Polynomial Spline-Approximation of Clarke's Model

Yuriy V. Zakharov, *Member, IEEE*, Tim C. Tozer, *Member, IEEE*, and Jonathan F. Adlard

***Abstract*—We investigate polynomial spline approximation of stationary random processes on a uniform grid applied to Clarke's model of time variations of path amplitudes in multipath fading channels with Doppler scattering. The integral mean square error (MSE) for optimal and interpolation splines is presented as a series of spectral moments. The optimal splines outperform the interpolation splines; however, as the sampling factor increases, the optimal and interpolation splines of even order tend to provide the same accuracy. To build such splines, the process to be approximated needs to be known for all time, which is impractical. Local splines, on the other hand, may be used where the process is known only over a finite interval. We first consider local splines with quasioptimal spline coefficients. Then, we derive optimal spline coefficients and investigate the error for different sets of samples used for calculating the spline coefficients. In practice, approximation with a low processing delay is of interest; we investigate local spline extrapolation with a zero-processing delay. The results of our investigation show that local spline approximation is attractive for implementation from viewpoints of both low processing delay and small approximation error; the error can be very close to the minimum error provided by optimal splines. Thus, local splines can be effectively used for channel estimation in multipath fast fading channels.**

***Index Terms*—Multipath fading channel, random process, spectral moments, spline-approximation.**

I. INTRODUCTION

In many applications of signal processing, polynomial spline approximation is attractive in terms of both low approximation error and simple implementation [1]. The application of splines for signal processing has been popularized by [2]–[4]. In particular, splines have been used for speech processing [5], [6], image processing [4], imitation of and compensation for Doppler distortion [7], identification of frequency-selective channels [8], estimation of time-varying communication channels [9], and other applications. In many cases, signals can be considered to be realizations of random processes; therefore, the investigation of spline approximation of random processes is of interest.

In [10], optimal splines were proposed, providing a minimum of the integral mean square error (MSE)

$$\varepsilon^2 = \frac{1}{\sigma^2 T}\int_0^T E\left\{[x(t) - \hat{x}(t)]^2\right\} dt \tag{1}$$

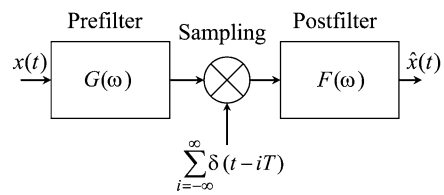

Fig. 1. Prefilter-sampling-postfilter scheme describing spline approximation of the process $x(t)$.

where $\hat{x}(t)$ is a spline, approximating a wide-sense stationary random process $x(t)$ with zero mean and variance σ^2, T is the sampling interval, and $E\{\cdot\}$ denotes statistical expectation. The optimal splines are built on the basis of the "prefilter-sampling-postfilter" scheme shown in Fig. 1, where $G(\omega)$ and $F(\omega)$ are transfer functions of the prefilter and postfilter, respectively, and $\delta(t)$ is the Dirac delta function [2], [11], [12]. If the impulse response of the postfilter is a B-spline, then the impulse response of the optimal prefilter is, in general, nonzero for all time $t \in (-\infty, \infty)$, which is impractical for many real-time applications because of an infinite processing delay.

In [13], the error ε^2 was derived for spline interpolation, i.e., when $\hat{x}(iT) = x(iT)$ and iT (i is an integer) are sampling instants. In the interpolation problem, the prefilter impulse response is the Dirac delta function, whereas the postfilter impulse response is a fundamental spline [14] (also called a cardinal spline [3]); if the spline order $n > 1$, then the postfilter impulse response is nonzero for all time $t \in (-\infty, \infty)$. It means that all the samples $x(iT)$ (past and future) will be involved in approximating the process $x(t)$ on the interval $[0, T]$, which is also impractical due to the high processing delay.

In practice, local splines are of interest. The term "local" means that the spline $\hat{x}(t)$ on the interval $[0, T]$ depends only on $x(t)$ over a relatively short interval $[-\Theta_1, \Theta_2]$. Here, Θ_2 is the processing delay, meaning that in order to represent the process $x(t)$ at the moment t, we need to know this process up to the moment $(t+\Theta_2)$. In real-time systems, the processing delay should be as small as possible. Optimal local splines, minimizing the error (1) under the constraint on the processing delay and based on discrete samples $x(iT)$, were obtained in [15].

In mobile communications, the Doppler effect causes time-varying fading of multipath components. This fading is well described by Clarke's model [16]–[18] (which is also known as Jakes's or the "classical" model [18]–[20]). A path amplitude is a stationary random process $x(t)$ with the correlation function

$$r(\tau) = \sigma^2 J_0(2\pi f_m \tau) \tag{2}$$

where f_m is the Doppler spread, σ^2 the variance, and $J_0(\cdot)$ the zero-order Bessel function of the first kind. Harmonic and polynomial approximations of path amplitudes are used to estimate and model fast fading channels [8], [20]–[22]. However, such

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approximations require the process $x(t)$ to be known at every instant within the approximation interval $T, t \in [0, T]$. In many communication scenarios, for example in the UMTS downlink [23], [24], a short periodically transmitted pilot signal is used for channel estimation. At a high signal-to-noise ratio, the path amplitude estimate over the pilot interval can be approximately considered to be a sample $x(iT)$, where T is the period of transmitting the pilots; as a result, we have regular sampling of the channel with the step T. In such scenarios, splines can be effectively used for channel approximation [9].

In this paper, we compare optimal, interpolation, and local splines of $n = 0, \ldots, 3$ orders in application to approximation of Clarke's model. In Section II, we describe optimal splines and present the minimum integral MSE for spline approximation of Clarke's model. Section III presents interpolation splines. Algorithms of local spline approximation are presented in Section IV; here, we consider Schoenberg's splines, local splines with quasioptimal and optimal spline coefficients, and local extrapolation splines. Finally, Section V contains conclusions.

II. Optimal Spline Approximation

An optimal spline of order n, approximating the random process $x(t)$, is a spline providing a minimum to the integral MSE (1). A spline of order n can be represented as

$$\hat{x}(t) = \sum_{i=-\infty}^{\infty} c_i b_n(t - iT) \tag{3}$$

where $b_n(t)$ is the B-spline of order n, and c_i are spline coefficients. B-spline $b_n(t)$ is a $(n+1)$-fold convolution of the B-spline of zero order [11]:

$$b_0(t) = \begin{cases} 1, & -\frac{T}{2} < |t| < \frac{T}{2} \\ \frac{1}{2}, & |t| = \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

B-splines $b_n(t)$ are conveniently described by the Fourier transform

$$B_n(\omega) = \int_{-\infty}^{\infty} b_n(t) e^{-j\omega t} = T \left[\frac{sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \right]^{n+1}. \tag{5}$$

For the optimal splines, the spline coefficients c_i are linear combinations

$$c_i = \sum_{k=-\infty}^{\infty} a_k \xi_{i-k} \tag{6}$$

of integral samples

$$\xi_k = \int_{-\infty}^{\infty} x(t) b_n(t - kT) dt. \tag{7}$$

The weight coefficients a_k are defined by an infinite system of linear equations; for example, for linear splines $n = 1$, we have $a_k = \sqrt{3}(\sqrt{3} - 2)^{|k|}$ [2], [12].

