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Poisson cluster measures: quasi-invariance, integration by parts and equilibrium stochastic dynamics

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Abstract

The distribution $\mu_{\rm cl}$ of a Poisson cluster process in $X = \mathbb{R}^d$ (with i.i.d. clusters) is studied via an auxiliary Poisson measure on the space of configurations in $\mathfrak{X} = \bigsqcup_n X^n$, with intensity measure defined as a convolution of the background intensity of cluster centres and the probability distribution of a generic cluster. We show that the measure $\mu_{\rm cl}$ is quasi-invariant with respect to the group of compactly supported diffeomorphisms of X and prove an integration-by-parts formula for $\mu_{\rm cl}$. The corresponding equilibrium stochastic dynamics is then constructed using the method of Dirichlet forms.

Key words: cluster point process; Poisson measure; configuration space; quasi-invariance; integration by parts; Dirichlet form; stochastic dynamics

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1 Introduction

In the mathematical modelling of multi-component stochastic systems, it is conventional to describe their behaviour in terms of random configurations of "particles" whose spatio-temporal dynamics is driven by interaction of particles with each other and the environment. Examples are ubiquitous and include various models in statistical mechanics, quantum physics, astrophysics, chemical physics, biology, computer science, economics, finance, etc. (see [16] and the extensive bibliography therein).

Initiated in statistical physics and theory of point processes, the development of a general mathematical framework for suitable classes of configurations was over

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decades a recurrent research theme fostered by widespread applications. More recently, there has been a boost of more specific interest in the *analysis* and *geometry* of configuration spaces. In the seminal papers [5,6], an approach was proposed to configuration spaces as *infinite-dimensional manifolds*. This is far from straightforward, since configuration spaces are not vector spaces and do not possess any natural structure of Hilbert or Banach manifolds. However, many "manifold-like" structures can be introduced, which appear to be nontrivial even in the Euclidean case. We refer the reader to papers [2,6,7,25,29] and references therein for further discussion of various aspects of analysis on configuration spaces and applications.

Historically, the approach in [5,6] was motivated by the theory of representations of diffeomorphism groups (see [17,20,33]). To introduce some notation, let $\varGamma_X$ be the space of countable subsets (*configurations*) without accumulation points in a topological space X (e.g., Euclidean space $\mathbb{R}^d$). Any probability measure μ on $\varGamma_X$, quasi-invariant with respect to the action of the group $\mathrm{Diff}_0(X)$ of compactly supported diffeomorphisms of X (lifted pointwise to transformations of $\varGamma_X$), generates a canonical unitary representation of $\mathrm{Diff}_0(X)$ in $L^2(\varGamma_X, \mu)$. It has been proved in [33] that this representation is irreducible if and only if μ is $\mathrm{Diff}_0(X)$-ergodic. Representations of such type are instrumental in the general theory of representations of diffeomorphism groups [33] and in quantum field theory [17,18].

According to a general paradigm described in [5,6], configuration space analysis is determined by the choice of a suitable probability measure μ on $\varGamma_X$ (quasi-invariant with respect to $\mathrm{Diff}_0(X)$). It can be shown that such a measure μ satisfies a certain integration-by-parts formula, which enables one to construct, via the theory of Dirichlet forms, the associated equilibrium dynamics (stochastic process) on $\varGamma_X$ such that μ is its invariant measure [5,6,27]. In turn, the equilibrium process plays an important role in the asymptotic analysis of statistical-mechanical systems whose spatial distribution is controlled by the measure μ; for instance, this process is a natural candidate for being an asymptotic "attractor" for motions started from a perturbed (non-equilibrium) configuration.

This programme has been successfully implemented in [5] for the Poisson measure, which is the simplest and most well-studied example of a $\mathrm{Diff}_0(X)$-quasi-invariant measure on $\varGamma_X$, and in [6] for a wider class of Gibbs measures, which appear in statistical mechanics of classical continuous gases. In particular, it has been shown that in the Poisson case, the equilibrium dynamics amounts to the well-known independent particle process, that is, an infinite family of independent (distorted) Brownian motions started at the points of a random Poisson configuration. In the Gibbsian case, the dynamics is much more complex due to interaction between the particles.

The Gibbsian class (containing the Poisson measure as a simple "interaction-free" case) is essentially the sole example so far that has been fully amenable to such analysis. In the present paper, our aim is to develop a similar framework for a different class of random spatial structures, namely the well-known *cluster point processes* (see, e.g., [14,16]). Cluster process is a simple model to describe effects of grouping ("clustering") in a sample configuration. The intuitive idea is to assume that the random configuration has a hierarchical structure, whereby independent clusters of points are distributed around a certain (random) configuration of invisible "centres". The simplest model of such a kind is the *Poisson cluster process*, obtained by choosing a Poisson point process as the background configuration of the cluster centres.

Cluster models have been very popular in numerous practical applications ranging from neurophysiology (nerve impulses) and ecology (spatial distribution of offspring around the parents) to seismology (statistics of earthquakes) and cosmology (formation of constellations and galaxies). More recent examples include applications to trapping models of diffusion-limited reactions in chemical kinetics [1,9,12], where clusterization may arise due to binding of traps to a substrate (e.g., a polymer chain) or trap generation (e.g., by radiation damage). An exciting range of new applications in physics and biology is related to the dynamics of clusters consisting of a few to hundreds of atoms or molecules. Investigation of such "mesoscopic" structures, intermediate between bulk matter and individual atoms or molecules, is of paramount importance in the modern nanoscience and nanotechnology (for an authoritative account of the state of the art in this area, see a recent review [15] and further references therein).

In the present work, we consider Poisson cluster processes in $X = \mathbb{R}^d$. We prove the $\mathrm{Diff}_0(X)$-quasi-invariance of the Poisson cluster measure μ_{cl} and establish the integration-by-parts formula. We then construct an associated Dirichlet form, which implies in a standard way the existence of equilibrium stochastic dynamics on the configuration space $\varGamma_X$. Our technique is based on the representation of μ_{cl} as a natural "projection" image of a certain Poisson measure on an auxiliary configuration space $\varGamma_{\mathfrak{X}}$ over a disjoint union $\mathfrak{X} = \bigsqcup_n X^n$, comprising configurations of "droplets" representing individual clusters of variable size. A suitable intensity measure on $\mathfrak{X}$ is obtained as a convolution of the background intensity $\lambda(\mathrm{d}x)$ (of cluster centres) with the probability distribution $\eta(\mathrm{d}\bar{y})$ of a generic cluster. This approach enables one to apply the well-developed apparatus of Poisson measures to the study of the Poisson cluster measure μ_{cl}.

Let us point out that the projection construction of the Poisson cluster measure is very general, and in particular it works even in the case when "generalized" configurations (with possible accumulation or multiple points) are allowed. However, to be able to construct a well-defined differentiable structure on cluster configurations, we need to restrict ourselves to the space $\varGamma_X$ of "proper" (i.e., locally finite and simple) configurations. Using the technique of Laplace functionals, we obtain necessary and sufficient conditions of almost sure (a.s.) properness for Poisson cluster configurations, set out in terms of the background intensity $\lambda(\mathrm{d}x)$ of cluster centres and the in-cluster distribution $\eta(\mathrm{d}\bar{y})$. To the best of our knowledge, these conditions appear to be new (cf., e.g., [16, § 6.3]) and may be of interest for the general theory of cluster point processes.

Some of the results of this paper have been sketched in [11] (in the case of clusters of fixed size). We anticipate that the projection approach developed in the present paper can be applied to the study of more general cluster measures on configurations spaces, especially Gibbs cluster measure (see [10] for the case of fixed-size clusters). Such models, and related functional-analytic issues, will be addressed in our future work.

The paper is organized as follows. In Section 2.1, we set out a general framework of probability measures in the space of generalized configurations $\Gamma_X^\sharp$. In Section 2.2, we recall the definition and discuss the construction and some basic properties of the Poisson measure on the space $\Gamma_X^\sharp$, while Section 2.3 goes on to describe the Poisson cluster measure. In Section 2.4, we discuss criteria for Poisson cluster configurations to be a.s. locally finite and simple (Theorem 2.7, the proof of which is deferred to the Appendix). An auxiliary intensity measure $\lambda^\star$ on the space $\mathfrak{X} = \bigsqcup_n X^n$ is introduced and discussed in Section 3.1, which allows us to define the corresponding Poisson measure $\pi_{\lambda^\star}$ on the configuration space $\Gamma_{\mathfrak{X}}^\sharp$ (Section 3.2). Theorem 3.6 of Section 3.3 shows that the Poisson cluster measure μ_{cl} can be obtained as a push-forward of the Poisson measure $\pi_{\lambda^\star}$ on $\Gamma_{\mathfrak{X}}^\sharp$ under the "unpacking" map $\mathfrak{X} \ni \bar{x} \mapsto \mathfrak{p}(\bar{x}) := \bigsqcup_{x_i \in \bar{x}} \{x_i\} \in \Gamma_{\mathfrak{X}}^\sharp$. In Section 3.4, we describe a more general construction of μ_{cl} using another Poisson measure defined on the space $\Gamma_{X\times\mathfrak{X}}^\sharp$ of configurations of pairs $(x, \bar{y})$ ($x =$ cluster centre, $\bar{y} =$ in-cluster configuration), with the product intensity measure $\lambda(\mathrm{d}x) \otimes \eta(\mathrm{d}\bar{y})$. Following a brief compendium on differentiable functions in configuration spaces (Section 4.1), Section 4.2 deals with the property of quasi-invariance of the measure μ_{cl} with respect to the diffeomorphism group $\mathrm{Diff}_0(X)$ (Theorem 4.3). Further on, an integration-by-parts formula for μ_{cl} is established in Section 4.3 (Theorem 4.5). The Dirichlet form $\mathcal{E}_{\mu_{\mathrm{cl}}}$ associated with μ_{cl} is defined and studied in Section 5.1, which enables us to construct in Section 5.2 the canonical equilibrium dynamics (i.e., diffusion on the configuration space with invariant measure μ_{cl}). In addition, we show that the form $\mathcal{E}_{\mu_{\mathrm{cl}}}$ is irreducible (Theorem 5.4, Section 5.3). Finally, the Appendix includes the proof of Theorem 2.7 (Section 6.1) and the proof of a well-known general result on quasi-invariance of Poisson measures, adapted to our purposes (Section 6.2).

2 Poisson and Poisson cluster measures in configuration spaces

In this section, we fix some notations and describe the setting of configuration spaces that we shall use. As compared to a standard exposition (see, e.g., [14,16]), we adopt a more general standpoint by allowing configurations with multiple points and/or accumulation points. With this modification in mind, we recall the definition and some properties of Poisson point process (as a probability measure in the generalized configuration space $\Gamma_X^\sharp$). We then proceed to introduce the main object of the paper, the cluster Poisson point process and the corresponding measure μ_{cl} in $\Gamma_X^\sharp$. The central result of this section is the projection constriction showing that μ_{cl} can be obtained as a push-forward of a suitable Poisson measure in the auxiliary "vector" configuration space $\Gamma_{\mathfrak{X}}^\sharp$, where $\mathfrak{X} = \bigsqcup_n X^n$.

Let X be a Polish space (i.e., separable completely metrizable topological space), equipped with the Borel σ-algebra $\mathcal{B}(X)$ generated by the open sets. Denote $\overline{\mathbb{Z}}_+ := \mathbb{Z}_+ \cup \{\infty\}$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, and consider the space $\mathfrak{X}$ built from Cartesian powers of X, that is, a disjoint union $\mathfrak{X} := \bigsqcup_{n \in \overline{\mathbb{Z}}_+} X^n$ including $X^0 = \{\emptyset\}$ and the space X^∞ of infinite sequences $(x_1, x_2, \dots)$. That is to say, $\bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}$ if and only if $\bar{x} \in X^n$ for some $n \in \overline{\mathbb{Z}}_+$. For simplicity of notation, we take the liberty to write $x_i \in \bar{x}$ if x_i is a coordinate of the vector $\bar{x}$.

Each space X^n is equipped with the product topology induced by X, that is, the coarsest topology in which all coordinate projections $(x_1, \dots, x_n) \mapsto x_i$ are continuous ($i = 1, \dots, n$). Hence, the space $\mathfrak{X}$ is endowed with the natural disjoint union topology, that is, the finest topology in which the canonical injections $\mathbf{j}_n : X^n \to \mathfrak{X}$ are continuous ($n \in \overline{\mathbb{Z}}_+$). In other words, a set $U \subset \mathfrak{X}$ is open in this topology whenever $U = \bigsqcup_{n \in \overline{\mathbb{Z}}_+} U_n$, where each U_n is an open subset in X^n ($n \in \overline{\mathbb{Z}}_+$). Hence, the Borel σ-algebra on $\mathfrak{X}$ is given by $\mathcal{B}(\mathfrak{X}) = \bigoplus_{n \in \overline{\mathbb{Z}}_+} \mathcal{B}(X^n)$, that is, consists of sets of the form $B = \bigsqcup_{n \in \overline{\mathbb{Z}}_+} B_n$, where $B_n \in \mathcal{B}(X^n)$, $n \in \overline{\mathbb{Z}}_+$.

Remark 2.1. Note that a set $K \subset \mathfrak{X}$ is compact if and only if $K = \bigsqcup_{n=0}^{N} K_n$, where $N < \infty$ and K_n are compact subsets of X^n, respectively. This becomes clear by considering an open cover of K by the sets $U_n = X^n$, $n \in \overline{\mathbb{Z}}_+$.

Denote by $\mathcal{N}(X)$ the space of $\overline{\mathbb{Z}}_+$-valued measures $N(\cdot)$ on $\mathcal{B}(X)$ with countable (i.e., finite or countably infinite) support $\operatorname{supp} N := \{x \in X : N\{x\} > 0\}$ (here and below, we use $N\{x\}$ as a shorthand for a more accurate $N(\{x\})$; the same convention applies to other measures). Consider the natural projection

$$\mathfrak{X} \ni \bar{x} \mapsto \mathfrak{p}(\bar{x}) := \sum_{x_i \in \bar{x}} \delta_{x_i} \in \mathcal{N}(X), \tag{2.1}$$

where δ_x is Dirac measure at point $x \in X$. Gathering any coinciding points $x_i \in \bar{x}$, the measure $N = \sum_{x_i \in \bar{x}} \delta_{x_i}$ in (2.1) can be written down as $N = \sum_{x_i^* \in \operatorname{supp} N} k_i \delta_{x_i^*}$, where $k_i = N\{x_i^*\} > 0$ is the "multiplicity" (possibly infinite) of the point $x_i^* \in \operatorname{supp} N$. Any such measure N can be conveniently associated with a *generalized configuration* γ of points in X,

$$N \leftrightarrow \gamma := \bigsqcup_{x_i^* \in \operatorname{supp} N} \underbrace{\{x_i^*\} \sqcup \dots \sqcup \{x_i^*\}}_{k_i},$$

where the disjoint union $\{x^*\} \sqcup \dots \sqcup \{x^*\}$ signifies the inclusion of several distinct copies of point $x^* \in \operatorname{supp} N$. Thus, the mapping (2.1) can be symbolically rewritten as

$$\mathfrak{p}(\bar{x}) = \gamma := \bigsqcup_{x_i \in \bar{x}} \{x_i\}, \qquad \bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}. \tag{2.2}$$

That is to say, under the projection mapping $\mathfrak{p}$ each vector from $\mathfrak{X}$ is "unpacked" into distinct components, resulting in a countable aggregate of points in X (with possible multiple points), which we interpret as a generalized configuration γ. Note that, formally, $\bar{x}$ may be from the "trivial" component $X^0 = \{\emptyset\}$, in which case the union in (2.2) (as well as the sum in (2.1)) is vacuous and hence corresponds to the empty configuration, $\gamma = \emptyset$.

Even though generalized configurations are not, strictly speaking, subsets of X (due to possible multiple points), it is convenient to keep using set-theoretic notations, which should not cause any confusion. For instance, we write $\gamma_B := \gamma \cap B$ for the restriction of configuration γ to a subset $B \in \mathcal{B}(X)$. Similarly, for a function $f : X \to \mathbb{R}$ we denote

$$\langle f, \gamma \rangle := \sum_{x_i \in \gamma} f(x_i) \equiv \sum_{x_i^* \in \operatorname{supp} N} N\{x_i^*\}\, f(x_i^*) = \int_X f(x)\, N(\mathrm{d}x). \tag{2.3}$$

This formula motivates the following convention that will be used throughout: if $\gamma = \emptyset$ then $\sum_{x \in \gamma} f(x) := 0$.

In what follows, we shall identify generalized configurations γ with the corresponding measures $N = \sum_{x_i \in \gamma} \delta_{x_i}$, and we shall opt to interpret the notation γ either as an aggregate of (multiple) points in X or as a $\overline{\mathbb{Z}}_+$-valued measure or both, depending on the context. For example, if $\mathbf{1}_B(x)$ is the indicator function of a set $B \in \mathcal{B}(X)$ then $\langle \mathbf{1}_B, \gamma \rangle = \gamma(B)$ is the total number of points (counted with their multiplicities) in the restriction γ_B of the configuration γ to B.

Definition 2.1. *Configuration space* $\Gamma_X^\sharp$ is the set of generalized configurations γ in X, endowed with the *cylinder σ-algebra* $\mathcal{B}(\Gamma_X^\sharp)$ generated by the class of cylinder sets $C_B^n := \{\gamma \in \Gamma_X^\sharp : \gamma(B) = n\}$, $B \in \mathcal{B}(X)$, $n \in \mathbb{Z}_+$.

Remark 2.2. Note that the set $C_B^\infty = \{\gamma \in \Gamma_X^\sharp : \gamma(B) = \infty\}$ is measurable:

$$C_B^\infty = \bigcap_{n=0}^{\infty} \{\gamma \in \Gamma_X^\sharp : \gamma(B) \ge n\} = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} C_B^k \in \mathcal{B}(\Gamma_X^\sharp).$$

The mapping $\mathfrak{p} : \mathfrak{X} \to \Gamma_X^\sharp$ defined by formula (2.2) is measurable, since for any cylinder set $C_B^n \in \mathcal{B}(\Gamma_X^\sharp)$ we have

$$\mathfrak{p}^{-1}(C_B^n) = D_B^n := \left\{ \bar{x} \in \mathfrak{X} : \sum_{x_i \in \bar{x}} \mathbf{1}_B(x_i) = n \right\} \in \mathcal{B}(\mathfrak{X}). \tag{2.4}$$

As already mentioned, conventional theory of point processes (and their distributions as probability measures on configuration spaces) usually rules out the possibility of accumulation points or multiple points (see, e.g., [16]).

Definition 2.2. Configuration $\gamma \in \Gamma_X^\sharp$ is said to be *locally finite* if $\gamma(K) < \infty$ for any compact set $K \subset X$. Configuration $\gamma \in \Gamma_X^\sharp$ is called *simple* if $\gamma\{x\} \le 1$ for each $x \in X$. Configuration $\gamma \in \Gamma_X^\sharp$ is called *proper* if it is both locally finite and simple. The set of proper configurations will be denoted by Γ_X and called the *proper configuration space* over X. The corresponding σ-algebra $\mathcal{B}(\Gamma_X)$ is generated by the cylinder sets $\{\gamma \in \Gamma_X : \gamma(B) = n\}$ $(B \in \mathcal{B}(X),\ n \in \mathbb{Z}_+)$.

Like in the standard theory for proper configuration spaces (see, e.g., [16, § 6.1]), every measure μ on the generalized configuration space $\Gamma_X^\sharp$ can be characterized by its Laplace functional

$$L_\mu[f] := \int_{\Gamma_X^\sharp} \mathrm{e}^{-\langle f,\gamma\rangle}\,\mu(\mathrm{d}\gamma), \qquad f \in \mathrm{M}_+(X), \tag{2.5}$$

where $\mathrm{M}_+(X)$ is the set of measurable non-negative functions on X (so that the integral in (2.5) is well defined since $0 \le \mathrm{e}^{-\langle f,\gamma\rangle} \le 1$). To see why $L_\mu[\cdot]$ completely determines the measure μ on $\mathcal{B}(\Gamma_X^\sharp)$, note that if $B \in \mathcal{B}(X)$ then $L_\mu[s\mathbf{1}_B]$ as a function of $s > 0$ gives the Laplace–Stieltjes transform of the distribution of the random variable $\gamma(B)$ and as such determines the values of the measure μ on the cylinder sets $C_B^n \in \mathcal{B}(\Gamma_X^\sharp)$ ($n \in \mathbb{Z}_+$). In particular, $L_\mu[s\mathbf{1}_B] = 0$ if and only if $\gamma(B) = \infty$ (μ-a.s.). Similarly, using linear combinations $\sum_{i=1}^k s_i \mathbf{1}_{B_i}$ we can recover the values of μ on the cylinder sets

$$C_{B_1,\dots,B_k}^{n_1,\dots,n_k} := \bigcap_{i=1}^k C_{B_i}^{n_i} = \{\gamma \in \Gamma_X^\sharp : \gamma(B_i) = n_i,\ i = 1,\dots,k\}$$

and hence on the ring $\mathcal{C}(X)$ of finite disjoint unions of such sets. Since the ring $\mathcal{C}(X)$ generates the cylinder σ-algebra $\mathcal{B}(\Gamma_X^\sharp)$, the extension theorem (see, e.g., [19, § 13, Theorem A] or [16, Theorem A1.3.III]) ensures that the measure μ on $\mathcal{B}(\Gamma_X^\sharp)$ is determined uniquely.

2.2 *Poisson measure*

We recall here some basic facts about Poisson measures in configuration spaces. As compared to the customary treatment, another difference, apart from working in the space of generalized configurations $\Gamma_X^\sharp$, is that we use a *σ-finite* intensity measure rather than a *locally finite* one.

Poisson measure on the configuration space $\Gamma_X^\sharp$ is defined descriptively as follows (cf. [16, § 2.4]).

Definition 2.3. Let λ be a σ-finite measure in $(X, \mathcal{B}(X))$ (not necessarily infinite, i.e., $\lambda(X) \le \infty$). The *Poisson measure π_λ with intensity* λ is a probability measure on $\mathcal{B}(\Gamma_X^\sharp)$ satisfying the following condition: for any disjoint sets $B_1,\dots,B_k \in \mathcal{B}(X)$ (i.e., $B_i \cap B_j = \emptyset$ for $i \ne j$), such that $\lambda(B_i) < \infty$ ($i = 1,\dots,k$), and any $n_1,\dots,n_k \in \mathbb{Z}_+$, the value of π_λ on the cylinder set $C_{B_1,\dots,B_k}^{n_1,\dots,n_k}$ is given by

$$\pi_\lambda\Bigl(C_{B_1,\dots,B_k}^{n_1,\dots,n_k}\Bigr) = \prod_{i=1}^k \frac{\lambda(B_i)^{n_i}\,\mathrm{e}^{-\lambda(B_i)}}{n_i!} \tag{2.6}$$

(with the convention $0^0 := 1$). That is, for disjoint sets B_i the values $\gamma(B_i)$ are mutually independent Poisson random variables with parameters $\lambda(B_i)$, respectively.

