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# One dimensional stable probability density functions for rational index $0 < \alpha \leq 2$

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## Abstract

Fox's H-function provide a unified and elegant framework to tackle several physical phenomena. We solve the space fractional diffusion equation on the real line equipped with a delta distribution initial condition and identify the corresponding H-function by studying the small  $x$  expansion of the solution. The asymptotic expansions near zero and infinity are expressed, for rational values of the index  $\alpha$ , in terms of a finite series of generalized hypergeometric functions. In  $x$ -space, the  $\alpha = 1$  stable law is also derived by solving the anomalous diffusion equation with an appropriately chosen infinitesimal generator for time translations. We propose a new classification scheme of stable laws according to which a stable law is now characterized by a generating probability density function. Knowing this elementary probability density function and bearing in mind the infinitely divisible property we can reconstruct the corresponding stable law. Finally, using the asymptotic behavior of H-function in terms of hypergeometric functions we can compute closed expressions for the probability density functions depending on their parameters  $\alpha, \beta, c, \tau$ . Known cases are then reproduced and new probability density functions are presented. Stable symmetric laws; H-function; Hypergeometric functions; Fractional diffusion equation.  
subclass60E07; 33C60; 33C20.

## 1 INTRODUCTION

The notion of stable distributions was first introduced by Lévy [1] in the study of Generalized Central Limit Theorem, and there is a nice early account of the theory in [2]. A stable law is a direct generalization of the Gaussian distribution and in fact includes the Gaussian as a limiting case. The main difference between the stable and the Gaussian distributions is that the tails of the stable density are heavier than those of the Gaussian density. This characteristic is one of the main reasons why stable laws are suitable for modelling a plethora of phenomena such as laser cooling [3], turbulence [4], dynamical systems [5], statistical mechanics, signal processing [6], biology [7] and mathematical finance [8].

The goal of the present paper is to give analytic expressions for the probability density functions (p.d.f.'s) of the stable laws, in terms of known functions. These were lacking from the literature apart from a handful of well known ones. To accomplish our task we organize the paper as follows:

In *Section 2* we briefly review the definition of infinite divisible laws in terms of characteristic functions (or the Fourier transform of the probability measure) and limit our investigation to the

subclass of stable laws. We give the characteristic exponent for the general one-dimensional case and comment on the role and the essential properties of the parameters which are involved.

In *Section 3* we exploit the definition of the Fox's  $H$ -function as a Mellin-Barnes path integral and performing the integration on the appropriate contour. This allows us to write its asymptotic expansion at zero and infinity (see expressions (16) and (19)). The reason for choosing the specific values  $m = n = 1$  and  $p = q = 2$  for the  $H$ -function is its close relation to the solution of the free space fractional diffusion equation. Using the Gauss's multiplication formula as well as standard properties of the gamma function we resum the series for rational values of the index  $\alpha$  and produce general closed expressions containing the generalized hypergeometric functions. These expressions can be manipulated, for different values of the parameters, using a simple computer program running under Maple software.

In *Section 4* we solve the anomalous spatial diffusion equation on the real line with a fractional Laplacian consisting of Weyl derivatives and a Dirac delta distribution as initial condition. In this way we determine the most general form of the p.d.f. for a stable law. We establish the connection with the corresponding Fox function and find its asymptotics. Finally, we express them as finite sums of generalized hypergeometric functions.

In *Section 5* we demonstrate that the diffusion equation has as infinitesimal generator of time translations the spatial derivative of the convolution of two terms, which under Fourier transformation they reproduce the characteristic function of the  $\alpha = 1$  stable law. The corresponding integral cannot be performed exactly, unless  $\beta = 0$  in which case we recover the shifted Cauchy p.d.f. Only in the small  $x, \beta$  regime one can provide a triple series expansion.

In *Section 6* we establish a new way to classify stable laws by exploiting their infinitely divisible property. It is possible to write a formula that determines the p.f.d. of the stable law as an infinite limit of the  $m$ -fold convolution of a generating p.d.f. This expression although it gives new insight, from calculational view point is cumbersome due to its complexity.

In *Section 7* we present a sample of our results in the subdiffusion regime ( $\alpha < 1$ ) while  $\alpha$  takes values on the Farey series <sup>1</sup>  $\mathcal{F}_n$  of order  $n = 5$ . In the superdiffusion regime ( $\alpha > 1$ ) we recover all previously known results and also give new ones for general rational  $\alpha$ .

## 2 PRELIMINARIES ON $\alpha$ -STABLE LAWS

Consider a probability measure  $\mu$  on  $R^n$  and its characteristic function [9, 10, 11]

$$\hat{\mu}(p) = \mathcal{F}[\mu](p) = \int_{R^n} e^{i\langle p, x \rangle} \mu(dx), \quad p \in R^n. \quad (1)$$

A probability measure  $\mu$  on  $R^n$  is called *infinitely divisible* if

$$\forall m \in N, \exists \mu_m, \mathcal{F}[\mu_m] : \mathcal{F}[\mu](p) = (\mathcal{F}[\mu_m](p))^m$$

where  $\mu = \mu_m * \dots * \mu_m$  is the  $m$ -fold convolution of  $\mu_m$  with itself. If the measure  $\mu$  is infinitely divisible then there exists a unique continuous function  $\psi : R^n \rightarrow C$ , called the characteristic exponent of  $\mu$ , such that  $\psi(0) = 0$  and

$$\mathcal{F}[\mu](p) = e^{\Psi(p)}, \quad p \in R^n.$$

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<sup>1</sup>The Farey series  $\mathcal{F}_n$  of order  $n$  is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed  $n$ . Thus  $\alpha = \frac{p}{q}$  belongs in  $\mathcal{F}_n$  if

$$0 \leq p \leq q \leq n, \quad (p, q) = 1$$

where  $(,)$  denotes the highest common divisor of two integers.

The *Lévy-Khintchine representation* or *Lévy-Khintchine formula* states that a probability measure  $\mu$  on  $R^n$  is infinitely divisible iff we can write the characteristic exponent in the form

$$\psi(p) = i \langle p, \tau \rangle - \frac{1}{2} \langle p, Ap \rangle + \int_{R^n} \left( e^{i \langle p, x \rangle} - 1 - i \langle p, x \rangle 1_{|x| \leq 1} \right) \nu(dx) \quad (2)$$

where  $\tau \in R^n$ ,  $A$  is a symmetric nonnegative-definite  $n \times n$  matrix, called the *Gaussian covariance matrix*, and  $\nu$  is a  $\sigma$ -finite Borel measure on  $R_0^n := R^n / \{0\}$ , called the *Lévy measure*, such that

$$\int_{R_0^n} \min\{1, \|x\|^2\} \nu(dx) < \infty. \quad (3)$$

The triplet  $[\tau, A, \nu]$  is unique and will be called the generating triplet of the infinitely divisible probability measure  $\mu$ . If  $A = 0$  then  $\mu$  is said to be purely non-Gaussian.

A subclass of infinitely divisible laws is the stable laws class. Suppose that  $\mathbf{X}, \mathbf{X}_1, \dots, \mathbf{X}_m$  denote mutually independent random variables with a common distribution  $F$  and  $\mathbf{S}_m = \sum_{i=1}^m \mathbf{X}_i$ . The distribution  $F$  is stable if for each  $m \in N$  there exist constants  $c_m > 0$  and  $\tau_m \in R$  such that

$$\mathbf{S}_m \stackrel{d}{=} c_m \mathbf{X} + \tau_m \quad (4)$$

and  $F$  is not concentrated at one point.  $F$  is stable in the strict sense if  $\tau_m = 0$ . The symbol  $\stackrel{d}{=}$  means that the distributions of  $\mathbf{S}_m$  and  $\mathbf{X}$  are identical up to scale and location parameters. The norming constants are of the form  $c_m = m^{\frac{1}{\alpha}}$  with  $0 < \alpha \leq 2$  and the constant  $\alpha$  is called *characteristic exponent* of  $F$  or *index* of the stable law.

Let  $0 < \alpha < 2$  and  $\mu$  be an infinitely divisible and non-trivial on  $R^n$  probability measure with generating triplet  $[\tau, A, \nu]$ . If  $\mu$  is  $\alpha$ -stable then there is a finite non-zero measure  $\lambda$  on the unit sphere  $S = \{x \in R^n : |x| = 1\}$  such that

- (i)  $A = 0$  and  $\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{d\xi}{r^{1+\alpha}}$  for  $B \in \mathcal{B}(R^n)$ .
- (ii)  $\hat{\mu}(p) = \exp \left[ - \int_S | \langle p, \xi \rangle |^\alpha (1 - i \tan(\frac{\pi\alpha}{2})) \operatorname{sgn} \langle p, \xi \rangle \lambda(d\xi) + i \langle \tau, p \rangle \right]$  for  $\alpha \neq 1$  and  $\tau \in R^n$
- (iii)  $\hat{\mu}(p) = \exp \left[ - \int_S (| \langle p, \xi \rangle | + i \frac{2}{\pi} \langle p, \xi \rangle \ln | \langle p, \xi \rangle |) \lambda(d\xi) + i \langle \tau, p \rangle \right]$  for  $\alpha = 1$  and  $\tau \in R^n$ .

In the one-dimensional case ( $n = 1$ ) one can prove that the characteristic exponent has the form

$$\psi(p) = i\tau p - \begin{cases} c|p|^\alpha \left[ 1 - i\beta \operatorname{sgn}(p) \tan(\frac{\pi\alpha}{2}) \right] & \text{if } \alpha \neq 1, 2 \\ c|p| \left[ 1 + i\beta \frac{2}{\pi} \operatorname{sgn}(p) \ln(|p|) \right] & \text{if } \alpha = 1 \end{cases} \quad (5)$$

where

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}, \quad \text{and} \quad c = \frac{\pi}{2\Gamma(1+\alpha)} \frac{1}{\sin(\frac{\pi\alpha}{2})} (c_+ + c_-), \quad \text{for } \alpha \neq 2 \quad (6)$$

with  $c_+, c_- \geq 0$  and  $c_+ + c_- > 0$ <sup>2</sup>. Note that when  $\alpha = 2$  then  $\nu = 0$ .

The collection of the four parameters  $(\alpha, \beta, c, \tau)$  is called the *stable law parameters* and completely determines the distribution as follows:

<sup>2</sup>The same result can be recovered if one uses the absolutely continuous Lévy measure

$$\nu(dx) = (c_+ 1_{x>0} + c_- 1_{x<0}) |x|^{-1-\alpha} dx.$$

*Characteristic exponent  $\alpha$ .* This parameter determines the degree of leptokurtosis and the fatness of the tails. For a stable real-valued random variable  $X$  it can be shown that

$$E|X| < \infty \iff 1 < \alpha \leq 2.$$

When  $\alpha \leq 1$  the means becomes infinite. The variance for  $\alpha \in (0, 2)$  becomes infinite or undefined while all moments of a random variable  $X$  become finite iff  $\alpha = 2$ . Also the moments of order less than  $\alpha$  are positive and have a finite limit, namely  $E|X|^k < \infty$ ,  $0 < k < \alpha$ <sup>3</sup>.

*Skewness parameter  $\beta$ .* This parameter characterizes the degree of asymmetry of the Lévy measure and takes values in the interval  $[-1, 1]$ . The measure  $\nu$  is called symmetric if  $\beta = 0$  (or  $c_+ = c_-$ ) and the  $\alpha$ -stable distribution is called stable  $\alpha$ -symmetric<sup>4</sup>.

*Scale parameter  $c$ .* This parameter ranges into the interval  $(0, \infty)$  and measures scale in place of standard deviation.

*Location parameter  $\tau$ .* This parameter saturates the set of real numbers and shifts the distribution to the left or right. If  $1 < \alpha < 2$  then  $\tau$  equals to the mean of  $\mu$ . When  $0 < \alpha < 1$ , although, the mean is infinite it serves as an index of the location of the peak of the stable distribution and is identical to the drift of  $\mu$ .