It is convenient to describe the optimal spline approximation in terms of the "prefilter-sampling-postfilter" scheme shown in Fig. 1. The postfilter transfer function is the Fourier transform of the B-spline $b_n(t)$, $F(\omega) = B_n(\omega)$, whereas the prefilter has the transfer function (see Appendix)

$$G(\omega) = \left[\left(\frac{\omega T}{2}\right) \sin\left(\frac{\omega T}{2}\right) \right]^{-n-1} \times \left[\sum_{k=-\infty}^{\infty} \left(\frac{\omega T}{2} + \pi k \right)^{-2n-2} \right]^{-1}. \tag{8}$$

Let the random process $x(t)$ have spectral moments

$$\lambda_\nu = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^\nu R(\omega) d\omega \tag{9}$$

where

$$R(\omega) = \int_{-\infty}^{\infty} \rho(\tau) e^{-j\omega\tau} d\tau \tag{10}$$

is the normalized spectral density, and $\rho(\tau) = r(\tau)/\sigma^2 = E\{x(t)x(t+\tau)\}/\sigma^2$ is the normalized correlation function of the process $x(t)$. For Clarke's model, we have $\rho(\tau) = J_0(2\pi f_m \tau)$

$$R(\omega) = \begin{cases} \left[\pi f_m \sqrt{1 - \left(\frac{\omega}{2\pi f_m}\right)^2} \right]^{-1}, & \omega < 2\pi f_m \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

and

$$\lambda_\nu = \begin{cases} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (\nu - 1)}{2 \cdot 4 \cdot \ldots \cdot \nu} (2\pi f_m)^\nu, & \text{if } \nu \text{ is even} \\ 0, & \text{if } \nu \text{ is odd.} \end{cases} \tag{12}$$

For a differentiable random process $x(t)$, the approximation error (1) for optimal splines can be represented as a series of the spectral moments λ_ν (see Appendix):

$$\varepsilon^2 = \sum_{\nu=2n+2}^{\infty} \lambda_\nu T^\nu \sum \frac{(-1)^m m!}{[(2n+1)!]^m \, i_{2n+2}! \ldots i_l!} \times \prod_{\eta=2n+2}^{l} \left[\frac{B_\eta}{(\eta - 2n - 2)! \eta} \right]^{i_\eta} \tag{13}$$

where B_η are Bernoulli numbers [25]; the second sum and the product are taken over all solutions in positive integers $\{i_\eta\}, \eta = 2n+2, \ldots, l$, to the equation $(2n+2)i_{2n+2} + (2n+4)i_{2n+4} + \ldots + l i_l = \nu$; l is even, $l \le \nu$; and $m = i_{2n+2} + i_{2n+4} + \ldots + i_l$. From (13), we obtain the following approximate formula for the error in a case of $n > 0$:

$$\varepsilon^2 = \frac{B_{2n+2}}{(2n+2)!} \lambda_{2n+2} T^{2n+2} + \frac{B_{2n+4}}{(2n+1)! 2(2n+4)} \times \lambda_{2n+4} T^{2n+4} + O(\lambda_{2n+6} T^{2n+6}). \tag{14}$$

For $n = 0$, the first term in (14) gives an accurate result.

Note that Clarke's random process is differentiable. Therefore, we can use (14) and (12) to calculate the error of approximation of Clarke's model by the optimal splines of an arbitrary order n, and we obtain

$$\varepsilon^2 \approx \frac{\pi^{2n+2} B_{2n+2}}{[(n+1)!]^2 \gamma^{2n+2}} + \frac{\pi^{2n+4}(n+1)(2n+3)B_{2n+4}}{[(n+2)!]^2 \gamma^{2n+4}} \tag{15}$$

where the sampling factor $\gamma = 1/(f_m T)$. The expression (15) will be used below as a lower bound for comparison with errors of spline interpolation and local spline approximation.

III. Spline Interpolation

We now consider the spline interpolation problem where the spline $\hat{x}(t)$ goes through the data points $x(iT)$ exactly, i.e., $\hat{x}(iT) = x(iT)$ [11]. A spline, interpolating the process $x(t)$, can be represented as [14]

$$\hat{x}(t) = \sum_{i=-\infty}^{\infty} x(iT) l_n(t - iT) \tag{16}$$

where $l_n(t)$ is the fundamental spline (or the cardinal spline [3]) of order n. In any interval $[(i+n/2)T - T/2, (i+n/2)T + T/2]$, the function $l_n(t)$ is a polynomial of order n, it has continuous derivatives up to order $(n-1)$, and its n^{th} derivative has a discontinuity at points $t = [i + (n+1)/2]T$; the interpolation spline $\hat{x}(t)$ obviously has the same properties. In addition, the fundamental spline has the property

$$l_n(t) = \begin{cases} 1, & t = 0 \\ 0, & t = iT, i \text{ is integer and } |i| > 0. \end{cases} \tag{17}$$

This property makes fundamental splines convenient for describing the spline interpolation.

In terms of the "prefilter-sampling-postfilter" scheme, for spline interpolation, we have the postfilter transfer function $F(\omega) = L_n(\omega)$, where the Fourier transform of the fundamental spline is [14]

$$\begin{aligned} L_n(\omega) &= \int_{-\infty}^{\infty} l_n(t) e^{-j\omega t} dt \\ &= \left[\left(\frac{\omega T}{2} \right)^{n+1} \sum_{k=-\infty}^{\infty} (-1)^{k(n+1)} \right. \\ &\qquad \left. \times \left(\frac{\omega T}{2} + \pi k \right)^{-n-1} \right]^{-1}. \end{aligned} \tag{18}$$

Note that by using formula (36) in the Appendix, we can find an optimal prefilter; it is easy to show that the optimal prefilter transfer function is $G(\omega) = L_{2n+1}(\omega)/L_n(\omega)$. Such a prefilter provides a minimum approximation error when fundamental splines are used for spline approximation. This minimum error is exactly the same as when B-splines are used; this follows from comparison (40) with (37) after the substitution $F(\omega) = L_n(\omega)$. However, we are now interested in spline interpolation according to (16); therefore, we consider the prefilter transfer function $G(\omega) = 1$. Obviously, the interpolation error should be higher than the error due to the optimal spline approximation.

The integral MSE (1) for the interpolation of differentiable random processes can also be calculated as a series of spectral moments by using the approach applied in Appendix to the optimal spline approximation. As a result, we obtain the following equation [13]:

$$\begin{aligned} \varepsilon^2 = \sum_{\nu=2n+2}^{\infty} \lambda_\nu T^\nu \Bigg\{ & \frac{B_\nu}{\nu(2n+1)!(\nu-2n-2)!} \\ & + \sum_l \frac{(-1)^m m!(m-1)}{i_s! \ldots i_l! (n!)^m} \\ & \times \prod_{\eta=s}^{l} \left[\frac{D_\eta}{(\eta-n-1)!\eta} \right]^{i_\eta} \\ & + \sum_{r=2n+2}^{\nu-2} \frac{B_r}{r(2n+1)!(r-2n-2)!} \\ & \times \sum_p \frac{(-1)^q (q+1)!}{i_s! \ldots i_p! (n!)^q} \\ & \times \prod_{\eta=s}^{p} \left[\frac{D_\eta}{(\eta-n-1)!\eta} \right]^{i_\eta} \Bigg\} \end{aligned} \tag{19}$$

where ν are even; the sum $\sum_l$ is taken over all solutions in positive integers with even indices $i_s, \ldots, i_l$ to the equation $si_s + \ldots + li_l = \nu$; $m = i_s + \ldots + i_l$; the sum $\sum_p$ is taken over all solutions in positive integers with even indices $i_s, \ldots, i_p$ to the equation $si_s + \ldots + pi_p = \nu - r$; $q = i_s + \ldots + i_p$; $s = n+1$ for odd n and $s = n+2$ for even n; $D_\eta = B_\eta(1 - 2^{1-\eta})$. From (19), we obtain the following approximate formulae.

a) For splines of order $n = 1$

$$\varepsilon^2 = \frac{1}{120} \lambda_4 T^4 + \frac{11}{30\,240} \lambda_6 T^6 + O(\lambda_8 T^8). \tag{20}$$

b) For splines of even order $n \geq 2$

$$\begin{aligned} \varepsilon^2 = & \frac{B_{2n+2}}{(2n+2)!} \lambda_{2n+2} T^{2n+2} \\ & + \left\{ \frac{D_{n+2}^2}{(n!)^2 (n+2)^2} + \frac{D_{2n+4}}{(2n+1)! 2(2n+4)} \right\} \lambda_{2n+4} T^{2n+4} \\ & + O(\lambda_{2n+6} T^{2n+6}). \end{aligned} \tag{21}$$

c) For splines of odd order $n \geq 3$

$$\begin{aligned} \varepsilon^2 = & \left\{ \frac{B_{2n+2}}{(2n+2)!} + \frac{B_{n+1}^2}{[(n+1)!]^2} \right\} \lambda_{2n+2} T^{2n+2} \\ & + \left\{ \frac{B_{n+1} B_{n+3}}{(n!)^2 (n+1)(n+3)} + \frac{B_{2n+4}}{(2n+1)! 2(2n+4)} \right\} \\ & \times \lambda_{2n+4} T^{2n+4} + O(\lambda_{2n+6} T^{2n+6}). \end{aligned} \tag{22}$$