A well-known "explicit" construction of the Poisson measure π_λ is as follows (cf. [5,31]). For a fixed set $\Lambda \in \mathcal{B}(X)$ such that $\lambda(\Lambda) < \infty$, consider the restriction mapping p_Λ,

$$\Gamma_X^\sharp \ni \gamma \mapsto p_\Lambda\gamma = \gamma \cap \Lambda \equiv \gamma_\Lambda \in \Gamma_\Lambda^\sharp.$$

Clearly, $p_\Lambda(C_\Lambda^n) = \{\tilde\gamma \in \Gamma_\Lambda^\sharp : \tilde\gamma(\Lambda) = n\}$. For $A \in \mathcal{B}(\Gamma_\Lambda^\sharp)$ and $n \in \mathbb{Z}_+$, let $A_{\Lambda,n} := A \cap p_\Lambda(C_\Lambda^n) \in \mathcal{B}(\Gamma_\Lambda^\sharp)$ and define the measure

$$\pi_\lambda^\Lambda(A) := \mathrm{e}^{-\lambda(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!}\, \lambda^{\otimes n} \circ \mathfrak{p}^{-1}(A_{\Lambda,n}), \qquad A \in \mathcal{B}(\Gamma_\Lambda^\sharp), \tag{2.7}$$

where $\lambda^{\otimes n} = \underbrace{\lambda \otimes \cdots \otimes \lambda}_{n}$ is the product measure in $(X^n, \mathcal{B}(X^n))$ (we formally set $\lambda^{\otimes 0} := \delta_{\{\emptyset\}}$) and $\mathfrak{p}$ is the projection operator defined in (2.2). In particular, (2.7) implies that π_λ^Λ is a probability measure on $\Gamma_\Lambda^\sharp$. It is easy to check that the "cylindrical" measure $\pi_\lambda^\Lambda \circ p_\Lambda$ in $\Gamma_X^\sharp$ (in fact, supported on $\bigcup_{n=0}^\infty C_\Lambda^n$) satisfies equation (2.6) for any disjoint Borel sets $B_i \subset \Lambda$. It is also clear that the family $\{\pi_\lambda^\Lambda,\ \Lambda \subset X\}$ is consistent, that is, the restriction of the measure π_λ^Λ to a smaller configuration space $\Gamma_{\Lambda'}^\sharp$ (with $\Lambda' \subset \Lambda$) coincides with $\pi_\lambda^{\Lambda'}$, that is, $\pi_\lambda^\Lambda \circ (p_\Lambda p_{\Lambda'}^{-1}) = \pi_\lambda^{\Lambda'}$.

Existence (and uniqueness) of a measure π_λ in $(\Gamma_X^\sharp, \mathcal{B}(\Gamma_X^\sharp))$ such that, for any $\Lambda \in \mathcal{B}(X)$, the push-forward measure $p_\Lambda^*\pi_\lambda \equiv \pi_\lambda \circ p_\Lambda^{-1}$ coincides with π_λ^Λ (which implies that π_λ satisfies Definition 2.3 and is therefore a Poisson measure on the configuration space $\Gamma_X^\sharp$), now follows by a projective version of the fundamental Kolmogorov extension theorem (see, e.g., [16, § A1.5] or [28, Ch. 5]). More precisely, recall that the measure λ on X is σ-finite, hence there is a countable family of sets $B_k \in \mathcal{B}(X)$ such that $\lambda(B_k) < \infty$ and $\bigcup_{k=1}^\infty B_k = X$. Then $\Lambda_m := \bigcup_{k=1}^m B_k \in \mathcal{B}(X)$ ($m \in \mathbb{N}$) is a monotone increasing sequence of sets such that $\lambda(\Lambda_m) < \infty$ and $\bigcup_{m=1}^\infty \Lambda_m = X$. By the construction (2.7), we obtain a consistent family of probability measures $\pi_\lambda^{\Lambda_m}$ on the configuration spaces $\Gamma_{\Lambda_m}^\sharp$, respectively. Using the metric in X (which is assumed to be a Polish space, see Section 2.1), one can define a suitable distance between finite configurations in each space $\Gamma_{\Lambda_m}^\sharp$ and thus convert $\Gamma_{\Lambda_m}^\sharp$ into a Polish space (see [31]), which ensures that the Kolmogorov extension theorem is applicable.

Remark 2.3. Even though the paper [31] deals with simple configurations only, its methods may be easily extended to a more general case of configurations with multiple points. However, finiteness of configurations in each Λ_m is essential.

Remark 2.4. The requirement that X is a Polish space (see Section 2.1) is only needed in order to equip the spaces of finite configurations in the sets Λ_m with the structure of a Polish space and thus to be able to apply the Kolmogorov extension theorem as explained above (see [31]). This assumption may be replaced by a more general condition that $(X, \mathcal{B}(X))$ is a standard Borel space (i.e., Borel isomorphic to a Borel subset of a Polish space, see [21,28]).

Remark 2.5. Formula (2.7), rewritten in the form

$$\pi_\lambda^\Lambda(A) = \sum_{n=0}^{\infty} \frac{\lambda(\Lambda)^n\, \mathrm{e}^{-\lambda(\Lambda)}}{n!} \cdot \frac{\lambda^{\otimes n} \circ \mathfrak{p}^{-1}(A_{\Lambda,n})}{\lambda(\Lambda)^n}\,,$$

gives an explicit way of sampling a Poisson configuration γ_Λ in the set Λ: first, a random value of $\gamma(\Lambda)$ is sampled as a Poisson random variable with parameter $\lambda(\Lambda) < \infty$, and then, conditioned on the event $\{\gamma(\Lambda) = n\}$ $(n \in \mathbb{Z}_+)$, the n points are distributed over Λ independently of each other, with probability distribution $\lambda(\mathrm{d}x)/\lambda(\Lambda)$ each (cf. [22, §2.4]).

Decomposition (2.7) implies that if $F(\gamma) \equiv F(\gamma_\Lambda)$ for some set $\Lambda \in \mathcal{B}(X)$ such that $\lambda(\Lambda) < \infty$, then

$$\begin{aligned}\int_{\Gamma_X^\sharp} F(\gamma)\, \pi_\lambda(\mathrm{d}\gamma) &= \int_{\Gamma_X^\sharp} F(p_\Lambda \gamma)\, \pi_\lambda(\mathrm{d}\gamma) = \int_{\Gamma_\Lambda^\sharp} F(\gamma)\, \pi_\lambda^\Lambda(\mathrm{d}\gamma)\\ &= \mathrm{e}^{-\lambda(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} F(\{x_1, \dots, x_n\})\, \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_n). \qquad (2.8)\end{aligned}$$

A well-known formula for the Laplace functional of a Poisson point process without accumulation points (see, e.g., [5,16]]) is easily verified in the case of generalized configurations.

Proposition 2.1. *The Laplace functional* $L_{\pi_\lambda}[f] := \int_{\Gamma_X^\sharp} \mathrm{e}^{-\langle f, \gamma\rangle}\, \pi_\lambda(\mathrm{d}\gamma)$ *of the Poisson measure* π_λ *on the configuration space* $\Gamma_X^\sharp$ *is given by*

$$L_{\pi_\lambda}[f] = \exp\left\{-\int_X \left(1 - \mathrm{e}^{-f(x)}\right) \lambda(\mathrm{d}x)\right\}, \qquad f \in \mathrm{M}_+(X). \qquad (2.9)$$

Proof. Repeating a standard derivation, suppose that $\lambda(\Lambda) < \infty$ and set $f_\Lambda := f \cdot \mathbf{1}_\Lambda$. Applying formula (2.8) we have

$$\begin{aligned}\int_{\Gamma_X^\sharp} \mathrm{e}^{-\langle f_\Lambda, \gamma\rangle}\, \pi_\lambda(\mathrm{d}\gamma) &= \mathrm{e}^{-\lambda(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \exp\left\{-\sum_{i=1}^{n} f_\Lambda(x_i)\right\} \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_n)\\ &= \mathrm{e}^{-\lambda(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_\Lambda \mathrm{e}^{-f_\Lambda(x)}\, \lambda(\mathrm{d}x)\right)^n\\ &= \exp\left\{-\int_X \left(1 - \mathrm{e}^{-f_\Lambda(x)}\right) \lambda(\mathrm{d}x)\right\}. \qquad (2.10)\end{aligned}$$

Since $f_\Lambda(x) \uparrow f(x)$ as $\Lambda \uparrow X$ (more precisely, setting $\Lambda = \Lambda_m$ as in the above construction of π_λ and passing to the limit as $m \to \infty$), by applying the monotone convergence theorem to both sides of (2.10) we obtain (2.9). $\square$

Formula (2.6) implies that if $B_1 \cap B_2 = \emptyset$ then the restricted configurations γ_{B_1} and γ_{B_2} are independent under the Poisson measure π_λ. That is, if $B := B_1 \cup B_2$ then the distribution $\pi_\lambda^B = p_B^* \pi_\lambda$ of composite configurations $\gamma_B = \gamma_{B_1} \sqcup \gamma_{B_2}$ coincides with the product measure $\pi_\lambda^{B_1} \otimes \pi_\lambda^{B_2}$ $(\pi_\lambda^{B_i} = p_{B_i}^* \pi_\lambda)$. Building on this observation, we obtain the following useful result.

Proposition 2.2. *Suppose that* $(X_n, \mathcal{B}(X_n))$ $(n \in \mathbb{N})$ *is a family of disjoint measurable spaces (i.e.,* $X_i \cap X_j = \emptyset$, $i \neq j$*), with measures* λ_n*, respectively, and let* π_{λ_n} *be the corresponding Poisson measures on the configuration spaces* $\Gamma_{X_n}^\sharp$ $(n \in \mathbb{N})$*. Consider the disjoint-union space* $X = \bigsqcup_{n=1}^\infty X_n$ *endowed with the* σ*-algebra* $\mathcal{B}(X) = \bigoplus_{n=1}^\infty \mathcal{B}(X_n)$ *and measure* $\lambda = \bigoplus_{n=1}^\infty \lambda_n$*. Then the product mea-*

sure $\pi_\lambda = \bigotimes_{n=1}^{\infty} \pi_{\lambda_n}$ *exists and is a Poisson measure on the configuration space* $\varGamma_X^\sharp$ *with intensity measure* λ.

Proof. Note that $\varGamma_X^\sharp$ is a Cartesian product space, $\varGamma_X^\sharp = \bigtimes_{n=1}^{\infty} \varGamma_{X_n}^\sharp$, endowed with the product σ-algebra $\mathcal{B}(\varGamma_X^\sharp) = \bigotimes_{n=1}^{\infty} \mathcal{B}(\varGamma_{X_n}^\sharp)$. The existence of the product measure $\pi_\lambda := \bigotimes_{n=1}^{\infty} \pi_{\lambda_n}$ on $(\varGamma_X^\sharp, \mathcal{B}(\varGamma_X^\sharp))$ now follows by a standard result for infinite products of probability measures (see, e.g., [19, § 38, Theorem B] or [21, Corollary 5.17]). Let us point out that this theorem is valid without any regularity conditions on the spaces X_n.

To show that π_λ is a Poisson measure, one could check the cylinder condition (2.7), but it is easier to compute its Laplace functional. Note that each function $f \in \mathrm{M}_+(X)$ is decomposed as $f = \sum_{n=1}^{\infty} f_{X_n} \cdot \mathbf{1}_{X_n}$, where $f_{X_n} \in \mathrm{M}_+(X_n)$ is the restriction of f to X_n; similarly, each configuration $\gamma \in \varGamma_X^\sharp$ may be represented as $\gamma = \bigsqcup_{n=1}^{\infty} \gamma_{X_n}$, where $\gamma_{X_n} = p_{X_n}\gamma \in \varGamma_{X_n}^\sharp$. Hence, $\langle f, \gamma\rangle = \sum_{n=1}^{\infty} \langle f_{X_n}, \gamma_{X_n}\rangle$ and, using Proposition 2.1 for each π_{λ_n}, we obtain

$$
\begin{aligned}
\int_{\varGamma_X^\sharp} \mathrm{e}^{-\langle f,\gamma\rangle}\, \pi_\lambda(\mathrm{d}\gamma) &= \int_{\bigtimes_{n=1}^{\infty} \varGamma_{X_n}^\sharp} \exp\left\{-\sum_{n=1}^{\infty} \langle f_{X_n}, \gamma_n\rangle\right\} \bigotimes_{n=1}^{\infty} \pi_{\lambda_n}(\mathrm{d}\gamma_n) \\
&= \prod_{n=1}^{\infty} \int_{\varGamma_{X_n}^\sharp} \mathrm{e}^{-\langle f_{X_n}, \gamma_n\rangle}\, \pi_{\lambda_n}(\mathrm{d}\gamma_n) \\
&= \exp\left\{-\sum_{n=1}^{\infty} \int_{X_n} \left(1 - \mathrm{e}^{-f_{X_n}(x_n)}\right) \lambda_n(\mathrm{d}x_n)\right\} \\
&= \exp\left\{-\int_X \left(1 - \mathrm{e}^{-f(x)}\right) \lambda(\mathrm{d}x)\right\},
\end{aligned}
$$

and it follows, according to formula (2.9), that π_λ is a Poisson measure. □

Remark 2.6. Using Proposition 2.2, one can give a construction of a Poisson measure π_λ on the configuration space $\varGamma_X^\sharp$ avoiding any additional topological conditions upon the space X (e.g., that X is a Polish space) that are needed for the sake of the Kolmogorov extension theorem (similar ideas are developed in [22,23] in the context of proper configuration spaces). To do so, recall that the measure λ is σ-finite and define $X_n := \varLambda_n \setminus \varLambda_{n-1}$ ($n \in \mathbb{N}$), where the sets $\emptyset = \varLambda_0 \subset \varLambda_1 \subset \cdots \subset \varLambda_n \subset \cdots \subset X$, such that $\lambda(\varLambda_n) < \infty$ and $\bigcup_{n=1}^{\infty} \varLambda_n = X$, were considered above. Then the family of sets (X_n) is a disjoint partition of X (i.e., $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_{n=1}^{\infty} X_n = X$), such that $\lambda(X_n) < \infty$ for all $n \in \mathbb{N}$. Using formula (2.6), we construct the Poisson measures $\pi_{\lambda_n} \equiv p_{X_n}\pi_\lambda$ on each $\varGamma_{X_n}^\sharp$, where $\lambda_n = \lambda_{X_n}$ is the restriction of the measure λ to the set X_n. Now, it follows by Proposition 2.2 that the product measure $\pi_\lambda = \bigotimes_{n=1}^{\infty} \pi_{\lambda_n}$ is the required Poisson measure on $\varGamma_X^\sharp$.

Remark 2.7. Although not necessary for the *existence* of the Poisson measure, in order to develop a sensible theory one needs to ensure that there are enough measurable sets and in particular any singleton set $\{x\}$ is measurable. To this end, it is suitable to assume (see [22, § 2.1]) that the diagonal set $\{x = y\}$ is measurable in the product space $X^2 = X \times X$, that is,

$$D := \{(x,y) \in X^2 : x = y\} \in \mathcal{B}(X^2). \tag{2.11}$$

This condition readily implies that $\{x\} \in \mathcal{B}(X)$ for each $x \in X$. Note that if X is a Polish space, condition (2.11) is automatically satisfied because then the diagonal D is a closed set in X^2.

Let us also record one useful general result known as the Mapping Theorem (see [22, § 2.3], where configurations are assumed proper and the mapping is one-to-one). Let $\varphi : X \to Y$ be a measurable mapping (not necessarily one-to-one) of X to another (or the same) measurable space Y endowed with Borel σ-algebra $\mathcal{B}(Y)$. The mapping φ can be lifted to a measurable "diagonal" mapping (denoted by the same letter) between the configuration spaces $\Gamma_X^\sharp$ and $\Gamma_Y^\sharp$:

$$\Gamma_X^\sharp \ni \gamma \mapsto \varphi(\gamma) := \bigsqcup_{x\in\gamma} \{\varphi(x)\} \in \Gamma_Y^\sharp. \tag{2.12}$$

Proposition 2.3 (Mapping Theorem). *If π_λ is a Poisson measure on $\Gamma_X^\sharp$ with intensity measure λ, then under the mapping* (2.12) *the push-forward measure $\varphi^*\pi_\lambda \equiv \pi_\lambda \circ \varphi^{-1}$ is a Poisson measure on $\Gamma_Y^\sharp$ with intensity measure $\varphi^*\lambda \equiv \lambda \circ \varphi^{-1}$.*

Proof. It suffices to compute the Laplace functional of $\varphi^*\pi_\lambda$. Using Proposition 2.1, for any $f \in \mathrm{M}_+(Y)$ we have

$$\begin{aligned}
L_{\varphi^*\pi_\lambda}[f] &= \int_{\Gamma_Y^\sharp} \mathrm{e}^{-\langle f,\gamma\rangle}\,(\varphi^*\pi_\lambda)(\mathrm{d}\gamma) = \int_{\Gamma_X^\sharp} \mathrm{e}^{-\langle f,\varphi(\gamma)\rangle}\,\pi_\lambda(\mathrm{d}\gamma)\\
&= \exp\left\{-\int_X \left(1-\mathrm{e}^{-f(\varphi(x))}\right)\lambda(\mathrm{d}x)\right\}\\
&= \exp\left\{-\int_Y \left(1-\mathrm{e}^{-f(y)}\right)(\varphi^*\lambda)(\mathrm{d}y)\right\} = L_{\pi_{\varphi^*\lambda}}[f],
\end{aligned}$$

and the proof is complete. $\square$

We conclude this section with necessary and sufficient conditions in order that π_λ-almost all (a.a.) configurations $\gamma \in \Gamma_X^\sharp$ be proper (see Definition 2.2). Although being apparently well-known folklore, these criteria are not always proved or even stated explicitly in the literature, most often being mixed up with various sufficient conditions, e.g., using the property of orderliness etc. (see, e.g., [14,16,22]). We do not include the proof here, as the result follows from a more general statement for the Poisson cluster measure (see Theorem 2.7 below).

Proposition 2.4. (a) *If $B \in \mathcal{B}(X)$ then $\gamma(B) < \infty$ (π_λ-a.s.) if and only if $\lambda(B) < \infty$. In particular, in order that π_λ-a.a. configurations $\gamma \in \Gamma_X^\sharp$ be locally finite, it is necessary and sufficient that $\lambda(K) < \infty$ for any compact set $K \in \mathcal{B}(X)$.*

(b) *In order that π_λ-a.a. configurations $\gamma \in \Gamma_X^\sharp$ be simple, it is necessary and sufficient that the measure λ be non-atomic, that is, $\lambda\{x\} = 0$ for each $x \in X$.*

Let us first recall the notion of a general cluster point process (CPP). The intuitive idea is to construct its realizations in two steps: (i) take a background random configuration of (invisible) "centres" obtained as a realization of some point process γ_{c} governed by a probability measure μ_{c} on $\Gamma_X^\sharp$, and (ii) relative to each centre $x \in \gamma_{\mathrm{c}}$, generate a set of observable secondary points (referred to as a *cluster centred at* x) according to a point process γ'_x with probability measure μ_x on $\Gamma_X^\sharp$ ($x \in X$).

The resulting (countable) assembly of random points, called the *cluster point process*, can be symbolically expressed as

$$\gamma = \bigsqcup_{x\in\gamma_{\mathrm{c}}} \gamma'_x \in \Gamma_X^\sharp,$$

where the disjoint union signifies that multiplicities of points should be taken into account. More precisely, assuming that the family of secondary processes $\gamma'_x(\cdot)$ is measurable as a function of $x \in X$, the integer-valued measure corresponding to a CPP realization γ is given by

$$\gamma(B) = \int_X \gamma'_x(B)\,\gamma_{\mathrm{c}}(\mathrm{d}x) = \sum_{x\in\gamma_{\mathrm{c}}} \gamma'_x(B) = \sum_{x\in\gamma_{\mathrm{c}}}\sum_{y\in\gamma'_x} \delta_y(B), \qquad B \in \mathcal{B}(X). \tag{2.13}$$

A tractable model of such a kind is obtained when (i) X is a linear space so that translations $X \ni y \mapsto y + x \in X$ are defined, and (ii) random clusters are independent and identically distributed (i.i.d.), being governed by the same probability law translated to the cluster centres,

$$\mu_x(A) = \mu_0(A - x), \qquad A \in \mathcal{B}(\Gamma_X^\sharp). \tag{2.14}$$

From now on, we make both of these assumptions.

Remark 2.8. Unlike the standard theory of CPPs whose sample configurations are *presumed* to be a.s. locally finite (see, e.g., [16, Definition 6.3.I]), the description of the CPP given above only implies that its configurations γ are countable aggregates in X, but possibly with multiple and/or accumulation points, even if the background point process γ_{c} is proper. Therefore, the distribution μ of the CPP (2.13) is a probability measure defined on the space $\Gamma_X^\sharp$ of *generalized* configurations. It is a matter of interest to obtain conditions in order that μ be actually supported on the proper configuration space Γ_X, and we shall address this issue in Section 2.4 below in the case of Poisson CPPs.

Let $\nu_x := \gamma'_x(X)$ be the total (random) number of points in a cluster γ'_x centred at point $x \in X$ (referred to as the *cluster size*). According to our assumptions, the random variables ν_x are i.i.d. for different x, with common distribution

$$p_n := \mu_0\{\nu_0 = n\} \qquad (n \in \overline{\mathbb{Z}}_+) \tag{2.15}$$

(so in principle the event $\{\nu_0 = \infty\}$ may have a positive probability, $p_\infty \geq 0$).

Remark 2.9. One might argue that allowing for vacuous clusters (i.e., with $\nu_x = 0$) is superfluous since these are not visible in a sample configuration, and in particular the probability p_0 cannot be estimated statistically [16, Corollary 6.3.VI]. In fact, the possibility of vacuous cluster may be ruled out without loss of generality, at the expense of rescaling the background intensity measure, $\lambda \mapsto (1-p_0)\,\lambda$. However, we keep this possibility in our model in order to provide a suitable framework for evolutionary cluster point processes with annihilation and creation of particles, which we intend to study elsewhere.

The following fact is well known in the case of CPPs without accumulation points (see, e.g., [16, § 6.3]).