A stable law generated by  $(\alpha, \beta, c, \tau)$  is often denoted by  $S_\alpha(\beta, c, \tau)$ . In the present work our law will be first generated by  $S_\alpha(c)$  and then we will study the most general case.

The parameter space  $(\alpha, b)$  of stable p.d.f.'s with the centering constant  $b$  restricted in the region [13]

$$|b| \leq \begin{cases} \alpha, & 0 < \alpha < 1 \\ 2 - \alpha, & 1 \leq \alpha \leq 2 \end{cases} \quad (7)$$

is depicted in the following figure

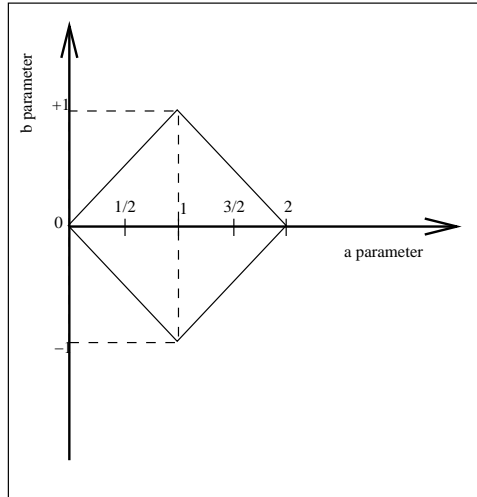


Figure 1: The parameter space of all stable p.d.f.'s on the  $\tau = 0$  plane. The points of the axis  $\beta = 0$  represent the stable  $\alpha$ -symmetric p.d.f.'s with the property  $f_{\alpha, \beta=0}(x) = f_{\alpha, \beta=0}(-x)$ . On this axis are located the familiar Cauchy ( $\alpha = 1$ ), Holtsmark ( $\alpha = \frac{3}{2}$ ) and Gaussian ( $\alpha = 2$ ) distributions.

<sup>3</sup>It is also true that a symmetric  $\alpha$ -stable random variable has finite negative-order moments  $-1 < k < 0$  [12].

<sup>4</sup>In general a measure  $\mu$  is symmetric when  $\mu(B) = \mu(-B)$  for  $B \in \mathcal{B}(R^n)$ . In  $n = 1$  the rotation invariance is tantamount to symmetry.

### 3 FOX'S $H$ -FUNCTION AND GENERALIZED HYPERGEOMETRIC FUNCTIONS FOR THE LAW $S_\alpha(c = K_\alpha)$ WITH RATIONAL $\alpha$

Fox [14, 15] defined the  $H$ -function in his studies of symmetrical Fourier kernels as the Mellin-Barnes path integral

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_1, A_1), (a_2, A_2), \dots, (a_p, A_p) \\ (b_1, B_1), (b_2, B_2), \dots, (b_q, B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{C}} \chi(s) z^s ds \quad (8)$$

where the integral density  $\chi(s)$  is given by

$$\chi(s) = \frac{\prod_{i=1}^m \Gamma(b_i - B_i s) \prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{i=m+1}^q \Gamma(1 - b_i + B_i s) \prod_{j=n+1}^p \Gamma(a_j - A_j s)}, \quad (9)$$

$m, n, p, q$  are integers satisfying

$$0 \leq n \leq p, \quad 0 \leq m \leq q,$$

$B_i, A_j$  are positive numbers and  $b_i, a_j$  are complex numbers such that

$$A_j(b_h + \nu) \neq B_h(a_j - 1 - \lambda), \quad \nu, \lambda = 0, 1, \dots; \quad h = 1, \dots, m; \quad j = 1, \dots, n.$$

This condition implies that the poles of  $\Gamma(b_i - B_i s)$  and  $\Gamma(1 - a_j + A_j s)$  form two disjoint sets.  $\mathcal{C}$  is a contour in the complex  $s$ -plane which runs from  $s = \infty - ik$  to  $s = \infty + ik$  with  $k > \frac{|\text{Im} b_j|}{B_j}$ ,  $j = 1, \dots, n$  and which encloses the poles

$$s = \frac{(b_i + \nu)}{B_i} \quad i = 1, \dots, m$$

but none of the poles

$$s = \frac{(a_j - 1 - \nu)}{A_j} \quad j = 1, \dots, n.$$

$H(z)$  makes sense and defines an analytic function of  $z$  in the following two cases:

(i) If

$$M = \sum_{i=1}^q B_i - \sum_{j=1}^p A_j > 0, \quad \forall z \neq 0 \quad (10)$$

(ii) If

$$M = 0 \quad \text{and} \quad 0 < |z| < R \quad \text{with} \quad R = \prod_{i=1}^q (B_i)^{B_i} \prod_{j=1}^p (A_j)^{-A_j}. \quad (11)$$

Schneider [16] has introduced into physics the Fox's  $H$ -function as analytic representations for the Lévy distributions in  $x$ -space, and as solutions of the fractional diffusion equation<sup>5</sup>. If one tries to solve the one-dimensional anomalous diffusion equation equipped with the following initial and boundary conditions<sup>6</sup>

$$\begin{aligned} \frac{\partial f(x, t)}{\partial t} &= K_\alpha \hat{\mathcal{A}}(\alpha) f(x, t), \quad x \in \mathbb{R}, t > 0, \alpha \in (0, 2) \\ \lim_{t \downarrow 0} f(x, t) &= \delta(x), \quad \lim_{|x| \rightarrow \infty} f(x, t) = 0 \end{aligned} \quad (12)$$

<sup>5</sup>In nature there is a diversity of diffusion processes for which the  $k$ -moment of the displacement grows not linearly with time but follows a power-law pattern of the form  $E|X(t)|^k \sim t^{\frac{k}{\alpha}}$ ,  $0 < k < \alpha$ .

<sup>6</sup>In equation (12)  $K_\alpha$  is the generalized diffusion constant having dimensions  $[L]^\alpha [T]^{-1}$  and  $\hat{\mathcal{A}}$  is the operator (33) with  $\beta = 0$  and  $\tau = 0$ . This is the generalization of the Laplacian to a fractional order.

then the solution (or propagator) reads

$$f(x, t; \alpha) = \frac{1}{\alpha|x|} H_{2,2}^{1,1} \left[ \frac{|x|}{(K_\alpha t)^{\frac{1}{\alpha}}} \left| \begin{matrix} (1, \frac{1}{\alpha}), & (1, \frac{1}{2}) \\ (1, 1), & (1, \frac{1}{2}) \end{matrix} \right. \right] \quad (13)$$

which expresses the  $\alpha$ -symmetric stable p.d.f. in terms of the Fox's  $H$ -function. Thus our study will be focused on the  $H$ -function

$$H(z) = H_{2,2}^{1,1} \left[ z \left| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{2}) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(1-s)\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})} z^s ds, \quad z = \frac{|x|}{(K_\alpha t)^{\frac{1}{\alpha}}}. \quad (14)$$

The simple poles of  $\Gamma(\frac{s}{\alpha})$  and  $\Gamma(1-s)$  are given by the disjoint sets of points

$$P(s) = \{s_\nu = -\alpha\nu, \quad \nu = 0, 1, \dots\}$$

$$Q(s) = \{s_\nu = 1 + \nu, \quad \nu = 0, 1, \dots\}$$

We distinguish the following two cases:

(i) Asymptotic expansion of  $H(z)$  near the point  $z = \infty$ . Applying the residue theorem clockwise we find

$$H(z) = \sum_{m=1}^{\infty} Res\{\chi(s)z^s; s_m \in P(s)\} + \frac{1}{2\pi i} \int_{\mathcal{C}_1} \chi(s)z^s ds - \frac{1}{2\pi i} \int_{\mathcal{C}_2} \chi(s)z^s ds \quad (15)$$

where the contour  $\mathcal{C}$  has been replaced by the rectilinear contours  $\mathcal{C}_1, \mathcal{C}_2$  running from  $\sigma \rightarrow \sigma + ik \rightarrow -\infty + ik$  and  $\sigma \rightarrow \sigma - ik \rightarrow -\infty - ik$  respectively. It can be proved that the error terms on the right-hand side of (15) vanish. In this case we obtain the algebraic asymptotic expansion

$$H_{2,2}^{1,1} \left[ z \left| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{2}) \end{matrix} \right. \right] = \frac{\alpha}{\pi} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\Gamma(1+m\alpha)}{\Gamma(m+1)} \sin(\frac{\pi}{2}m\alpha) z^{-m\alpha}. \quad (16)$$

It is worth noting that all the points of  $P(s)$  contribute in the large  $z$  limit. For the series (16), applying the ratio test, one has

$$\rho_1 = \lim_{m \rightarrow \infty} \left| \frac{\Gamma((m+1)\alpha) \sin(\frac{\pi}{2}(m+1)\alpha)}{m\Gamma(m\alpha) \sin(\frac{\pi}{2}m\alpha)} \right| \leq \lim_{m \rightarrow \infty} \left| \frac{\Gamma((m+1)\alpha)}{m\Gamma(m\alpha)} \right| = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } 1 < \alpha \leq 2. \end{cases} \quad (17)$$

Thus the series converges absolutely for every value of  $z \neq 0$  in the interval

$$(-R_1, R_1) = (-\infty, \infty) \quad \text{if } 0 < \alpha < 1. \quad (18)$$

(ii) Asymptotic expansion of  $H(z)$  near the point  $z = 0$ . The function  $H(z)$  is analytic in the  $s$ -plane for  $\alpha \in (1, 2]$  since then  $M = 1 - \frac{1}{\alpha} > 0$ ,  $\forall z \neq 0$ . Also for  $\alpha = 1$ ,  $M = 0$ , and  $H$  is analytic for  $0 < |z| < 1$ . In this case we find

$$\begin{aligned} H_{2,2}^{1,1} \left[ z \left| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{2}) \end{matrix} \right. \right] &= - \sum_{m=1}^{\infty} Res\{\chi(s)z^s; s_m \in Q(s)\} = \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\frac{2m+1}{\alpha})}{\Gamma(2m+1)} z^{2m+1} \\ &= \frac{\alpha}{\pi} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\Gamma(1+\frac{m}{\alpha})}{\Gamma(1+m)} \sin(\frac{\pi}{2}m) z^m. \end{aligned} \quad (19)$$

In this case the even numbers of  $Q(s)$  give a vanishing result and

$$\rho_2 = \lim_{m \rightarrow \infty} \left| \frac{\Gamma(\frac{2m+3}{\alpha})}{\Gamma(2m+3)} \right| = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ 0 & \text{if } 1 < \alpha \leq 2. \end{cases} \quad (20)$$

The series converges absolutely for every value of  $z$  in the intervals

$$(-R_2, R_2) = \begin{cases} (-1, 1) & \text{if } \alpha = 1 \\ (-\infty, \infty) & \text{if } 1 < \alpha \leq 2. \end{cases} \quad (21)$$

We assume now that  $\alpha$  is a positive rational number thus it can be written as:  $\alpha = \frac{p}{q}$ ,  $p, q \in \mathbb{Z}^+$  (the symbols  $p, q$  should not be confused with those used in the definition of the  $H$ -function). Relation (16), using the substitution  $m = nq + l$ ,  $l = 0, \dots, q-1$ , can be casted into the form

$$H(z) = -\frac{\alpha}{\pi} \sum_{l=0}^{q-1} e^{i\pi l} z^{-l\frac{p}{q}} \left( \sum_{n=0}^{\infty} \frac{\Gamma(np + \frac{lp}{q} + 1)}{\Gamma(nq + l + 1)} \sin\left(\frac{\pi}{2}\left(n + \frac{l}{q}\right)p\right) e^{i\pi nq} z^{-np} \right). \quad (22)$$

