^\tau, \overline{\boldsymbol{\omega}}_1], \quad \widetilde{\boldsymbol{h}}_2 = [\widetilde{\boldsymbol{u}}_0^\tau, \overline{\boldsymbol{h}}_1], \quad \overline{\boldsymbol{\omega}}_2 = \boxed{?}, \quad \overline{\boldsymbol{h}}_2 = \boxed{?}. \tag{4.3}$$

The third-order equations that take into account (3.11), (4.1) and (4.2) are

$$\widetilde{\boldsymbol{\omega}}_{3\tau} + \overline{\boldsymbol{\omega}}_{1s} + [\widehat{\boldsymbol{\omega}}_2, \widetilde{\boldsymbol{u}}_0] + [\overline{\boldsymbol{\omega}}_1, \widehat{\boldsymbol{u}}_1] - [\overline{\boldsymbol{j}}_1, \overline{\boldsymbol{h}}_1] = 0, \tag{4.4a}$$

$$\widetilde{\boldsymbol{h}}_{3\tau} + \overline{\boldsymbol{h}}_{1s} + [\widehat{\boldsymbol{h}}_2, \widetilde{\boldsymbol{u}}_0] + [\overline{\boldsymbol{h}}_1, \widehat{\boldsymbol{u}}_1] = 0. \tag{4.4b}$$

The bar part (2.2) (or the averaged part) of this system is

$$\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_1, \overline{\boldsymbol{u}}_1] - [\overline{\boldsymbol{j}}_1, \overline{\boldsymbol{h}}_1] + \langle [\widetilde{\boldsymbol{\omega}}_2, \widetilde{\boldsymbol{u}}_0] \rangle = 0, \tag{4.5a}$$

$$\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_1] + \langle [\widetilde{\boldsymbol{h}}_2, \widetilde{\boldsymbol{u}}_0] \rangle = 0, \tag{4.5b}$$

which can be transformed with the use of (4.3) and (2.10) into the final form

$$\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_1, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] - [\overline{\boldsymbol{j}}_1, \overline{\boldsymbol{h}}_1] = 0, \tag{4.6}$$

$$\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] = 0, \tag{4.7}$$

where the drift velocity is

$$\overline{\boldsymbol{V}}_0 \equiv \langle [\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{u}}_0^\tau] \rangle / 2. \tag{4.8}$$

If one uses (4.6) as a closed mathematical model, then all the subscripts and bars can be deleted:

$$\boldsymbol{\omega}_s + [\boldsymbol{\omega}, \boldsymbol{u} + \boldsymbol{V}] - [\boldsymbol{j}, \boldsymbol{h}] = 0, \quad \boldsymbol{j} = \operatorname{curl} \boldsymbol{h}, \tag{4.9a}$$

$$\boldsymbol{h}_s + [\boldsymbol{h}, \boldsymbol{u} + \boldsymbol{V}] = 0, \quad \operatorname{div} \boldsymbol{u} = 0, \quad \operatorname{div} \boldsymbol{h} = 0, \tag{4.9b}$$

with $\boldsymbol{V} \equiv \overline{\boldsymbol{V}}_0$, where the equation for the zero-order oscillating velocity $\widetilde{\boldsymbol{u}}_0$ is absent. There are only two restrictions: $\widetilde{\boldsymbol{u}}_0$ is incompressible and potential. Hence the drift velocity $\boldsymbol{V}$ represents a function that is 'external' to the equations. We call (4.9) the *MHD drift equations*. This system of equations looks similar to the original one (3.1). One might think that (4.9) describes 'just' an additional advection of vorticity and a magnetic field. However, the fact that the averaged vorticity is additionally transported by the drift is highly non-trivial; in particular, it contains the possibility of Langmuir circulations (which can also be called 'vortex dynamo'), which we consider below.