We can use (20)–(22) to calculate the error of interpolation of an arbitrary differentiable random process by polynomial splines of an arbitrary order n. Taking (12) into account, after some algebra, we obtain, for Clarke's model, the following.

a) For splines of order $n = 1$

$$\varepsilon^2 \approx \frac{\pi^4}{20\gamma^4} + \frac{11\pi^6}{1512\gamma^6}. \tag{23}$$

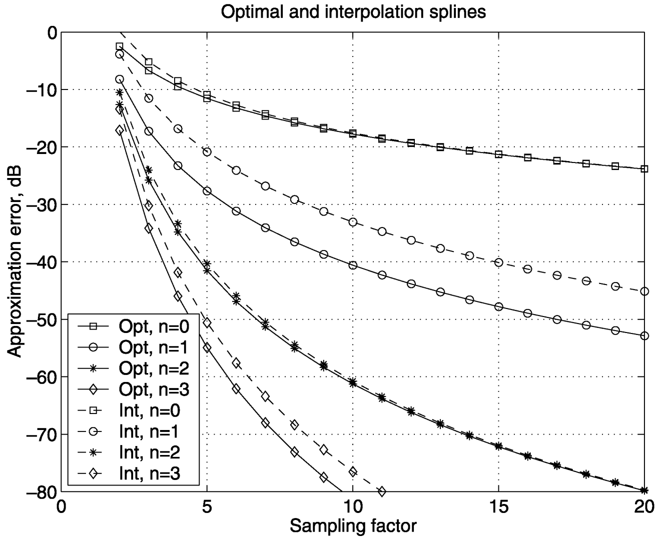


Fig. 2. Optimal (Opt) and interpolation (Int) splines.

b) For splines of even order $n \geq 2$

$$\varepsilon^2 \approx \frac{\pi^{2n+2} B_{2n+2}}{[(n+1)!]^2 \gamma^{2n+2}} + \left\{ \frac{(n+1)(2n+4)! D_{n+2}^2}{[(n+2)!]^2} + \frac{D_{2n+4}}{2^n} \right\} \times \frac{\pi^{2n+4}(n+1)}{[(n+2)!]^2 \gamma^{2n+4}}. \quad (24)$$

c) For splines of odd order $n \geq 3$

$$\varepsilon^2 \approx \left\{ B_{2n+2} + \frac{(2n+2)! B_{n+1}^2}{[(n+1)!]^2} \right\} \frac{\pi^{2n+2}}{[(n+1)!]^2 \gamma^{2n+2}} + \left\{ \frac{(2n+4)! B_{n+1} B_{n+3}}{[(n+1)!]^2 (n+3)} + \frac{B_{2n+4}}{2^n} \right\} \frac{\pi^{2n+4}(n+1)}{[(n+2)!]^2 \gamma^{2n+4}}. \quad (25)$$

Fig. 2 shows dependencies of the error for optimal and interpolation splines of $n = 0$, 1, 2, and 3 orders on the sampling factor $\gamma = 1/(f_m T)$ for Clarke's model. Note that the sampling factor $\gamma = 2$ corresponds to Nyquist sampling. In principle, there are techniques leading to a zero error for $\gamma > 2$; however, this requires an infinite spline order [2]. In practice, the spline order is often chosen less than or equal to $n = 3$. For a fixed n, the optimal splines outperform the interpolation splines. For an even n, these errors asymptotically coincide as $\gamma \to \infty$; this can be seen by comparing the first terms in formulae (14) and (21). For an odd n, as $\gamma \to \infty$, the ratio of the errors is between 3 (for $n \to \infty$) and 6 (for $n = 1$) [13]. As n increases, the error is reduced, and the dependence tends to a vertical line $\gamma = 2$, corresponding to Nyquist sampling.

For the zero-order approximation, an error of -40 dB is achievable only at $\gamma > 57$; this shows that in many cases, such an approximation is not of practical interest. Linear interpolation is most often used in practice [1]. We see from Fig. 2 that for an error of -40 dB, linear interpolation requires a sampling factor of $\gamma \approx 15$. The optimal linear spline, compared with the linear interpolation spline, allows the sampling rate to be decreased as much as a factor of 1.5 ($\gamma \approx 10$). To achieve this error when using the optimal splines, the sampling factors $\gamma \approx 10$, 5, and 3.5 are required for $n = 1$, 2, and 3, respectively, which means that the increase of the spline order from $n = 1$ to $n = 3$ allows a decrease in the sampling rate by a factor of about three. This means that in a communication system with a periodic pilot signal used to estimate a time-varying channel, the use of optimal cubic splines instead of linear interpolation allows the period between pilot signals to be increased by a factor of 4.5, which leads to a higher spectral efficiency of the communication system.

IV. Local Spline Approximation

A local spline of an order n is represented as

$$\hat{x}(t) = \sum_{i=-\infty}^{\infty} c_i b_n(t - iT) \quad (26)$$

where the spline coefficients c_i are linear combinations of a finite number $(L_1 + L_2 + 1)$ of instant samples $x(iT)$:

$$c_i = \sum_{k=-L_1}^{L_2} a_k x(iT - kT). \quad (27)$$

In terms of the "prefilter-sampling-postfilter" scheme in Fig. 1, we have the postfilter transfer function as $F(\omega) = B_n(\omega)$, whereas the prefilter transfer function is

$$G(\omega) = \sum_{k=-L_1}^{L_2} a_k e^{-j\omega kT}.$$

Substituting (26) and (27) in (1), we get the expression

$$\varepsilon^2 = 1 - 2 \sum_{k=-L_1}^{L_2} a_k \gamma_n(k) + \sum_{k=-L_1}^{L_2} \sum_{l=-L_1}^{L_2} A_{kl} a_k a_l \quad (28)$$

where

$$\gamma_n(k) = \frac{1}{T} \int_{-(n+1)\frac{T}{2}}^{(n+1)\frac{T}{2}} b_n(\tau) \rho(\tau + kT) d\tau \quad (29)$$

$$A_{kl} = \sum_{p=-n}^{n} b_{2n+1}(pT) \rho(pT - lT + kT). \quad (30)$$

Formula (28) holds for any weight coefficients a_k. To find the optimal coefficients $a_k^{(opt)}$, which minimize the error ε^2, we differentiate the right side of (28) with respect to a_s and make it equal to zero. As a result, we obtain a linear system of equations

$$\sum_{k=-L_1}^{L_2} a_k^{(opt)} A_{ks} = \gamma_n(s), \quad s = -L_1, \ldots, L_2. \quad (31)$$

The solution to the system (31) is a vector of optimal weight coefficients. Substituting (31) in (28) gives a minimum error for the local splines with the optimal coefficients $a_k^{(opt)}$

$$\varepsilon^2 = 1 - \sum_{k=-L_1}^{L_2} a_k^{(opt)} \gamma_n(k). \quad (32)$$

This is similar to the result in [15], with the only difference being that now, we can use $L_1 \neq L_2$, in particular, $L_1 \leq 0$, so that

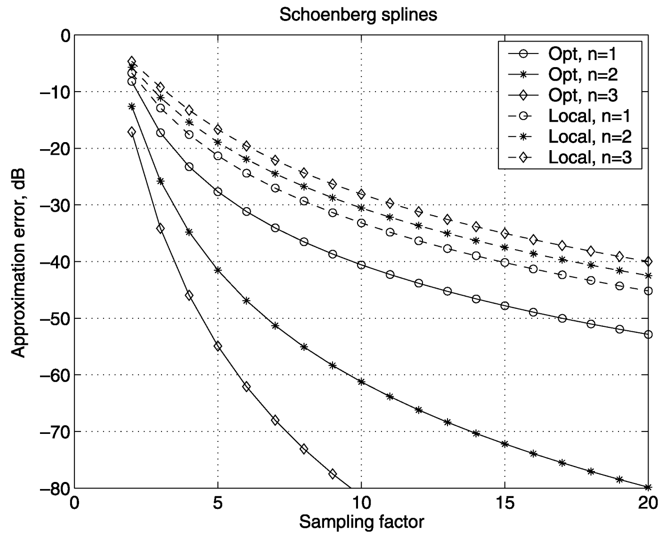


Fig. 3. Optimal (Opt) and Schoenberg's (Local) splines.