Using the multiplication theorem of Gauss <sup>7</sup> one can write the ratio of the gamma functions as

$$\frac{\Gamma(np + \frac{lp}{q} + 1)}{\Gamma(nq + l + 1)} = (2\pi)^{\frac{(q-p)}{2}} \sqrt{\frac{p}{q}} \left(\frac{p^p}{q^q}\right)^{n+\frac{l}{q}} \frac{\prod_{k=0}^{p-1} \Gamma(\frac{l}{q} + \frac{k+1}{p}) \prod_{k=0}^{p-1} (a_k)_n}{\prod_{s=0}^{q-1} \Gamma(\frac{l+s+1}{q}) \prod_{s=0}^{q-1} (b_s)_n} \quad (23)$$

where  $(a)_n$  is the Pochhammer's symbol  $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$  and

$$a_k = \frac{k+1}{p} + \frac{l}{q}, \quad b_s = \frac{l+s+1}{q}. \quad (24)$$

Combining (22) with (23) and assuming that  $p$  is odd, we obtain

$$\begin{aligned} H(z) &= -\frac{\alpha^{\frac{3}{2}}}{\pi} (2\pi)^{\frac{(q-p)}{2}} \sum_{l=0}^{q-1} \left(\frac{p^p e^{i\pi q}}{q^q z^p}\right)^{\frac{l}{q}} \frac{\prod_{k=0}^{p-1} \Gamma(a_k)}{\prod_{s=0}^{q-1} \Gamma(b_s)} \left( \sin\left(\frac{\pi lp}{2q}\right) \sum_{m=0}^{\infty} \frac{\prod_{k=0}^{p-1} (a_k)_{2m}}{\prod_{s=0}^{q-1} (b_s)_{2m}} \left(\frac{p^p e^{i\pi(q+\frac{p}{2})}}{q^q z^p}\right)^{2m} \right. \\ &+ \left. e^{-i\frac{\pi}{2}p} \cos\left(\frac{\pi lp}{2q}\right) \sin\left(\frac{\pi}{2}p\right) \sum_{m=0}^{\infty} \frac{\prod_{k=0}^{p-1} (a_k)_{2m+1}}{\prod_{s=0}^{q-1} (b_s)_{2m+1}} \left(\frac{p^p e^{i\pi(q+\frac{p}{2})}}{q^q z^p}\right)^{2m+1} \right) \\ &= -\frac{\alpha^{\frac{3}{2}}}{\pi} (2\pi)^{\frac{(q-p)}{2}} \sum_{l=0}^{q-1} \left(\frac{p^p e^{i\pi q}}{q^q z^p}\right)^{\frac{l}{q}} \frac{\prod_{k=0}^{p-1} \Gamma(a_k)}{\prod_{s=0}^{q-1} \Gamma(b_s)} \left( \sin\left(\frac{\pi lp}{2q}\right) + e^{-i\frac{\pi}{2}p} \cos\left(\frac{\pi lp}{2q}\right) \sin\left(\frac{\pi}{2}p\right) \left(\frac{p^p e^{i\pi(q+\frac{p}{2})}}{q^q z^p}\right) \right) \\ &\times {}_{p+1}F_q \left( 1, a_0, \dots, a_{p-1}; b_0, \dots, b_{q-1}; \left(\frac{p^p e^{i\pi(q+\frac{p}{2})}}{q^q z^p}\right)^2 \right). \quad (25) \end{aligned}$$

The same result holds for  $p$  even and the only difference is the swapping of the two terms in the parentheses. Of course the hypergeometric series converges absolutely when  $0 < \alpha \leq 1 - \frac{1}{q}$  which implies that  $\alpha \in (0, 1)$ .

For the case  $1 \leq \alpha \leq 2$ , following similar steps, we can write the ratio of gamma functions as

$$\frac{\Gamma(\frac{q(2m+1)}{p})}{\Gamma(2m+1)} = \frac{(2\pi)^{p-q}}{2^{(2l+1)(1-\frac{q}{p})}} \left(\frac{p}{q}\right)^{\frac{1}{2}} \left(\frac{q^q}{p^p}\right)^{2l+1} \left(\frac{(2q)^{2qn}}{(2p)^{2pn}}\right) \frac{\prod_{k=1}^{2q-1} \Gamma(a_k) \prod_{k=1}^{2q-1} (a_k)_n}{\prod_{s=1}^{2p-1} \Gamma(b_s) \prod_{s=1}^{2p-1} (b_s)_n} \quad (26)$$

<sup>7</sup>The multiplication theorem of Gauss states that  $\Gamma(mz)(2\pi)^{(m-1)/2} = m^{mz-1/2} \Gamma(z) \Gamma(z+\frac{1}{m}) \dots \Gamma(z+\frac{m-1}{m})$ ,  $\forall m \in \mathbb{N}$ .



where  $a_k = \frac{1}{2p}(2l+1) + \frac{k}{2q}$  and  $b_s = \frac{1}{2p}(2l+1) + \frac{s}{2p}$ . The relation (19) then becomes

$$\begin{aligned} H(z) &= z(2\pi)^{p-q-1} 2^{\frac{q}{p}} \frac{q^{\frac{q}{p}-\frac{1}{2}}}{p^{\frac{1}{2}}} \sum_{l=0}^{p-1} (-1)^l \left( \frac{q^q z^p}{p^p 2^{p-q}} \right)^{\frac{2l}{p}} \frac{\prod_{k=1}^{2q-1} \Gamma(a_k)}{\prod_{s=1}^{2p-1} \Gamma(b_s)} \\ &\times {}_{2q}F_{2p-1} \left( 1, a_1, \dots, a_{2q-1}; b_1, \dots, b_{2p-1}; \frac{(2q)^{2q}}{(2p)^{2p}} e^{i\pi p} z^{2p} \right). \end{aligned} \quad (27)$$

The hypergeometric series  ${}_{2q}F_{2p-1}$  converges absolutely if  $2q \leq 2p-1$  or equivalently when

$$\alpha \geq 1 + \frac{1}{2q}, \quad q \in Z^+ \quad (28)$$

which implies that  $\alpha \in (1, 2]$ . It also converges when  $2q = (2p-1) + 1 \Rightarrow \alpha = 1$  provided that  $|z| < 1$ .

#### 4 FOX'S H-FUNCTION FOR THE LAW $S_\alpha(\beta, c = K_\alpha, \tau)$ , $\alpha \in (0, 2]$ AND $\alpha \neq 1$

We consider the generalized one-dimensional anomalous and anisotropic diffusion problem with the following initial and boundary conditions

$$\begin{aligned} \frac{\partial f(x, t)}{\partial t} &= -\frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)} (K^- {}_{-\infty}\mathcal{D}_x^\alpha + K^+ {}_x\mathcal{D}_\infty^\alpha) f(x, t) - \tau \frac{\partial f(x, t)}{\partial x} \equiv \hat{\mathcal{A}}(\alpha, \beta, \tau) f(x, t), \\ x \in R, t \in (0, \infty), \quad \lim_{t \downarrow 0} f(x, t) &= \delta(x), \quad \lim_{|x| \rightarrow \infty} f(x, t) = 0 \end{aligned} \quad (29)$$

where  $K^\pm$  are diffusion constants satisfying  $K^\pm \geq 0, K^+ + K^- > 0$  and the dispersion term is proportional to the constant  $\tau$  having dimensions  $[L]/[T]$ <sup>8</sup>. Also by definition [21, 22]

$$({}_{-\infty}\mathcal{D}_x^\alpha + {}_x\mathcal{D}_\infty^\alpha) f(x, t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(m-\alpha)} \left( \frac{\partial}{\partial x} \right)^m \left( \int_{-\infty}^x \frac{f(y, t)}{(x-y)^{\alpha-m+1}} dy + (-1)^m \int_x^\infty \frac{f(y, t)}{(y-x)^{\alpha-m+1}} dy \right) \quad (30)$$

are the nonlocal fractional left-handed (right-handed) Weyl derivatives. In (30)  $m = [\alpha] + 1$  with  $[\alpha]$  representing the integral part of  $\alpha$ . Note that when  $\alpha$  is an even integer then the two derivatives are localized and equal while for odd integer values of  $\alpha$  both derivatives appear opposite in signs.

The Fourier transform of  $\hat{\mathcal{A}}(\alpha, \beta) f(x, t)$  for fixed  $t \in (0, \infty)$  and  $f \in \mathcal{S}(R)$ <sup>9</sup> is given by (see proof at Appendix A)

$$\mathcal{F} \left[ \hat{\mathcal{A}}(\alpha, \beta) f \right] (q, t) = \left[ -K_\alpha |q|^\alpha \left( 1 - i\beta \text{sign}(q) \tan\left(\frac{\alpha\pi}{2}\right) \right) + i\tau q \right] \hat{f}(q, t) \quad (31)$$

where

$$K_\alpha = K^+ + K^- > 0, \quad \beta = \frac{K^+ - K^-}{K^- + K^+} \in [-1, 1], \quad \tau \in R. \quad (32)$$

The operator  $\hat{\mathcal{A}}(\alpha, \beta)$  could be casted into the equivalent form

$$\hat{\mathcal{A}}(\alpha, \beta) = -\frac{K_\alpha}{2 \cos\left(\frac{\alpha\pi}{2}\right)} ((1-\beta) {}_{-\infty}\mathcal{D}_x^\alpha + (1+\beta) {}_x\mathcal{D}_\infty^\alpha) - \tau \frac{\partial}{\partial x}. \quad (33)$$

<sup>8</sup>One might wonder if it is legitimate to add a Laplacian term to the operator  $\hat{\mathcal{A}}(\alpha, \beta)$ . This suggestion is prohibited by the fact that we consider only  $\alpha$ -stable laws.

<sup>9</sup> $\mathcal{S}(R)$  is the set of all infinitely differentiable and rapidly decreasing functions on  $R$ , namely  $\sup_{x \in R} |x^n (D^m f)(x)| < \infty, \forall m, n = 0, 1, \dots$ . This space is usually called the Schwartz space.

The  $\hat{f}(q, t)$  satisfies the Fourier transformed initial value problem

$$\begin{aligned}\frac{\partial \hat{f}(q, t)}{\partial t} &= \left[ -K_\alpha |q|^\alpha \left( 1 - i\beta \text{sign}(q) \tan\left(\frac{\alpha\pi}{2}\right) \right) + i\tau q \right] \hat{f}(q, t) \\ \hat{f}(q, 0) &= 1.\end{aligned}\quad (34)$$

with solution

$$\hat{f}(q, t) = e^{-K_\alpha |q|^\alpha t (1 - i\beta \text{sign}(q) \tan(\frac{\alpha\pi}{2})) + i\tau q t}.\quad (35)$$

The propagator is thus given by

$$f(x, t) = \mathcal{F}^{-1}[\hat{f}](x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iq(x-\tau t)} e^{-K_\alpha |q|^\alpha (1 - i\beta \text{sign}(q) \tan(\frac{\alpha\pi}{2}))} dq.\quad (36)$$

We will first study the small  $x - \tau t$  expansion. We expand  $\cos(qx), \sin(qx)$  in finite Taylor series

$$\begin{aligned}\cos(qx) &= \sum_{n=0}^m \frac{(-q^2 x^2)^n}{(2n)!} + \frac{g(qx)(qx)^{2m+1}}{(2m+1)!}, \\ \sin(qx) &= \sum_{n=0}^m \frac{(-1)^n (qx)^{2n+1}}{(2n+1)!} + \frac{h(qx)(qx)^{2m+2}}{(2m+2)!}\end{aligned}\quad (37)$$

where, by the generalized mean value theorem, the functions  $g(qx), h(qx)$  are bounded by the extreme values of the  $(2m+1)$ th ( $(2m+2)$ th) derivative of the cosine (sine), thus  $|g(qx)|, |h(qx)| < 1$ . By substituting these series into (36), integrating term-by-term and taking the limit  $m \rightarrow \infty$  we obtain the complete asymptotic expansion<sup>10</sup>

$$f(x, t; \alpha, \beta, \tau) = \frac{1}{\pi z (K_\alpha t)^{\frac{1}{\alpha}} (1 + \gamma^2)^{\frac{1}{2\alpha}}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(1 + \frac{n}{\alpha})}{\Gamma(n+1)} \sin\left(\frac{n\pi}{2}\delta\right) z^n\quad (38)$$

where

$$\gamma = \beta \tan\left(\frac{\alpha\pi}{2}\right), \quad \delta = 1 + \frac{2}{\alpha\pi} \arctan \gamma, \quad \text{and} \quad z = \frac{|x - \tau t|}{(K_\alpha t)^{\frac{1}{\alpha}} (1 + \gamma^2)^{\frac{1}{2\alpha}}}.\quad (39)$$