Eulerian drift (4.8) differs from a classical (Lagrangian) one, and therefore we first demonstrate the match of (4.8) with the Stokes drift. Let the velocity field be $\widetilde{\boldsymbol{u}}_0 = \overline{\boldsymbol{p}} \sin \tau + \overline{\boldsymbol{q}} \cos \tau$, with arbitrary functions $\overline{\boldsymbol{p}}(\boldsymbol{x}, s)$ and $\overline{\boldsymbol{q}}(\boldsymbol{x}, s)$. The calculations give the drift velocity (4.8) as $\overline{\boldsymbol{V}}_0 = [\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}]/2$. The *dimensional* exact solution for a plane potential harmonic travelling wave is

$$\widehat{\boldsymbol{u}}^*_0 = U\widetilde{\boldsymbol{u}}_0, \qquad \widetilde{\boldsymbol{u}}_0 = \exp(k^*z^*)\begin{pmatrix}\cos(k^*x^* - \tau)\\ \sin(k^*x^* - \tau)\end{pmatrix}, \tag{4.10}$$

where (x^*, z^*) are Cartesian coordinates and $k^* \equiv 1/L$ is a wavenumber. (In Stokes (1847), Lamb (1932) and Debnath (1994) one can see that $U = k^*g^*a^*/\sigma^*$ where a^* and g^* are dimensional spatial wave amplitude and gravity; however these physical details are excessive for our analysis.) The dimensionless velocity field (4.10) is

$$\widetilde{\boldsymbol{u}}_0 = \mathrm{e}^z\begin{pmatrix}\cos(x-\tau)\\ \sin(x-\tau)\end{pmatrix}, \quad \overline{\boldsymbol{p}} = A\mathrm{e}^z\begin{pmatrix}\sin x\\ -\cos x\end{pmatrix}, \quad \overline{\boldsymbol{q}} = A\mathrm{e}^z\begin{pmatrix}\cos x\\ \sin x\end{pmatrix}, \tag{4.11}$$

where all fields are unbounded as $z \to \infty$, but that is not essential for our purposes. The calculations yield the dimensionless and dimensional form of a drift velocity

$$\overline{\boldsymbol{V}}_0 = \mathrm{e}^{2z}\begin{pmatrix}1\\ 0\end{pmatrix}, \quad \overline{\boldsymbol{V}}^*_0 = \frac{U^2k^*}{\sigma^*}\mathrm{e}^{2k^*z^*}\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{4.12}$$

which coincides with the classical expression for a drift velocity given by Stokes (1847), Lamb (1932) and Debnath (1994). To obtain $\overline{\boldsymbol{V}}^*_0$ from $\overline{\boldsymbol{V}}_0$, one should take into account the difference between t and $s = t/\sigma$. The point is that $\overline{\boldsymbol{V}}_0$ appears in (4.9), which describes the averaged fluid motion in terms of s, not t.

5. Averaged Euler's equations and Langmuir circulations

For $\boldsymbol{h} \equiv 0$ the MHD drift equations (4.9) are reduced to the averaged Euler's equations (or the Craik–Leibovich equations):

$$\boldsymbol{\omega}_s + [\boldsymbol{\omega}, \boldsymbol{u} + \boldsymbol{V}] = 0, \quad \operatorname{div}\boldsymbol{u} = 0. \tag{5.1}$$

In order to show the link with Langmuir circulations, we note that the first of (5.1) can be integrated in space as $\boldsymbol{u}_s + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{V} = -\nabla p$, where p is modified pressure. Let the zero-order flow (4.1) represent a travelling plane potential gravity wave (4.11) with the drift velocity (4.12). Let Cartesian coordinates (x, y, z) be such that $\boldsymbol{V} = (U, 0, 0)$, $U = \mathrm{e}^{2z}$ and $\boldsymbol{u} = (u, v, w)$, where all the components are x-independent (translationally invariant) and the x, z variables coincide with those in (4.11). Then the component form of (5.1) can be rewritten (see Vladimirov 1985*a*,*b*) as

$$v_s + vv_y + wv_z = -P_y - \rho\Phi_y, \quad w_s + vw_y + ww_z = -P_z - \rho\Phi_z, \tag{5.2a}$$