Proposition 2.5. *The Laplace functional $L_\mu[\cdot]$ of the probability measure μ on $\varGamma_X^\sharp$ corresponding to the CPP* (2.13) *is given, for all functions $f \in \mathrm{M}_+(X)$, by*

$$L_\mu[f] = L_{\mu_\mathrm{c}}\Bigl(-\ln L_{\mu_x}[f]\Bigr) = L_{\mu_\mathrm{c}}\Bigl(-\ln L_{\mu_0}[f(\,\cdot+x)]\Bigr), \tag{2.16}$$

where L_{μ_c} acts in variable x.

Proof. The representation (2.13) of cluster configurations γ implies that

$$\langle f, \gamma\rangle = \sum_{z\in\gamma} f(z) = \sum_{x\in\gamma_\mathrm{c}} \sum_{y\in\gamma'_x} f(y).$$

Conditioning on the background configuration γ_c and using the independence of the clusters γ'_x for different x, we obtain

$$\begin{aligned}\int_{\varGamma_X^\sharp} \mathrm{e}^{-\langle f,\gamma\rangle}\,\mu(\mathrm{d}\gamma) &= \int_{\varGamma_X^\sharp} \prod_{x\in\gamma_\mathrm{c}} \left(\int_{\varGamma_X^\sharp} \mathrm{e}^{-\sum_{y\in\gamma'_x} f(y)}\,\mu_x(\mathrm{d}\gamma'_x)\right) \mu_\mathrm{c}(\mathrm{d}\gamma_\mathrm{c})\\ &= \int_{\varGamma_X^\sharp} \exp\biggl\{\sum_{x\in\gamma_\mathrm{c}} \ln\left(L_{\mu_x}[f]\right)\biggr\}\,\mu_\mathrm{c}(\mathrm{d}\gamma_\mathrm{c}) = L_{\mu_\mathrm{c}}\Bigl(-\ln L_{\mu_x}[f]\Bigr),\end{aligned}$$

which proves the first formula in (2.16). The second one easily follows by shifting the measure μ_x to the origin using (2.14). □

In this paper, we are mostly concerned with the *Poisson CPPs*, which are specified by assuming that μ_c is a Poisson measure on configurations, with some intensity measure λ. The corresponding probability measure on the configuration space $\varGamma_X^\sharp$ will be denoted by μ_cl and called the *Poisson cluster measure*.

The combination of (2.9) and (2.16) yields a formula for the Laplace functional of the measure μ_cl.

Proposition 2.6. *The Laplace functional $L_{\mu_\mathrm{cl}}[f]$ of the Poisson cluster measure μ_cl on $\varGamma_X^\sharp$ is given, for all $f \in \mathrm{M}_+(X)$, by*

$$L_{\mu_\mathrm{cl}}[f] = \exp\left\{-\int_X \left(\int_{\varGamma_X^\sharp} \left(1 - \mathrm{e}^{-\sum_{y\in\gamma'_0} f(y+x)}\right) \mu_0(\mathrm{d}\gamma'_0)\right) \lambda(\mathrm{d}x)\right\}. \tag{2.17}$$

According to the convention made in Section 2.1 (see after equation (2.3)), if $\gamma_0' = \emptyset$ then the function under the internal integral in (2.17) vanishes, so the integral over $\Gamma_X^\sharp$ is reduced to that over the subset $\{\gamma_0' \in \Gamma_X^\sharp : \gamma_0' \neq \emptyset\}$.

2.4 Criteria of local finiteness and simplicity

In this section, we give criteria for the Poisson CPP to be locally finite and simple. As mentioned in the Introduction, these results appear to be new (e.g., a general criterion of local finiteness in [16, Lemma 6.3.II and Proposition 6.3.III] is merely a more formal rewording of the finiteness condition).

For a given set $B \in \mathcal{B}(X)$ and each in-cluster configuration γ_0' centred at the origin, consider the set (referred to as *droplet cluster*)

$$D_B(\gamma_0') := \bigcup_{y \in \gamma_0'} (B - y), \tag{2.18}$$

which is a set-theoretic union of "droplets" of shape B shifted to the centrally reflected points of γ_0'.

Theorem 2.7. *Let $\mu_{\rm cl}$ be a Poisson cluster measure on the generalized configuration space $\Gamma_X^\sharp$.*

(a) *In order that $\mu_{\rm cl}$-a.a. configurations $\gamma \in \Gamma_X^\sharp$ be locally finite, it is necessary and sufficient that the following two conditions hold:*

(a-i) *in-cluster configurations γ_0' are a.s. locally finite, that is, for any compact set $K \in \mathcal{B}(X)$,*

$$\gamma_0'(K) < \infty \qquad (\mu_0\text{-}a.s.) \tag{2.19}$$

(a-ii) *for any compact set $K \in \mathcal{B}(X)$, the mean λ-measure of the droplet cluster $D_K(\gamma_0')$ is finite,*

$$\int_{\Gamma_X^\sharp} \lambda\Big(D_K(\gamma_0')\Big)\, \mu_0(\mathrm{d}\gamma_0') < \infty. \tag{2.20}$$

(b) *In order that $\mu_{\rm cl}$-a.a. configurations $\gamma \in \Gamma_X^\sharp$ be simple, it is necessary and sufficient that the following two conditions hold:*

(b-i) *in-cluster configurations γ_0' are a.s. simple,*

$$\sup_{x \in X} \gamma_0'\{x\} \le 1 \qquad (\mu_0\text{-}a.s.) \tag{2.21}$$

(b-ii) *for any $x \in X$, the "point" droplet cluster $D_{\{x\}}(\gamma_0')$ has a.s. zero λ-measure,*

$$\lambda\Big(D_{\{x\}}(\gamma_0')\Big) = 0 \qquad (\mu_0\text{-}a.s.) \tag{2.22}$$

The proof of Theorem 2.7 is deferred to the Appendix (Section 6.1).

Let us discuss the conditions of properness. First of all, the interesting question is whether the local finiteness of the Poisson CPP is compatible with the possibility that the number of points in a cluster, $\nu_0 = \gamma_0'(X)$, is infinite (see (2.15)). The next proposition describes a simple situation where this is not the case.

Proposition 2.8. *Let both conditions* (a-i) *and* (a-ii) *be satisfied, and suppose that for any compact set $K \in \mathcal{B}(X)$, the λ-measure of its translations is uniformly bounded from below,*

$$c_K := \inf_{x\in X} \lambda(K + x) > 0. \tag{2.23}$$

Then $\nu_0 < \infty$ (μ_0-a.s.).

Proof. Suppose that γ_0' is an infinite configuration. Due to (a-i), γ_0' must be locally finite (μ_0-a.s.), which implies that there is an infinite subset of points $y_j \in \gamma_0'$ such that the sets $K - y_j$ are disjoint ($j \in \mathbb{N}$). Hence, using (2.23) we get

$$\lambda\Big(D_K(\gamma_0')\Big) \ge \sum_{j=1}^{\infty} \lambda(K - y_j) = \infty,$$

which, according to condition (a-ii), may occur only with zero probability. □

On the other hand, it is easy to construct examples of locally finite Poisson CPPs with a.s.-infinite clusters.

Example 2.1. Let $X = \mathbb{R}^d$ and choose a measure λ such that, for any compact set $K \subset \mathbb{R}^d$, $\lambda(K - x) \sim C_d\, \lambda(K)\, |x|^{-\alpha}$ as $x \to \infty$, where $\alpha > 0$ (e.g., take $\lambda(\mathrm{d}x) = (1 + |x|)^{-\alpha-d+1}\, \mathrm{d}x$). Suppose now that the in-cluster configurations $\gamma_0' = \{x_n\}$ are such that $n^{2/\alpha} < |x_n| \le (n + 1)^{2/\alpha}$, $n \in \mathbb{N}$ (μ_0-a.s.). Then for any compact set K

$$\lambda\Big(D_K(\gamma_0')\Big) \le \sum_{x_n\in\gamma_0'} \lambda(K - x_n) < \infty,$$

because $\lambda(K - x_n) \sim C_d\, \lambda(K)\, |x_n|^{-\alpha} = O(n^{-2})$ as $n \to \infty$.

It is easy to give conditions sufficient for (a-ii). The first set of conditions below is expressed in terms of the intensity measure λ and the mean number of points in a cluster, while the second condition focuses on the location of in-cluster points.

Proposition 2.9. *Suppose that $\nu_0 < \infty$ (μ_0-a.s.). Then either of the following conditions is sufficient for condition* (a-ii) *in Theorem* 2.7.

(a-ii′) *For any compact set $K \in \mathcal{B}(X)$, the λ-measure of its translations is uniformly bounded from above,*

$$C_K := \sup_{x\in X} \lambda(K + x) < \infty, \tag{2.24}$$

and, moreover, the mean number of in-cluster points is finite,

$$\int_{\Gamma_X^\sharp} \gamma_0'(X)\, \mu_0(\mathrm{d}\gamma_0') = \sum_{n\in\overline{\mathbb{Z}}_+} n p_n < \infty \tag{2.25}$$

(*this necessarily implies that $p_\infty = 0$*).

(a-ii$''$) *In-cluster configuration* γ_0' *as a set in* X *is* μ_0*-a.s. bounded, that is, there exists a compact set* $K_0 \in \mathcal{B}(X)$ *such that* $\gamma_0' \subset K_0$ *(*μ_0*-a.s.).*

Proof. From (2.18) and (2.24) we obtain

$$\lambda\Big(D_K(\gamma_0')\Big) \le \sum_{y\in\gamma_0'} \lambda(K-y) \le C_K\,\gamma_0'(X) = C_K\,\nu_0,$$

and condition (a-ii) follows by (2.25),

$$\int_{\Gamma_X^\sharp} \lambda\Big(D_K(\gamma_0')\Big)\,\mu_0(\mathrm{d}\gamma_0') \le C_K \int_{\Gamma_X^\sharp} \gamma_0'(X)\,\mu_0(\mathrm{d}\gamma_0') < \infty.$$

If condition (a-ii$''$) holds then

$$D_K(\gamma_0') \subset \bigcup_{y\in K_0} (K-y) =: K - K_0,$$

where the set $K - K_0$ is compact. Therefore,

$$\int_{\Gamma_X^\sharp} \lambda\Big(D_K(\gamma_0')\Big)\,\mu_0(\mathrm{d}\gamma_0') \le \lambda(K-K_0) \int_{\Gamma_X^\sharp} \mu_0(\mathrm{d}\gamma_0') = \lambda(K-K_0) < \infty,$$

and condition (a-ii) follows. □

The impact of conditions (a-ii$'$) and (a-ii$''$) is clear: (a-ii$'$) imposes a bound on the *number* of points which can be contributed from remote clusters, while (a-ii$''$) restricts the *range* of such contribution.

Similarly, one can work out simple sufficient conditions for (b-ii). The first condition below is set in terms of the measure λ, whereas the second one exploits the in-cluster distribution μ_0.

Proposition 2.10. *Suppose that* $\nu_0 < \infty$ *(*μ_0*-a.s.). Then either of the following conditions is sufficient for condition* (b-ii) *of Theorem* 2.7.

(b-ii$'$) *The measure* λ *is non-atomic, that is,* $\lambda\{x\} = 0$ *for each* $x \in X$.

(b-ii$''$) *In-cluster configurations* γ_0' *have no fixed points, that is,* $\mu_0\{\gamma_0' \in \Gamma_X^\sharp : x \in \gamma_0'\} = 0$ *for each* $x \in X$.

Proof. Condition (b-ii$'$) readily implies (b-ii):

$$0 \le \lambda\Big(D_{\{x\}}(\gamma_0')\Big) \le \sum_{y\in\gamma_0'} \lambda\{x-y\} = 0.$$

Further, if condition (b-ii$''$) holds then

$$\begin{aligned}\int_{\Gamma_X^\sharp} \lambda\Big(D_{\{x\}}(\gamma_0')\Big)\,\mu_0(\mathrm{d}\gamma_0') &= \int_X \left(\int_{\Gamma_X^\sharp} \mathbf{1}_{\cup_{y\in\gamma_0'}\{x-y\}}(z)\,\mu_0(\mathrm{d}\gamma_0')\right)\lambda(\mathrm{d}z)\\ &= \int_X \left(\int_{\Gamma_X^\sharp} \mathbf{1}_{\gamma_0'}(z-x)\,\mu_0(\mathrm{d}\gamma_0')\right)\lambda(\mathrm{d}z)\\ &= \int_X \mu_0\{\gamma_0'\in\Gamma_X^\sharp : z-x\in\gamma_0'\}\,\lambda(\mathrm{d}z) = 0, \qquad (2.26)\end{aligned}$$

and condition (b-ii) follows. □

3 Poisson cluster processes via Poisson measures

In this section, we construct an auxiliary Poisson measure $\pi_{\lambda^\star}$ on the "vector" configuration space $\mathfrak{X}$ and prove that the Poisson cluster measure μ_{cl} coincides with the projection of $\pi_{\lambda^\star}$ onto the configuration space $\Gamma_X^\sharp$ (Theorem 3.6). This furnishes a useful description of Poisson cluster measures that will enable us to apply to their study the well-developed calculus on Poisson configuration spaces.

3.1 An auxiliary intensity measure $\lambda^\star$

Recall that the space $\mathfrak{X} = \bigsqcup_{n\in\overline{\mathbb{Z}}_+} X^n$ of finite or infinite vectors $\bar{x} = (x_1, x_2, \dots)$ was introduced in Section 2.1 The probability distribution μ_0 of a generic cluster γ_0' centred at the origin (see Section 2.3) determines a probability measure η in $\mathfrak{X}$ which is symmetric with respect to permutations of coordinates. Conversely, μ_0 is a push-forward of the measure η under the projection mapping $\mathfrak{p} : \mathfrak{X} \to \Gamma_X^\sharp$ defined by (2.2), that is,

$$\mu_0 = \mathfrak{p}^*\eta \equiv \eta\circ\mathfrak{p}^{-1}. \qquad (3.1)$$

Conditional measure induced by η on the space X^n via the condition $\gamma_0'(X) = n$ will be denoted η_n ($n\in\overline{\mathbb{Z}}_+$); in particular, $\eta_0 = \delta_{\{\emptyset\}}$. Hence (recall (2.15)),

$$\eta(B) = \sum_{n\in\overline{\mathbb{Z}}_+} p_n\eta_n(B\cap X^n), \qquad B\in\mathcal{B}(\mathfrak{X}). \qquad (3.2)$$

Note that if $p_n = \eta\{\gamma_0'(X) = n\} = 0$ then η_n is not well defined; however, this is immaterial since the corresponding term vanishes from the sum (3.2) (cf. also the decomposition (3.5) below).

The following definition is fundamental for our construction.

Definition 3.1. We introduce the measure $\lambda^\star$ on $\mathfrak{X}$ as a special "convolution" of the measures η and λ:

$$\lambda^\star(B) := \int_X \eta(B-x)\,\lambda(\mathrm{d}x), \qquad B\in\mathcal{B}(\mathfrak{X}); \qquad (3.3)$$

equivalently, if $\mathrm{M}_+(\mathfrak{X})$ is the set of all non-negative measurable functions on $\mathfrak{X}$ then, for any $f \in \mathrm{M}_+(\mathfrak{X})$,

$$\int_{\mathfrak{X}} f(\bar{y})\,\lambda^\star(\mathrm{d}\bar{y}) = \int_X \left(\int_{\mathfrak{X}} f(\bar{y}+x)\,\eta(\mathrm{d}\bar{y})\right)\lambda(\mathrm{d}x). \tag{3.4}$$

Here and below, we use the shift notation

$$\bar{y} + x := (y_1 + x, y_2 + x, \dots), \qquad \bar{y} = (y_1, y_2, \dots) \in \mathfrak{X}, \quad x \in X.$$

Using the decomposition (3.2), the measure $\lambda^\star$ on $\mathfrak{X}$ can be represented as a weighted sum of contributions from the constituent spaces X^n:

$$\lambda^\star(B) = \sum_{n\in\overline{\mathbb{Z}}_+} p_n \lambda^\star_n(B\cap X^n), \qquad B \in \mathcal{B}(\mathfrak{X}), \tag{3.5}$$

where, for each $n \in \overline{\mathbb{Z}}_+$,

$$\lambda^\star_n(B_n) := \int_X \eta_n(B_n - x)\,\lambda(\mathrm{d}x), \qquad B_n \in \mathcal{B}(X^n). \tag{3.6}$$

Remark 3.1 (Case $n = 0$)**.** Recall that $X^0 = \{\emptyset\}$ and $\mathcal{B}(X^0) = \{\emptyset, X^0\} = \{\emptyset, \{\emptyset\}\}$. Since $\emptyset - x = \emptyset$, $\{\emptyset\} - x = \{\emptyset\}$ ($x \in X$) and $\eta_0 = \delta_{\{\emptyset\}}$, formula (3.6) for $n = 0$ must be interpreted as follows:

$$\begin{aligned} \lambda^\star_0(\emptyset) &= \int_X \eta_0(\emptyset)\,\lambda(\mathrm{d}x) = 0,\\ \lambda^\star_0(\{\emptyset\}) &= \int_X \eta_0(\{\emptyset\})\,\lambda(\mathrm{d}x) = \int_X \lambda(\mathrm{d}x) = \lambda(X) = \infty. \end{aligned} \tag{3.7}$$

If $p_\infty = 0$ (i.e., clusters are a.s. finite) and $X = \mathbb{R}^d$, then in order that the measure η be absolutely continuous (a.c.) with respect to the "Lebesgue measure" $\mathrm{d}\bar{y} = \delta_{\{\emptyset\}}(\mathrm{d}\bar{y}) \oplus \bigoplus_{n=1}^\infty \mathrm{d}y_1 \otimes \cdots \otimes \mathrm{d}y_n$ on $\mathfrak{X} = \bigsqcup_{n=0}^\infty X^n$, with some density h,

$$\eta(\mathrm{d}\bar{y}) = h(\bar{y})\,\mathrm{d}\bar{y}, \qquad \bar{y} \in \mathfrak{X}, \tag{3.8}$$

it is necessary and sufficient that each measure η_n is a.c. with respect to Lebesgue measure on X^n, that is, $\eta_n(\mathrm{d}\bar{y}) = h_n(\bar{y})\,\mathrm{d}\bar{y}$, $\bar{y} \in X^n$ ($n \in \mathbb{Z}_+$); in this case, the density h is decomposed as

$$h(\bar{y}) = \sum_{n=0}^\infty p_n h_n(\bar{y})\,\mathbf{1}_{X^n}(\bar{y}), \qquad \bar{y} \in \mathfrak{X}. \tag{3.9}$$

Moreover, it follows that the measures $\lambda^\star$ and $\lambda^\star_n$ ($n \in \mathbb{Z}_+$) are also a.c., with the corresponding densities

$$\begin{aligned} s(\bar{y}) &= \frac{\lambda^\star(\mathrm{d}\bar{y})}{\mathrm{d}\bar{y}} = \int_X h(\bar{y}-x)\,\lambda(\mathrm{d}x), \qquad &\bar{y} \in \mathfrak{X},\\ s_n(\bar{y}) &= \frac{\lambda^\star_n(\mathrm{d}\bar{y})}{\mathrm{d}\bar{y}} = \int_X h_n(\bar{y}-x)\,\lambda(\mathrm{d}x), \qquad &\bar{y} \in X^n, \end{aligned} \tag{3.10}$$

related by the equation (cf. (3.5), (3.9))

$$s(\bar{y}) = \sum_{n=0}^{\infty} p_n s_n(\bar{y})\, \mathbf{1}_{X^n}(\bar{y}), \qquad \bar{y} \in \mathfrak{X}. \tag{3.11}$$

Remark 3.2. In the case $n = 1$, the definition (3.6) is reduced to

$$\lambda_1^\star(B_1) = \int_X \eta_1(B_1 - x)\, \lambda(\mathrm{d}x) = \int_X \lambda(B_1 - x)\, \eta_1(\mathrm{d}x), \qquad B_1 \in \mathcal{B}(X). \tag{3.12}$$

In particular, if λ is translation invariant (i.e., $\lambda(B_1 - x) = \lambda(B_1)$ for each $B_1 \in \mathcal{B}(X)$ and any $x \in X$), then $\lambda_1^\star$ coincides with λ.

Remark 3.3. There is a possibility that the measure $\lambda_n^\star$ defined by (3.6) is not σ-finite (even if λ is), and moreover, $\lambda_n^\star$ may appear to be locally infinite, in that $\lambda_n^\star(B) = \infty$ for any compact set $B \subset \mathbb{R}^n$ with non-empty interior, as in the following example.

Example 3.1. Let $X = \mathbb{R}$, and for $n \geq 1$ set

$$\lambda(\mathrm{d}x) := \mathrm{e}^{|x|}\, \mathrm{d}x, \qquad \eta_1(\mathrm{d}x) := \frac{|x|\, \mathrm{d}x}{(x^2+1)^2} \qquad (x \in \mathbb{R}),$$

and $\eta_n(\mathrm{d}\bar{x}) := \eta_1(\mathrm{d}x_1) \otimes \cdots \otimes \eta_1(\mathrm{d}x_n)$, $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Note that for $a < b$ and any $x \notin [a, b]$,

$$\eta_1[a - x, b - x] = \frac{(b-a)\,|a + b - 2x|}{2\,((a-x)^2+1)\,((b-x)^2+1)} \sim \frac{b-a}{|x|^3} \qquad (x \to \infty),$$

so, for any rectangle $B = \mathsf{X}_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ ($a_i < b_i$), by (3.12) we obtain

$$\lambda_1^\star(B) = \int_{-\infty}^{\infty} \prod_{i=1}^n \eta_1[a_i - x, b_i - x]\, \mathrm{e}^{|x|}\, \mathrm{d}x = \infty.$$

The next example illustrates a non-pathological situation.

Example 3.2. Let $X = \mathbb{R}$, and for $n \geq 1$ set

$$h_n(\bar{y}) = \frac{1}{(2\pi)^{n/2}}\, \mathrm{e}^{-\|\bar{y}\|^2/2}, \qquad \bar{y} = (y_1, \dots, y_n) \in \mathbb{R}^n,$$

where $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$. Thus, η_n is a standard Gaussian measure on $\mathbb{R}^n$. Assume that λ is the Lebesgue measure on $\mathbb{R}$, $\lambda(\mathrm{d}x) = \mathrm{d}x$. For $n = 1$, from equation (3.10) we obtain

$$s_1(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(y-x)^2/2}\, \mathrm{d}x = 1,$$

hence $\lambda_1^\star = \lambda$, in accord with Remark 3.2. If $n = 2$ then from (3.10) we get

$$s_2(y_1, y_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-((y_1-x)^2 + (y_2-x)^2)/2}\, \mathrm{d}x = \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{-(y_1-y_2)^2/4}.$$

Via the orthogonal transformation $z_1 = (y_1 + y_2)/\sqrt{2}$, $z_2 = (y_1 - y_2)/\sqrt{2}$, the measure $\lambda_2^\star$ is reduced to

$$\lambda_2^\star(\mathrm{d}z_1, \mathrm{d}z_2) = \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{-z_2^2/2}\, \mathrm{d}z_1\, \mathrm{d}z_2,$$

which is a product of the standard Gaussian measure (along the coordinate axis z_1) and the scaled Lebesgue measure $\mathrm{d}z_2/\sqrt{2}$. Note that $\lambda_2^\star(\mathbb{R}^2) = \infty$, but any vertical or horizontal strip of finite width (in coordinates $\bar{y}$) has finite $\lambda_2^\star$-measure.