It is evident from (39) that  $\delta \in [0, 2]$  and can acquire the three integer values  $\{0, 1, 2\}$  provided that  $\beta \in \{-1, 0, 1\}$ . The p.d.f. (38) can be reproduced by the following H-function

$$H(z) = H_{2,2}^{1,1} \left[ z \left| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{\delta}{2}) \\ (1, 1) & (1, \frac{\delta}{2}) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_C \frac{\Gamma(1-s)\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{\delta s}{2})\Gamma(1-\frac{\delta s}{2})} z^s ds, \quad z = \frac{|x - \tau t|}{(K_\alpha t)^{\frac{1}{\alpha}} (1 + \gamma^2)^{\frac{1}{2\alpha}}}\quad (40)$$

with asymptotic expansion near the point  $z = 0$  given by

$$H(z) = \frac{\alpha}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(1 + \frac{n}{\alpha})}{\Gamma(n+1)} \sin\left(\frac{n\pi}{2}\delta\right) z^n.\quad (41)$$

In contrast to (19) all the points of  $Q(s)$  now contribute. The asymptotic expansion of  $H(z)$  near the point  $z = \infty$  is

$$H(z) = \frac{\alpha}{\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\Gamma(1 + n\alpha)}{\Gamma(1+n)} \sin\left(\frac{\pi n\alpha}{2}\delta\right) |z|^{-n\alpha}.\quad (42)$$

<sup>10</sup>The derivation of this expression is based on the integral formula

$$\int_0^{\infty} x^{\mu-1} e^{-\nu x} \sin(ax) dx = \frac{\Gamma(\mu)}{(\nu^2 + a^2)^{\frac{\mu}{2}}} \sin\left(\mu \arctan \frac{a}{\nu}\right), \quad \text{Re}\mu > -1, \text{Re}\nu > |\text{Im}a|$$

and a similar result for the cosine, provided that  $\text{Re}\mu > 0, \text{Re}\nu > |\text{Im}a|$ .

For the symmetric  $\beta = 0$  case we recover expressions (16) and (19). When  $\alpha$  is rational we can also express (41) and (42) in terms of hypergeometric functions as follows

$$\begin{aligned}
H(z) &= -\sqrt{\frac{p}{q}}(2\pi)^{\frac{p-q}{2}-1} \sum_{l=0}^{p-1} \left( \frac{q^{\frac{q}{p}} e^{i\pi} z}{p} \right)^l \frac{\prod_{k=1}^{q-1} \Gamma(a_k)}{\prod_{s=1}^{p-1} \Gamma(b_s)} \\
&\times \left[ e^{i\frac{\pi}{2}(l\delta-1)} {}_qF_{p-1} \left( 1, a_1, \dots, a_{q-1}; b_1, \dots, b_{p-1}; \frac{e^{i\pi p(1+\frac{\delta}{2})} q^q z^p}{p^p} \right) \right. \\
&\left. - e^{-i\frac{\pi}{2}(l\delta+1)} {}_qF_{p-1} \left( 1, a_1, \dots, a_{q-1}; b_1, \dots, b_{p-1}; \frac{e^{i\pi p(1-\frac{\delta}{2})} q^q z^p}{p^p} \right) \right] \quad (43)
\end{aligned}$$

with  $a_k = \frac{l}{p} + \frac{k}{q}$  and  $b_s = \frac{l}{p} + \frac{s}{p}$ . Similarly for the large  $z$  expansion we obtain

$$\begin{aligned}
H(z) &= -\left(\frac{p}{q}\right)^{\frac{3}{2}} (2\pi)^{\frac{q-p}{2}-1} \sum_{l=0}^{q-1} \left( \frac{p^p e^{i\pi} z}{q^q |z|^p} \right)^{\frac{l}{q}} \frac{\prod_{k=0}^{p-1} \Gamma(a_k)}{\prod_{s=0}^{q-1} \Gamma(b_s)} \\
&\times \left[ e^{i\frac{\pi}{2}(\frac{lp}{q}\delta-1)} {}_{p+1}F_q \left( 1, a_0, \dots, a_{p-1}; b_0, \dots, b_{q-1}; \frac{e^{i\pi(q+\frac{p\delta}{2})} p^p}{q^q |z|^p} \right) \right. \\
&\left. - e^{-i\frac{\pi}{2}(\frac{lp}{q}\delta+1)} {}_{p+1}F_q \left( 1, a_0, \dots, a_{p-1}; b_0, \dots, b_{q-1}; \frac{e^{i\pi(q-\frac{p\delta}{2})} p^p}{q^q |z|^p} \right) \right] \quad (44)
\end{aligned}$$

where  $a_k, b_s$  are given by (24).

## 5 THE LAW $S_{\alpha=1}(\beta, c = K, \tau)$

The operator we consider in this case is

$$\mathcal{A}(\alpha = 1, \beta) f(x, t) = -\frac{d}{dx} (K_1(s + \beta h) * f(t))(x) \quad (45)$$

where

$$s(x) = \frac{1}{2\pi^2 x}, \quad h(x) = \frac{1}{2\pi^2} \left( \frac{1}{|x|} + 2C_\gamma \delta(x) \right). \quad (46)$$

The first convolution is the Hilbert transform of the function  $f$  defined by

$$Hf(x, t) = (s * f(t))(x) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} f(y, t) s(x-y) dt = \frac{1}{2\pi^2} P.V. \left( \int_{-\infty}^{\infty} \frac{f(y, t)}{x-y} dy \right) \quad (47)$$

with  $P.V.$  representing the Cauchy principal value of the integral. The Fourier transform <sup>11</sup> of (47) is

$$\mathcal{F}[Hf](x) = \mathcal{F}[s](x) \mathcal{F}[f](x) = \frac{i}{2\pi} \text{sign}(q) \hat{f}(q, t). \quad (48)$$

The second convolution term in (45) has Fourier transform (see Appendix B for the proof)

$$\mathcal{F}[h * f](x) = \mathcal{F}[h](x) \mathcal{F}[f](x) = -\frac{1}{\pi^2} \ln(|q|) f(q, t). \quad (49)$$

<sup>11</sup>We have absorbed the coefficients of the Fourier transform into the exponential thus defining

$$\mathcal{F}[f] = \hat{f}(p) = \int_{-\infty}^{\infty} e^{2i\pi p x} f(x) dx, \quad \mathcal{F}^{-1}[\hat{f}] = f(x) = \int_{-\infty}^{\infty} e^{-2i\pi p x} \hat{f}(p) dp.$$

Hence, the initial value problem is equivalent to the Fourier transformed

$$\begin{aligned}\frac{\partial \hat{f}(q, t)}{\partial t} &= \left[ -K|q| \left( 1 + i\beta \frac{2}{\pi} \text{sign}(q) \ln(|q|) \right) + 2i\pi q\tau \right] \hat{f}(q, t) \\ \hat{f}(q, 0) &= 1\end{aligned}\tag{50}$$

with solution

$$\hat{f}(q, t) = e^{-K|q|t(1+i\beta\frac{2}{\pi}\text{sign}(q)\ln(|q|))+2i\pi q\tau t}.\tag{51}$$

The propagator is then given by the absolutely convergent integral

$$f(x, t) = \int_{-\infty}^{\infty} e^{-2i\pi q(x-\tau t)} e^{-K|q|t(1+i\beta\frac{2}{\pi}\text{sign}(q)\ln(|q|))} dq.\tag{52}$$

If we set  $\beta = 0$  in (52) we recover the shifted Cauchy p.f.d.

$$f(x, t) = \frac{2K_1 t}{(K_1 t)^2 + 4\pi^2(x - \tau t)^2}.\tag{53}$$

## 6 AN ALTERNATIVE WAY OF CLASSIFYING STABLE LAWS

The symbol of the operator (33) is given by

$$\eta(q) = -K_\alpha |q|^\alpha (1 - i\text{sign}(q)\gamma) + i\tau q.\tag{54}$$

From (54) the real part  $\text{Re}(\eta(q)) \leq 0$ ,  $\forall q \in R$ , thus we define  $h_\lambda : R \rightarrow C$ ,  $\lambda > 0$  by

$$h_\lambda(q) = \mathcal{L}[e^{t\eta(q)}](\lambda) = \int_0^\infty e^{-t\lambda} e^{t\eta(q)} dt = \frac{1}{\lambda - \eta(q)}\tag{55}$$

which is positive definite. The mapping  $q \rightarrow h_\lambda(q)$  is continuous and applying Bochner's theorem there exists a finite measure on  $\mathcal{B}(R)$  such that

$$h_\lambda(q) = \widehat{\mu_\lambda}(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iqx} \mu_\lambda(dx).\tag{56}$$

Hence,

$$\mu_\lambda(x) = \int_{-\infty}^{\infty} e^{iqx} h_\lambda(q) dq.\tag{57}$$

It can be shown that the operator  $\hat{\mathcal{A}}(\alpha, \beta, \tau)$  is the infinitesimal generator of a strongly continuous semigroup of operators and its resolvent  $\mathcal{R}_\lambda(\hat{\mathcal{A}})$  satisfies [11]

$$\mathcal{R}_\lambda(\hat{\mathcal{A}})\psi = \mu_\lambda * \psi,\tag{58}$$

or equivalently

$$\left( I - \frac{1}{\lambda} \hat{\mathcal{A}} \right)^{-1} \psi = \lambda \mu_\lambda * \psi.\tag{59}$$

By applying on the righthand side of (59)  $m$ -times the operator  $\left( I - \frac{1}{\lambda} \hat{\mathcal{A}} \right)^{-1}$ , setting  $\lambda = \frac{m}{t}$  and taking the limit  $m \rightarrow \infty$  we have

$$\begin{aligned}\lim_{m \rightarrow \infty} \left( I - \frac{t}{m} \hat{\mathcal{A}} \right)^{-m} \psi &= e^{-t\hat{\mathcal{A}}}\psi = \lim_{m \rightarrow \infty} \left( \left( \frac{m}{t} \right)^m \underbrace{\mu_{m/t} * \dots * \mu_{m/t}}_{m\text{-times}} \right) * \psi \\ &= f * \psi\end{aligned}\tag{60}$$

where  $f$  is the p.d.f. of the stable law  $S_\alpha(\beta, K_\alpha, \tau)$ . Thus, a stable law can be determined by  $\mu_\lambda$  instead of  $f$ , using relations (55) and (57). As an example we consider the law  $S_2(0, K_2, 0)$ . A simple calculation gives the p.d.f.

$$\mu_\lambda(x) = \frac{\pi}{2\sqrt{K_2\lambda}} e^{-\sqrt{\frac{\lambda}{K_2}}x}, \quad x > 0. \quad (61)$$

Although (57) seems to be elegant it is unattractive for calculations since the corresponding convergent integral only exceptionally gives a closed expression.