$L_1 + L_2 \geq 0$. This allows us to derive algorithms with different processing delays, in particular, with a zero processing delay, i.e., spline extrapolation algorithms.

A. Schoenberg's Splines

The case of $L_1 = L_2 = 0$ and $a_0 = 1$ corresponds to Schoenberg's splines [26]. The error calculated by using (28)–(30) for Clarke's model for different γ is shown in Fig. 3 for $n = 1$, 2, and 3. It can be seen that for Schoenberg's splines, a minimum error is provided by the linear spline $n = 1$ (which, in this case, is also the linear interpolation spline), whereas, as the spline order n increases, the error is also increased. This is in accordance with the known result that for Schoenberg's splines, the error is minimum for linear splines [26]. The error can be decreased if a_0 is [as follows from (31)]

$$a_0 = \frac{\gamma_n(0)}{\sum_{p=-n}^{n} \rho(pT) b_{2n+1}(pT)} \tag{33}$$

where $\gamma_n(0)$ is defined in (29). We will call such splines optimal Schoenberg's splines.

Fig. 4 shows the weight a_0 from (33) for optimal Schoenberg's splines as a function of the sampling factor γ. For small γ, the weight exceeds 1 and increases as the spline order n increases. As the sampling factor increases, the weight approaches 1. The error for the optimal Schoenberg's splines is shown in Fig. 5. The optimal choice of the coefficient a_0 minimizes the error. However, this performance improvement is not large; for linear splines and an error of -40 dB, the optimal coefficient a_0 allows a reduction of the sampling factor γ from 15 to 12 (compare Figs. 3 and 5).

In order to decrease the approximation error further, we need to increase the number of samples L_1 and L_2 used to calculate the spline coefficients.

B. Local Splines With Quasi-Optimal Spline Coefficients

In the case of $L_1 = L_2 = 1$, $a_1 = a_{-1} = -(n+1)/24$, and $a_0 = 1 - 2a_1$ for $n \geq 1$, the local splines provide a zero approximation error for polynomials of an order less than or equal to n [27]. We will call the corresponding spline coefficients quasioptimal. For $n = 1$, 2, and 3, we have $[a_{-1}, a_0, a_1] = [-1/12, 7/6, -1/12]$, $[-1/8, 5/4, -1/8]$, and $[-1/6, 4/3, -1/6]$, respectively.

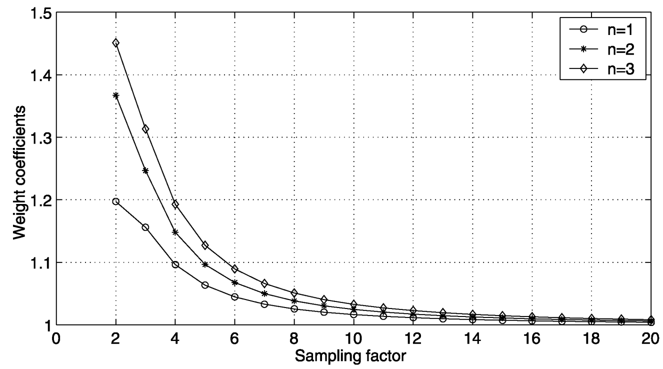


Fig. 4. Weight coefficients for optimal Schoenberg's splines; $L = 0$.

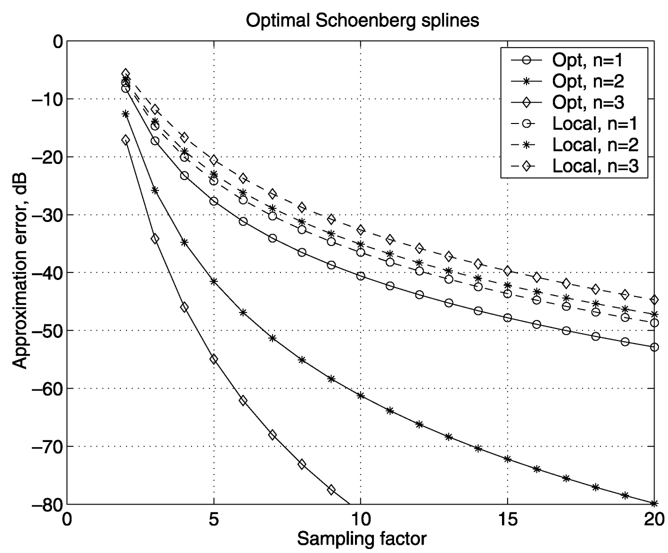


Fig. 5. Optimal splines (Opt) and optimal Schoenberg's splines (Local).

Fig. 6 shows the approximation error for local splines with quasioptimal spline coefficients. It can be seen that the local spline of the first order provides nearly the same accuracy as the optimal spline; a small difference is seen only at low sampling factors. Although for $n = 2$ and $n = 3$ we obtain a lower error than for $n = 1$, this error is significantly higher than that of the optimal splines. Thus, local splines with quasioptimal spline coefficients provide an error that is close to the corresponding minimum error only for $n = 1$ and significantly higher for $n > 1$.

We can also see that the increase of the spline order from $n = 2$ to $n = 3$ results in an increasing approximation error. This is similar to the result for Schoenberg's splines in Fig. 3, where the increase of the spline order from $n = 1$ to $n = 2$ and, further $n = 3$, also results in an increasing error. The use of additional instant samples (a "better" prefiltering) by the local splines with quasioptimal spline coefficients has allowed us to obtain a lower error with respect to Schoenberg's splines. Additional improvement of the approximation accuracy can be achieved by local splines with optimal spline coefficients.

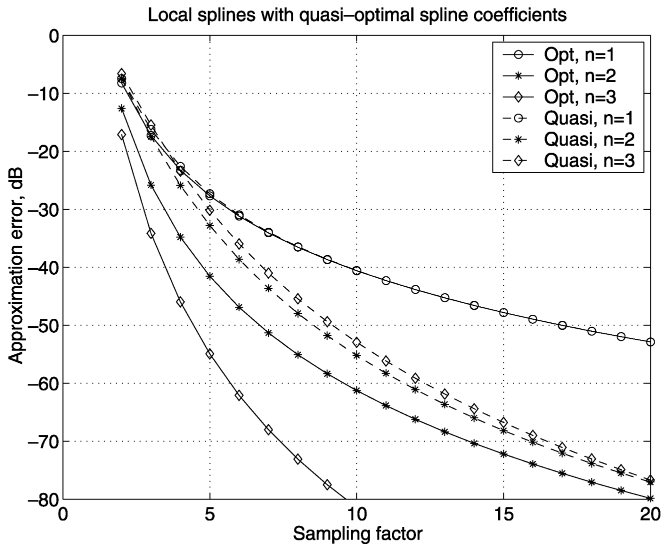


Fig. 6. Optimal splines (Opt) and local splines with quasioptimal spline coefficients (Quasi).

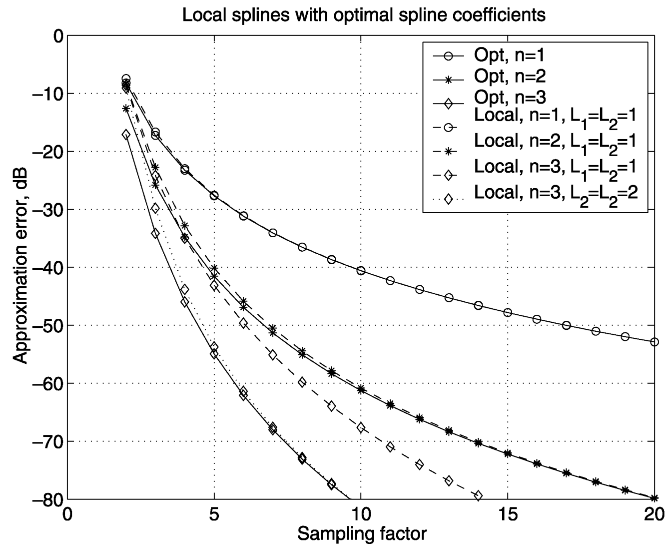


Fig. 8. Optimal splines (Opt) and local splines with optimal spline coefficients (Local).