In general ($n \geq 2$), integration in (3.10) yields

$$s_n(\bar{y}) = \frac{1}{(\sqrt{2\pi})^{n-1}\sqrt{n}}\, \exp\left\{-\frac{1}{2}\left(\|\bar{y}\|^2 - n^{-1}|y_1 + \cdots + y_n|^2\right)\right\}, \qquad \bar{y} \in \mathbb{R}^n,$$

It is easy to check that after an orthogonal transformation $\bar{z} = \bar{y}U$ such that $z_1 = n^{-1/2}(y_1 + \cdots + y_n)$, the measure $\lambda_n^\star$ takes the form

$$\lambda_n^\star(\mathrm{d}\bar{z}) = \frac{\mathrm{d}z_1}{\sqrt{n}} \cdot \frac{1}{(\sqrt{2\pi})^{n-1}}\, \mathrm{e}^{-(z_2^2+\cdots+z_n^2)/2}\, \mathrm{d}z_2 \cdots \mathrm{d}z_n\,, \qquad \bar{z} = (z_1, \ldots, z_n).$$

That is, $\lambda_n^\star(\mathrm{d}\bar{z})$ is a product of the scaled Lebesgue measure $\mathrm{d}z_1/\sqrt{n}$ and the standard Gaussian measure in coordinates $z_2, \ldots, z_n$. Hence $\lambda_n^\star(\mathbb{R}^n) = \infty$, but for any coordinate strip $C_i = \{\bar{y} \in \mathbb{R}^n : |y_i| \leq c\}$ we have $\lambda_n^\star(C_i) < \infty$.

Example 3.2 can be generalized as follows.

Proposition 3.1. *Suppose that $p_\infty = 0$ and $X = \mathbb{R}^d$. For each $n \geq 1$, consider an orthogonal linear transformation $\bar{z} = \bar{y}U_n$ of the space X^n such that*

$$z_1 = \frac{y_1 + \cdots + y_n}{\sqrt{n}}, \qquad \bar{z} = (z_1, \ldots, z_n), \quad \bar{y} = (y_1, \ldots, y_n). \tag{3.13}$$

Set $\bar{z}' := (z_2, \ldots, z_n)$ and consider the measures

$$\eta_n'(B') := \int_X \eta_n(\mathrm{d}z_1, B') = \eta_n(X \times B'), \qquad B' \in \mathcal{B}(X^{n-1}), \tag{3.14}$$

$$\tilde{\lambda}_n(B_1|\,\bar{z}') := \int_X \lambda\left(\frac{B_1 - z_1}{\sqrt{n}}\right) \eta_n(\mathrm{d}z_1|\,\bar{z}'), \qquad B_1 \in \mathcal{B}(X), \tag{3.15}$$

where $\eta_n(\mathrm{d}z_1|\,\bar{z}')$ is the measure on X obtained from η_n via conditioning on $\bar{z}'$. Then the measure $\lambda^\star$ can be decomposed as

$$\lambda^\star(\mathrm{d}\bar{z}) = p_0\lambda_0^\star(\mathrm{d}\bar{z}) + \sum_{n=1}^{\infty} p_n \tilde{\lambda}_n(\mathrm{d}z_1|\,\bar{z}')\, \eta_n'(\mathrm{d}\bar{z}'), \tag{3.16}$$

where $\lambda_0^\star$ is defined in (3.7). *In particular, if the measure λ on $X = \mathbb{R}^d$ is translation invariant then*

$$\lambda^\star(\mathrm{d}\bar{z}) = p_0\lambda_0^\star(\mathrm{d}\bar{z}) + \sum_{n=1}^{\infty} p_n \frac{\lambda(\mathrm{d}z_1)}{n^{d/2}}\, \eta_n'(\mathrm{d}\bar{z}'). \tag{3.17}$$

Proof. For a fixed $n \ge 1$, let $\bar z = \bar y U_n$ and consider a Borel set in X^n of the form $B_n = \{\bar y \in X^n : z_1 \in B_1,\ \bar z' \in B_n'\}$. By equation (3.13) and orthogonality of U_n, we have $B_n - x = \{\bar z \in X^n : z_1 \in B_1 - x\sqrt{n},\ \bar z' \in B_n'\}$. Therefore, from (3.6) we obtain

$$
\begin{aligned}
\lambda_n^\star(B_n) &= \int_X \left(\int_{X^n} \mathbf{1}_{(B_1 - x\sqrt{n}) \times B_n'}(\bar z)\, \eta_n(\mathrm{d}\bar z) \right) \lambda(\mathrm{d}x) \\
&= \int_{X^n} \left(\int_X \mathbf{1}_{B_1 - x\sqrt{n}}(z_1)\, \lambda(\mathrm{d}x) \right) \mathbf{1}_{B_n'}(\bar z')\, \eta_n(\mathrm{d}\bar z) \\
&= \int_{X \times X^{n-1}} \left(\int_X \mathbf{1}_{(B_1 - z_1)/\sqrt{n}}(x)\, \lambda(\mathrm{d}x) \right) \mathbf{1}_{B_n'}(\bar z')\, \eta_n(\mathrm{d}z_1 |\, \bar z')\, \eta_n'(\mathrm{d}\bar z') \\
&= \int_{B_n'} \left(\int_X \lambda\left((B_1 - z_1)/\sqrt{n}\, \right) \eta_n(\mathrm{d}z_1 |\, \bar z') \right) \eta_n'(\mathrm{d}\bar z') \\
&= \int_{B_n'} \tilde\lambda_n(B_1 |\, \bar z')\, \eta_n'(\mathrm{d}\bar z'),
\end{aligned}
$$

and by inserting this into equation (3.5) we get (3.16). Finally, the translation invariance of λ implies that $\lambda((B_1 - z_1)/\sqrt{n}\,) = n^{-d/2}\lambda(B_1)$. Formula (3.15) then gives $\tilde\lambda_n(B_1 |\, \bar z') = n^{-d/2}\lambda(B_1)$, and (3.17) readily follows from (3.16). $\square$

Using decomposition (3.16), it is easy to obtain the following criterion of absolute continuity of the measure $\lambda^\star$.

Corollary 3.2. *Suppose that $p_\infty = 0$ and $X = \mathbb{R}^d$. Then the measure $\lambda^\star(\mathrm{d}\bar x)$ on $\mathfrak{X}$ is a.c. with respect to the Lebesgue measure $\mathrm{d}\bar x = \delta_{\{\emptyset\}}(\mathrm{d}\bar x) \oplus \bigoplus_{n=1}^\infty \mathrm{d}x_1 \otimes \cdots \otimes \mathrm{d}x_n$ if and only if the following two conditions hold:*

- (i) *for each $n \ge 1$, the measure $\eta_n'(\mathrm{d}\bar z')$ is a.c. with respect to the Lebesgue measure $\mathrm{d}\bar z'$ on X^{n-1};*
- (ii) *for a.a. $\bar z'$, the measure $\tilde\lambda_n(\mathrm{d}z_1 |\, \bar z')$ is a.c. with respect to the Lebesgue measure $\mathrm{d}z_1$ on X.*

In particular, if λ is translation invariant then condition (ii) *is automatically fulfilled and hence condition* (i) *alone is necessary and sufficient for the absolute continuity of $\lambda^\star$.*

Remark 3.4. The absolute continuity of η is sufficient (cf. (3.8), (3.10)), but not necessary, for condition (i). This is illustrated by the following example:

$$
\eta(\mathrm{d}y_1, \mathrm{d}y_2) = \frac{1}{2}\, \delta_{\{1\}}(\mathrm{d}y_1) f(y_2)\, \mathrm{d}y_2 + \frac{1}{2}\, \delta_{\{1\}}(\mathrm{d}y_2) f(y_1)\, \mathrm{d}y_1, \qquad (y_1, y_2) \in \mathbb{R}^2,
$$

where $f(y)$ ($y \in \mathbb{R}$) is some probability density function. Then the projection measure η' on $\mathbb{R}$ (see (3.14)) is given by

$$
\eta'(\mathrm{d}z') = \frac{\sqrt{2}}{2} \left(f(1 - \sqrt{2}\, z') + f(1 + \sqrt{2}\, z') \right) \mathrm{d}z', \qquad z' = \frac{y_1 - y_2}{\sqrt{2}},
$$

and so $\eta'(\mathrm{d}z')$ is absolutely continuous.

The next result shows that the absolute continuity of $\lambda^\star$ implies that the Poisson cluster process a.s. has no multiple points (see Definition 2.2).

Proposition 3.3. *Suppose that* $p_\infty = 0$, $X = \mathbb{R}^d$, *and the measure* $\lambda^\star(\mathrm{d}\bar{x})$ *on* $\mathfrak{X}$ *is a.c. with respect to the Lebesgue measure* $\mathrm{d}\bar{x}$. *Then* μ_{cl}*-a.a. configurations* $\gamma \in \Gamma_X^\sharp$ *are simple.*

Proof. By Theorem 2.7, it suffices to check conditions (b-i) and (b-ii). First, note that if condition (b-i) is not satisfied (i.e., if the set of points $\bar{y} \in \mathfrak{X}$ with two or more coinciding coordinates has positive η-measure), than the projected measure $\eta'(\mathrm{d}\bar{z}')$ charges a hyperplane (of codimension 1) in the space $\mathfrak{X}'$ spanned over the coordinates $\bar{z}'$. But this contradicts the absolute continuity of $\lambda^\star$, since such hyperplanes have zero Lebesgue measure.

Furthermore, similarly to (2.26) and using the definition (3.3), for each $x \in X$ we obtain

$$\begin{aligned}\int_{\mathfrak{X}} \lambda\Big(\textstyle\bigcup_{y_i \in \bar{y}} \{x - y_i\}\Big)\, \eta(\mathrm{d}\bar{y}) &= \int_X \eta\{\bar{y} \in \mathfrak{X} : z - x \in \bar{y}\}\, \lambda(\mathrm{d}z)\\ &= \lambda^\star\{\bar{y} \in \mathfrak{X} : -x \in \mathfrak{p}(\bar{y})\} = 0,\end{aligned}$$

by the absolute continuity of $\lambda^\star$. Hence, $\lambda\Big(\bigcup_{y_i \in \bar{y}}\{x - y_i\}\Big) = 0$ (η-a.s.) and condition (b-ii) follows. $\square$

3.2 An auxiliary Poisson measure $\pi_{\lambda^\star}$

Recall that the "unpacking" map $\mathfrak{p} : \mathfrak{X} \to \Gamma_X^\sharp$ is defined in (2.2). For any Borel subset $B \in \mathcal{B}(X)$, denote

$$\mathfrak{X}_B := \{\bar{x} \in \mathfrak{X} :\ \mathfrak{p}(\bar{x}) \cap B \neq \emptyset\} \in \mathcal{B}(\Gamma_X^\sharp). \tag{3.18}$$

The following result is crucial for our purposes (cf. Example 3.2).

Proposition 3.4. *Let* $B \in \mathcal{B}(X)$ *be a set such that* $\lambda(B) < \infty$. *Then condition* (2.20) *of Theorem* 2.7(a) (*i.e., that the mean* λ*-measure of the droplet cluster* D_B *is finite*) *is necessary and sufficient in order that* $\lambda^\star(\mathfrak{X}_B) < \infty$, *or equivalently,* $\bar{\gamma}(\mathfrak{X}_B) < \infty$ *for* $\pi_{\lambda^\star}$*-a.a.* $\bar{\gamma} \in \Gamma_{\mathfrak{X}}$.

Proof. Using (3.3) we obtain

$$\lambda^\star(\mathfrak{X}_B) = \int_X \eta(\mathfrak{X}_B - x)\, \lambda(\mathrm{d}x) = \int_{\mathfrak{X}} \Big(\int_X \mathbf{1}_{\mathfrak{X}_B}(\bar{y} + x)\, \lambda(\mathrm{d}x)\Big)\, \eta(\mathrm{d}\bar{y}). \tag{3.19}$$

By definition (3.18), $\bar{y} + x \in \mathfrak{X}_B$ if and only if $x \in \bigcup_{y_i \in \bar{y}} (B - y_i) \equiv D_B(\bar{y})$ (see (2.18)). Hence, (3.19) can be rewritten as

$$\begin{aligned}\lambda^\star(\mathfrak{X}_B) &= \int_{\mathfrak{X}} \Big(\int_X \mathbf{1}_{D_B(\bar{y})}(x)\, \lambda(\mathrm{d}x)\Big)\, \eta(\mathrm{d}\bar{y})\\ &= \int_{\mathfrak{X}} \lambda\Big(D_B(\bar{y})\Big)\, \eta(\mathrm{d}\bar{y}) = \int_{\Gamma_X^\sharp} \lambda\Big(D_B(\gamma_0')\Big)\, \mu_0(\mathrm{d}\gamma_0'),\end{aligned}$$

by the change of measure (3.1). Thus, the bound $\lambda^\star(\mathfrak{X}_B) < \infty$ is nothing else but condition (2.20) applied to B. The second part follows by Proposition 2.4(a). $\square$

Let us consider the cluster configuration space $\Gamma^\sharp_{\mathfrak{X}}$ over the space $\mathfrak{X}$ with generic elements $\bar\gamma \in \Gamma^\sharp_{\mathfrak{X}}$. Our next goal is to define a Poisson measure $\pi_{\lambda^\star}$ on $\Gamma^\sharp_{\mathfrak{X}}$ with intensity $\lambda^\star$. However, as Remark 3.3 and Example 3.1 indicate, the measure $\lambda^\star$ may not be σ-finite, in which case a general construction of the Poisson measure as developed in Section 2.2 would not be applicable. It turns out that Proposition 3.4 provides a suitable basis for a good theory.

Proposition 3.5. *Suppose that condition* (2.20) *of Theorem* 2.7(a) *is fulfilled for any set $B \in \mathcal{B}(X)$ such that $\lambda(B) < \infty$. Then the measure $\lambda^\star$ on $\mathfrak{X}$ is σ-finite.*

Proof. Since the measure λ on X is σ-finite, there is a sequence of sets $B_k \in \mathcal{B}(X)$ $(k \in \mathbb{N})$ such that $\lambda(B_k) < \infty$ and $\bigcup_{k=1}^\infty B_k = X$. Hence, by Proposition 3.4, $\lambda^\star(\mathfrak{X}_{B_k}) < \infty$ for each B_k, and from the definition (3.18) it is clear that $\bigcup_{k=1}^\infty \mathfrak{X}_{B_k} = \mathfrak{X}$. $\square$

By virtue of Proposition 3.5 and according to the discussion in Section 2.2, the Poisson measure $\pi_{\lambda^\star}$ on the configuration space $\Gamma^\sharp_{\mathfrak{X}}$ does exist. Moreover, due to Remark 2.6, this is true even without any extra topological assumptions, except that of σ-finiteness of the basic intensity measure λ. The construction of $\pi_{\lambda^\star}$ may be elaborated further by applying Proposition 2.2 to $\mathfrak{X} = \bigsqcup_{n\in\overline{\mathbb{Z}}_+} X^n$ and $\lambda^\star = \bigoplus_{n\in\overline{\mathbb{Z}}_+} p_n \lambda_n^\star$; namely, one first defines the Poisson measures $\pi_{p_n\lambda_n^\star}$ on the constituent configuration spaces $\Gamma^\sharp_{X^n}$ (of course, the measures $\lambda_n^\star$ are σ-finite together with $\lambda^\star$) and then constructs the Poisson measure $\pi_{\lambda^\star}$ on $\Gamma^\sharp_{\mathfrak{X}} = \bigtimes_{n\in\overline{\mathbb{Z}}_+} \Gamma^\sharp_{X^n}$ as a product measure, $\pi_{\lambda^\star} = \bigotimes_{n\in\overline{\mathbb{Z}}_+} \pi_{p_n\lambda_n^\star}$.

Remark 3.5. A degenerate Poisson measure $\pi_{p_0\lambda_0^\star}$ on $\Gamma^\sharp_{X^0}$ is defined as $\pi_{p_0\lambda_0^\star} := \delta_{\{\bar\gamma_\infty\}}$, where $\bar\gamma_\infty = (\{\emptyset\}, \{\emptyset\}, \dots)$, i.e., $\bar\gamma_\infty(X^0) = \infty$. The component $\pi_{p_0\lambda_0^\star}$ is actually irrelevant in the projection construction described in the next section.

3.3 Poisson cluster measure via the Poisson measure $\pi_{\lambda^\star}$

We can lift the projection mapping (2.2) to the configuration space $\Gamma^\sharp_{\mathfrak{X}}$ by setting

$$\Gamma^\sharp_{\mathfrak{X}} \ni \bar\gamma \mapsto \mathfrak{p}(\bar\gamma) := \bigsqcup_{\bar x\in\bar\gamma} \mathfrak{p}(\bar x) \in \Gamma^\sharp_X. \tag{3.20}$$

Disjoint union in (3.20) highlights the fact that $\mathfrak{p}(\bar\gamma)$ may have multiple points, even if $\bar\gamma$ is proper. It is not difficult to see that (3.20) is a measurable mapping. Indeed, using the sets D_B^n introduced in (2.4), for any cylinder set $C_B^n \subset \Gamma^\sharp_X$ $(B \in \mathcal{B}(X)$,

$n \in \mathbb{Z}_+$) we have $\mathfrak{p}^{-1}(C_B^n) = A_B^n \in \mathcal{B}(\varGamma_{\mathfrak{X}}^\sharp)$, where, for instance,

$$\begin{aligned}
A_B^0 &= \{\bar\gamma \in \varGamma_{\mathfrak{X}}^\sharp :\ \bar\gamma(\mathfrak{X} \setminus D_B^0) = 0\},\\
A_B^1 &= \{\bar\gamma \in \varGamma_{\mathfrak{X}}^\sharp :\ \bar\gamma(D_B^1) = 1\},\\
A_B^2 &= \{\bar\gamma \in \varGamma_{\mathfrak{X}}^\sharp :\ \bar\gamma(D_B^2) = 1 \ \text{ or } \ \bar\gamma(D_B^1) = 2\},
\end{aligned}$$

and, more generally, $A_B^n = \bigcup_{(n_k)} \bigcap_{k=1}^\infty \{\bar\gamma \in \varGamma_{\mathfrak{X}}^\sharp :\ \bar\gamma(D_B^k) = n_k\}$, where the union is taken over integer arrays $(n_k) = (n_1, n_2, \dots)$ such that $n_k > 0$ and $\sum_k k n_k = n$.

Finally, we introduce the measure μ on $\varGamma_X^\sharp$ as a push-forward of the Poisson measure $\pi_{\lambda^\star}$ under the mapping $\mathfrak{p}$,

$$\mu(A) := (\mathfrak{p}^* \pi_{\lambda^\star})(A) \equiv \pi_{\lambda^\star}(\mathfrak{p}^{-1}(A)), \qquad A \in \mathcal{B}(\varGamma_X^\sharp). \tag{3.21}$$

The next theorem is the main result of this section.

Theorem 3.6. *The measure $\mu = \mathfrak{p}^*\pi_{\lambda^\star}$ on $\varGamma_X^\sharp$ defined by* (3.21) *coincides with the Poisson cluster measure μ_{cl}.*

Proof. According to Section 2.1, it is sufficient to compute the Laplace functional of the measure μ. For any $f \in \mathrm{M}_+(X)$, by the change of measure (3.21) we have

$$\int_{\varGamma_X^\sharp} \mathrm{e}^{-\langle f, \gamma\rangle}\, \mu(\mathrm{d}\gamma) = \int_{\varGamma_{\mathfrak{X}}} \mathrm{e}^{-\langle f, \mathfrak{p}(\bar\gamma)\rangle}\, \pi_{\lambda^\star}(\mathrm{d}\bar\gamma) = \int_{\varGamma_{\mathfrak{X}}} \mathrm{e}^{-\langle \tilde f, \bar\gamma\rangle}\, \pi_{\lambda^\star}(\mathrm{d}\bar\gamma), \tag{3.22}$$

where $\tilde f(\bar y) := \sum_{y_i \in \bar y} f(y_i) \in \mathrm{M}_+(\mathfrak{X})$. According to (2.9) and (3.4), the right-hand side of (3.22) takes the form

$$\begin{aligned}
\exp\left\{-\int_{\mathfrak{X}} \left(1 - \mathrm{e}^{-\tilde f(\bar y)}\right) \lambda^\star(\mathrm{d}\bar y)\right\} &= \exp\left\{-\int_X \int_{\mathfrak{X}} \left(1 - \mathrm{e}^{-\tilde f(\bar y + x)}\right) \eta(\mathrm{d}\bar y)\, \lambda(\mathrm{d}x)\right\}\\
&= \exp\left\{-\int_X \left(\int_{\mathfrak{X}} \left(1 - \mathrm{e}^{-\sum_{y_i \in \bar y} f(y_i + x)}\right) \eta(\mathrm{d}\bar y)\right) \lambda(\mathrm{d}x)\right\},
\end{aligned}$$

which, after the change of measure (3.1), coincides with the expression (2.17) for the Laplace functional of the Poisson cluster measure μ_{cl}. □

Remark 3.6. As an elegant application of the technique developed here, let us give a transparent proof of Theorem 2.7(a) (cf. the Appendix, Section 6.1). Indeed, in order that a given compact set $K \subset X$ contain finitely many points of configuration $\gamma = \mathfrak{p}(\bar\gamma)$, it is necessary and sufficient that (i) each cluster "point" $\bar x \in \bar\gamma$ is locally finite, which is equivalent to the condition (a-i), and (ii) there are finitely many points $\bar x \in \bar\gamma$ which contribute to the set K under the mapping $\mathfrak{p}$, the latter being equivalent to condition (a-ii) by Proposition 3.4.