$$v_y + w_z = 0, \quad \rho_s + u\rho_x + v\rho_y = 0, \tag{5.2b}$$

where $\rho \equiv u$, $\Phi \equiv U = \mathrm{e}^{2z}$ and P is the new modified pressure. One can see that (5.2) is mathematically equivalent to the system of equations for an incompressible stratified fluid, written in Boussinesq's approximation. The effective 'gravity field' $\boldsymbol{g} = -\nabla\Phi = (0, 0, -2\mathrm{e}^{2z})$ is non-homogeneous, which makes the analogy with a 'standard' stratified fluid non-complete. Nevertheless, one can see that any increasing function $\rho(z) \equiv u(z)$ (taken from the shear flow $(u, v, w) = (u(z), 0, 0)$) produces 'Taylor's instability of an inversely stratified equilibrium'. This leads to the growth of longitudinal vortices and can be connected to Langmuir circulations (see Craik & Leibovich 1976; Leibovich 1983; Craik 1985; Thorpe 2004).

6. Discussion

Our approach (based on the two-timing method, time average and distinguished limit) is different from the mainstream of classical MHD dynamo theory, which is based on mean-field electrodynamics (see Moffatt 1978, 1983; Roberts & Soward 1992; Hughes & Proctor 2010) and/or the multi-scale (homogenization) theory (see Moffatt 1978; Frisch 1985; Zheligovsky 2009). Therefore, we hope that solving the MHD drift equations can bring qualitatively new results.

The notion of a drift is intensely used in this paper. It is known that a drift velocity can appear from Lagrangian, Eulerian, or hybrid (Euler–Lagrange) considerations. In our study, we use *Eulerian drift*, which appears after Eulerian averaging of the governing PDEs, without directly addressing the motion of particles (see Craik & Leibovich 1976; Craik 1985; Riley 2001; Vladimirov 2010, 2011; Ilin & Morgulis 2011). *Lagrangian drift* appears as the average motion of Lagrangian particles and its theory is based on the averaging of ordinary differential equations or their solutions (see Stokes 1847; Lamb 1932; Longuet-Higgins 1953; Batchelor 1967); the hybrid drift coincides with the Lagrangian one (see Andrews & McIntyre 1978; Craik 1985; Soward & Roberts 2010; Vladimirov 2010). It is also known that Lagrangian and Eulerian drifts are not identical to each other, but the leading term $\overline{\boldsymbol{V}}_0$ (4.8) is the same (see Vladimirov 2010).

The consideration of this paper is based on the assumption that the enforced frequency σ^* (3.3) of oscillations is higher than all intrinsic frequencies; σ^* appears in our theory via the prescribed zero-order term $\widetilde{\boldsymbol{u}}_0$, which is potential due to the choice of the class of flows with the vanishing zero-order mean velocity (3.11). We emphasize that the zero-order velocity represents the only potential term, while the resulting model (4.9) describes the first-order vorticity dynamics. The oscillatory zero-order velocity $\widetilde{\boldsymbol{u}}_0$ can be caused by different factors. For example, it can be produced by the prescribed oscillations of boundaries or appear in the full viscous theory due to the matching of an external flow with a boundary-layer solution. The latter option often appears in applications (see Riley 2001; Vladimirov 2008; Ilin & Morgulis 2011).

The s-independent version of (4.6) was first discovered in the studies of Langmuir circulations by Craik & Leibovich (1976), who used the asymptotic procedure based on the smallness of an amplitude. Craik & Leibovich (1976) did not use the distinguished limit, since they considered only steady flows. However, since our equation is the same mathematically, we call (4.6) *the Craik–Leibovich equation*, as it is accepted in the theory of Langmuir circulations. Two different s-independent versions of (4.6) were obtained in the steady streaming problems by Riley (2001) and Ilin & Morgulis (2011), who employed different (from ours) and more cumbersome methods. This equation can also be obtained as a special case from the general Lagrange–Euler (hybrid) consideration of Soward & Roberts (2010).

In the previous remark, we emphasized the difference between the high-frequency asymptotic solutions and small-amplitude asymptotic solutions. The relations between these classes of solutions were studied in Vladimirov (2011); in particular, for purely periodic non-modulated solutions, these classes are isomorphic (any solutions from one class can be transformed into a solution from another class).