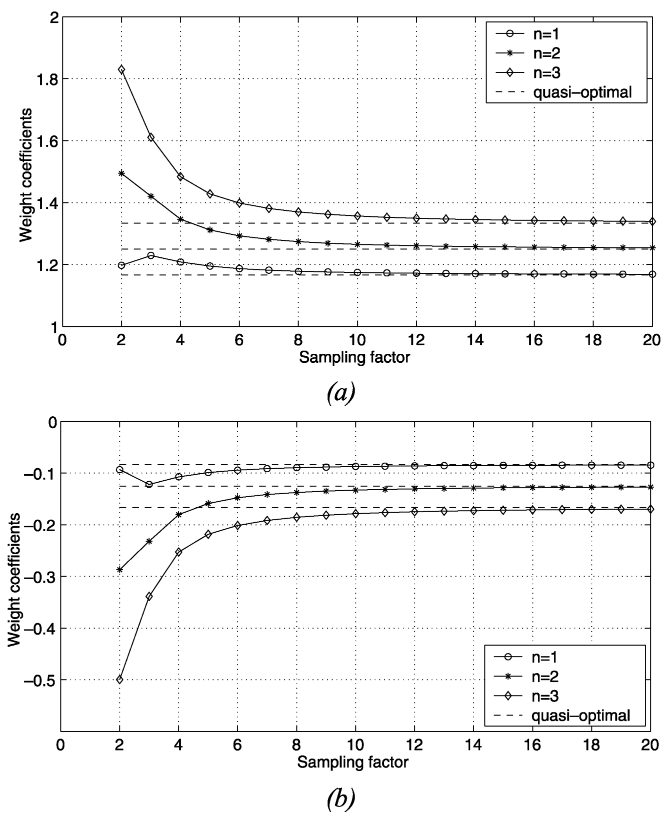


Fig. 7. Weight coefficients for optimal local splines; $L = 1$. (a) a_0. (b) a_{-1} and a_1.

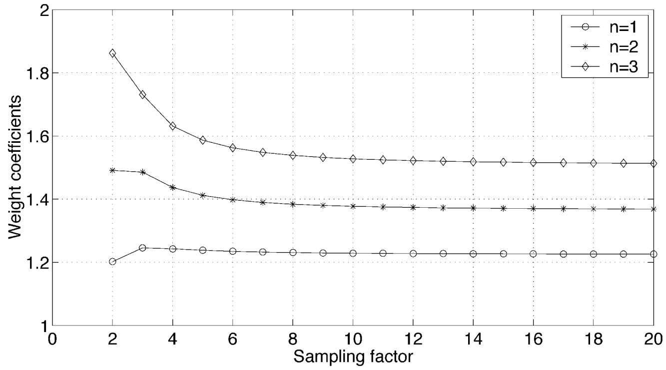
C. Local Splines with Optimal Spline Coefficients

Fig. 7 shows the weight coefficients a_0 and $a_1 = a_{-1}$ from (31) required for calculating the optimal spline coefficients in the case of $L_1 = L_2 = 1$. As the sampling factor γ increases, the optimal weights become close to the quasioptimal weights considered in the previous subsection. For small γ, the optimal weights significantly differ from the quasioptimal weights. Fig. 8 shows the approximation error for local splines with optimal spline coefficients shown in Fig. 7. The local linear spline demonstrates nearly optimal performance. The local parabolic spline also provides an error close to that of the optimal parabolic spline, i.e., the accuracy of the local parabolic spline with optimal spline coefficients is significantly better than that of the local parabolic spline with quasioptimal coefficients (see Fig. 6). However, for $n = 3$, there is still a significant difference between the errors. If we take $L_1 = L_2 = 2$, the errors for the optimal cubic spline and the local cubic spline with optimal spline coefficients nearly coincide. Thus, local splines with optimal spline coefficients, calculated according to (31), provide near minimum errors for $L_1 = L_2 = 1, n \leq 2$ and for $L_1 = L_2 = 2$, $n = 3$. Fig. 9 shows optimal weights for the case of $L_1 = L_2 = 2$. Again, we see that the weights approach steady-state values as the sampling factor increases, whereas for low γ, the weights differ for different γ. In practice, small γ are of interest; therefore, optimal weight coefficients calculated for specific values of γ are preferable in practice.

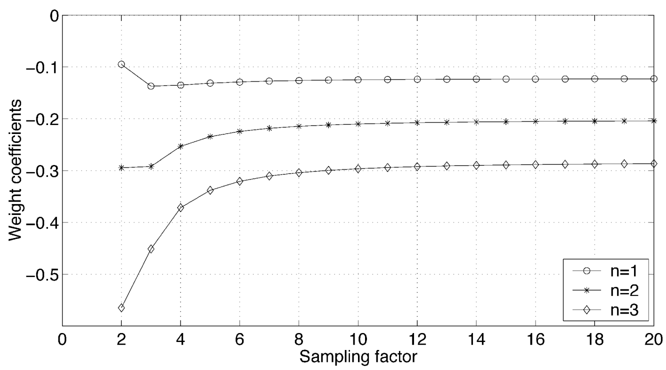
The processing delay for local splines is defined as $\Theta_2 = (n+1)T/2 + L_1 T$; for example, we have

1) $\Theta_2 = T/2$ for $L_1 = L_2 = 0$, $n = 0$, zero order spline;
2) $\Theta_2 = T$ for $L_1 = L_2 = 0$, $n = 1$, linear interpolation spline;
3) $\Theta_2 = 2T$ for $L_1 = L_2 = 1$, $n = 1$, local linear splines with quasioptimal or optimal spline coefficients;
4) $\Theta_2 = 2.5T$ for $L_1 = L_2 = 1$, $n = 2$, local parabolic splines with quasioptimal or optimal spline coefficients;
5) $\Theta_2 = 4T$ for $L_1 = L_2 = 2$, $n = 3$, local cubic spline with optimal spline coefficients.

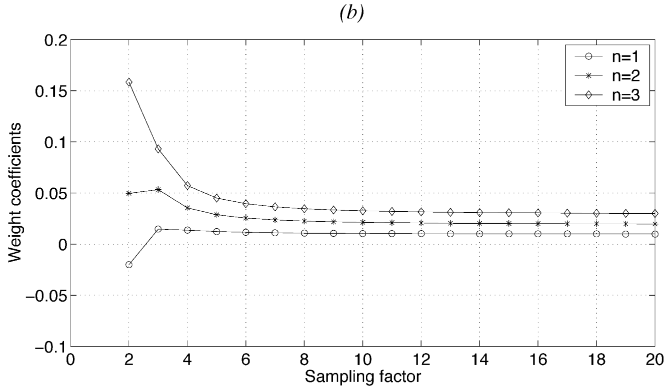
The first case corresponds to a zero-order spline approximation, which provides a small processing delay but results in a large approximation error. The second case corresponds to the linear interpolation with a processing delay of T. For example, if a receiver estimates a channel impulse response with the pilot period T, the receiver has to store the received signal in a memory for a time T and delay the data demodulation by T s. For the cases 3)–5), we have to further increase the memory



(a)



(b)



(c)

Fig. 9. Weight coefficients for optimal local splines; $L = 2$. (a) a_0. (b) a_{-1} and a_1. (c) a_{-2} and a_2.

and the demodulation delay up to $2T$, $2.5T$, and $4T$, which can be unacceptable for real-time systems.