3.4 An alternative construction of the measures $\pi_{\lambda^\star}$ and μ_{cl}

The measure $\pi_{\lambda^\star}$ was introduced in the previous section as a Poisson measure on the configuration space $\varGamma_{\mathfrak{X}}$ with a certain intensity measure $\lambda^\star$ prescribed *ad hoc* by equation (3.3). In this section, we show that $\pi_{\lambda^\star}$ can be obtained in a more natural way as a suitable skew projection of a canonical Poisson measure $\widehat{\pi}$ defined on a bigger configuration space $\Gamma^\sharp_{X\times\mathfrak{X}}$, with the product intensity measure $\lambda\otimes\eta$.

More specifically, given a Poisson measure π_λ in $\Gamma^\sharp_X$, let us construct a new measure $\widehat{\mu}$ in $\Gamma^\sharp_{X\times\mathfrak{X}}$ as the probability distribution of random configurations $\widehat{\gamma}\in\Gamma^\sharp_{X\times\mathfrak{X}}$ obtained from Poisson configurations $\gamma\in\Gamma^\sharp_X$ by the rule

$$\gamma\mapsto\widehat{\gamma}:=\{(x,\bar{y}_x): x\in\gamma,\ \bar{y}_x\in\mathfrak{X}\},\tag{3.23}$$

where the random vectors $\{\bar{y}_x\}$ are i.i.d., with common distribution $\eta(\mathrm{d}\bar{y})$. Geometrically, such a construction may be viewed as pointwise i.i.d. translations of the Poisson configuration $\gamma\in X$ into the space $X\times\mathfrak{X}$,

$$X\ni x\leftrightarrow(x,0)\mapsto(x,\bar{y}_x)\in X\times\mathfrak{X}.$$

Remark 3.7. Vector $\bar{y}_x$ in each pair $(x,\bar{y}_x)\in X\times\mathfrak{X}$ can be interpreted as a *mark* attached to the point $x\in X$, so that $\widehat{\gamma}$ becomes a marked configuration, with the mark space $\mathfrak{X}$ (see [16,24]).

Theorem 3.7. *The probability distribution $\widehat{\mu}$ of random configurations $\widehat{\gamma}\in\Gamma^\sharp_{X\times\mathfrak{X}}$ constructed in* (3.23) *is given by the Poisson measure $\pi_{\widehat{\lambda}}$ on the configuration space $\Gamma^\sharp_{X\times\mathfrak{X}}$, with the product intensity measure $\widehat{\lambda}:=\lambda\otimes\eta$.*

Proof. Let us check that, for any non-negative measurable function $f(x,\bar{y})$ on $X\times\mathfrak{X}$, the Laplace functional of the measure $\widehat{\mu}$ is given by formula (2.9). Using independence of the vectors $\bar{y}_x$ corresponding to different x, we obtain

$$\begin{aligned}\int_{\Gamma^\sharp_{X\times\mathfrak{X}}}\mathrm{e}^{-\langle f,\widehat{\gamma}\rangle}\,\widehat{\mu}(\mathrm{d}\widehat{\gamma})&=\int_{\Gamma^\sharp_X}\prod_{x\in\gamma}\Bigl(\int_{\mathfrak{X}}\mathrm{e}^{-f(x,\bar{y})}\,\eta(\mathrm{d}\bar{y})\Bigr)\,\pi_\lambda(\mathrm{d}\gamma)\\&=\exp\Bigl\{-\int_X\Bigl(1-\int_{\mathfrak{X}}\mathrm{e}^{-f(x,\bar{y})}\,\eta(\mathrm{d}\bar{y})\Bigr)\,\lambda(\mathrm{d}x)\Bigr\}\\&=\exp\Bigl\{-\int_X\int_{\mathfrak{X}}\bigl(1-\mathrm{e}^{-f(x,\bar{y})}\bigr)\,\lambda(\mathrm{d}x)\,\eta(\mathrm{d}\bar{y})\Bigr\}\\&=\exp\Bigl\{-\int_{X\times\mathfrak{X}}\bigl(1-\mathrm{e}^{-f(x,\bar{y})}\bigr)\,\widehat{\lambda}(\mathrm{d}x,\mathrm{d}\bar{y})\Bigr\}\\&=\int_{\Gamma^\sharp_{X\times\mathfrak{X}}}\mathrm{e}^{-\langle f,\widehat{\gamma}\rangle}\,\pi_{\widehat{\lambda}}(\mathrm{d}\widehat{\gamma}),\end{aligned}$$

where we have applied formula (2.9) for the Laplace functional of the Poisson measure π_λ with the function $\tilde{f}(x)=-\ln\bigl(\int_{\mathfrak{X}}\mathrm{e}^{-f(x,\bar{y})}\,\eta(\mathrm{d}\bar{y})\bigr)\in\mathrm{M}_+(X)$. $\square$

Remark 3.8. The measure $\widehat{\mu}$, originally defined on configurations $\widehat{\gamma}$ of the form (3.23), naturally extends to a probability measure on the entire space $\Gamma^\sharp_{X\times\mathfrak{X}}$.

Remark 3.9. Theorem 3.7 can be regarded as a generalization of the well-known invariance property of Poisson measures under random i.i.d. translations (see, e.g., [14,16,22]). A novel element here is that starting from a Poisson point process in

X, random translations create a new (Poisson) point process in a bigger space, $X \times \mathfrak{X}$, with the product intensity measure. On the other hand, note that the pointwise coordinate projection $X \times \mathfrak{X} \ni (x, \bar{y}_x) \mapsto x \in X$ recovers the original Poisson measure π_λ, in accord with the Mapping Theorem (see Proposition 2.3). Therefore, Theorem 3.7 provides a converse counterpart to the Mapping Theorem. To the best of our knowledge, these interesting properties of Poisson measures have not been pointed out in the literature so far.

Theorem 3.7 can be easily extended to more general (skew) translations.

Theorem 3.8. *Suppose that random configurations $\widehat{\gamma}_+ \in \Gamma^\sharp_{X\times\mathfrak{X}}$ are obtained from Poisson configurations $\gamma \in \Gamma^\sharp_X$ by pointwise translations $x \mapsto (x, \bar{y}_x + x)$, where $\bar{y}_x \in \mathfrak{X}$ $(x \in X)$ are i.i.d. with common distribution $\eta(\mathrm{d}\bar{y})$. Then the corresponding probability measure $\widehat{\mu}_+$ on $\Gamma^\sharp_{X\times\mathfrak{X}}$ coincides with the Poisson measure of intensity*

$$\widehat{\lambda}_+(\mathrm{d}x, \mathrm{d}\bar{y}) := \lambda(\mathrm{d}x)\,\eta(\mathrm{d}\bar{y} - x). \tag{3.24}$$

Corollary 3.9. *Under the pointwise projection $(x, \bar{y}) \mapsto \bar{y}$ applied to configurations $\widehat{\gamma}_+ \in \Gamma^\sharp_{X\times\mathfrak{X}}$, the Poisson measure $\widehat{\mu}_+$ of Theorem* 3.8 *is pushed forward to the Poisson measure $\pi_{\lambda^\star}$ on $\Gamma^\sharp_{\mathfrak{X}}$ with intensity measure $\lambda^\star$ defined in* (3.3).

Proof. By the Mapping Theorem (see Proposition 2.3), the image of the measure $\widehat{\mu}_+$ under the projection $(x, \bar{y} + x) \mapsto \bar{y} + x$ is a Poisson measure with intensity given by the push-forward of the measure (3.24), that is,

$$\int_X \widehat{\lambda}_+(\mathrm{d}x, B) = \int_X \eta(B - x)\,\lambda(\mathrm{d}x) = \lambda^\star(B), \qquad B \in \mathcal{B}(\mathfrak{X}),$$

according to the definition (3.3). $\square$

Remark 3.10. According to Corollary 3.9, σ-finiteness of the intensity measure $\lambda^\star$ (see Proposition 3.5) is not necessary for the existence of the Poisson measure $\pi_{\lambda^\star}$.

Finally, combining Theorems 3.7, 3.8 and Corollary 3.9 with Theorem 3.6, we arrive at the following result.

Theorem 3.10. *Suppose that all the conditions of Theorems* 3.7 *and* 3.8 *are fulfilled. Then, under the composition mapping*

$$\tilde{\mathfrak{p}} : (x, \bar{y}) \mapsto (x, \bar{y} + x) \mapsto \bar{y} + x \mapsto \mathfrak{p}(\bar{y} + x),$$

the Poisson measure $\pi_{\widehat{\lambda}}$ constructed in Theorem 3.7 *is pushed forward from the space $\Gamma^\sharp_{X\times\mathfrak{X}}$ directly to the space $\Gamma^\sharp_X$ where it coincides with the prescribed Poisson cluster measure μ_{cl},*

$$(\tilde{\mathfrak{p}}^*\pi_{\widehat{\lambda}})(A) \equiv \pi_{\widehat{\lambda}}(\tilde{\mathfrak{p}}^{-1}(A)) = \mu_{\mathrm{cl}}(A), \qquad A \in \mathcal{B}(\Gamma^\sharp_X).$$

Remark 3.11. The construction used in Theorem 3.10 may prove instrumental for more complex (e.g., Gibbs) cluster processes, as it enables one to avoid the intermediate space $\Gamma^\sharp_{\mathfrak{X}}$ where the push-forward measure (analogous to $\pi_{\lambda^\star}$) may have no explicit description.

4 Quasi-invariance and integration by parts

From now on, we restrict ourselves to the case where $X = \mathbb{R}^d$. We shall assume throughout that conditions (a-i) and (a-ii) of Theorem 2.7 are fulfilled, so that μ_{cl}-a.a. configurations $\gamma \in \Gamma_X^\sharp$ are locally finite. Furthermore, all clusters are assumed to be a.s. finite, hence $p_\infty \equiv \mu_0\{\nu_0 = \infty\} = 0$ and the component X^∞ may be dropped from the disjoint union $\mathfrak{X} = \bigsqcup_n X^n$. We shall also require the absolute continuity of the measure $\lambda^\star$ (see the corresponding necessary and sufficient conditions in Corollary 3.2). By Proposition 3.3, this implies that configurations γ are μ_{cl}-a.s. simple (i.e., have no multiple points). In particular, these assumptions ensure that μ_{cl}-a.a. configurations γ belong to the proper configuration space Γ_X.

Under these conditions, in this section we prove the quasi-invariance of the measure μ_{cl} with respect to the action of compactly supported diffeomorphisms of X and establish an integration-by-parts formula. We begin with a brief description of some convenient "manifold-like" concepts and notations first introduced in [5], which provide the suitable framework for analysis on configuration spaces.

4.1 Differentiable functions on configuration spaces

Let T_xX be the tangent space of $X = \mathbb{R}^d$ at point $x \in X$. It can be identified in the natural way with $\mathbb{R}^d$, with the corresponding (canonical) inner product denoted by a "fat" dot $\boldsymbol{\cdot}$. The gradient on X is denoted by ∇. Following [5], we define the "tangent space" of the configuration space Γ_X at $\gamma \in \Gamma_X$ as the Hilbert space $T_\gamma\Gamma_X := L^2(X \to TX;\, \mathrm{d}\gamma)$, or equivalently $T_\gamma\Gamma_X = \bigoplus_{x\in\gamma} T_xX$. The scalar product in $T_\gamma\Gamma_X$ is denoted by $\langle\cdot,\cdot\rangle_\gamma$. A vector field V over Γ_X is a mapping $\Gamma_X \ni \gamma \mapsto V(\gamma) = (V(\gamma)_x)_{x\in\gamma} \in T_\gamma\Gamma_X$. Thus, for vector fields V_1, V_2 over Γ_X we have

$$\langle V_1(\gamma), V_2(\gamma)\rangle_\gamma = \sum_{x\in\gamma} V_1(\gamma)_x \boldsymbol{\cdot} V_2(\gamma)_x\,, \qquad \gamma \in \Gamma_X.$$

For $\gamma \in \Gamma_X$ and $x \in \gamma$, denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighbourhood of x in X such that $\mathcal{O}_{\gamma,x} \cap \gamma = \{x\}$. For any measurable function $F : \Gamma_X \to \mathbb{R}$, define the function $F_x(\gamma,\cdot) : \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by $F_x(\gamma,y) := F((\gamma\setminus\{x\})\cup\{y\})$, and set

$$\nabla_x F(\gamma) := \nabla F_x(\gamma,y)\big|_{y=x}\,, \qquad x \in X,$$

provided $F_x(\gamma,\cdot)$ is differentiable at x.

Denote by $\mathcal{FC}(\Gamma_X)$ the class of functions on Γ_X of the form

$$F(\gamma) = f(\langle\phi_1,\gamma\rangle,\dots,\langle\phi_k,\gamma\rangle), \qquad \gamma \in \Gamma_X, \tag{4.1}$$

where $k \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^k)$ (:= the set of C^∞-functions on $\mathbb{R}^k$ bounded together with all their derivatives), and $\phi_1,\dots,\phi_k \in C_0^\infty(X)$ (:= the set of C^∞-functions on X with compact support). Each $F \in \mathcal{FC}(\Gamma_X)$ is local, that is, there is a compact set $K \subset X$ (which may depend on F) such that $F(\gamma) = F(\gamma_K)$ for all $\gamma \in \Gamma_X$. Thus, for a fixed γ there are only finitely many non-zero derivatives $\nabla_x F(\gamma)$.

For a function $F \in \mathcal{FC}(\Gamma_X)$, its Γ-gradient $\nabla^\Gamma F$ is defined as follows:

$$\nabla^\Gamma F(\gamma) := (\nabla_x F(\gamma))_{x\in\gamma} \in T_\gamma \Gamma_X, \qquad \gamma \in \Gamma_X, \tag{4.2}$$

so the directional derivative of F along a vector field V is given by

$$\nabla^\Gamma_V F(\gamma) := \langle \nabla^\Gamma F(\gamma), V(\gamma)\rangle_\gamma = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot V(\gamma)_x, \qquad \gamma \in \Gamma_X.$$

Note that the sum on the right-hand side contains only finitely many non-zero terms. Further, let $\mathcal{FV}(\Gamma_X)$ be the class of cylinder vector fields V on Γ_X of the form

$$V(\gamma)_x = \sum_{i=1}^k A_i(\gamma)\, v_i(x) \in T_x X, \qquad x \in X, \tag{4.3}$$

where $A_i \in \mathcal{FC}(\Gamma_X)$ and $v_i \in \mathrm{Vect}_0(X)$ (:= the space of compactly supported C^∞-smooth vector fields on X), $i = 1, \dots, k$ ($k \in \mathbb{N}$). Any vector filed $v \in \mathrm{Vect}_0(X)$ generates a constant vector field V on Γ_X defined by $V(\gamma)_x := v(x)$. We shall preserve the notation v for it. Thus,

$$\nabla^\Gamma_v F(\gamma) = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x), \qquad \gamma \in \Gamma_X. \tag{4.4}$$

Recall (see Proposition 2.4(a)) that if $\lambda(\Lambda) < \infty$ then $\gamma(\Lambda) < \infty$ for π_λ-a.a. $\gamma \in \Gamma_X$. This motivates the definition of the class $\mathcal{FC}_\lambda(\Gamma_X)$ of functions on Γ_X of the form (4.1), where $\phi_1, \dots, \phi_k$ are C^∞-functions with $\lambda(\operatorname{supp}\phi_i) < \infty$, $i = 1, \dots, k$. Any function $F \in \mathcal{FC}_\lambda(\Gamma_X)$ is local in the sense that there exists a set $B \in \mathcal{B}(X)$ (depending on F) such that $\lambda(B) < \infty$ and $F(\gamma) = F(\gamma_B)$ for all $\gamma \in \Gamma_X$. As in the case of functions from $\mathcal{FC}(\Gamma_X)$, for a fixed γ there are only finitely many non-zero derivatives $\nabla_x F(\gamma)$.

The approach based on "lifting" the differential structure from the underlying space X to the configuration space Γ_X as described above can also be applied to the spaces $\mathfrak{X} = \bigsqcup_{n=0}^\infty X^n$ and $\Gamma_{\mathfrak{X}}$. First of all, the space $\mathfrak{X}$ is endowed with the natural differential structure inherited from the constituent spaces X^n. Namely, the tangent space of $\mathfrak{X}$ at point $\bar{x} \in \mathfrak{X}$ is defined piecewise as $T_{\bar{x}}\mathfrak{X} := T_{\bar{x}}X^n$ for $\bar{x} \in X^n$ ($n \in \mathbb{Z}_+$), with the scalar product in $T_{\bar{x}}\mathfrak{X}$ induced from the tangent spaces $T_{\bar{x}}X^n$ and again denoted by the dot $\cdot$; furthermore, for a function $f : \mathfrak{X} \to \mathbb{R}$ its gradient ∇f acts on each space X^n as $\nabla f(\bar{x}) = (\nabla_{x_1} f(\bar{x}), \dots, \nabla_{x_n} f(\bar{x})) \in T_{\bar{x}}X^n$, where ∇_{x_i} is the "partial" gradient with respect to the component $x_i \in \bar{x} \in X^n$. A vector field on $\mathfrak{X}$ is a map $\mathfrak{X} \ni \bar{x} \mapsto V(\bar{x}) \in T_{\bar{x}}\mathfrak{X}$; in other words, the restriction of V to X^n is a vector field on X^n ($n \in \mathbb{Z}_+$). The derivative of a function $f : \mathfrak{X} \to \mathbb{R}$ along a vector field V on $\mathfrak{X}$ is then defined by $\nabla_V f(\bar{x}) := \nabla f(\bar{x}) \cdot V(\bar{x})$ ($\bar{x} \in \mathfrak{X}$).

The functional class $C^\infty(\mathfrak{X})$ is defined, as usual, as the set of C^∞-functions $f : \mathfrak{X} \to \mathbb{R}$; similarly, $C_0^\infty(\mathfrak{X})$ is the subclass of $C^\infty(\mathfrak{X})$ consisting of functions with compact support. Since differentiability is a local property, $C^\infty(\mathfrak{X})$ admits a component-wise description: $f \in C^\infty(\mathfrak{X})$ if and only if for each $n \in \mathbb{Z}_+$ the restriction of f to X^n is in $C^\infty(X^n)$. However, this is not true for the class $C_0^k(\mathfrak{X})$ which, according to Remark 2.1, involves a stronger condition that $f(\bar{x}) \equiv \bar{x}$ ($\bar{x} \in X^n$) for all large enough n.

Now, lifting this differentiable structure from the space $\mathfrak{X}$ to the configuration space $\mathit{\Gamma}_{\mathfrak{X}}$ can be done by repeating the same constructions as before with only obvious modifications, so we do not dwell on details. This way, we introduce the tangent space $T_{\bar{\gamma}}\mathit{\Gamma}_{\mathfrak{X}} = \bigoplus_{\bar{x}\in\bar{\gamma}} T_{\bar{x}}\mathfrak{X}$, vector fields V over $\mathit{\Gamma}_{\mathfrak{X}}$, and differentiable functions $\mathit{\Phi} : \mathit{\Gamma}_{\mathfrak{X}} \to \mathbb{R}$. Similarly to (4.1) and (4.3) one can define the spaces $\mathcal{FC}(\mathit{\Gamma}_{\mathfrak{X}})$, $\mathcal{FC}_{\lambda^\star}(\mathit{\Gamma}_{\mathfrak{X}})$ and $\mathcal{FV}(\mathit{\Gamma}_{\mathfrak{X}})$ of C^∞-smooth local functions and vector fields on $\mathfrak{X}$, and we shall use these notations without further explanation.

4.2 Diff_0*-quasi-invariance*

In this section, we discuss the property of quasi-invariance of the measure μ_{cl} with respect to diffeomorphisms of X. Let us start by describing how diffeomorphisms of X act on configuration spaces. For a measurable mapping $\varphi : X \to X$, its *support* $\operatorname{supp}\varphi$ is defined as the smallest closed set containing all $x \in X$ such that $\varphi(x) \neq x$. Let $\mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X with *compact support*. For any $\varphi \in \mathrm{Diff}_0(X)$, we define the "diagonal" diffeomorphism $\bar{\varphi} : \mathfrak{X} \to \mathfrak{X}$ acting on each space X^n ($n \in \mathbb{Z}_+$) as follows:

$$X^n \ni \bar{x} = (x_1, \dots, x_n) \mapsto \bar{\varphi}(\bar{x}) := (\varphi(x_1), \dots, \varphi(x_n)) \in X^n.$$

Remark 4.1. Although $K := \operatorname{supp}\varphi$ is compact in X, note that the support of the diffeomorphism $\bar{\varphi}$ (again defined as the closure of the set $\{\bar{x} \in \mathfrak{X} : \varphi(\bar{x}) \neq \bar{x}\}$) is given by $\operatorname{supp}\bar{\varphi} = \mathfrak{X}_K$ (see (3.18)) and hence is *not* compact in the topology of $\mathfrak{X}$ (see Remark 2.1). However, $\lambda^\star(\mathfrak{X}_K) < \infty$ (by Proposition 3.4), which is sufficient for our purposes.

The mappings φ and $\bar{\varphi}$ can be lifted to measurable "diagonal" transformations (denoted by the same letters) of the configuration spaces $\mathit{\Gamma}_X$ and $\mathit{\Gamma}_{\mathfrak{X}}$, respectively:

$$\begin{aligned} \mathit{\Gamma}_X \ni \gamma &\mapsto \varphi(\gamma) := \{\varphi(x),\ x \in \gamma\} \in \mathit{\Gamma}_X, \\ \mathit{\Gamma}_{\mathfrak{X}} \ni \bar{\gamma} &\mapsto \bar{\varphi}(\bar{\gamma}) := \{\bar{\varphi}(\bar{x}),\ \bar{x} \in \bar{\gamma}\} \in \mathit{\Gamma}_{\mathfrak{X}}. \end{aligned} \tag{4.5}$$

Let $\mathcal{I} : L^2(\mathit{\Gamma}_X, \mu_{\mathrm{cl}}) \to L^2(\mathit{\Gamma}_{\mathfrak{X}}, \pi_{\lambda^\star})$ be the isometry defined by the projection $\mathfrak{p}$,

$$(\mathcal{I}F)(\bar{\gamma}) := F(\mathfrak{p}(\bar{\gamma})), \qquad \bar{\gamma} \in \mathit{\Gamma}_{\mathfrak{X}}, \tag{4.6}$$

and let $\mathcal{I}^* : L^2(\mathit{\Gamma}_{\mathfrak{X}}, \pi_{\lambda^\star}) \to L^2(\mathit{\Gamma}_X, \mu_{\mathrm{cl}})$ be the adjoint operator.

Remark 4.2. The definition implies that $\mathcal{I}^*\mathcal{I}$ is the identity operator in $L^2(\mathit{\Gamma}_X, \mu_{\mathrm{cl}})$. However, the operator $\mathcal{I}\mathcal{I}^*$ acting in the space $L^2(\mathit{\Gamma}_{\mathfrak{X}}, \pi_{\lambda^\star})$ is a non-trivial orthogonal projection, which plays the role of an infinite particle symmetrization operator.