## 7 KNOWN AND UNKNOWN RESULTS

Expressions (25), (27), (43) and (44) can be used to calculate p.d.f.'s for arbitrary rational values of  $\alpha \in (0, 2]$  in terms of generalized hypergeometric functions. In particular, a sample of stable symmetric p.d.f.'s is given in Table 1 for the subdiffusion regime with  $\alpha \in (0, 1)$  and in Table 2 for the superdiffusion regime with  $\alpha \in (1, 2]$ . Previous known results are reproduced and new ones are also presented.

### ( $\alpha$ ) The law $S_{1/2}(0, K_{1/2}, 0)$

Let us first consider the particular case  $(p, q) = (1, 2)$  that gives the index value  $\alpha = 1/2$ . From equation (25) one can easily show that

$$\begin{aligned} H(z) &= -\frac{1}{2\sqrt{\pi}} \sum_{l=0}^{\infty} \left( \frac{e^{i\pi}}{2\sqrt{z}} \right)^l \left( \sum_{n=0}^{\infty} \frac{\sin(\frac{\pi}{2}(n + \frac{l}{2}))}{\Gamma(n + \frac{(l+1)}{2})} \frac{1}{(4z)^n} \right) \\ &= -\frac{1}{4\pi z} \left[ \cos\left(\frac{1}{4z}\right) {}_1F_2\left(\frac{1}{4}; \frac{1}{2}, \frac{5}{4}, -\frac{1}{64z^2}\right) + \frac{1}{12z} \sin\left(\frac{1}{4z}\right) {}_1F_2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}, -\frac{1}{64z^2}\right) \right] \\ &\quad + \frac{\sqrt{2}}{8\sqrt{\pi z}} \left[ \frac{1}{4z} {}_0F_1\left(\frac{3}{2}; -\frac{1}{64z^2}\right) + {}_0F_1\left(\frac{1}{2}; -\frac{1}{64z^2}\right) \right] \\ &= -\frac{1}{2\sqrt{2\pi|z|}} \left[ \cos\left(\frac{1}{4z}\right) C\left(\frac{1}{\sqrt{2\pi z}}\right) + \sin\left(\frac{1}{4z}\right) S\left(\frac{1}{\sqrt{2\pi z}}\right) \right] \\ &\quad + \frac{1}{4\sqrt{2\pi z}} \left[ \cos\left(\frac{1}{4z}\right) + \sin\left(\frac{1}{4z}\right) \right]. \end{aligned} \quad (62)$$

Where  $C(x), S(x)$  are the cosine and sine Fresnel integrals given by

$$\begin{aligned} C(x) &= \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt = x {}_1F_2\left(\frac{1}{4}; \frac{1}{2}, \frac{5}{4}, -\frac{\pi^2}{16}x^4\right) \\ &= x \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})^2} \int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{3}{4}} {}_0F_1\left(\frac{5}{4}, -\frac{\pi^2}{16}x^4t\right)dt \\ S(x) &= \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt = \frac{\pi}{6}x^3 {}_1F_2\left(\frac{3}{4}; \frac{3}{2}, \frac{7}{4}, -\frac{\pi^2}{16}x^4\right) \\ &= x \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4})^2} \int_0^1 t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}} {}_0F_1\left(\frac{7}{4}, -\frac{\pi^2}{16}x^4t\right)dt \end{aligned} \quad (63)$$

which are odd functions of  $x$ . The function  $f(z) = \frac{1}{\alpha z} H(z)$  is indeed a p.d.f. since it is positive definite  $\forall z \in (-\infty, \infty)$ , integrable and satisfies

$$\int_{-\infty}^{\infty} f(z)dz = 1. \quad (64)$$

The more general one-sided law  $S_{1/2}(1, K_{1/2}, \tau)$  is found to correspond to the p.d.f.

$$f(z) = \frac{1}{2z^{\frac{3}{2}}\sqrt{\pi}} e^{-\frac{1}{4z}}, \quad z > 0. \quad (65)$$

The case  $\beta = -1$  gives a vanishing p.d.f. since  $\delta = 0$ . This result holds independently of the value of  $\alpha$ .

**(\beta) The Cauchy law  $S_1(\mathbf{0}, \mathbf{K}_1, \mathbf{0})$**

Setting in (27),  $p = q$ , yields

$$\begin{aligned} H(z) &= \frac{z}{\pi} {}_1F_0(1; 0; (-z^2)^q) \sum_{l=0}^{q-1} (-1)^l z^{2l} \\ &= \frac{z}{\pi} \left( \frac{1 - (-z^2)^q}{1 + z^2} \right) {}_1F_0(1; 0; (-z^2)^q). \end{aligned} \quad (66)$$

When  $q = 1$  this corresponds to the Cauchy p.d.f.

$$f(z) = \frac{1}{\alpha z} H(z) = \frac{1}{\pi} \frac{1}{(1 + z^2)}, \quad z \in (-\infty, \infty). \quad (67)$$

with applications, e.g. in spectroscopy. For the general law  $S_1(0, K_1, \tau)$  the p.d.f. is given by (53).

**(\gamma) The Holtsmark law  $S_{3/2}(\mathbf{0}, \mathbf{K}_{3/2}, \mathbf{0})$**

In this case we set  $(p, q) = (3, 2)$  in (27) and arrive at

$$\begin{aligned} H(z) &= \frac{z 2^{\frac{5}{6}}}{\sqrt{3}} \sum_{l=0}^2 (-1)^l \left( \frac{2}{27} z^3 \right)^{\frac{2l}{3}} \frac{\prod_{k=1}^3 \Gamma(\frac{1}{6}(2l+1) + \frac{k}{4})}{\prod_{s=1}^5 \Gamma(\frac{1}{6}(2l+1) + \frac{s}{6})} \\ &\times {}_4F_5 \left( 1, \frac{1}{6}(2l+1) + \frac{1}{4}, \dots, \frac{1}{6}(2l+1) + \frac{3}{4}; \frac{1}{6}(2l+1) + \frac{1}{6}, \dots, \frac{1}{6}(2l+1) + \frac{5}{6} \right. \\ &\left. ; \frac{4^4}{6^6} e^{3i\pi} z^6 \right) \\ &= \frac{z}{\pi} \left[ \Gamma\left(\frac{2}{3}\right) {}_2F_3\left(\frac{5}{12}, \frac{11}{12}; \frac{1}{3}, \frac{1}{2}, \frac{5}{6}; -\frac{4}{729} z^6\right) - \frac{z^2}{2} {}_3F_4\left(1, \frac{3}{4}, \frac{5}{4}; \frac{5}{6}, \frac{2}{3}, \frac{4}{3}, \frac{7}{6}; -\frac{4}{729} z^6\right) \right. \\ &\left. + \frac{14\sqrt{3}}{486} \frac{\pi}{\Gamma(\frac{2}{3})} z^4 {}_2F_3\left(\frac{19}{12}, \frac{13}{12}; \frac{3}{2}, \frac{7}{6}, \frac{5}{3}; -\frac{4}{729} z^6\right) \right]. \end{aligned} \quad (68)$$

The corresponding p.d.f. was discovered in physics by the Danish astronomer Holtsmark in 1919 [24, 25]. It was the outcome of his efforts to study the stationary distribution of the force acting on a star, per unit mass, due to the gravitational attraction of the neighboring stars.

**(\delta) The Gaussian law  $S_2(\mathbf{1}, \mathbf{K}_2, \tau)$**

Substituting  $p = 2q$  in (27) we get

$$\begin{aligned} H(z) &= \frac{z(2\pi)^{q-1}}{\sqrt{q}} \sum_{l=0}^{2q-1} (-1)^l \left( \frac{z^2}{8q} \right)^l \frac{\prod_{k=1}^{2q-1} \Gamma(\frac{2(l+k)+1}{4q})}{\prod_{s=1}^{4q-1} \Gamma(\frac{2l+s+1}{4q})} \\ &\times {}_{2q}F_{4q-1} \left( 1, a_1, \dots, a_{2q-1}; b_1, \dots, b_{2q-1}; \frac{(2q)^{2q}}{(4q)^{4q}} e^{2i\pi q} z^{4q} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{z(2\pi)^{q-1}}{\sqrt{q}} \sum_{l=0}^{2q-1} (-1)^l \left(\frac{z^2}{8q}\right)^l \frac{1}{\prod_{s=0}^{2q-1} \Gamma\left(\frac{l+s+1}{2q}\right)} \\
&\times {}_2qF_{4q-1}\left(1, a_1, \dots, a_{2q-1}; b_1, \dots, b_{2q-1}; \frac{1}{(4^3 q^2)^q} z^{4q}\right)
\end{aligned} \tag{69}$$

where  $a_k = \frac{2l+1}{4q} + \frac{k}{2q}$  and  $b_s = \frac{2l+1}{2p} + \frac{s}{2p}$ . Setting  $q = 1$  in (69) we have

$$\begin{aligned}
H(z) &= \frac{z}{\sqrt{\pi}} \left[ {}_0F_1\left(\frac{1}{2}; \frac{z^4}{64}\right) - \frac{z^2}{4} {}_0F_1\left(\frac{3}{2}; \frac{z^4}{64}\right) \right] = \frac{z}{\sqrt{\pi}} \left( \cosh\left(\frac{z^2}{4}\right) - \sinh\left(\frac{z^2}{4}\right) \right) \\
&= \frac{z}{\sqrt{\pi}} e^{-\frac{z^2}{4}}.
\end{aligned} \tag{70}$$

The associated p.d.f. is

$$f(z) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}}$$

and changing variable to  $z = \frac{|x-\tau t|}{\sqrt{K_2 t}}$  we recover the traditional one

$$f(x, t) = \frac{1}{\sqrt{4\pi K_2 t}} e^{-\frac{(x-\tau t)^2}{4K_2 t}}.$$

## 8 CONCLUSIONS

In this article we presented analytic expressions for the  $\alpha$ -stable p.d.f.'s for rational values of the index  $\alpha \in (0, 2]$ . These p.d.f.'s can be viewed as solutions of the spatial anomalous diffusion equation subjected to a Dirac delta initial condition. We established their connection to the Fox's  $H$ -function for the most general law  $S_\alpha(\beta, K_\alpha, \tau)$ . The characteristic function of the  $\alpha = 1$  stable law was also reproduced by solving a suitably chosen fractional diffusion equation. An alternative way of classification which captures the infinite divisible character of stable laws was proposed. The rationality of the index allows us to write closed expressions for the p.d.f.'s in terms of generalized hypergeometric functions. This method recovers known results, such as the Cauchy p.d.f. for  $\alpha = 1$ , the Holtmark p.d.f. for  $\alpha = \frac{3}{2}$  and the normal p.d.f. for  $\alpha = 2$ . When  $\alpha$  takes an arbitrary rational value in  $(0, 2]$  new p.d.f.'s are derived generalizing and unifying previous results.

## Appendix A

The proof of (31) is straightforward provided that we first show that

(i) If  $f$  and  $g$  belong to the space  $AC^{[\alpha]}(R)$ <sup>12</sup> with boundary conditions  $\lim_{|x| \rightarrow \infty} f^{(k)}(x) = 0 = \lim_{|x| \rightarrow \infty} g^{(k)}(x)$ ,  $k = 0, \dots, [\alpha] - 1$  then

$$\int_I (-\infty \mathcal{D}_x^\alpha f(x)) g(x) dx = \int_I f(x) ({}_x \mathcal{D}_\infty^\alpha g(x)) dx \quad (\text{A.1})$$

where  $I = (-\infty, \infty)$ .