D. Local Extrapolation Splines

In order to obtain a lower processing delay, we need to decrease the parameter L_1. For a zero-processing delay, the parameter L_1 should take on values $L_1 = -1$, $L_1 = -2$, and $L_1 = -2$, or less, for $n = 1, 2$, and 3, respectively. Fig. 10 compares errors for optimal splines and extrapolation splines, providing a zero processing delay, with different values of the parameter L_2. The linear extrapolation spline provides a near-optimal accuracy at $\gamma > 6$ by processing, for each spline coefficient, $L_1 + L_2 + 1 = 7$ samples of the function $x(t)$. For quasioptimal (see Fig. 6) and optimal (see Fig. 8) local splines, three samples $(L_1 + L_2 + 1 = 3)$ are sufficient. The increase in the number of samples processed is a penalty for the zero

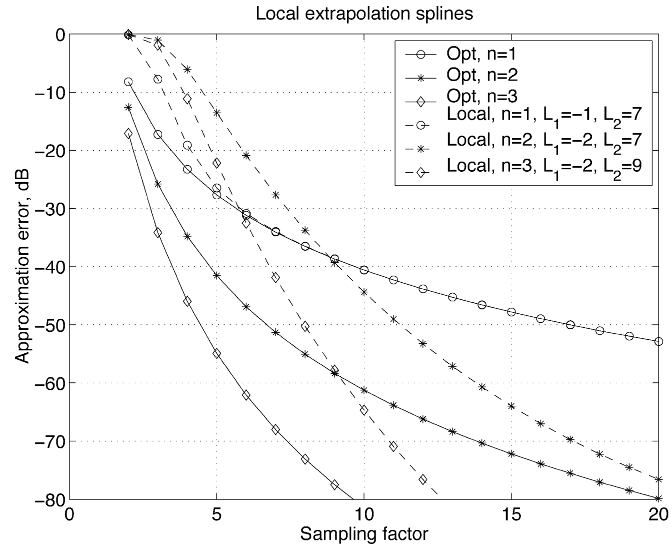
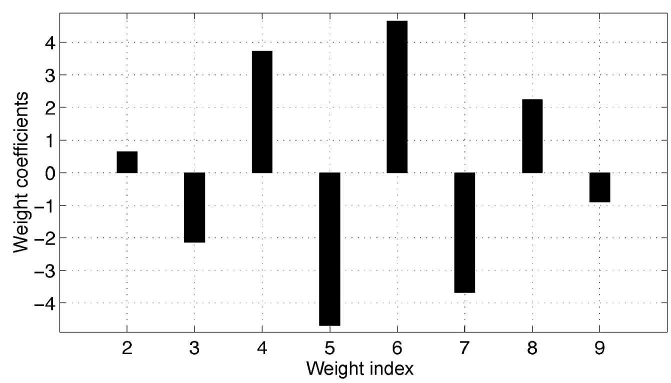
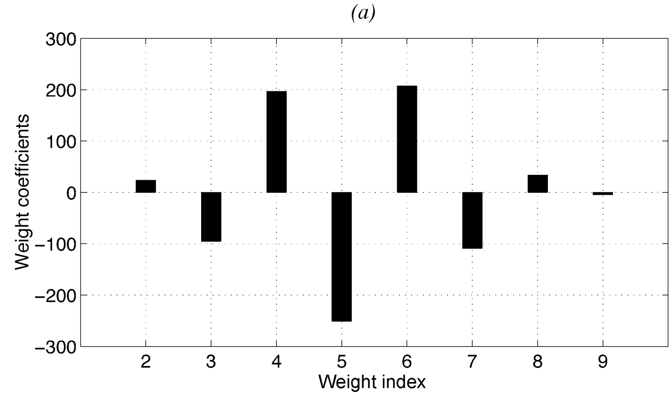


Fig. 10. Optimal splines (Opt) and local extrapolation splines (Local).



(a)



(b)

Fig. 11. Weight coefficients for local extrapolation cubic splines: $L_1 = -2$, $L_2 = 9$. (a) $\gamma = 3$. (b) $\gamma = 10$.

processing delay. We also note that the error for local extrapolation splines of higher orders $n = 2$ and $n = 3$ approaches that of optimal splines only at very high sampling factors γ, and at low sampling factors, the linear spline outperforms the splines of higher orders. Another disadvantage of extrapolation splines is that the weight coefficients become large, which can lead to an unstable approximation. Fig. 11 demonstrates this for cubic splines with parameters $L_1 = -2$ and $L_2 = 9$ for two values of

TABLE I
APPROXIMATION ERROR FOR THE UMTS SCENARIO WITH A DOPPLER SPREAD OF $f_m = 220$ HZ

	Spline order	$n = 1$	$n = 2$	$n = 3$
1	Optimal splines	-34 dB	-51 dB	-68 dB
2	Interpolation splines	-27 dB	-50 dB	-63 dB
3	Schoenberg's splines	-27 dB	-25 dB	-22 dB
4	Optimal Schoenberg's splines	-30 dB	-29 dB	-26 dB
5	Local splines with quasi-optimal coefficients	-34 dB	-44 dB	-41 dB
6	Local splines with optimal coefficients	-34 dB	-50 dB	-68 dB
7	Extrapolation splines	-34 dB	-28 dB	-42 dB

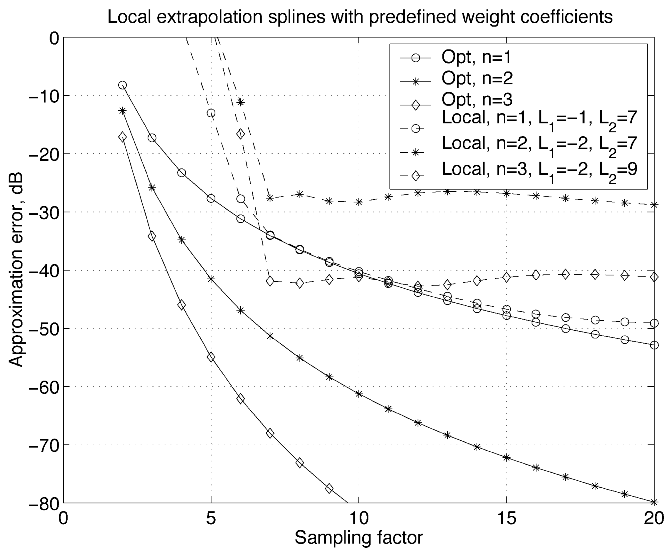


Fig. 12. Optimal splines (Opt) and local extrapolation splines with weight coefficients that are optimal for $\gamma = 7$ (Local).

γ: $\gamma = 3$ and $\gamma = 10$; we see that the growth of the weight coefficients is significant at high sampling factors, where the cubic spline outperforms the linear and parabolic splines.

Local splines with optimal spline coefficients require the Doppler spread f_m to be known or estimated; this is an additional disadvantage of optimal local splines. However, we can avoid estimating f_m by precalculating the weight coefficients a_k for the lowest possible f_m and using these weights for the other f_m. For example, in UMTS, the Doppler spread $f_m =$ 220 Hz corresponds to a maximum mobile speed of 120 km/h, and the pilot period is $T =$1/1500 s [23]; this gives a maximum sampling factor $\gamma = 1/(f_m T) \approx 7$. Fig. 12 shows the approximation error for extrapolation splines with weight coefficients that are optimal for $\gamma = 7$. It is interesting that for $\gamma > 7$, the error is approximately constant for $n = 2$ and $n = 3$. We see that in this scenario, the linear spline outperforms splines of the higher orders (excluding a range of $\gamma = 7 \ldots 10$ for $n = 3$).

Table I summarizes results of calculation of the approximation error for the UMTS scenario with a Doppler spread of $f_m = 220$ Hz (the sampling factor $\gamma = 7$). It can be seen that the linear optimal splines allow the error to be reduced by 7 dB with respect to the linear interpolation most often used in practice. Moreover, this improvement can be achieved by the extrapolation splines possessing a zero processing delay. For parabolic $(n = 2)$ and cubic $(n = 3)$ splines, the accuracy depends heavily on the approximation algorithm. For $n = 2$ and 3, the local splines with optimal spline coefficients allow us to obtain the accuracy that is close to that of the optimal splines. For parabolic splines, the extrapolation algorithm demonstrates a relatively low performance; the error is even higher than for the linear extrapolation splines and close to the error of the linear interpolation. For cubic splines, the extrapolation algorithm results in an error as small as -42 dB; however, this is higher by 26 dB, compared with the optimal cubic splines and local cubic splines with optimal spline coefficients.