Unfortunately, general explicit form of the operators $\mathcal{I}^*$ and $\mathcal{I}\mathcal{I}^*$ is not known, and may be hard to obtain.

By the next lemma, the action of $\mathrm{Diff}_0(X)$ commutes with the operators $\mathfrak{p}$ and $\mathcal{I}$.

Lemma 4.1. *For any $\varphi \in \mathrm{Diff}_0(X)$, we have $\varphi \circ \mathfrak{p} = \mathfrak{p} \circ \bar{\varphi}$ and furthermore, $\mathcal{I}(F \circ \varphi) = (\mathcal{I}F) \circ \bar{\varphi}$ for any $F \in L^2(\Gamma_X, \mu_{\mathrm{cl}})$.*

Proof. The first statement follows from the definition (3.20) of the mapping $\mathfrak{p}$ and the diagonal form of $\bar{\varphi}$ (see (4.5)). The second statement then readily follows by the definition (4.6) of the operator $\mathcal{I}$. □

Let us now consider the configuration space $\Gamma_{\mathfrak{X}}$ equipped with the Poisson measure $\pi_{\lambda^\star}$ introduced in Section 3.2. As already mentioned, we assume that the intensity measure $\lambda^\star$ is a.c. with respect to the Lebesgue measure on $\mathfrak{X}$ and, moreover,

$$s(\bar{x}) := \frac{\lambda^\star(\mathrm{d}\bar{x})}{\mathrm{d}\bar{x}} > 0 \qquad \text{for a.a. } \bar{x} \in \mathfrak{X}. \tag{4.7}$$

This implies that the measure $\lambda^\star$ is quasi-invariant with respect to the action of diagonal transformations $\bar{\varphi} : \mathfrak{X} \to \mathfrak{X}$ ($\varphi \in \mathrm{Diff}_0(X)$) and the corresponding Radon–Nikodym derivative is given by

$$\rho_{\lambda^\star}^{\bar{\varphi}}(\bar{x}) = \frac{s(\bar{\varphi}^{-1}(\bar{x}))}{s(\bar{x})}\, J_{\bar{\varphi}}(\bar{x})^{-1} \qquad \text{for } \lambda^\star\text{-a.a. } \bar{x}, \tag{4.8}$$

where $J_{\bar{\varphi}}$ is the Jacobian determinant of $\bar{\varphi}$ (we set $\rho_{\lambda^\star}^{\bar{\varphi}}(\bar{x}) = 1$ if $s(\bar{x}) = 0$ or $s(\bar{\varphi}^{-1}(\bar{x})) = 0$).

Proposition 4.2. *The Poisson measure $\pi_{\lambda^\star}$ is quasi-invariant with respect to the action of diagonal diffeomorphisms $\bar{\varphi} : \Gamma_{\mathfrak{X}} \to \Gamma_{\mathfrak{X}}$ ($\varphi \in \mathrm{Diff}_0(X)$). The corresponding Radon–Nikodym density $R_{\pi_{\lambda^\star}}^{\bar{\varphi}} := \mathrm{d}(\bar{\varphi}^*\pi_{\lambda^\star})/\mathrm{d}\pi_{\lambda^\star}$ is given by*

$$R_{\pi_{\lambda^\star}}^{\bar{\varphi}}(\bar{\gamma}) = \exp\left\{\int_{\mathfrak{X}} \Bigl(1 - \rho_{\lambda^\star}^{\bar{\varphi}}(\bar{x})\Bigr)\, \lambda^\star(\mathrm{d}\bar{x})\right\} \cdot \prod_{\bar{x}\in\bar{\gamma}} \rho_{\lambda^\star}^{\bar{\varphi}}(\bar{x}), \qquad \bar{\gamma} \in \Gamma_{\mathfrak{X}}, \tag{4.9}$$

where $\rho_{\lambda^\star}^{\bar{\varphi}}$ is defined in (4.8).

Proof. The result follows from Remark 4.1 and Proposition 6.1 in the Appendix below (applied to the space $\mathfrak{X}$ with measure $\lambda^\star$ and mapping $\bar{\varphi}$). □

Remark 4.3. The function $R_{\pi_{\lambda^\star}}^{\bar{\varphi}}$ is local in the sense that, for $\pi_{\lambda^\star}$-a.a. $\bar{\gamma} \in \Gamma_{\mathfrak{X}}$, we have $R_{\pi_{\lambda^\star}}^{\bar{\varphi}}(\bar{\gamma}) = R_{\pi_{\lambda^\star}}^{\bar{\varphi}}(\bar{\gamma} \cap \mathfrak{X}_K)$, where $K := \operatorname{supp} \varphi$.

Remark 4.4 (Explicit form of $R_{\pi_{\lambda^\star}}^{\bar{\varphi}}$)**.** Let the measure $\eta(\mathrm{d}\bar{y})$ be a.c. with respect to Lebesgue measure $\mathrm{d}\bar{y}$ on $\mathfrak{X}$, with density $h(\bar{y})$ (see (3.8)). According to (4.8),

$$\rho_{\lambda^\star}^{\bar{\varphi}}(\bar{y}) = \frac{\int_X h(\varphi^{-1}(y_1) - x, \dots, \varphi^{-1}(y_n) - x)\, \lambda(\mathrm{d}x)}{\int_X h(y_1 - x, \dots, y_n - x)\, \lambda(\mathrm{d}x)} \prod_{i=1}^n J_\varphi(y_i)^{-1}, \qquad \bar{y} \in X^n,$$

where $J_\varphi(\bar y) = \det(\partial\varphi_i/\partial y_j)$ is the Jacobian determinant of φ (note that $J_{\bar\varphi}(\bar y) = \prod_{i=1}^n J_\varphi(y_i)$ for $\bar y \in X^n$). Then $R^{\bar\varphi}_{\pi_{\lambda^\star}}(\bar\gamma)$ can be calculated using formula (4.9). In particular, if clusters have i.i.d. points, so that $h(\bar y) = \prod_{i=1}^n h_0(y_i)$, then

$$\rho^{\bar\varphi}_{\lambda^\star}(\bar y) = \frac{\int_X \prod_{i=1}^n J_\varphi(y_i)^{-1} h_0(\varphi^{-1}(y_i) - x)\,\lambda(\mathrm{d}x)}{\int_X \prod_{i=1}^n h_0(y_i - x)\,\lambda(\mathrm{d}x)}, \qquad \bar y = (y_1, \dots, y_n) \in X^n,$$

and

$$R^{\bar\varphi}_{\pi_{\lambda^\star}}(\bar\gamma) = C \prod_{\bar y \in \bar\gamma} \frac{\int_X \prod_{y \in \bar y} J_\varphi(y)^{-1} h_0(\varphi^{-1}(y) - x)\,\lambda(\mathrm{d}x)}{\int_X \prod_{y \in \bar y} h_0(y - x)\,\lambda(\mathrm{d}x)}, \qquad \bar\gamma \in \Gamma_{\mathfrak{X}},$$

where $C := \exp\left\{\int_{\mathfrak{X}} (1 - \rho^{\bar\varphi}_{\lambda^\star}(\bar y))\,\lambda^\star(\mathrm{d}\bar y)\right\}$ is a normalizing constant.

Now we can prove the main result of this section.

Theorem 4.3. *Under condition* (4.7)*, the Poisson cluster measure* μ_{cl} *on* Γ_X *is quasi-invariant with respect to the action of* $\mathrm{Diff}_0(X)$ *on* Γ_X*. The Radon–Nikodym density* $R^\varphi_{\mu_{\mathrm{cl}}} := \mathrm{d}(\varphi^*\mu_{\mathrm{cl}})/\mathrm{d}\mu_{\mathrm{cl}}$ *is given by* $R^\varphi_{\mu_{\mathrm{cl}}} = \mathcal{I}^* R^{\bar\varphi}_{\pi_{\lambda^\star}}$*, where the density* $R^{\bar\varphi}_{\pi_{\lambda^\star}} = \mathrm{d}(\bar\varphi^*\pi_{\lambda^\star})/\mathrm{d}\pi_{\lambda^\star}$ *is defined in* (4.9).

Proof. According to Theorem 3.6 (see (3.21)) and Lemma 4.1,

$$\begin{aligned} \varphi^*\mu_{\mathrm{cl}} &= (\mathfrak{p}^*\pi_{\lambda^\star}) \circ \varphi^{-1} = \pi_{\lambda^\star} \circ (\varphi \circ \mathfrak{p})^{-1} \\ &= \pi_{\lambda^\star} \circ (\mathfrak{p} \circ \bar\varphi)^{-1} = (\bar\varphi^*\pi_{\lambda^\star}) \circ \mathfrak{p}^{-1} = \mathfrak{p}^*(\bar\varphi^*\pi_{\lambda^\star}). \end{aligned}$$

Hence, by the change of variables $\gamma = \mathfrak{p}(\bar\gamma)$, for any non-negative measurable function F on Γ_X we obtain

$$\begin{aligned} \int_{\Gamma_X} F(\gamma)\,(\varphi^*\mu_{\mathrm{cl}})(\mathrm{d}\gamma) &= \int_{\Gamma_X} F(\gamma)\,\mathfrak{p}^*(\bar\varphi^*\pi_{\lambda^\star})(\mathrm{d}\gamma) = \int_{\Gamma_{\mathfrak{X}}} \mathcal{I}F(\bar\gamma)\,(\bar\varphi^*\,\pi_{\lambda^\star})(\mathrm{d}\bar\gamma) \\ &= \int_{\Gamma_{\mathfrak{X}}} \mathcal{I}F(\bar\gamma)\,R^{\bar\varphi}_{\pi_{\lambda^\star}}(\bar\gamma)\,\pi_{\lambda^\star}(\mathrm{d}\bar\gamma) = \int_{\Gamma_X} F(\gamma)\,(\mathcal{I}^* R^{\bar\varphi}_{\pi_{\lambda^\star}})(\gamma)\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma), \end{aligned}$$

where we have also used formula (4.6) and Proposition 4.2. Thus, the measure $\varphi^*\mu_{\mathrm{cl}}$ is a.c. with respect to the measure μ_{cl}, with the Radon–Nikodym density $R^\varphi_{\mu_{\mathrm{cl}}} = \mathcal{I}^* R^{\bar\varphi}_{\pi_{\lambda^\star}}$, and the theorem is proved. □

Remark 4.5. We do not know an explicit form of the density $R^\varphi_{\mu_{\mathrm{cl}}}$ (cf. Remark 4.2).

Remark 4.6. The Poisson cluster measure μ_{cl} on the configuration space Γ_X can be used to construct the canonical unitary representation U of the diffeomorphism group $\mathrm{Diff}_0(X)$ by operators in $L^2(\Gamma_X, \mu_{\mathrm{cl}})$, given by the formula

$$U_\varphi F(\gamma) = \sqrt{R^\varphi_{\mu_{\mathrm{cl}}}(\gamma)}\,F(\varphi^{-1}(\gamma)), \qquad F \in L^2(\Gamma_X, \mu_{\mathrm{cl}}).$$

Such representations, which can be defined for arbitrary quasi-invariant measures on Γ_X, play a significant role in the representation theory of the diffeomorphism group $\mathrm{Diff}_0(X)$ [20,33] and quantum field theory [17,18]. An important question is whether the representation U is irreducible. According to [33], this is equivalent to the $\mathrm{Diff}_0(X)$-ergodicity of the measure μ_{cl}, which in our case is equivalent to the ergodicity of the measure $\pi_{\lambda^\star}$ with respect to the group of transformations $\bar{\varphi}$, where $\varphi \in \mathrm{Diff}_0(X)$. The latter is an open question.

4.3 Integration-by-parts formula

The main objective of this section is to establish an integration-by-parts (IBP) formula for the Poisson cluster measure μ_{cl}, in the spirit of the IBP formula for Poisson measures proved in [5]. To this end, we shall use the projection operator $\mathfrak{p}$ and the properties of the auxiliary Poisson measure $\pi_{\lambda^\star}$. Since our framework is somewhat different from that in [5], we give a proof of the IBP formula for $\pi_{\lambda^\star}$.

First, recall that the classical IBP formula for a Borel measure ϖ on a Euclidean space $\mathbb{R}^m$ (see, e.g., [13, Ch. 5]) is expressed by the following identity that should hold for any vector field $v \in \mathrm{Vect}_0(\mathbb{R}^m)$ and all functions $f, g \in C_0^\infty(\mathbb{R}^m)$:

$$\begin{aligned} \int_{\mathbb{R}^m} f(y)\,\nabla_v\, g(y)\,\varpi(\mathrm{d}y) = &-\int_{\mathbb{R}^m} g(y)\,\nabla_v f(y)\,\varpi(\mathrm{d}y) \\ &-\int_{\mathbb{R}^m} f(y)\,g(y)\,\beta_\varpi^v(y)\,\varpi(\mathrm{d}y), \end{aligned} \tag{4.10}$$

where $\nabla_v \phi(y)$ is the derivative of ϕ along v at point $y \in Y$ and $\beta_\varpi^v \in L^1_{\mathrm{loc}}(\mathbb{R}^m, \varpi)$ is a measurable function called the *logarithmic derivative* of ϖ along the vector field v. It is easy to see that β_ϖ^v can be represented in the form

$$\beta_\varpi^v(y) = \beta_\varpi(y) \cdot v(y) + \mathrm{div}\, v(y),$$

where the corresponding mapping $\beta_\varpi : \mathbb{R}^m \to \mathbb{R}^m$ is called *vector logarithmic derivative* of ϖ. Suppose that the measure ϖ is a.c. with respect to the Lebesgue measure $\mathrm{d}y$, with density w such that $w^{1/2} \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^m)$ (:= the local Sobolev space of order 1 in $L^2(\mathbb{R}^m; \mathrm{d}y)$, i.e., the space of functions on $\mathbb{R}^m$ whose first-order partial derivatives are locally square integrable). Then the measure ϖ satisfies the IBP formula (4.10) with the vector logarithmic derivative $\beta_\varpi(y) = w(y)^{-1}\nabla w(y)$ (note that $w(y) \neq 0$ for ϖ-a.a. $y \in \mathbb{R}^m$).

Assume that the density $s(\bar{x}) = \lambda^\star(\mathrm{d}\bar{x})/\mathrm{d}\bar{x}$ $(\bar{x} \in \mathfrak{X})$ satisfies the condition $s^{1/2} \in H^{1,2}_{\mathrm{loc}}(\mathfrak{X})$ (:= the local Sobolev space of order 1 in $L^2(\mathfrak{X}; \mathrm{d}\bar{x})$). By formula (3.10) and decompositions (3.5) and (3.11), the latter condition is equivalent to the set of analogous conditions for the restrictions of $s(\bar{x})$ to the spaces X^n. That is, assuming without loss of generality that $p_n \neq 0$, for each $s_n(\bar{x}) = \lambda_n^\star(\mathrm{d}\bar{x})/\mathrm{d}\bar{x}$ $(\bar{x} \in X^n)$ we have $s_n^{1/2} \in H^{1,2}_{\mathrm{loc}}(X^n)$. By the general result alluded to above, this ensures that the IBP formula holds for each measure $\lambda_n^\star$, with the vector logarithmic derivative $\beta_{\lambda_n^\star}(\bar{x}) = (\beta_1(\bar{x}), \dots, \beta_n(\bar{x}))$ $(\bar{x} \in X^n)$, where

$$\beta_i(\bar{x}) := \frac{\nabla_i\, s_n(\bar{x})}{s_n(\bar{x})} = \frac{\int_X \nabla_i\, h_n(x_1 - x, \dots, x_n - x)\,\lambda(\mathrm{d}x)}{\int_X h_n(x_1 - x, \dots, x_n - x)\,\lambda(\mathrm{d}x)} \tag{4.11}$$

if $s_n(\bar{x}) \neq 0$ and $\beta_i(\bar{x}) := 0$ if $s_n(\bar{x}) = 0$.

For any $v \in \mathrm{Vect}_0(X)$, let us define the vector field $\bar{v}$ on $\mathfrak{X}$ by setting

$$\bar{v}(\bar{x}) := (v(x_1), \dots, v(x_n)), \quad \bar{x} = (x_1, \dots, x_n) \in X^n \quad (n \in \mathbb{Z}_+). \tag{4.12}$$

The logarithmic derivative of the measure $\lambda_n^\star$ along the vector field $\bar{v}$ is given by

$$\beta_{\lambda_n^\star}^{\bar{v}}(\bar{x}) = \sum_{x_i \in \bar{x}} \Bigl(\beta_i(\bar{x}) \cdot v(x_i) + \operatorname{div} v(x_i)\Bigr), \qquad \bar{x} \in X^n. \tag{4.13}$$

Proposition 4.4. *The measure $\lambda^\star$ satisfies the following IBP formula:*

$$\begin{aligned}\int_{\mathfrak{X}} f(\bar{x})\, \nabla_{\bar{v}}\, g(\bar{x})\, \lambda^\star(\mathrm{d}\bar{x}) = &- \int_{\mathfrak{X}} g(\bar{x})\, \nabla_{\bar{v}} f(\bar{x})\, \lambda^\star(\mathrm{d}\bar{x}) \\ &- \int_{\mathfrak{X}} f(\bar{x})\, g(\bar{x})\, \beta_{\lambda^\star}^{\bar{v}}(\bar{x})\, \lambda^\star(\mathrm{d}\bar{x}),\end{aligned} \tag{4.14}$$

where $f, g \in C_0^\infty(\mathfrak{X})$ and $\beta_{\lambda^\star}^{\bar{v}}(\bar{x}) = \beta_{\lambda_n^\star}^{\bar{v}}(\bar{x})$ if $\bar{x} \in X^n$ $(n \in \mathbb{Z}_+)$.

Proof. The result easily follows from the decomposition (3.5) of the measure $\lambda^\star$ and the IBP formula for each measure $\lambda_n^\star$ such that $p_n \neq 0$ $(n \in \mathbb{Z}_+)$. $\square$

Remark 4.7. Formula (4.14) can be rewritten in the form

$$\begin{aligned}\int_{\mathfrak{X}} f(\bar{x}) \sum_{x \in \mathfrak{p}(\bar{x})} \Bigl(\nabla_x\, g(\bar{x}) \cdot v(x)\Bigr)\, \lambda^\star(\mathrm{d}\bar{x}) = &- \int_{\mathfrak{X}} g(\bar{x}) \sum_{x \in \mathfrak{p}(\bar{x})} \Bigl(\nabla_x f(\bar{x}) \cdot v(x)\Bigr)\, \lambda^\star(\mathrm{d}\bar{x}) \\ &- \int_{\mathfrak{X}} f(\bar{x})\, g(\bar{x})\, \beta_{\lambda^\star}^{\bar{v}}(\bar{x})\, \lambda^\star(\mathrm{d}\bar{x}).\end{aligned}$$

Recall that the functional classes $\mathcal{FC}(\Gamma_X)$, $\mathcal{FC}(\Gamma_{\mathfrak{X}})$, and $\mathcal{FC}_{\lambda^\star}(\Gamma_{\mathfrak{X}})$ of local functions on the configuration spaces Γ_X and $\Gamma_{\mathfrak{X}}$ are defined in Section 4.1.

Theorem 4.5. *For each $v \in \mathrm{Vect}_0(X)$ and any $F, G \in \mathcal{FC}(\Gamma_X)$, the following IBP formula holds:*

$$\begin{aligned}\int_{\Gamma_X} F(\gamma)\, \nabla_v^\Gamma G(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) = &- \int_{\Gamma_X} G(\gamma)\, \nabla_v^\Gamma F(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) \\ &- \int_{\Gamma_X} F(\gamma)\, G(\gamma)\, B_{\mu_{\mathrm{cl}}}^v(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma),\end{aligned} \tag{4.15}$$

where ∇_v^Γ is the Γ-gradient along the vector field v defined by (4.4)*, $B_{\mu_{\mathrm{cl}}}^v(\gamma) := \mathcal{I}^*\langle \beta_{\lambda^\star}^{\bar{v}}, \bar{\gamma}\rangle$, and $\beta_{\lambda^\star}^{\bar{v}}$ is the logarithmic derivative of $\lambda^\star$ along the corresponding vector field $\bar{v}$ (see* (4.12)*).*

Proof. Denote

$$Q(\gamma) := F(\gamma)\, \nabla_v^\Gamma G(\gamma) = F(\gamma) \sum_{x \in \gamma} \nabla_x G(\gamma) \cdot v(x),$$

then

$$(\mathcal{I}Q)(\bar{\gamma}) = (\mathcal{I}F)(\bar{\gamma}) \sum_{x \in \mathfrak{p}(\bar{\gamma})} \nabla_x G(\mathfrak{p}(\bar{\gamma})) \cdot v(x). \tag{4.16}$$

Note that $\mathcal{I}Q \in \mathcal{FC}_{\lambda^\star}(\varGamma_{\mathfrak{X}})$, so we can use (2.8) in order to integrate $\mathcal{I}Q$ with respect to $\pi_{\lambda^\star}$. Using Theorem 3.6 (see (3.21)) and formula (4.16), we obtain

$$\begin{aligned}
&\int_{\varGamma_X} F(\gamma)\,\nabla_v^\varGamma G(\gamma)\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\varGamma_{\mathfrak{X}}} (\mathcal{I}F)(\bar\gamma) \sum_{x\in\mathfrak{p}(\bar\gamma)} \nabla_x G(\mathfrak{p}(\bar\gamma))\cdot v(x)\,\pi_{\lambda^\star}(\mathrm{d}\bar\gamma)\\
&\quad = \mathrm{e}^{-\lambda^\star(\mathfrak{X}_K)} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(\mathfrak{X}_K)^m} F(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\\
&\qquad\times \sum_{i=1}^{m} \sum_{x\in\mathfrak{p}(\bar x_i)} \nabla_x G(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\cdot v(x) \bigotimes_{i=1}^{m} \lambda^\star(\mathrm{d}\bar x_i)\\
&\quad = \mathrm{e}^{-\lambda^\star(\mathfrak{X}_K)} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i=1}^{m} \int_{(\mathfrak{X}_K)^{m-1}} \biggl(\int_{\mathfrak{X}_K} F(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\\
&\qquad\times \sum_{x\in\mathfrak{p}(\bar x_i)} \nabla_x G(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\cdot v(x)\,\lambda^\star(\mathrm{d}\bar x_i)\biggr) \bigotimes_{j\ne i} \lambda^\star(\mathrm{d}\bar x_j).
\end{aligned}\tag{4.17}$$