*Proof of (A.1)*

$$\begin{aligned} \int_I (-\infty \mathcal{D}_x^\alpha f(x)) g(x) dx &= \frac{1}{\Gamma([\alpha] + 1 - \alpha)} \int_I \left( \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} \int_0^\infty \frac{f(x-u)}{u^{\alpha-[\alpha]}} du \right) g(x) dx \\ &= \frac{(-1)^{[\alpha]+1}}{\Gamma([\alpha] + 1 - \alpha)} \int_I \left( \int_0^\infty \frac{f(x-u)}{u^{\alpha-[\alpha]}} du \right) \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} g(x) dx \\ &= \frac{(-1)^{[\alpha]+1}}{\Gamma([\alpha] + 1 - \alpha)} \int_I f(s) \left( \frac{d^{[\alpha]+1}}{ds^{[\alpha]+1}} \int_s^\infty \frac{g(x)}{(x-s)^{\alpha-[\alpha]}} dx \right) ds \\ &= \int_I f(x) ({}_x \mathcal{D}_\infty^\alpha g(x)) dx. \end{aligned} \quad (\text{A.2})$$

(ii) The two identities hold

$$\begin{aligned} -\infty \mathcal{D}_x^\alpha e^{ipx} &= (ip)^\alpha e^{ipx} = |p|^\alpha e^{\frac{i\alpha\pi}{2} \text{sign}(p)} e^{ipx}, \\ {}_x \mathcal{D}_\infty^\alpha e^{ipx} &= (-ip)^\alpha e^{ipx} = |p|^\alpha e^{-\frac{i\alpha\pi}{2} \text{sign}(p)} e^{ipx}. \end{aligned} \quad (\text{A.3})$$

*Proof of (A.3)*

Using (30) we have

$$\begin{aligned} -\infty \mathcal{D}_x^\alpha e^{ipx} &= \frac{1}{\Gamma([\alpha] + 1 - \alpha)} \left( \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} e^{ipx} \right) \int_0^\infty \frac{e^{-ipu}}{u^{\alpha-[\alpha]}} du \\ &= (ip)^\alpha e^{ipx}, \quad p > 0. \end{aligned} \quad (\text{A.4})$$

<sup>12</sup>This space consists of all functions  $f$  which have continuous derivatives up to order  $[\alpha] - 1$  on  $R$  with  $f^{([\alpha]-1)}(x) \in AC(R)$ . We also recall that  $AC(R) \subset S(R)$ .



In the derivation of (A.4) we have used the integral formulas [23]

$$\begin{aligned}\int_0^\infty x^{\mu-1} \cos px dx &= \Gamma(\mu) \cos\left(\frac{\pi\mu}{2}\right) \\ \int_0^\infty x^{\mu-1} \sin px dx &= \Gamma(\mu) \sin\left(\frac{\pi\mu}{2}\right), \quad \text{for } p > 0 \text{ and } 0 < \mu < 1.\end{aligned}\tag{A.5}$$

## Appendix B

We first prove the identities

(1)

$$\lim_{\epsilon \rightarrow 0^+} \left( |x|^\epsilon - \frac{2}{\epsilon} \delta(x) \right) = \frac{1}{|x|}.\tag{B.1}$$

*Proof of (B.1)*

Differentiating with respect to  $x$  the expression

$$\ln(|x|)\text{sign}(x) = \lim_{\epsilon \rightarrow 0^+} \left( \frac{|x|^\epsilon - 1}{\epsilon} \right) \text{sign}(x)\tag{B.2}$$

and taking into account that  $\frac{d\text{sign}(x)}{dx} = 2\delta(x)$  we easily derive (B.1).

(2)

$$\mathcal{F}[|x|^\alpha] = \frac{2}{(2\pi|p|)^{1+\alpha}} \Gamma(1+\alpha) \cos\left(\frac{\pi}{2}(1+\alpha)\right), \quad \alpha > -1.\tag{B.3}$$

*Proof of (B.3)*

This is a direct consequence of (A.5).

The Fourier transform of  $\ln(|x|)$  is

$$\begin{aligned}\mathcal{F}[\ln(|x|)] &= \lim_{\epsilon \rightarrow 0^+} \mathcal{F}\left(\frac{1 - |x|^{-\epsilon}}{\epsilon}\right) = \lim_{\epsilon \rightarrow 0^+} \left( \frac{\delta(p)}{\epsilon} - \frac{2}{(2\pi|p|)^{1-\epsilon}} \Gamma(1-\epsilon) \frac{\sin\left(\frac{\pi}{2}\epsilon\right)}{\epsilon} \right) \\ &= -\frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \left( |p|^{\epsilon-1} - 2 \frac{\delta(p)}{\epsilon} - C_\gamma \epsilon |p|^{\epsilon-1} \right) \\ &= -\frac{1}{2} \frac{1}{|p|} + C_\gamma \delta(p)\end{aligned}\tag{B.4}$$

where  $C_\gamma$  is the Euler-Mascheroni constant. In the derivation of (B.4) we made use of the  $\Gamma(1+z)$ ,  $|z| < 1$  [25] expansion

$$\Gamma(1+z) = \sum_{n=0}^{\infty} a_n z^n\tag{B.5}$$

where the coefficients are given by

$$a_0 = 1, \quad na_n = -\gamma a_{n-1} + \sum_{k=2}^n (-1)^k a_{n-k} \zeta(k).\tag{B.6}$$

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Table 1: The Fox's  $H$ -function for the Farey series  $\mathcal{F}_n$  of order  $n = 5$  excluding the first  $\frac{0}{1}$  and the last  $\frac{1}{1}$  member of the series. The functions  $csc(z)$  and  $I_\nu(z)$  are the cosecant of  $z$  and the modified Bessel function of the first kind given by:  $I_\nu(z) = e^{-i\pi\frac{\nu}{2}} J_\nu(e^{i\frac{\pi}{2}} z) = \frac{1}{\Gamma(\nu+1)} (\frac{z}{2})^\nu {}_0F_1(\nu+1; \frac{z^2}{4})$ .

$\alpha$	$H(z) = \alpha z f(z)$
$\frac{1}{5}$	$\begin{aligned} & \frac{1}{25} \frac{csc(\frac{\pi}{5}) \sin(\frac{\pi}{10})}{z^{\frac{1}{5}} \Gamma(\frac{4}{5})} {}_0F_7(\frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & - \frac{1}{25} \frac{csc(\frac{2\pi}{5}) \sin(\frac{\pi}{5})}{z^{\frac{2}{5}} \Gamma(\frac{3}{5})} {}_0F_7(\frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & + \frac{1}{50} \frac{\Gamma(\frac{3}{5}) \sin(\frac{3\pi}{10})}{z^{\frac{3}{5}} \pi} {}_0F_7(\frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & - \frac{1}{150} \frac{\Gamma(\frac{4}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{4}{5}} \pi} {}_0F_7(\frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & + \frac{1}{600} \frac{1}{z\pi} {}_1F_8(1; \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & - \frac{1}{15000} \frac{csc(\frac{\pi}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{5}{5}} \Gamma(\frac{4}{5})} {}_0F_7(\frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & + \frac{1}{45000} \frac{csc(\frac{2\pi}{5}) \sin(\frac{3\pi}{10})}{z^{\frac{7}{5}} \Gamma(\frac{3}{5})} {}_0F_7(\frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & - \frac{1}{210000} \frac{\Gamma(\frac{3}{5}) \sin(\frac{\pi}{5})}{z^{\frac{8}{5}} \pi} {}_0F_7(\frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}; -\frac{1}{25 \cdot 10^8 z^2}) \\ & + \frac{1}{1260000} \frac{\Gamma(\frac{4}{5}) \sin(\frac{\pi}{10})}{z^{\frac{9}{5}} \pi} {}_0F_7(\frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}, \frac{9}{5}; -\frac{1}{25 \cdot 10^8 z^2}) \end{aligned}$
$\frac{1}{4}$	$\begin{aligned} & \frac{1}{16} \frac{\sqrt{2} \sin(\frac{\pi}{8})}{z^{\frac{1}{4}} \Gamma(\frac{3}{4})} {}_0F_5(\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}; -\frac{1}{4194304z^2}) - \frac{1}{32} \frac{\sqrt{2}}{\sqrt{z}\sqrt{\pi}} {}_0F_5(\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}; -\frac{1}{4194304z^2}) \\ & + \frac{1}{32} \frac{\Gamma(\frac{3}{4}) \sin(\frac{3\pi}{8})}{z^{\frac{3}{4}} \pi} {}_0F_5(\frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{9}{8}, \frac{5}{4}; -\frac{1}{4194304z^2}) - \frac{1}{96} \frac{1}{z\pi} {}_1F_6(1; \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}; -\frac{1}{4194304z^2}) \\ & + \frac{1}{1536} \frac{\sqrt{2} \sin(\frac{3\pi}{8})}{z^{\frac{5}{4}} \Gamma(\frac{3}{4})} {}_0F_5(\frac{3}{4}, \frac{7}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}; -\frac{1}{4194304z^2}) - \frac{1}{7680} \frac{\sqrt{2}}{z^{\frac{3}{2}} \sqrt{\pi}} {}_0F_5(\frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{3}{2}, \frac{13}{8}; -\frac{1}{4194304z^2}) \\ & + \frac{1}{15360} \frac{\Gamma(\frac{3}{4}) \sin(\frac{\pi}{8})}{z^{\frac{7}{4}} \pi} {}_0F_5(\frac{9}{8}, \frac{5}{4}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}; -\frac{1}{4194304z^2}) \end{aligned}$
$\frac{1}{3}$	$\begin{aligned} & \frac{1}{27} \frac{\sqrt{3}}{z^{\frac{1}{3}} \Gamma(\frac{2}{3})} {}_0F_3(\frac{1}{3}, \frac{1}{2}, \frac{5}{6}; -\frac{1}{11664z^2}) - \frac{1}{18} \frac{\sqrt{3}\Gamma(\frac{2}{3})}{z^{\frac{2}{3}} \pi} {}_0F_3(\frac{1}{2}, \frac{2}{3}, \frac{7}{6}; -\frac{1}{11664z^2}) \\ & + \frac{1}{18} \frac{1}{z\pi} {}_1F_4(1; \frac{2}{3}, \frac{5}{6}, \frac{7}{6}, \frac{4}{3}; -\frac{1}{11664z^2}) \\ & - \frac{1}{162} \frac{1}{z^{\frac{4}{3}} \Gamma(\frac{2}{3})} {}_0F_3(\frac{5}{6}, \frac{4}{3}, \frac{3}{2}; -\frac{1}{11664z^2}) + \frac{1}{648} \frac{\Gamma(\frac{2}{3})}{z^{\frac{5}{3}} \pi} {}_0F_3(\frac{7}{6}, \frac{3}{2}, \frac{5}{3}; -\frac{1}{11664z^2}) \end{aligned}$
$\frac{2}{5}$	$\begin{aligned} & \frac{4}{25} \frac{1}{z^{\frac{2}{5}} \Gamma(\frac{3}{5})} csc(\frac{2\pi}{5}) \sin(\frac{\pi}{5}) {}_2F_7(\frac{7}{20}, \frac{17}{20}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}; \frac{1}{39062500z^4}) \\ & - \frac{4}{25} \frac{\Gamma(\frac{4}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{4}{5}} \pi} \frac{1}{4} {}_2F_7(\frac{9}{20}, \frac{19}{20}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}; \frac{1}{39062500z^4}) \\ & + \frac{2}{125} \frac{csc(\frac{\pi}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{6}{5}} \Gamma(\frac{4}{5})} {}_2F_7(\frac{11}{20}, \frac{21}{20}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}; \frac{1}{39062500z^4}) \\ & - \frac{2}{125} \frac{\Gamma(\frac{3}{5}) \sin(\frac{\pi}{5})}{z^{\frac{8}{5}} \pi} {}_2F_7(\frac{13}{20}, \frac{23}{20}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}; \frac{1}{39062500z^4}) \\ & + \frac{7}{9375} \frac{csc(\frac{2\pi}{5}) \sin(\frac{\pi}{5})}{z^{\frac{12}{5}} \Gamma(\frac{3}{5})} {}_2F_7(\frac{17}{20}, \frac{27}{20}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}; \frac{1}{39062500z^4}) \\ & - \frac{1}{3125} \frac{\Gamma(\frac{4}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{14}{5}} \pi} {}_2F_7(\frac{19}{20}, \frac{29}{20}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}; \frac{1}{39062500z^4}) \\ & + \frac{11}{656250} \frac{csc(\frac{\pi}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{16}{5}} \Gamma(\frac{4}{5})} {}_2F_7(\frac{21}{20}, \frac{31}{20}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}; \frac{1}{39062500z^4}) \\ & - \frac{13}{1312500} \frac{\Gamma(\frac{3}{5}) \sin(\frac{\pi}{5})}{z^{\frac{18}{5}} \pi} {}_2F_7(\frac{23}{20}, \frac{33}{20}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}, \frac{9}{5}; \frac{1}{39062500z^4}) \end{aligned}$
$\frac{1}{2}$	$-\frac{1}{2\sqrt{2\pi z}} \left[ \cos(\frac{1}{4z}) C(\frac{1}{\sqrt{2\pi z}}) + \sin(\frac{1}{4z}) S(\frac{1}{\sqrt{2\pi z}}) \right] + \frac{1}{4\sqrt{2\pi z}} \left[ \cos(\frac{1}{4z}) + \sin(\frac{1}{4z}) \right].$
$\frac{3}{5}$	$\begin{aligned} & \frac{9}{25} \frac{\Gamma(\frac{3}{5}) \sin(\frac{3\pi}{10})}{z^{\frac{3}{5}} \pi} {}_4F_7(\frac{4}{15}, \frac{13}{30}, \frac{23}{30}, \frac{14}{15}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & - \frac{9}{125} \frac{csc(\frac{\pi}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{6}{5}} \Gamma(\frac{4}{5})} {}_4F_7(\frac{11}{30}, \frac{8}{15}, \frac{13}{15}, \frac{31}{30}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & + \frac{18}{125} \frac{\Gamma(\frac{4}{5}) \sin(\frac{\pi}{10})}{z^{\frac{9}{5}} \pi} {}_4F_7(\frac{7}{15}, \frac{19}{30}, \frac{29}{30}, \frac{17}{15}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & + \frac{21}{625} \frac{csc(\frac{2\pi}{5}) \sin(\frac{\pi}{5})}{z^{\frac{12}{5}} \Gamma(\frac{3}{5})} {}_4F_7(\frac{17}{30}, \frac{11}{15}, \frac{16}{15}, \frac{37}{30}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & - \frac{3}{100} \frac{1}{z^{\frac{3}{5}} \pi} {}_5F_8(\frac{2}{3}, \frac{5}{6}, \frac{1}{6}, \frac{4}{3}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & + \frac{117}{15625} \frac{\Gamma(\frac{3}{5}) \sin(\frac{\pi}{5})}{z^{\frac{18}{5}} \pi} {}_4F_7(\frac{23}{30}, \frac{14}{15}, \frac{19}{15}, \frac{43}{30}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & + \frac{66}{78125} \frac{csc(\frac{\pi}{5}) \sin(\frac{\pi}{10})}{z^{\frac{21}{5}} \Gamma(\frac{4}{5})} {}_4F_7(\frac{13}{15}, \frac{31}{30}, \frac{41}{30}, \frac{23}{15}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}; -\frac{729}{15625 \cdot 10^4 z^6}) \\ & - \frac{171}{1562500} \frac{\Gamma(\frac{4}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{24}{5}} \pi} {}_4F_7(\frac{17}{15}, \frac{29}{30}, \frac{39}{30}, \frac{49}{30}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}; -\frac{729}{15625 \cdot 10^4 z^6}) \end{aligned}$