The averaged kinematic equation (4.7) was obtained independently by Vladimirov (2010, formula (8.69)) and by Herreman & Lesaffre (2011). There are four differences between these results to be mentioned here: (i) the former paper employs only standard asymptotic methods, while the latter uses deep physical analysis, physical constructions and suggestions; (ii) these papers consider different functional classes/forms of solutions and, as a consequence, different definitions of averaging operations; (iii) in Vladimirov (2010) the zero-, first- and third-order approximations of the averaged equations have been calculated, while Herreman & Lesaffre (2011) have obtained only the zero-order term; and (iv) the former paper aimed to obtain the averaged equation and to describe and justify the asymptotic procedure, while the latter paper also shows the existence of the Stokes drift dynamo.

In both Vladimirov (2010) and Herreman & Lesaffre (2011) the leading term of a magnetic field is of zero order, while in this paper it is of first order. This important difference appears due to the choice of the class of flows (3.11); if the restrictions (3.11) are abolished, then the final model is different from (4.9), as derived in Vladimirov (2011).

The mathematical justification of (4.6) by the estimation of the error in the original equation (3.1) can be performed in a way similar to Vladimirov (2010, 2011). Such a justification is not complete mathematically, since it produces only the errors (residuals) of the exact equations and requires more calculations to obtain the errors of the solutions.

One can also derive the higher approximations of the averaged equation (4.6) as was done in Vladimirov (2010, 2011). They are especially useful for the study of motions with $\overline{\boldsymbol{V}}_0 \equiv 0$ (see Vladimirov 2010). In this case one can show that Langmuir circulations can still be generated by a similar mechanism.

Viscosity and diffusivity can be routinely incorporated in (4.6) and (4.7) as the right-hand side terms $\nu\nabla^2\overline{\boldsymbol{\omega}}_1$ and $\kappa\nabla^2\overline{\boldsymbol{h}}_1$. In fact, this has already been done for Langmuir circulations (see Craik & Leibovich 1976; Leibovich 1983). However, some additional small parameters might appear in the list (3.6), and therefore the distinguished limit should be studied independently from (3.8).

In this paper, we consider periodic (in fast time variable τ) functions. The studies of oscillations that are non-periodic in τ might represent the next stage of research. In fact, such a generalization has already been done for Langmuir circulations (see Craik & Leibovich 1976; Leibovich 1983; Craik 1985).

The consideration of translationally invariant MHD motions in (4.9) can use the isomorphism between MHD flows and stratified flows, established by Vladimirov, Moffatt & Ilin (1996). In particular, one can consider models generalizing Langmuir circulations of § 5.

If the boundary $\partial\mathscr{D}$ of the flow domain $\mathscr{D}$ is finite and oscillating with time, then the Eulerian average (2.2) works only if $\boldsymbol{x} \in \mathscr{D}$ at any instant. If at some instant $\boldsymbol{x} \notin \mathscr{D}$, then the theory can be based on the 'Taylor's series projection' of the boundary condition on a mean position of boundary $\overline{\partial\mathscr{D}}$. Such a consideration requires the smallness of the amplitude a^*/L of oscillatory displacements of fluid particles. One can see that $a^* \sim \boldsymbol{u}^*/\sigma^*$ and hence $a^*/L \sim 1/\sigma \equiv \varepsilon$ (3.6). Therefore, the consideration of a flow domain with oscillating walls does not contain any new small parameter, and the distinguished limit (3.8) will stay the same.

One can make the conjecture that, since (4.6) for $\boldsymbol{h} \equiv 0$ describes a mechanism of Langmuir circulations (or 'vortex dynamo'), and the mathematical structures of the averaged equations for $\boldsymbol{h} \equiv 0$ and $\boldsymbol{h} \neq 0$ (4.9) are similar, then the equations with $\boldsymbol{h} \neq 0$ could also lead to instability and a new mechanism of MHD dynamo. The recent demonstration of the existence of the *kinematic* Stokes drift dynamo by Herreman & Lesaffre (2011) can be regarded as an additional argument to support this conjecture.

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