By the IBP formula for $\lambda^\star$, the inner integral in (4.17) can be rewritten as

$$\begin{aligned}
-\int_{\mathfrak{X}_K} G(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\}) \biggl(&\sum_{x\in\mathfrak{p}(\bar x_i)} \nabla_x F(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\cdot v(x)\\
&+ F(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\,\beta_{\lambda^\star}^{\bar v}(\bar x_i)\biggr)\,\lambda^\star(\mathrm{d}\bar x_i).
\end{aligned}$$

Hence, the right-hand side of (4.17) is reduced to

$$\begin{aligned}
&-\mathrm{e}^{-\lambda^\star(\mathfrak{X}_K)} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{(\mathfrak{X}_K)^m} G(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\\
&\qquad\times \biggl(\sum_{x\in\mathfrak{p}(\{\bar x_1,\dots,\bar x_m\})} \nabla_x F(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\cdot v(x)\\
&\qquad\qquad + F(\{\mathfrak{p}(\bar x_1),\dots,\mathfrak{p}(\bar x_m)\})\,B_{\pi_{\lambda^\star}}^{\bar v}(\{\bar x_1,\dots,\bar x_m\})\biggr) \bigotimes_{i=1}^{m} \lambda^\star(\mathrm{d}\bar x_i)\\
&= -\int_{\varGamma_{\mathfrak{X}}} G(\mathfrak{p}(\bar\gamma)) \biggl(\sum_{x\in\mathfrak{p}(\bar\gamma)} \nabla_x F(\mathfrak{p}(\bar\gamma))\cdot v(x) + F(\mathfrak{p}(\bar\gamma))\,B_{\pi_{\lambda^\star}}^{\bar v}(\bar\gamma)\biggr)\,\pi_{\lambda^\star}(\mathrm{d}\bar\gamma)\\
&= -\int_{\varGamma_X} G(\gamma)\,\nabla_v^\varGamma F(\gamma)\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma) - \int_{\varGamma_X} F(\gamma) G(\gamma)\,B_{\mu_{\mathrm{cl}}}^{v}(\gamma)\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma),
\end{aligned}$$

where

$$B_{\pi_{\lambda^\star}}^{\bar v}(\bar\gamma) := \sum_{\bar x\in\bar\gamma} \beta_{\lambda^\star}^{\bar v}(\bar x) = \langle \beta_{\lambda^\star}^{\bar v}, \bar\gamma\rangle, \qquad \bar\gamma\in\varGamma_{\mathfrak{X}}, \tag{4.18}$$

and $B_{\mu_{\mathrm{cl}}}^{v} := \mathcal{I}^* B_{\pi_{\lambda^\star}}^{\bar v}$. Note that $B_{\pi_{\lambda^\star}}^{\bar v}$ is well defined since $\lambda^\star(\operatorname{supp}\bar v) < \infty$, so there are only finitely many non-zero terms in the sum (4.18). Moreover, finiteness of the first and second moments of $\pi_{\lambda^\star}$ implies that $B_{\pi_{\lambda^\star}}^{\bar v} \in L^2(\varGamma_{\mathfrak{X}}, \pi_{\lambda^\star})$. $\square$

Remark 4.8. The logarithmic derivative $B^{\bar v}_{\pi_{\lambda^\star}}$ can be written in the form (cf. (4.11))

$$
\begin{aligned}
B^{\bar v}_{\pi_{\lambda^\star}}(\bar\gamma) &= \sum_{\bar x\in\bar\gamma}\sum_{x_i\in\bar x}\Bigl(\beta_i(\bar x)\cdot v(x_i)+\operatorname{div} v(x_i)\Bigr)\\
&= \sum_{\bar x\in\bar\gamma}\Bigl(\beta_{\lambda^\star}(\bar x)\cdot \bar v(\bar x)+\operatorname{div}\bar v(\bar x)\Bigr), \qquad \bar\gamma\in\Gamma_{\mathfrak X}.
\end{aligned}
$$

Formula (4.15) can be extended to more general vector fields on Γ_X. For any vector field $V\in\mathcal{FV}(\Gamma_X)$ of the form (4.3), we set

$$
B^V_{\mu_{\rm cl}}(\gamma):=\sum_{i=1}^k\Biggl(A_i(\gamma)B^{v_i}_\mu(\gamma)+\sum_{x\in\gamma}\nabla_x A_i(\gamma)\cdot v_i(x)\Biggr), \qquad \gamma\in\Gamma_X.
$$

Theorem 4.6. *For any $V\in\mathcal{FV}(\Gamma_X)$ and all $F,G\in\mathcal{FC}(\Gamma_X)$, we have*

$$
\begin{aligned}
\int_{\Gamma_X}F(\gamma)\,\nabla^\Gamma_V G(\gamma)\,\mu_{\rm cl}(\mathrm d\gamma) = &-\int_{\Gamma_X}G(\gamma)\,\nabla^\Gamma_V F(\gamma)\,\mu_{\rm cl}(\mathrm d\gamma)\\
&-\int_{\Gamma_X}F(\gamma)\,G(\gamma)\,B^V_{\mu_{\rm cl}}(\gamma)\,\mu_{\rm cl}(\mathrm d\gamma).
\end{aligned} \tag{4.19}
$$

Proof. The result readily follows from Theorem 4.5 and linearity of the right-hand side of (4.13) with respect to v. □

Remark 4.9. An explicit form of $B^V_{\mu_{\rm cl}}$ is not known (cf. Remarks 4.2 and 4.5).

Remark 4.10. The logarithmic derivative $B^V_{\mu_{\rm cl}}$ can be represented in the form $B^V_{\mu_{\rm cl}}=\mathcal I^* B^{\mathcal I V}_{\pi_{\lambda^\star}}$, where $B^{\mathcal I V}_{\pi_{\lambda^\star}}$ is the logarithmic derivative of $\pi_{\lambda^\star}$ along the vector field $\mathcal I V(\bar\gamma):=V(\mathfrak p(\bar\gamma))$. Note that the equality

$$
T_{\bar\gamma}\Gamma_{\mathfrak X}=\bigoplus_{\bar x\in\bar\gamma}T_{\bar x}\mathfrak X=\bigoplus_{\bar x\in\bar\gamma}\bigoplus_{x_i\in\bar x}T_{x_i}X=\bigoplus_{x\in\mathfrak p(\bar\gamma)}T_xX=T_{\mathfrak p(\bar\gamma)}\Gamma_X
$$

implies that $V(\mathfrak p(\bar\gamma))\in T_{\bar\gamma}\Gamma_{\mathfrak X}$, and thus $\mathcal I V(\bar\gamma)$ is a vector field on $\Gamma_{\mathfrak X}$.

5 Dirichlet forms and equilibrium stochastic dynamics

In this section, we construct a Dirichlet form $\mathcal E_{\mu_{\rm cl}}$ associated with the Poisson cluster measure $\mu_{\rm cl}$ and prove the existence of the corresponding equilibrium stochastic dynamics on the configuration space. We also show that the Dirichlet form $\mathcal E_{\mu_{\rm cl}}$ is irreducible. We assume throughout that the measure $\lambda^\star$ satisfies all the conditions set out at the beginning of Section 4 and in Section 4.3.

Let us introduce the pre-Dirichlet form $\mathcal{E}_{\mu_{\mathrm{cl}}}$ associated with the Poisson cluster measure μ_{cl}, defined on $\mathcal{FC}(\varGamma_X) \subset L^2(\varGamma_X, \mu_{\mathrm{cl}})$ by

$$\mathcal{E}_{\mu_{\mathrm{cl}}}(F, G) := \int_{\varGamma_X} \langle \nabla^{\varGamma} F(\gamma), \nabla^{\varGamma} G(\gamma)\rangle_\gamma \, \mu_{\mathrm{cl}}(\mathrm{d}\gamma), \qquad F, G \in \mathcal{FC}(\varGamma_X), \tag{5.1}$$

where $\nabla^{\varGamma}$ is the $\varGamma$-gradient on the configuration space $\varGamma_X$ (see (4.2)). The next proposition shows that the form $\mathcal{E}_{\mu_{\mathrm{cl}}}$ is well defined.

Proposition 5.1. *For any $F, G \in \mathcal{FC}(\varGamma_X)$, we have $\mathcal{E}_{\mu_{\mathrm{cl}}}(F, G) < \infty$.*

Proof. The statement follows from the existence of the first moments of μ_{cl}. Indeed, let $F, G \in \mathcal{FC}(\varGamma_X)$ have representations

$$F(\gamma) = f(\langle \phi_1, \gamma\rangle, \dots, \langle \phi_k, \gamma\rangle), \qquad G(\gamma) = g(\langle \psi_1, \gamma\rangle, \dots, \langle \psi_\ell, \gamma\rangle)$$

(see (4.1)), then a direct calculation shows that

$$\langle \nabla^{\varGamma} F(\gamma), \nabla^{\varGamma} G(\gamma)\rangle_\gamma = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot \nabla_x G(\gamma) = \sum_{i,j} Q_{ij}(\gamma) \langle q_{ij}, \gamma\rangle,$$

where $q_{ij}(x) := \nabla\phi_i(x) \cdot \nabla\psi_j(x) \in C_0(X)$ and

$$Q_{ij}(\gamma) := \nabla_i f(\langle \phi_1, \gamma\rangle, \dots, \langle \phi_k, \gamma\rangle)\, \nabla_j g(\langle \psi_1, \gamma\rangle, \dots, \langle \psi_\ell, \gamma\rangle) \in \mathcal{FC}(\varGamma_X).$$

Denoting for brevity $q(x) := q_{ij}(x)$ and setting $\tilde{q}(\bar{x}) := \sum_{x\in\bar{x}} q(x)$, by Theorem 3.6 we have

$$\begin{aligned}\int_{\varGamma_X} \langle q, \gamma\rangle \, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) &= \int_{\varGamma_{\mathfrak{X}}} \langle q, \mathfrak{p}(\bar{\gamma})\rangle \, \pi_{\lambda^\star}(\mathrm{d}\bar{\gamma}) \\ &= \int_{\varGamma_{\mathfrak{X}}} \langle \tilde{q}, \bar{\gamma}\rangle \, \pi_{\lambda^\star}(\mathrm{d}\bar{\gamma}) = \int_{\mathfrak{X}} \tilde{q}(\bar{y}) \, \lambda^\star(\mathrm{d}\bar{y}) < \infty,\end{aligned}$$

because $\lambda^\star(\operatorname{supp} \tilde{q}) = \lambda^\star(\mathfrak{X}_{\operatorname{supp} q}) < \infty$ by Proposition 3.4. Therefore, $\langle q, \gamma\rangle \in L^1(\varGamma_X, \mu_{\mathrm{cl}})$ and the required result follows. $\square$

Let us also consider the pre-Dirichlet form $\mathcal{E}_{\pi_{\lambda^\star}}$ associated with the Poisson measure $\pi_{\lambda^\star}$, defined on the space $\mathcal{FC}(\varGamma_{\mathfrak{X}}) \subset L^2(\varGamma_{\mathfrak{X}}, \pi_{\lambda^\star})$ by

$$\mathcal{E}_{\pi_{\lambda^\star}}(\varPhi, \varPsi) := \int_{\varGamma_{\mathfrak{X}}} \langle \nabla^{\varGamma} \varPhi(\bar{\gamma}), \nabla^{\varGamma} \varPsi(\bar{\gamma})\rangle_{\bar{\gamma}} \, \pi_{\lambda^\star}(\mathrm{d}\bar{\gamma}), \qquad \varPhi, \varPsi \in \mathcal{FC}(\varGamma_{\mathfrak{X}})$$

(here $\nabla^{\varGamma}$ is the $\varGamma$-gradient on the configuration space $\varGamma_{\mathfrak{X}}$, cf. (4.2)). Pre-Dirichlet forms of such type associated with general Poisson measures were introduced and studied in [5]. Finiteness of the first moments of the Poisson measure $\pi_{\lambda^\star}$ implies that $\mathcal{E}_{\pi_{\lambda^\star}}$ is well defined. It follows from the IBP formula for $\pi_{\lambda^\star}$ that

$$\mathcal{E}_{\pi_{\lambda^\star}}(\varPhi, \varPsi) = \int_{\varGamma_{\mathfrak{X}}} H_{\pi_{\lambda^\star}} \varPhi(\bar{\gamma})\, \varPsi(\bar{\gamma}) \, \pi_{\lambda^\star}(\mathrm{d}\bar{\gamma}), \qquad \varPhi, \varPsi \in \mathcal{FC}(\varGamma_{\mathfrak{X}}), \tag{5.2}$$

where $H_{\pi_{\lambda^\star}}$ is a symmetric non-negative operator in $L^2(\Gamma_{\mathfrak{X}}, \pi_{\lambda^\star})$ (called the Dirichlet operator of the Poisson measure $\pi_{\lambda^\star}$, see [5]) defined on the domain $\mathcal{FC}(\Gamma_{\mathfrak{X}})$ by

$$(H_{\pi_{\lambda^\star}}\Phi)(\bar\gamma) := -\sum_{\bar x\in\bar\gamma}\Big(\Delta_{\bar x}\Phi(\bar\gamma) + \nabla_{\bar x}\Phi(\bar\gamma)\cdot\beta_{\lambda^\star}(\bar x)\Big) \qquad (\bar\gamma\in\Gamma_{\mathfrak{X}}). \tag{5.3}$$

Since function $\Phi\in\mathcal{FC}(\Gamma_{\mathfrak{X}})$ is local (see Section 4.1), there are only finitely many non-zero terms in the sum (5.3).

Remark 5.1. Note that the operator $H_{\pi_{\lambda^\star}}$ is well defined by formula (5.3) on the bigger space $\mathcal{FC}_{\lambda^\star}(\Gamma_{\mathfrak{X}})$. Similar arguments as before show that the pre-Dirichlet form $\mathcal{E}_{\pi_{\lambda^\star}}(\Phi,\Psi)$ is well defined on $\mathcal{FC}_{\lambda^\star}(\Gamma_{\mathfrak{X}})$ and formula (5.2) holds for any $\Phi,\Psi\in\mathcal{FC}_{\lambda^\star}(\Gamma_{\mathfrak{X}})$.

Consider a symmetric operator in $L^2(\Gamma_X,\mu_{\mathrm{cl}})$ defined on $\mathcal{FC}(\Gamma_X)$ by the formula

$$H_{\mu_{\mathrm{cl}}} := \mathcal{I}^* H_{\pi_{\lambda^\star}}\mathcal{I}. \tag{5.4}$$

Note that the domain $\mathcal{FC}(\Gamma_X)$ is dense in $L^2(\Gamma_X,\mu_{\mathrm{cl}})$.

Theorem 5.2. *For any $F,G\in\mathcal{FC}(\Gamma_X)$, the form* (5.1) *satisfies the equality*

$$\mathcal{E}_{\mu_{\mathrm{cl}}}(F,G) = \int_{\Gamma_X} H_{\mu_{\mathrm{cl}}}F(\gamma)\,G(\gamma)\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma). \tag{5.5}$$

In particular, this implies that $H_{\mu_{\mathrm{cl}}}$ is a non-negative operator on $\mathcal{FC}(\Gamma_X)$.

Proof. Let us fix $F,G\in\mathcal{FC}(\Gamma_X)$ and set $Q(\gamma) := \langle\nabla^\Gamma F(\gamma),\nabla^\Gamma G(\gamma)\rangle_\gamma$. From the definition (4.6) of the operator $\mathcal{I}$, it readily follows that

$$(\mathcal{I}Q)(\bar\gamma) = \sum_{x\in\mathfrak{p}(\bar\gamma)}\nabla_x\mathcal{I}F(\bar\gamma)\cdot\nabla_x\mathcal{I}G(\bar\gamma) = \sum_{\bar x\in\bar\gamma}\nabla_{\bar x}\mathcal{I}F(\bar\gamma)\cdot\nabla_{\bar x}\mathcal{I}G(\bar\gamma), \tag{5.6}$$

where $\nabla_{\bar x} := (\nabla_{x_1},\dots,\nabla_{x_n})$ when $\bar x = (x_1,\dots,x_n)\in X^n$ $(n\in\mathbb{N})$. Thus, by Theorem 3.6 and formulas (4.6) and (5.6) we obtain

$$\begin{aligned}\mathcal{E}_{\mu_{\mathrm{cl}}}(F,G) &= \int_{\Gamma_X} Q(\gamma)\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_{\mathfrak{X}}}(\mathcal{I}Q)(\bar\gamma)\,\pi_{\lambda^\star}(\mathrm{d}\bar\gamma)\\ &= \int_{\Gamma_{\mathfrak{X}}}\sum_{\bar x\in\bar\gamma}\nabla_{\bar x}\mathcal{I}F(\bar\gamma)\cdot\nabla_{\bar x}\mathcal{I}G(\bar\gamma)\,\pi_{\lambda^\star}(\mathrm{d}\bar\gamma) = \mathcal{E}_{\pi_{\lambda^\star}}(\mathcal{I}F,\mathcal{I}G)\end{aligned} \tag{5.7}$$

(note that $\mathcal{I}F,\mathcal{I}G\in\mathcal{FC}_{\lambda^\star}(\Gamma_{\mathfrak{X}})\subset\mathcal{D}(\mathcal{E}_{\pi_{\lambda^\star}})$). Finally, combining (5.7) with formula (5.2) we get (5.5). □

Remark 5.2. The operator $H_{\mu_{\mathrm{cl}}}$ defined in (5.4) can be represented in the following form separating its diffusive and drift parts:

$$(H_{\mu_{\mathrm{cl}}}F)(\gamma) = -\sum_{x\in\gamma}\Delta_x F(\gamma) - (\mathcal{I}^*\Psi_F)(\gamma), \qquad F\in\mathcal{FC}(\Gamma_X), \tag{5.8}$$

where $\Psi_F(\bar\gamma) := \sum_{\bar x\in\bar\gamma}\nabla_{\bar x}\mathcal{I}F(\bar\gamma)\cdot\beta_{\lambda^\star}(\bar x)$ $(\bar\gamma\in\Gamma_{\mathfrak{X}})$.

Remark 5.3. Formulas (5.5) and (5.8) can also be obtained directly from the IBP formula (4.19).

5.2 The associated equilibrium stochastic dynamics

Formula (5.5) implies that the form $\mathcal{E}_{\mu_{\rm cl}}$ is closable on $L^2(\varGamma_X, \mu_{\rm cl})$, and we preserve the same notation for its closure. Its domain $\mathcal{D}(\mathcal{E}_{\mu_{\rm cl}})$ is obtained as a completion of $\mathcal{FC}(\varGamma_X)$ with respect to the norm

$$\|F\|_{\mathcal{E}_{\mu_{\rm cl}}} := \left(\mathcal{E}_{\mu_{\rm cl}}(F,F) + \int_{\varGamma_X} F^2 \,\mathrm{d}\mu_{\rm cl}\right)^{1/2}.$$

In the canonical way, the Dirichlet form $(\mathcal{E}_{\mu_{\rm cl}}, \mathcal{D}(\mathcal{E}_{\mu_{\rm cl}}))$ defines a non-negative self-adjoint operator in $L^2(\varGamma_X, \mu_{\rm cl})$ (i.e., the Friedrichs extension of $H_{\mu_{\rm cl}} = \mathcal{I}^* H_{\pi_{\lambda^\star}} \mathcal{I}$ from the domain $\mathcal{FC}(\varGamma_X)$), for which we keep the same notation $H_{\mu_{\rm cl}}$. In turn, this operator generates the semigroup $\exp(-tH_{\mu_{\rm cl}})$ in $L^2(\varGamma_X, \mu_{\rm cl})$.

According to a general result (see [27, § 4]), it follows that $\mathcal{E}_{\mu_{\rm cl}}$ is a quasi-regular local Dirichlet form on a bigger space $L^2(\ddot{\varGamma}_X, \mu_{\rm cl})$, where $\ddot{\varGamma}_X$ is the space of all locally finite configurations γ with possible multiple points (note that $\ddot{\varGamma}_X$ can be identified in the standard way with the space of $\mathbb{Z}_+$-valued Radon measures on X, cf. [5,27,30]). Then, by the general theory of Dirichlet forms (see [26]), we obtain the following result.

Theorem 5.3. *There exists a conservative diffusion process* $\mathbf{X} = (\mathbf{X}_t,\ t \ge 0)$ *on* $\ddot{\varGamma}_X$, *properly associated with the Dirichlet form* $\mathcal{E}_{\mu_{\rm cl}}$; *that is, for any function* $F \in L^2(\ddot{\varGamma}_X, \mu_{\rm cl})$ *and all* $t \ge 0$, *the mapping*

$$\ddot{\varGamma}_X \ni \gamma \mapsto p_t F(\gamma) := \int_{\varOmega} F(\mathbf{X}_t) \,\mathrm{d}P_\gamma$$

is an $\mathcal{E}_{\mu_{\rm cl}}$*-quasi-continuous version of* $\exp(-tH_{\mu_{\rm cl}})F$. *Here* $\varOmega$ *is the canonical sample space* (*of* $\ddot{\varGamma}_X$*-valued continuous functions on* $\mathbb{R}_+$) *and* $(P_\gamma,\ \gamma \in \ddot{\varGamma}_X)$ *is the family of probability distributions of the process* $\mathbf{X}$ *conditioned on the initial value* $\gamma = \mathbf{X}_0$. *The process* $\mathbf{X}$ *is unique up to* $\mu_{\rm cl}$*-equivalence. In particular,* $\mathbf{X}$ *is* $\mu_{\rm cl}$*-symmetric* (*i.e.,* $\int F p_t G \,\mathrm{d}\mu_{\rm cl} = \int G p_t F \,\mathrm{d}\mu_{\rm cl}$ *for all measurable functions* $F, G : \ddot{\varGamma}_X \to \mathbb{R}_+$) *and* $\mu_{\rm cl}$ *is its invariant measure.*

Remark 5.4. It can be proved that in the case of Poisson and Gibbs measures, under certain technical conditions the diffusion process $\mathbf{X}$ actually lives on the proper configuration space $\varGamma_X$ (see [30]). It is plausible that a similar result should be valid for the Poisson cluster measure, but this is an open problem.

Remark 5.5. Formula (5.2) implies that the "pre-projection" form $\mathcal{E}_{\pi_{\lambda^\star}}$ is closable. According to the general theory of Dirichlet forms [26,27], its closure is a quasi-regular local Dirichlet form on $\ddot{\varGamma}_{\mathfrak{X}}$ and as such generates a diffusion process $\bar{\mathbf{X}}$ on $\ddot{\varGamma}_{\mathfrak{X}}$. This process coincides with the independent infinite particle process, which amounts to independent distorted Brownian motions in $\mathfrak{X}$ with drift given by the vector logarithmic derivative of λ (see [5]). However, it is not clear in what sense the process $\mathbf{X}$ constructed in Theorem 5.3 can be obtained directly via the projection of $\bar{\mathbf{X}}$ from $\ddot{\Gamma}_{\mathfrak{X}}$ onto $\ddot{\Gamma}_X$.