$\frac{2}{3}$	$\frac{4}{9} \left( \cosh\left(\frac{2}{27z^2}\right) I_{\frac{1}{3}}\left(\frac{2}{27z^2}\right) + \frac{1}{9z^2} \sinh\left(\frac{2}{27z^2}\right) I_{\frac{1}{3}}\left(\frac{2}{27z^2}\right) + \frac{1}{9z^2} \cosh\left(\frac{2}{27z^2}\right) I_{\frac{4}{3}}\left(\frac{2}{27z^2}\right) \right)$ $- \frac{4}{27} \left( 6 \cosh\left(\frac{2}{27z^2}\right) I_{\frac{2}{3}}\left(\frac{2}{27z^2}\right) + \frac{1}{3z^2} \sinh\left(\frac{2}{27z^2}\right) I_{\frac{2}{3}}\left(\frac{2}{27z^2}\right) + \frac{1}{3z^2} \cosh\left(\frac{2}{27z^2}\right) I_{\frac{5}{3}}\left(\frac{2}{27z^2}\right) \right)$ $+ \frac{4}{81z^2} \left( \cosh\left(\frac{2}{27z^2}\right) I_{\frac{1}{3}}\left(\frac{2}{27z^2}\right) + 9z^2 \sinh\left(\frac{2}{27z^2}\right) I_{\frac{1}{3}}\left(\frac{2}{27z^2}\right) + \sinh\left(\frac{2}{27z^2}\right) I_{\frac{4}{3}}\left(\frac{2}{27z^2}\right) \right)$ $- \frac{4}{81z^2} \left( \cosh\left(\frac{2}{27z^2}\right) I_{\frac{2}{3}}\left(\frac{2}{27z^2}\right) + 18z^2 \sinh\left(\frac{2}{27z^2}\right) I_{\frac{2}{3}}\left(\frac{2}{27z^2}\right) + \sinh\left(\frac{2}{27z^2}\right) I_{\frac{5}{3}}\left(\frac{2}{27z^2}\right) \right)$
$\frac{3}{4}$	$\frac{9}{16} \frac{\Gamma(\frac{3}{4}) \sin(\frac{3\pi}{8})}{z^{\frac{3}{4}} \pi} {}_4F_5\left(\frac{7}{24}, \frac{11}{24}, \frac{19}{24}, \frac{23}{24}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, -\frac{729}{262144z^6}\right)$ $- \frac{9}{64} \frac{\sqrt{2}}{z^{\frac{3}{2}} \sqrt{\pi}} {}_4F_5\left(\frac{5}{12}, \frac{7}{12}, \frac{11}{12}, \frac{13}{12}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}, -\frac{729}{262144z^6}\right)$ $- \frac{45}{512} \frac{\sqrt{2} \sin(\frac{\pi}{8})}{z^{\frac{3}{4}} \Gamma(\frac{3}{4})} {}_4F_5\left(\frac{13}{24}, \frac{17}{24}, \frac{25}{24}, \frac{29}{24}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{9}{8}, \frac{5}{4}, -\frac{729}{262144z^6}\right)$ $+ \frac{3}{16} \frac{1}{z^{\frac{3}{2}} \pi} {}_5F_6\left(\frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}, \frac{4}{3}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, -\frac{729}{262144z^6}\right)$ $- \frac{693}{8192} \frac{\Gamma(\frac{3}{4}) \sin(\frac{\pi}{8})}{z^{\frac{15}{4}} \pi} {}_4F_5\left(\frac{19}{24}, \frac{23}{24}, \frac{31}{24}, \frac{35}{24}, \frac{3}{4}, \frac{7}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, -\frac{729}{262144z^6}\right)$ $- \frac{63}{4096} \frac{\sqrt{2}}{z^{\frac{9}{2}} \sqrt{\pi}} {}_4F_5\left(\frac{11}{12}, \frac{13}{12}, \frac{17}{12}, \frac{19}{12}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{3}{8}, \frac{13}{8}, -\frac{729}{262144z^6}\right)$ $+ \frac{1989}{262144} \frac{\sqrt{2} \sin(\frac{3\pi}{8})}{z^{\frac{21}{4}} \Gamma(\frac{3}{4})} {}_4F_5\left(\frac{25}{24}, \frac{29}{24}, \frac{37}{24}, \frac{41}{24}, \frac{9}{8}, \frac{5}{4}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}, -\frac{729}{262144z^6}\right)$
$\frac{4}{5}$	$\frac{16}{25} \frac{\Gamma(\frac{4}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{4}{5}} \pi} {}_6F_7\left(\frac{9}{40}, \frac{7}{40}, \frac{19}{40}, \frac{29}{40}, \frac{17}{40}, \frac{39}{40}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{16384}{9765625z^8}\right)$ $- \frac{48}{125} \frac{\Gamma(\frac{3}{5}) \sin(\frac{\pi}{5})}{z^{\frac{8}{5}} \pi} {}_6F_7\left(\frac{13}{40}, \frac{9}{40}, \frac{23}{40}, \frac{33}{40}, \frac{19}{40}, \frac{43}{40}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{16384}{9765625z^8}\right)$ $- \frac{112}{625} \frac{\csc(\frac{2\pi}{5}) \sin(\frac{\pi}{5})}{z^{\frac{12}{5}} \Gamma(\frac{3}{5})} {}_6F_7\left(\frac{17}{40}, \frac{11}{40}, \frac{27}{40}, \frac{37}{40}, \frac{21}{40}, \frac{47}{40}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{16384}{9765625z^8}\right)$ $+ \frac{176}{3125} \frac{\csc(\frac{\pi}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{16}{5}} \Gamma(\frac{4}{5})} {}_6F_7\left(\frac{21}{40}, \frac{13}{40}, \frac{31}{40}, \frac{41}{40}, \frac{23}{40}, \frac{51}{40}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{16384}{9765625z^8}\right)$ $- \frac{6384}{78125} \frac{\Gamma(\frac{4}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{24}{5}} \pi} {}_6F_7\left(\frac{29}{40}, \frac{17}{40}, \frac{39}{40}, \frac{49}{40}, \frac{27}{40}, \frac{59}{40}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}, \frac{16384}{9765625z^8}\right)$ $\frac{14352}{390625} \frac{\Gamma(\frac{3}{5}) \sin(\frac{\pi}{5})}{z^{\frac{28}{5}} \pi} {}_6F_7\left(\frac{33}{40}, \frac{19}{40}, \frac{43}{40}, \frac{53}{40}, \frac{29}{40}, \frac{63}{40}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{16384}{9765625z^8}\right)$ $+ \frac{26928}{1953125} \frac{\csc(\frac{2\pi}{5}) \sin(\frac{\pi}{5})}{z^{\frac{32}{5}} \Gamma(\frac{3}{5})} {}_6F_7\left(\frac{37}{40}, \frac{21}{40}, \frac{47}{40}, \frac{57}{40}, \frac{31}{40}, \frac{67}{40}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}, \frac{16384}{9765625z^8}\right)$ $- \frac{35464}{9765625} \frac{\csc(\frac{\pi}{5}) \sin(\frac{2\pi}{5})}{z^{\frac{36}{5}} \Gamma(\frac{4}{5})} {}_6F_7\left(\frac{41}{40}, \frac{23}{40}, \frac{51}{40}, \frac{61}{40}, \frac{33}{40}, \frac{71}{40}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}, \frac{9}{5}, \frac{16384}{9765625z^8}\right)$

Table 2: The Fox's  $H$ -function for  $1 < \alpha \leq 2$ .