V. CONCLUSIONS

We have investigated the integral MSE of polynomial spline approximation of random processes in application to Clarke's model used for modeling time variations of path amplitudes in multipath fading channels with Doppler scattering. We have been concerned mainly with splines of order $n \leq 3$ as these are simple for implementation and most frequently used in practice. The errors for optimal and interpolation splines have been presented as a series of spectral moments. The optimal splines outperform the interpolation splines; however, as the sampling factor increases, the error of spline interpolation of even order approaches the minimum error. To build optimal and interpolation splines, the function to be approximated needs to be known over an infinite time interval, which is impractical. Local splines may be used when the function is known over only a finite interval.

We have derived optimal spline coefficients for local splines when a fixed number of samples is used for approximation and investigated the error for different sets of the samples. We have considered Schoenberg's splines and found that these splines are not attractive for implementation (even when optimal weight coefficients are used) because of a relatively high approximation error. We have also considered local splines with quasioptimal spline coefficients calculated on a basis of three samples and guaranteeing a zero approximation error for polynomials of an order less than or equal to the spline order; such splines provide an error that is close to the corresponding minimum only for $n = 1$ and is significantly higher for $n > 1$. Local splines with optimal spline coefficients provide nearly optimal performance for $n \leq 2$ when three samples are used for calculating the spline coefficients, whereas $n = 3$ requires five samples for nearly optimal performance.

Approximation with low processing delay is of interest, allowing a decrease of demodulation delay in data transmission with a periodic pilot used for channel estimation. We have investigated spline extrapolation algorithms possessing zero processing delay. The results show that even for a zero processing delay, we can build linear splines providing an error that is close to that of the optimal linear spline. Extrapolation splines of second and third order allow reduction of the error with respect to linear extrapolation splines at high sampling factors γ. For the extrapolation algorithms considered, the reduction is seen at $\gamma \geq 9$ for $n = 2$ and $\gamma \geq 6$ for $n = 3$. For lower sampling factors, the linear extrapolation spline outperforms the extrapolation splines of higher orders. At large sampling factors, weight coefficients of local extrapolation splines increase, which can result in unstable implementation of spline approximation. Optimal weights depend on the sampling factor; in application to channel estimation, this requires the Doppler spread to be known or estimated. For scenarios where such an estimation is not possible, we have proposed calculating the optimal weights for a lowest possible γ and retaining them for the other sampling factors. We have shown that in such extrapolation algorithms, the error remains approximately constant for higher sampling factors.

The results of our investigation have shown that local spline approximation is attractive for implementation from viewpoints of complexity, a low processing delay, and a small approximation error and, thus, can be used effectively for channel estimation in multipath fast-fading channels. The results presented in this paper may readily be applied to other random processes, for example, to random processes describing the "flat" fading channel model [18].

APPENDIX
ERROR OF OPTIMAL SPLINE APPROXIMATION

In this Appendix, we show the derivation of the error (1) for optimal spline approximation of random processes. This derivation is based on [10] and [12].

Consider the "prefilter-sampling-postfilter" scheme shown in Fig. 1. Let $G(\omega)$ be a transfer function of the prefilter and $F(\omega)$ be a transfer function of the postfilter. The error (1) can be written in terms of $G(\omega)$ and $F(\omega)$ as

$$\varepsilon^2 = 1 - \frac{1}{\pi T}\int_{-\infty}^{\infty} R(\omega)F(\omega)G(\omega)d\omega + \frac{1}{2\pi T^2} \times \int_{-\infty}^{\infty} R(\omega)\left|G(\omega)\right|^2 \sum_{k=-\infty}^{\infty}\left|F\left(\omega+\frac{2\pi k}{T}\right)\right|^2 d\omega \quad (34)$$

where $R(\omega)$ is the normalized spectral density

$$R(\omega) = \int_{-\infty}^{\infty} \rho(\tau)e^{-j\omega\tau}d\tau. \quad (35)$$

For the postfilter transfer function $F(\omega)$, the error (34) is minimized if the prefilter transfer function is (see also [1])

$$G(\omega) = \frac{TF^*(\omega)}{\sum_{k=-\infty}^{\infty}\left|F\left(\omega+\frac{2\pi k}{T}\right)\right|^2} \quad (36)$$

where $(\cdot)^*$ denotes complex conjugate. The minimum error corresponding to such a prefilter is

$$\varepsilon^2 = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)\frac{\left|F(\omega)\right|^2}{\sum_{k=-\infty}^{\infty}\left|F\left(\omega+\frac{2\pi k}{T}\right)\right|^2}d\omega. \quad (37)$$

For spline approximation of order n, it is convenient to represent as a B-spline $b_n(t)$ the impulse response of the postfilter. Note that we can chose another postfilter impulse response whose time shifts form basis functions for representation of n-order splines (for example, it can be the fundamental spline [14]); the choice of basis functions does not affect the final result. For B-splines, the transfer function $F(\omega)$ is the Fourier transform

$$F(\omega) = B_n(\omega) = \int_{-\infty}^{\infty} b_n(t)e^{-j\omega t} = T\left[\frac{\sin(z)}{z}\right]^{n+1} \quad (38)$$

where $z = \omega T/2$; the optimal prefilter transfer function is

$$G(\omega) = [z\sin(z)]^{-n-1}\left[\sum_{k=-\infty}^{\infty}(z+\pi k)^{-2n-2}\right]^{-1}. \quad (39)$$

By using (38), the minimum error (37) can be written as

$$\varepsilon^2 = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)L_{2n+1}(\omega)d\omega \quad (40)$$

where

$$L_{2n+1}(\omega) = y_{2n+2}^{-1}(z) = f\left[y_{2n+2}(z)\right]$$

$f(y) = 1/y$, and

$$y_{2n+2}(z) = z^{2n+2}\sum_{k=-\infty}^{\infty}(z+\pi k)^{-2n-2}.$$

It is easy to see that when n increases, we have

$$\lim_{n\to\infty} L_{2n+1}(\omega) = \begin{cases} T, & |\omega| < \frac{\pi}{T} \\ \frac{T}{2}, & |\omega| = \frac{\pi}{T} \\ 0, & |\omega| > \frac{\pi}{T} \end{cases} \quad (41)$$

and the error approaches the value

$$\varepsilon^2 = \frac{1}{\pi}\int_{\frac{\pi}{T}}^{\infty} R(\omega)d\omega \quad (42)$$

corresponding to the ideal lowpass prefilter and postfilter with cutoff frequencies $\pm\pi/T$. The error (42) is equal to zero if the spectral density $R(\omega)$ is zero at frequencies $|\omega| > \pi/T$.

Now, we derive the error as a series of spectral moments

$$\lambda_\nu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^\nu R(\omega) d\omega \tag{43}$$

of the random process $x(t)$ by expanding the function $L_{2n+1}(\omega)$ in the Taylor series with respect to the frequency $\omega = 0$:

$$L_{2n+1}(\omega) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} L_{2n+1}^{(\nu)}(0)\omega^\nu. \tag{44}$$

Herein, we use the notation

$$L_{2n+1}^{(\nu)}(0) = \frac{d^\nu}{d\omega^\nu} L_{2n+1}(\omega)\Big|_{\omega=0}.$$

We assume that the spectral moments λ_ν are limited, which means that the random process $x(t)$ is differentiable. For Clarke's random processes, in which we are interested in this paper, this assumption is fulfilled. Derivatives of the function $L_{2n+1}(\omega)$ are found through derivatives of the composite function $f(y)$ (see [25, equation 0.430] for derivatives of a composite function):

$$L_{2n+1}^{(\nu)}(0) = \nu! \left(\frac{T}{2}\right)^\nu \sum f^{(m)}\left(y_{2n+2}(0)\right) \prod_{\eta=1}^{l} \frac{1}{i_\eta!}\left[\frac{y_n^{(\eta)}(0)}{\eta!}\right]^{i_\eta} \tag{45}$$

where the summation is performed over all solutions in nonnegative integers of the equation

$$i_1 + 2i_2 + \ldots + l i_l = \nu \tag{46}$$

and $m = i_1 + i_2 + \ldots + i_l$. It is easy to find that

$$f^{(m)}\left(y_{2n+2}(0)\right) = (-1)^m m!. \tag{47}$$

In [13], it has been shown that

$$y_{2n+2}^{(\eta)}(0) = \begin{cases} 1, & \eta = 0 \\ 0, & \eta \text{ is odd or } 0<\eta<2n+2 \\ 2\pi^{-\eta}\zeta(\eta)C_\eta^{2n+2}\left[(\eta-1)!\right], & \text{otherwise} \end{cases} \tag{48}$$

where $\zeta(\eta)$ is the zeta function [25], and $C_\eta^{2n+2} = (\eta!)/((2n+2)!(\eta-2n-2)!)$ is the binomial coefficient.