5.3 Irreducibility of the Dirichlet form $\mathcal{E}_{\mu_{\rm cl}}$

Let us recall that a Dirichlet form $\mathcal{E}$ is called *irreducible* if the condition $\mathcal{E}(F,F) = 0$ implies that $F = \text{const}$.

Theorem 5.4. *The Dirichlet form $(\mathcal{E}_{\mu_{\rm cl}}, \mathcal{D}(\mathcal{E}_{\mu_{\rm cl}}))$ is irreducible.*

Proof. For any $F \in \mathcal{D}(\mathcal{E}_{\mu_{\rm cl}})$, we have

$$\begin{aligned}\|F\|^2_{\mathcal{E}_{\mu_{\rm cl}}} &= \mathcal{E}_{\mu_{\rm cl}}(F,F) + \int_{\Gamma_X} F^2 \, \mathrm{d}\mu_{\rm cl} \\ &= \mathcal{E}_{\pi_{\lambda^\star}}(\mathcal{I}F, \mathcal{I}F) + \int_{\Gamma_X} (\mathcal{I}F)^2 \, \mathrm{d}\pi_{\lambda^\star} = \|\mathcal{I}F\|^2_{\mathcal{E}_{\pi_{\lambda^\star}}},\end{aligned}$$

which implies that $\mathcal{I}\mathcal{D}(\mathcal{E}_{\mu_{\rm cl}}) \subset \mathcal{D}(\mathcal{E}_{\pi_{\lambda^\star}})$. It is obvious that if $\mathcal{I}F = \text{const}$ ($\pi_{\lambda^\star}$-a.s.) then $F = \text{const}$ ($\mu_{\rm cl}$-a.s.). Therefore, according to formula (5.7), it suffices to prove that the Dirichlet form $(\mathcal{E}_{\pi_{\lambda^\star}}, \mathcal{D}(\mathcal{E}_{\pi_{\lambda^\star}}))$ is irreducible, which is established in Lemma 5.6 below. $\square$

We first need the following general result (see [3, Lemma 3.3]).

Lemma 5.5. *Let A and B be self-adjoint, non-negative operators in separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then* $\operatorname{Ker}(A \boxplus B) = \operatorname{Ker} A \otimes \operatorname{Ker} B$, *where $A \boxplus B$ is the closure of the operator $A \otimes I + I \otimes B$ from the algebraic tensor product of the domains of A and B.*

Proof. $\operatorname{Ker} A$ and $\operatorname{Ker} B$ are closed subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively, and so their tensor product $\operatorname{Ker} A \otimes \operatorname{Ker} B$ is a closed subspace of the space $\mathcal{H} \otimes \mathcal{K}$. The inclusion $\operatorname{Ker} A \otimes \operatorname{Ker} B \subset \operatorname{Ker}(A \boxplus B)$ is trivial. Let $f \in \operatorname{Ker}(A \boxplus B)$. Using the theory of operators admitting separation of variables (see, e.g., [8, Ch. 6]), we have

$$\begin{aligned}0 = (A \boxplus Bf, f) &= \int_{\mathbb{R}^2_+} (x_1 + x_2) \, \mathrm{d}(E(x_1,x_2)f,f) \\ &= \int_{\mathbb{R}^2_+} x_1 \, \mathrm{d}(E(x_1,x_2)f,f) + \int_{\mathbb{R}^2_+} x_2 \, \mathrm{d}(E(x_1,x_2)f,f) \\ &= (A \otimes If, f) + (I \otimes Bf, f), \end{aligned} \tag{5.9}$$

where E is a joint resolution of the identity of the commuting operators $A \otimes I$ and $I \otimes B$. Since both operators $A \otimes I$ and $I \otimes B$ are non-negative, we conclude from (5.9) that

$$f \in \operatorname{Ker}(A \otimes I) \cap \operatorname{Ker}(I \otimes B) = \operatorname{Ker} A \otimes \operatorname{Ker} B,$$

which completes the proof of the lemma. $\square$

Lemma 5.6. *The Dirichlet form $(\mathcal{E}_{\pi_{\lambda^\star}}, \mathcal{D}(\mathcal{E}_{\pi_{\lambda^\star}}))$ is irreducible.*

Remark 5.6. Irreducibility of Dirichlet forms associated with Poisson measures on configuration spaces of connected Riemannian manifolds was shown in [5]. However, the space $\mathfrak{X}$ consists of countably many disjoint connected components X^n, so we need to adapt the result of [5] to this situation.

Proof of Lemma 5.6. Let us recall that, according to the general theory (see, e.g., [4]), irreducibility of a Dirichlet form is equivalent to the condition that the kernel of its generator consists of constants (*uniqueness of the ground state*). Thus, it suffices to prove that $\operatorname{Ker} H_{\pi_{\lambda^\star}} = \{\mathrm{const}\}$.

Let us consider the "residual" spaces $\tilde{\mathfrak{X}}_n := \bigsqcup_{k=n}^\infty X^k$, $n \in \mathbb{Z}_+$, endowed with the measures $\tilde{\lambda}_n^\star := \sum_{k=n}^\infty p_k \lambda_k^\star$. Hence, $\mathfrak{X} = X^0 \sqcup X^1 \sqcup \cdots \sqcup X^n \sqcup \tilde{\mathfrak{X}}_{n+1}$, which implies that $\varGamma_{\mathfrak{X}} = \varGamma_{X^0} \times \varGamma_{X^1} \times \cdots \times \varGamma_{X^n} \times \varGamma_{\tilde{\mathfrak{X}}_{n+1}}$ and, according to Proposition 2.2, $\pi_{\lambda^\star} = \pi_0 \otimes \pi_1 \otimes \cdots \otimes \pi_n \otimes \tilde{\pi}_{n+1}$, where we use a shorthand notation $\pi_n := \pi_{p_n \lambda_n^\star}$, $\tilde{\pi}_n := \pi_{\tilde{\lambda}_n^\star}$. Therefore, there is an isomorphism of Hilbert spaces

$$L^2(\varGamma_{\mathfrak{X}}, \pi_{\lambda^\star}) \cong L^2(\varGamma_X, \pi_1) \otimes \cdots \otimes L^2(\varGamma_{X^n}, \pi_n) \otimes L^2(\varGamma_{\mathfrak{X}_{n+1}}, \tilde{\pi}_{n+1}).$$

Consequently, the Dirichlet operator $H_{\pi_{\lambda^\star}}$ can be decomposed as

$$H_{\pi_{\lambda^\star}} = H_{\pi_1} \boxplus \cdots \boxplus H_{\pi_n} \boxplus H_{\tilde{\pi}_{n+1}}. \tag{5.10}$$

Since all operators on the right-hand side of (5.10) are self-adjoint and non-negative, it follows by Lemma 5.5 that

$$\operatorname{Ker} H_{\pi_{\lambda^\star}} = \operatorname{Ker} H_{\pi_1} \otimes \cdots \otimes \operatorname{Ker} H_{\pi_n} \otimes \operatorname{Ker} H_{\tilde{\pi}_{n+1}}. \tag{5.11}$$

The Dirichlet forms of all measures π_k are irreducible (as Dirichlet forms of Poisson measures on connected manifolds), hence $\operatorname{Ker} H_{\pi_k} = \mathbb{R}$ and (5.11) implies that $\operatorname{Ker} H_{\pi_{\lambda^\star}} = \operatorname{Ker} H_{\tilde{\pi}_{n+1}}$. Since n is arbitrary, it follows that every function $F \in \operatorname{Ker} H_{\pi_{\lambda^\star}}$ does not depend on any finite number of variables, and thus $F = \mathrm{const}$ ($\pi_{\lambda^\star}$-a.s.). $\square$

Remark 5.7. The result of Lemma 5.6 (and the idea of its proof) can be viewed as a functional-analytic analogue of Kolmogorov's zero–one law (see, e.g., [21, Ch. 3]), stating that for a sequence of independent random variables (X_n), the corresponding tail σ-algebra $\mathcal{F}_\infty := \bigcap_n \mathcal{F}_{\ge n}$ is trivial (where $\mathcal{F}_{\ge n} := \sigma\{X_k : k \ge n\}$), and in particular, all $\mathcal{F}_\infty$-measurable random variables are a.s. constants.

Remark 5.8. According to the general theory of Dirichlet forms (see, e.g., [4]), the irreducibility of $\mathcal{E}_{\mu_{\mathrm{cl}}}$ is equivalent to each of the following properties:

(i) *The semigroup* $\mathrm{e}^{-tH_{\mu_{\mathrm{cl}}}}$ *is* L^2*-ergodic, that is, as* $t \to \infty$,

$$\int_{\varGamma_X} \left(\mathrm{e}^{-tH_{\mu_{\mathrm{cl}}}} F(\gamma) - \int_{\varGamma_X} F(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) \right)^2 \mu_{\mathrm{cl}}(\mathrm{d}\gamma) \to 0.$$

(ii) *If* $F \in \mathcal{D}(H_{\mu_{\mathrm{cl}}})$ *and* $H_{\mu_{\mathrm{cl}}} F = 0$ *then* $F = \mathrm{const}$.

6 Appendix

6.1 *Proof of Theorem 2.7*

Note that the droplet cluster $D_B(\gamma_0') = \bigcup_{y\in\gamma_0'}(B-y)$ (see (2.18)) can be decomposed into disjoint components according to the number of constituent "layers" (including infinitely many):

$$D_B(\gamma_0') = \bigcup_{1\le\ell\le\infty} D_B^\ell(\gamma_0'),$$

where

$$D_B^\ell(\gamma_0') := \{x\in X:\ \gamma_0'(B-x)=\ell\}, \qquad \ell\in\overline{\mathbb{Z}}_+.$$

(a) Set $f_q := -\ln q\cdot \mathbf{1}_K \in \mathrm{M}_+(X)$ $(0<q<1)$, then

$$\begin{aligned} L_{\mu_{\mathrm{cl}}}[f_q] = \int_{\Gamma_X^\sharp} q^{\gamma(K)}\,\mu_{\mathrm{cl}}(\mathrm{d}\gamma) &= \sum_{n=0}^\infty q^n \mu_{\mathrm{cl}}\{\gamma\in\Gamma_X^\sharp : \gamma(K)=n\} \\ &\to \mu_{\mathrm{cl}}\{\gamma\in\Gamma_X^\sharp : \gamma(K)<\infty\} \qquad (q\uparrow 1). \end{aligned} \tag{6.1}$$

Therefore, $\gamma(K)<\infty$ (μ_{cl}-a.s.) if and only if $\lim_{q\uparrow 1}\ln L_{\mu_{\mathrm{cl}}}[f_q]=0$.

Clearly, condition (2.19) is necessary for local finiteness of μ_{cl}-a.a. configurations $\gamma\in\Gamma_X^\sharp$. Furthermore, (2.19) implies that, for any compact set $K\subset X$ and any $x\in X$, we have $\gamma_0'(K-x)<\infty$ (μ_0-a.s.). Hence, according to (2.17),

$$\begin{aligned} -\ln L_{\mu_{\mathrm{cl}}}[f_q] &= \int_X\left(\int_{\Gamma_X^\sharp}\left(1-q^{\gamma_0'(K-x)}\right)\mu_0(\mathrm{d}\gamma_0')\right)\lambda(\mathrm{d}x) \\ &= \int_{\Gamma_X^\sharp}\left(\int_X \sum_{\ell=0}^\infty (1-q^\ell)\,\mathbf{1}_{D_K^\ell(\gamma_0')}(x)\,\lambda(\mathrm{d}x)\right)\mu_0(\mathrm{d}\gamma_0') \\ &= \int_{\Gamma_X^\sharp}\sum_{\ell=1}^\infty (1-q^\ell)\,\lambda\Big(D_K^\ell(\gamma_0')\Big)\,\mu_0(\mathrm{d}\gamma_0'). \end{aligned} \tag{6.2}$$

Note that, for $0<q<1$,

$$0\le \sum_{\ell=1}^\infty (1-q^\ell)\,\lambda\Big(D_K^\ell(\gamma_0')\Big) \le \sum_{\ell=1}^\infty \lambda\Big(D_K^\ell(\gamma_0')\Big) = \lambda\Big(D_K(\gamma_0')\Big),$$

so if condition (2.20) is satisfied then we can apply Lebesgue's dominated convergence theorem and pass termwise to the limit on the right-hand side of (6.2) as $q\uparrow 1$, which gives $\lim_{q\uparrow 1}\ln L_{\mu_{\mathrm{cl}}}[f_q]=0$, as required.

Conversely, since

$$\sum_{\ell=1}^\infty (1-q^\ell)\,\lambda\Big(D_K^\ell(\gamma_0')\Big) \ge (1-q)\sum_{\ell=1}^\infty \lambda\Big(D_K^\ell(\gamma_0')\Big) = (1-q)\,\lambda\Big(D_K(\gamma_0')\Big)\ge 0,$$

from (6.2) we must have

$$(1-q)\int_{\Gamma_X^\sharp}\lambda\Big(D_K(\gamma_0')\Big)\,\mu_0(\mathrm{d}\gamma_0')\to 0\qquad (q\uparrow 1),$$

which implies (2.20).

(b) Let us first prove the "only if" part. Clearly, condition (2.21) is necessary in order to avoid any in-cluster ties. Furthermore, each fixed $x_0\in X$ cannot belong to more than one cluster; in particular, for any $2\le\ell\le\infty$,

$$\lambda\left(D^{\ell}_{\{x_0\}}(\gamma_0')\right)=0\qquad (\mu_0\text{-a.s.})\tag{6.3}$$

Let $f_q:=-\ln q\cdot\mathbf{1}_{\{x_0\}}$ $(0<q<1)$. The expansion (6.1) then implies that in order for x_0 to be simple (μ_{cl}-a.s.), $L_{\mu_{\mathrm{cl}}}[f_q]$ must be a linear function of q. But from (6.2) and (6.3) we have

$$L_{\mu_{\mathrm{cl}}}[f_q]=\exp\Big\{-(1-q)\int_{\Gamma_X^\sharp}\lambda\Big(D^{\ell=1}_{\{x_0\}}(\gamma_0')\Big)\,\mu_0(\mathrm{d}\gamma_0')\Big\},$$

and it follows that $\lambda\Big(D^{\ell=1}_{\{x_0\}}(\gamma_0')\Big)=0$ (μ_0-a.s.). Together with (6.3), this gives

$$\lambda\Big(D_{\{x_0\}}(\gamma_0')\Big)=\sum_{1\le\ell\le\infty}\lambda\Big(D^{\ell}_{\{x_0\}}(\gamma_0')\Big)=0\qquad(\mu_0\text{-a.s.}),$$

and condition (2.22) follows.

To prove the "if" part, it suffices to show that, under conditions (2.21) and (2.22), with probability one there are no cross-ties between the clusters whose centres belong to a set $\Lambda\subset X$, $\lambda(\Lambda)<\infty$. Conditionally on the total number of cluster centres in Λ (which are then i.i.d. and have the distribution $\lambda(\cdot)/\lambda(\Lambda)$), the probability of a tie between a given pair of (independent) clusters is given by

$$\frac{1}{\lambda(\Lambda)^2}\int_{\Gamma_X^\sharp\times\Gamma_X^\sharp}\lambda^{\otimes 2}\Big(B_\Lambda(\gamma_1,\gamma_2)\Big)\,\mu_0(\mathrm{d}\gamma_1)\,\mu_0(\mathrm{d}\gamma_2),$$

where

$$B_\Lambda(\gamma_1,\gamma_2):=\{(x_1,x_2)\in\Lambda^2:\ x_1+y_1=x_2+y_2\ \text{ for some }\ y_1\in\gamma_1,\ y_2\in\gamma_2\}.$$

But

$$\begin{aligned}\lambda^{\otimes 2}\Big(B_\Lambda(\gamma_1,\gamma_2)\Big)&=\int_\Lambda\lambda\left(\textstyle\bigcup_{y_1\in\gamma_1}\bigcup_{y_2\in\gamma_2}\{x_1+y_1-y_2\}\right)\lambda(\mathrm{d}x_1)\\&\le\sum_{y_1\in\gamma_1}\int_\Lambda\lambda\left(\textstyle\bigcup_{y_2\in\gamma_2}\{x_1+y_1-y_2\}\right)\lambda(\mathrm{d}x_1)\\&=\sum_{y_1\in\gamma_1}\int_\Lambda\lambda\Big(D_{\{x_1+y_1\}}(\gamma_2)\Big)\,\lambda(\mathrm{d}x_1)=0\qquad(\mu_0\text{-a.s.}),\end{aligned}$$

since, by assumption (2.22), $\lambda\Big(D_{\{x_1+y_1\}}(\gamma_2)\Big)=0$ (μ_0-a.s.) and γ_1 is a countable set. Thus, the proof is complete.

The next general result is a direct consequence of Skorokhod's theorem [32] on the absolute continuity of Poisson measures (see also [5]). Although essentially well known, we give its simple proof adapted to our slightly more general setting, whereby transformations φ have support of finite measure rather than compact.

Suppose that π_λ is a Poisson measure on the configuration space $\varGamma_X$ with intensity measure λ. Let $\varphi : X \to X$ be a measurable mapping; as explained earlier (see (4.5)), it can be lifted to a (measurable) transformation of $\varGamma_X$:

$$\varGamma_X \ni \gamma \mapsto \varphi(\gamma) := \{\varphi(x),\ x \in \gamma\} \in \varGamma_X. \tag{6.4}$$

Proposition 6.1. *Let $\varphi : X \to X$ be a measurable bijection such that $\lambda(\operatorname{supp}\varphi) < \infty$. Assume that the measure λ is quasi-invariant with respect to φ, that is, the push-forward measure $\varphi^*\lambda \equiv \lambda \circ \varphi^{-1}$ is a.c. with respect to λ, with density*

$$\rho_\lambda^\varphi(x) := \frac{\varphi^*\lambda(\mathrm{d}x)}{\lambda(\mathrm{d}x)}, \qquad x \in X. \tag{6.5}$$

Then the measure π_λ is quasi-invariant with respect to the action (6.4)*, that is,*

$$\varphi^*\pi_\lambda(\mathrm{d}\gamma) = R_{\pi_\lambda}^\varphi(\gamma)\,\pi_\lambda(\mathrm{d}\gamma), \qquad \gamma \in \varGamma_X, \tag{6.6}$$

where the density $R_{\pi_\lambda}^\varphi$ is given by

$$R_{\pi_\lambda}^\varphi(\gamma) = \exp\left\{\int_X \Bigl(1 - \rho_\lambda^\varphi(x)\Bigr)\,\lambda(\mathrm{d}x)\right\} \cdot \prod_{x\in\gamma} \rho_\lambda^\varphi(x), \qquad \gamma \in \varGamma_X, \tag{6.7}$$

and moreover, $R_{\pi_\lambda}^\varphi \in L^2(\varGamma_X, \pi_\lambda)$.

Proof. Note that $\rho_\lambda^\varphi \equiv 1$ outside the set $K := \operatorname{supp}\varphi$. By Proposition 2.4(a), the condition $\lambda(K) < \infty$ implies that, for π_λ-a.a. $\gamma \in \varGamma_X$, there are only finitely many terms in the product $\prod_{x\in\gamma} \rho_\lambda^\varphi(x)$ not equal to 1, thus the right-hand side of equation (6.7) is well defined. Using formulas (6.5), (6.7) and Proposition 2.1, the Laplace functional of the measure $\pi_\lambda^\varphi := R_{\pi_\lambda}^\varphi \pi_\lambda$ is obtained as follows:

$$\begin{aligned}
L_{\pi_\lambda^\varphi}[f] &= \exp\left\{\int_X \Bigl(1 - \rho_\lambda^\varphi(x)\Bigr)\,\lambda(\mathrm{d}x)\right\} \cdot \int_{\varGamma_X} \mathrm{e}^{-\langle f,\gamma\rangle} \prod_{x\in\gamma} \rho_\lambda^\varphi(x)\,\pi_\lambda(\mathrm{d}\gamma)\\
&= \exp\left\{\int_X \Bigl(1 - \rho_\lambda^\varphi(x)\Bigr)\,\lambda(\mathrm{d}x)\right\} \cdot \exp\left\{-\int_X \Bigl(1 - \mathrm{e}^{-f(x)+\ln\rho_\lambda^\varphi(x)}\Bigr)\,\lambda(\mathrm{d}x)\right\}\\
&= \exp\left\{-\int_X \Bigl(1 - \mathrm{e}^{-f(x)}\Bigr)\,\rho_\lambda^\varphi(x)\,\lambda(\mathrm{d}x)\right\}\\
&= \exp\left\{-\int_X \Bigl(1 - \mathrm{e}^{-f(x)}\Bigr)\,\varphi^*\lambda(\mathrm{d}x)\right\} = L_{\pi_{\varphi^*\lambda}}[f],
\end{aligned}$$

and so $\pi_\lambda^\varphi = \pi_{\varphi^*\lambda}$. But, according to the Mapping Theorem (see Proposition 2.3), we have $\pi_{\varphi^*\lambda} = \varphi^*\pi_\lambda$, and formula (6.6) follows.

To check that $R^{\varphi}_{\pi_\lambda} \in L^2(\Gamma_X, \pi_\lambda)$, let us compute its L^2-norm:

$$\begin{aligned}\int_{\Gamma_X} |R^{\varphi}_{\pi_\lambda}(\gamma)|^2\, \pi_\lambda(\mathrm{d}\gamma) &= \exp\left\{\int_X \Big(1-\rho^{\varphi}_\lambda(x)\Big)\, \lambda(\mathrm{d}x)\right\} \cdot \int_{\Gamma_X} \mathrm{e}^{\langle 2\ln \rho^{\varphi}_\lambda, \gamma\rangle}\, \pi_\lambda(\mathrm{d}\gamma) \\ &= \exp\left\{\int_X \Big(1-\rho^{\varphi}_\lambda(x)\Big)\, \lambda(\mathrm{d}x)\right\} \cdot \exp\left\{-\int_X \Big(1-\mathrm{e}^{2\ln \rho^{\varphi}_\lambda(x)}\Big)\, \lambda(\mathrm{d}x)\right\} \\ &= \exp\left\{\int_X \Big(|\rho^{\varphi}_\lambda(x)|^2 - \rho^{\varphi}_\lambda(x)\Big)\, \lambda(\mathrm{d}x)\right\} < \infty,\end{aligned}$$

because $|\rho^{\varphi}_\lambda(x)|^2 - \rho^{\varphi}_\lambda(x) = 0$ outside the set $K = \operatorname{supp}\varphi$. $\square$

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