$\alpha$	$H(z) = \alpha z f(z)$
$\frac{4}{3}$	$\frac{6\frac{1}{4}}{\sqrt{\pi}} z \left( \frac{1}{2} \frac{\csc(\frac{7\pi}{24})\csc(\frac{11\pi}{24})}{\csc(\frac{3\pi}{8})} \frac{\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})\Gamma(\frac{5}{8})}{\Gamma(\frac{17}{24})\Gamma(\frac{13}{24})\Gamma(\frac{7}{8})} {}_4F_5\left(\frac{7}{24}, \frac{11}{24}, \frac{19}{24}, \frac{23}{24}; \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}; \frac{729z^8}{262144}\right) \right.$ $- \frac{5\sqrt{6}}{384} z^2 \frac{\csc(\frac{\pi}{24})\csc(\frac{5\pi}{24})}{\csc(\frac{\pi}{8})} \frac{\Gamma(\frac{13}{24})\Gamma(\frac{17}{24})\Gamma(\frac{7}{8})}{\Gamma(\frac{23}{24})\Gamma(\frac{19}{24})\Gamma(\frac{5}{8})} {}_4F_5\left(\frac{13}{24}, \frac{17}{24}, \frac{25}{24}, \frac{29}{24}; \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{9}{8}, \frac{5}{4}; \frac{729z^8}{262144}\right)$ $+ \frac{77}{1024} z^4 \frac{\csc(\frac{7\pi}{24})\csc(\frac{11\pi}{24})}{\csc(\frac{3\pi}{8})} \frac{\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})\Gamma(\frac{5}{8})}{\Gamma(\frac{17}{24})\Gamma(\frac{13}{24})\Gamma(\frac{7}{8})} {}_4F_5\left(\frac{19}{24}, \frac{23}{24}, \frac{31}{24}, \frac{35}{24}; \frac{3}{4}, \frac{7}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}; \frac{729z^8}{262144}\right)$ $\left. - \frac{221\sqrt{6}}{196608} z^6 \frac{\csc(\frac{\pi}{24})\csc(\frac{5\pi}{24})}{\csc(\frac{\pi}{8})} \frac{\Gamma(\frac{13}{24})\Gamma(\frac{17}{24})\Gamma(\frac{7}{8})}{\Gamma(\frac{23}{24})\Gamma(\frac{19}{24})\Gamma(\frac{5}{8})} {}_4F_5\left(\frac{25}{24}, \frac{29}{24}, \frac{37}{24}, \frac{41}{24}; \frac{9}{8}, \frac{5}{4}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}; \frac{729z^8}{262144}\right) \right)$
$\frac{5}{4}$	$\frac{2\frac{7}{5}}{\sqrt{5\pi}} z \left( \frac{\csc(\frac{9\pi}{40})\csc(\frac{7\pi}{20})\csc(\frac{19\pi}{40})}{\csc(\frac{3\pi}{10})\csc(\frac{2\pi}{5})} \frac{\Gamma(\frac{29}{40})\Gamma(\frac{17}{20})\Gamma(\frac{39}{40})\Gamma(\frac{3}{5})}{\Gamma(\frac{9}{10})\Gamma(\frac{31}{40})\Gamma(\frac{13}{20})\Gamma(\frac{21}{40})} {}_6F_7\left(\frac{9}{40}, \frac{7}{20}, \frac{19}{40}, \frac{29}{40}, \frac{17}{20}, \frac{39}{40}; \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}; -\frac{16384z^{10}}{9765625}\right) \right.$ $- \frac{72\frac{4}{5}}{100} z^2 \frac{\csc(\frac{17\pi}{40})\csc(\frac{\pi}{20})\csc(\frac{7\pi}{40})}{\csc(\frac{2\pi}{5})\csc(\frac{\pi}{10})\csc(\frac{\pi}{5})} \frac{\Gamma(\frac{11}{20})\Gamma(\frac{27}{40})\Gamma(\frac{37}{40})\Gamma(\frac{4}{5})}{\Gamma(\frac{23}{40})\Gamma(\frac{19}{20})\Gamma(\frac{33}{40})\Gamma(\frac{7}{10})} {}_6F_7\left(\frac{17}{40}, \frac{11}{20}, \frac{27}{40}, \frac{37}{40}, \frac{21}{20}, \frac{47}{40}; \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}; -\frac{16384z^{10}}{9765625}\right)$ $+ \frac{1}{2\frac{9}{10}} z^4 \frac{\csc(\frac{\pi}{8})\csc(\frac{3\pi}{8})}{\csc(\frac{\pi}{10})\csc(\frac{3\pi}{5})\csc(\frac{2\pi}{5})} {}_7F_8\left(\frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{5}, \frac{7}{10}, \frac{4}{9}, \frac{9}{10}, \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{7}{5}; -\frac{16384z^{10}}{9765625}\right)$ $- \frac{2\frac{5}{2}897}{31250} z^6 \frac{\csc(\frac{3\pi}{40})\csc(\frac{13\pi}{40})\csc(\frac{9\pi}{20})}{\csc(\frac{2\pi}{10})\csc(\frac{3\pi}{10})\csc(\frac{2\pi}{5})} \frac{\Gamma(\frac{33}{40})\Gamma(\frac{19}{20})\Gamma(\frac{23}{40})\Gamma(\frac{7}{10})}{\Gamma(\frac{37}{40})\Gamma(\frac{27}{40})\Gamma(\frac{11}{20})\Gamma(\frac{4}{5})} {}_6F_7\left(\frac{33}{40}, \frac{19}{20}, \frac{43}{40}, \frac{53}{40}, \frac{29}{20}, \frac{63}{40}; \frac{4}{5}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}, \frac{7}{5}, \frac{3}{2}, \frac{8}{5}; -\frac{16384z^{10}}{9765625}\right)$ $\left. + \frac{2\frac{1}{2}4433}{3125000} z^8 \frac{\csc(\frac{\pi}{40})\csc(\frac{3\pi}{20})\csc(\frac{11\pi}{40})}{\csc(\frac{\pi}{10})\csc(\frac{3\pi}{5})\csc(\frac{3\pi}{10})} \frac{\Gamma(\frac{21}{40})\Gamma(\frac{13}{20})\Gamma(\frac{31}{40})\Gamma(\frac{9}{10})}{\Gamma(\frac{39}{40})\Gamma(\frac{17}{20})\Gamma(\frac{29}{40})\Gamma(\frac{3}{5})} {}_6F_7\left(\frac{41}{40}, \frac{23}{20}, \frac{51}{40}, \frac{61}{40}, \frac{33}{20}, \frac{71}{40}; \frac{11}{10}, \frac{6}{5}, \frac{13}{10}, \frac{3}{2}, \frac{8}{5}, \frac{17}{10}, \frac{9}{5}; -\frac{16384z^{10}}{9765625}\right) \right)$
$\frac{6}{5}$	$\frac{2\frac{5}{6}5\frac{1}{3}}{\sqrt{\pi}} z \left( \frac{1}{8} \frac{\csc(\frac{11\pi}{60})\csc(\frac{17\pi}{60})\csc(\frac{23\pi}{60})\csc(\frac{29\pi}{60})}{\csc(\frac{5\pi}{12})} \frac{\Gamma(\frac{41}{60})\Gamma(\frac{47}{60})\Gamma(\frac{53}{60})\Gamma(\frac{59}{60})\Gamma(\frac{7}{12})}{\Gamma(\frac{49}{60})\Gamma(\frac{43}{60})\Gamma(\frac{37}{60})\Gamma(\frac{31}{60})\Gamma(\frac{11}{12})} \times \right.$ $\times {}_8F_9\left(\frac{11}{60}, \frac{17}{60}, \frac{23}{60}, \frac{29}{60}, \frac{41}{60}, \frac{47}{60}, \frac{53}{60}, \frac{59}{60}; \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{2}{3}, \frac{5}{4}, \frac{11}{12}; \frac{9765625z^{12}}{8707129344}\right)$ $- \frac{5\frac{2}{3}2\frac{1}{6}3}{320} z^2 \frac{\csc(\frac{7\pi}{20})\csc(\frac{9\pi}{20})\csc(\frac{\pi}{20})\csc(\frac{3\pi}{20})}{\csc(\frac{5\pi}{12})\csc(\frac{\pi}{12})} {}_8F_9\left(\frac{7}{20}, \frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{17}{20}, \frac{19}{20}, \frac{21}{20}, \frac{23}{20}; \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{5}{6}, \frac{11}{12}, \frac{13}{12}; \frac{9765625z^{12}}{8707129344}\right)$ $+ \frac{10\frac{1}{3}1729}{2488320} z^4 \frac{\csc(\frac{\pi}{60})\csc(\frac{7\pi}{60})\csc(\frac{13\pi}{60})\csc(\frac{19\pi}{60})}{\csc(\frac{\pi}{12})} \frac{\Gamma(\frac{31}{60})\Gamma(\frac{37}{60})\Gamma(\frac{43}{60})\Gamma(\frac{49}{60})\Gamma(\frac{11}{12})}{\Gamma(\frac{59}{60})\Gamma(\frac{53}{60})\Gamma(\frac{47}{60})\Gamma(\frac{41}{60})\Gamma(\frac{7}{12})} \times$ $\times {}_8F_9\left(\frac{31}{60}, \frac{37}{60}, \frac{43}{60}, \frac{49}{60}, \frac{61}{60}, \frac{67}{60}, \frac{73}{60}, \frac{79}{60}; \frac{1}{2}, \frac{7}{12}, \frac{3}{4}, \frac{6}{12}, \frac{5}{6}, \frac{4}{3}, \frac{13}{6}, \frac{7}{4}; \frac{9765625z^{12}}{8707129344}\right)$ $- \frac{124729}{8957952} z^6 \frac{\csc(\frac{11\pi}{60})\csc(\frac{17\pi}{60})\csc(\frac{23\pi}{60})\csc(\frac{29\pi}{60})}{\csc(\frac{5\pi}{12})} \frac{\Gamma(\frac{41}{60})\Gamma(\frac{47}{60})\Gamma(\frac{53}{60})\Gamma(\frac{59}{60})\Gamma(\frac{7}{12})}{\Gamma(\frac{49}{60})\Gamma(\frac{43}{60})\Gamma(\frac{37}{60})\Gamma(\frac{31}{60})\Gamma(\frac{11}{12})} \times$ $\times {}_8F_9\left(\frac{41}{60}, \frac{47}{60}, \frac{53}{60}, \frac{59}{60}, \frac{71}{60}, \frac{77}{60}, \frac{83}{60}, \frac{89}{60}; \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, \frac{7}{5}, \frac{4}{3}, \frac{17}{3}, \frac{9765625z^{12}}{8707129344}\right)$ $+ \frac{5\frac{2}{3}2\frac{1}{6}429}{655360} z^8 \frac{\csc(\frac{\pi}{20})\csc(\frac{3\pi}{20})\csc(\frac{7\pi}{20})\csc(\frac{9\pi}{20})}{\csc(\frac{\pi}{12})\csc(\frac{5\pi}{12})} {}_8F_9\left(\frac{17}{20}, \frac{19}{20}, \frac{21}{20}, \frac{23}{20}, \frac{27}{20}, \frac{29}{20}, \frac{31}{20}, \frac{33}{20}; \frac{5}{6}, \frac{11}{12}, \frac{13}{12}, \frac{7}{6}, \frac{4}{3}, \frac{17}{12}, \frac{3}{2}, \frac{19}{12}; \frac{9765625z^{12}}{8707129344}\right)$ $\left. - \frac{10\frac{1}{3}596932063}{16717688340480} z^{10} \frac{\csc(\frac{7\pi}{60})\csc(\frac{13\pi}{60})\csc(\frac{19\pi}{60})\csc(\frac{25\pi}{60})}{\csc(\frac{\pi}{12})} \frac{\Gamma(\frac{31}{60})\Gamma(\frac{37}{60})\Gamma(\frac{43}{60})\Gamma(\frac{49}{60})\Gamma(\frac{11}{12})}{\Gamma(\frac{59}{60})\Gamma(\frac{53}{60})\Gamma(\frac{47}{60})\Gamma(\frac{41}{60})\Gamma(\frac{7}{12})} \times \right.$ $\left. \times {}_8F_9\left(\frac{61}{60}, \frac{67}{60}, \frac{73}{60}, \frac{79}{60}, \frac{91}{60}, \frac{97}{60}, \frac{103}{60}, \frac{109}{60}; \frac{13}{12}, \frac{7}{6}, \frac{5}{4}, \frac{4}{3}, \frac{19}{12}, \frac{5}{4}, \frac{7}{6}, \frac{11}{6}; \frac{9765625z^{12}}{8707129344}\right) \right)$