Now, we will show that for odd ν, $L_{2n+1}^{(\nu)}(0) = 0$. First, we prove that for odd ν, in every solution of the (46), there exists at least one nonzero element i_η with odd index η. If the latter is not true, then all terms i_η with odd indices η in the left side of the (46) are zero, and we have a sum of even numbers, i.e., the sum is also even. However, due to our assumption, the right side is odd ν. Hence, in every solution of (46), there exists at least one nonzero element i_η with odd index η. Then, due to (48), in every component of the sum in (45), there exists a zero-factor $(0)^{i_\eta}$, i.e., all components of the sum are zero. The latter results in $L_{2n+1}^{(\nu)}(0) = 0$ for odd ν.

For even ν, due to (48), solutions of (46) with elements i_η, having odd indices η or indeces $\eta < 2n+2$, contribute zero components in the sum in (45). Then, (46) can be rewritten as

$$(2n+2)i_{2n+2} + (2n+4)i_{2n+4} + \ldots + l i_l = \nu \tag{49}$$

where l is even. Taking (47) and (48) into account, for even ν, we obtain

$$L_{2n+1}^{(\nu)}(0) = \nu!\left(\frac{T}{2}\right)^\nu \sum (-1)^m m!(2n+2)^m \times \prod_{\eta=2n+2}^{l} \frac{1}{i_\eta!}\left[\frac{2\zeta(\eta)}{\eta\pi^\eta}C_\eta^{2n+2}\right]^{i_\eta}. \tag{50}$$

Substituting (44) into (40), taking (43) and (50) into account, and using that $z = \omega T/2$ and $\zeta(\eta) = B_\eta(2\pi)^\eta/2\eta!$, where B_η are Bernoulli numbers [25], we can formulate the following

Theorem: A minimum integral MSE of spline approximation of a random process with spectral moments $\{\lambda_\nu\}$ can be represented as

$$\varepsilon^2 = \sum_{\nu=2n+2}^{\infty} \lambda_\nu T^\nu \sum \frac{(-1)^m m!}{\left[(2n+1)!\right]^m i_{2n+2}!\ldots i_l!} \times \prod_{\eta=2n+2}^{l}\left[\frac{B_\eta}{(\eta-2n-2)!\eta}\right]^{i_\eta} \tag{51}$$

where the second sum and the product are taken over all solutions in positive integers $\{i_\eta\}$, $\eta = 2n+2, \ldots, l$ to the equation $(2n+2)i_{2n+2} + (2n+4)i_{2n+4} + \ldots + l i_l = \nu$; l is even, $l \le \nu$, and $m = i_{2n+2} + i_{2n+4} + \ldots + i_l$.

REFERENCES

[1] M. Unser, "Sampling—50 years after Shannon," *Proc. IEEE*, vol. 88, pp. 569–587, Apr. 2000.

[2] M. Unser, A. Aldroubi, and M. Eden, "Polynomial spline signal approximations: filter design and asymptotic equivalence with Shannon's sampling theorem," *IEEE Trans. Inform. Theory*, vol. 38, pp. 95–103, Jan. 1992.

[3] ——, "B-spline signal processing: part I—theory," *IEEE Trans. Signal Processing*, vol. 41, pp. 821–833, Feb. 1993.

[4] ——, "B-spline signal processing: part II—efficient design and applications," *IEEE Trans. Signal Processing*, vol. 41, pp. 834–848, Feb. 1993.

[5] Y. V. Zakharov, "Cubic spline-approximation of spectral characteristics of speech signals" (in Russian), in *Collection of Research Papers*. Moscow, Russia: Moscow Power Eng. Inst., 1987.

[6] M. J. Flaherty, "Application of polynomial splines to a time-varying autoregressive model of speech," in *Proc. ICASSP*, 1988, pp. 2220–2223.

[7] Y. V. Zakharov and V. P. Kodanev, "Imitation of the Doppler effect for processing pseudo-noise signals by using spline-approximation" (in Russian), in *Signal Processing in Control and Data Transmission Systems*. Voronezh, Russia: Voronezh State Univ., 1987, pp. 27–31.

[8] ——, "Multipath-Doppler diversity of OFDM signals in an underwater acoustic channel," in *Proc. IEEE ICASSP*, vol. 5, June 5–9, 2000, pp. 2941–2944.

[9] N. Sugita, "Method and System for Demodulating Receive Signal Including Pilot Signal," Patent EP0 961 427, Dec. 1999.

[10] Y. V. Zakharov and E. A. Sidorov, "Limiting spline approximation of random functions" (in Russian), *Avtometriya*, no. 5, pp. 22–26, Sept.–Oct. 1982.

[11] M. Unser, "Splines: a perfect fit for signal and image processing," *IEEE Signal Processing Mag.*, vol. 16, no. 6, pp. 22–38, Nov. 1999.

[12] Y. V. Zakharov and E. A. Sidorov, "Optimum prefiltering of analog signals in PCM systems" (in Russian), *Avtometriya*, no. 4, pp. 88–93, July–Aug. 1979.

[13] Y. V. Zakharov, "Polynomial spline interpolation of differentiable random functions," *Comput. Maths. Math. Phys.*, vol. 35, no. 4, pp. 471–476, 1995.

[14] S. L. Lee, "Fourier transforms of B-splines and fundamental splines for cardinal Hermite interpolations," *Proc. AMS*, vol. 57, no. 2, pp. 291–296, 1976.

[15] Y. V. Zakharov, "Optimum approximation of samples of random processes by local splines," *Telecommun. Radio Eng., pt. 2 (Radio Eng.)*, vol. 46, no. 5, pp. 57–61, May 1991.

[16] R. H. Clarke, "A statistical theory of mobile radio reception," *Bell System Tech. J.*, vol. 47, pp. 957–1000, 1968.

[17] G. L. Stuber, *Principles of Mobile Communication*. Boston, MA: Kluwer, 1997.

[18] Recommendation ITU-R M.1225. Guidelines for Evaluation of Radio Transmission Technologies for IMT-2000, Jan. 1997.

[19] W. C. Jakes, *Microwave Mobile Communications*. New York: Wiley, 1974.

[20] M. Patzold, U. Killat, F. Laue, and Y. Li, "On the statistical properties of deterministic simulation models for mobile fading channel," *IEEE Trans. Veh. Technol.*, vol. 47, pp. 254–269, Feb. 1998.

[21] A. Sayeed and B. Aazhang, "Joint multipath-Doppler diversity in mobile wireless communications," *IEEE Trans. Commun.*, vol. 47, pp. 123–132, Jan. 1999.

[22] D. K. Borah and B. D. Hart, "Frequency-selective fading channel estimation with a polynomial time-varying channel model," *IEEE Trans. Commun.*, vol. 47, pp. 862–873, June 1999.

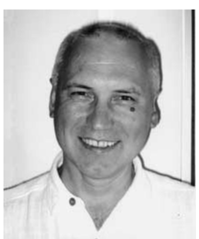
[23] H. Holma and A. Toskala, *WCDMA for UMTS Radio Access for Third Generation Mobile Communications*. New York: Wiley, 2000.

[24] S. Fukumoto, M. Sawahashi, and F. Adachi, "Performance comparison of forward link transmit diversity techniques for W-CDMA mobile radio," in *Proc. 10th IEEE Int. Symp. Personal Indoor Mobile Radio Commun.*, Sept. 1999.

[25] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Sixth ed. New York: Academic, 2000.

[26] V. A. Zheludev, "Local spline-approximation on a uniform grid," *Comput. Maths. Math. Phys.*, vol. 27, no. 9, pp. 1296–1310, 1987.

[27] ——, "Asymptotic formulae for local spline-approximation on a uniform grid" (in Russian), *Doclady Academii Nauk*, vol. 269, no. 4, pp. 797–802, 1983.



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