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HARDY INEQUALITY WITH THREE MEASURES ON MONOTONE FUNCTIONS

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(communicated by L.-E. Persson)

Abstract. Characterization of \( L^p_\nu[0,\infty) - L^q_\mu[0,\infty) \) boundedness of the general Hardy operator

\[
(H_s f)(x) = \left( \int_{[0,x]} f^s u d\lambda \right)^{\frac{1}{s}}
\]

restricted to monotone functions \( f \geq 0 \) for \( 0 < p, q, s < \infty \) with positive Borel \( \sigma \)-finite measures \( \lambda, \mu \) and \( \nu \) is obtained.

1. Introduction

Let \( \mathcal{M}^+ \) be the class consisting of all Borel functions \( f: [0, \infty) \to [0, +\infty] \) and \( \mathcal{M} \downarrow (\mathcal{M} \uparrow) \) be a subclass of \( \mathcal{M}^+ \) which consists of all non-increasing (non-decreasing) functions \( f \in \mathcal{M}^+ \). Suppose that \( \lambda, \mu \) and \( \nu \) are positive Borel \( \sigma \)-finite measures on \( [0, \infty) \) and \( u, v, w \in \mathcal{M}^+ \) are weight functions.

For \( 0 < p, q, s < \infty \) we study the problem when the Hardy inequality of the form

\[
\left( \int_{[0,\infty)} (H_s f)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p w d\nu \right)^{\frac{1}{p}},
\]

(1.1)

holds for all \( f \in \mathcal{M} \downarrow \) or for all \( f \in \mathcal{M} \uparrow \), where

\[
(H_s f)(x) := \left( \int_{[0,x]} f^s u d\lambda \right)^{\frac{1}{s}}.
\]

(1.2)

Since by the substitution \( f^s \to f \) the inequality (1.1) can be reduced to the equivalent inequality with new parameters \( p \) and \( q \) of the form

\[
\left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p w d\nu \right)^{\frac{1}{p}},
\]

(1.3)


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where

\[(Hf)(x) := (H_{\lambda}f)(x) = \int_{[0,x]} f \, u \, d\lambda\]  \hspace{1cm} (1.4)

we may and shall restrict our studies to the inequality (1.3). All the characterizations of (1.1) can be easily reproduced from the results for (1.3).

The weighted inequality (1.3) for \( f \in \mathcal{M} \downarrow \), when \( \lambda = \mu = \nu \) is the Lebesgue measure, was essentially characterized in [9] and [13] with the complement for the case \( 0 < q < 1 = p \) in [12] and recent contribution in [1] for the case \( 0 < q < p < 1 \). In fact, [9], [13], [12] and [1] deal with the case \( u(x) = 1 \), but a weight \( u \) can be incorporated with no change in the arguments. A piece of historical remarks and the literature can be found in ([3] and [4], Chapter 6). We summarize these results in the following

**Theorem 1.1.** Let \( \lambda = \mu = \nu \) be the Lebesgue measure. Then the inequality (1.3) holds for all \( f \in \mathcal{M} \downarrow \) if and only if:

(a) \( 1 < p \leq q < \infty \), \( \max (A_0, A_1) < \infty \), where

\[A_0 := \sup_{r>0} \left( \int_{r}^{\infty} \left( \int_{0}^{s} u \right)^{q} v(s) ds \right)^{\frac{1}{p'}} \left( \int_{0}^{r} w \right)^{-\frac{1}{p}} ,\]

and

\[A_1 := \sup_{r>0} \left( \int_{r}^{\infty} v \right)^{\frac{1}{p'}} \left( \int_{0}^{r} \left( \int_{0}^{s} u \right)^{p} v(s) ds \right)^{\frac{1}{p}} \left( \int_{0}^{r} w \right)^{-\frac{1}{p'}} w(s) ds \right)^{\frac{1}{p'}} \]

and \( C \approx A_0 + A_1 \).

(b) \( 0 < q < p < \infty \), \( 1 < p < \infty \), \( \frac{1}{r} := \frac{1}{q} - \frac{1}{p} \), \( \max (B_0, B_1) < \infty \), where

\[B_0 := \left( \int_{0}^{\infty} \left( \int_{0}^{t} w \right)^{-\frac{1}{p'}} \left( \int_{0}^{t} \left( \int_{0}^{s} u \right)^{q} v(s) ds \right)^{\frac{1}{p}} \left( \int_{0}^{t} u \right)^{q} v(t) dt \right)^{\frac{1}{r}} ,\]

and

\[B_1 := \left( \int_{0}^{\infty} \left( \int_{0}^{t} v \right)^{\frac{1}{p'}} \left( \int_{0}^{t} \left( \int_{0}^{s} u \right)^{p} w(s) ds \right)^{\frac{1}{p}} \left( \int_{0}^{t} u \right)^{q} w(t) dt \right)^{\frac{1}{r}} \]

and \( C \approx B_0 + B_1 \).

(c) \( 0 < q < p \leq 1 \). \( \max (B_0, B_1) < \infty \), where

\[B_1 := \left( \int_{0}^{\infty} \left( \int_{0}^{t} v \right)^{\frac{1}{p'}} \left( \int_{0}^{t} \left( \int_{0}^{s} u \right)^{p} w(s) ds \right)^{\frac{1}{p}} \left( \int_{0}^{t} u \right)^{q} v(t) dt \right)^{\frac{1}{r}} \]

and \( C \approx B_0 + B_1 \).

(d) \( 0 < p \leq q < \infty \), \( 0 < p \leq 1 \), \( \max (A_0, A_1) < \infty \), where

\[A_1 := \sup_{r>0} \left( \int_{0}^{r} u \right) \left( \int_{0}^{\infty} v \right)^{\frac{1}{q}} \left( \int_{0}^{r} w \right)^{-\frac{1}{p'}} \]

and \( C \approx A_0 + A_1 \).
It is important to note, that the weighted case of (1.3) for $1 < p, q < \infty$ was solved in [9] by proving the principle of duality which allows to reduce an inequality with a positive operator on monotone functions to an inequality with modified operator on non-negative functions. The other cases, when $p,q \not\in (1, \infty)$ were studied by different methods.

Our aim is twofold. First we study the inequality (1.3) in the case $0 < p \leq 1$ proving a complete analog of the parts (c) and (d) of Theorem 1.1 (Section 3). In the case $0 < q < p \leq 1$ our method is based on the characterization of the Hardy inequality on nonnegative functions in the case $0 < q < 1 = p$, which we establish in Section 3 (Theorem 3.1). This approach is direct and different from discretization methods of [1] and [2].

Hardy inequality (1.3) on monotone functions with two different measures was recently investigated by G. Sinnamon [11]. Namely, for $1 < p < \infty$ and $0 < q < \infty$ the author established the equivalence of (1.3) with $u \equiv v \equiv w \equiv 1$ and $d\lambda = d\nu$ for $f \in \mathcal{M}^{+}$ to the same inequality restricted to $f \in \mathcal{M}^{-}$. Moreover, such equivalence takes place also for more general operator than (1.4), that is for the operator $(Kf)(x) = \int_{[0,x]} k(x,y)f(y)d\lambda(y)$ with a kernel $k(x,y) \geq 0$, which is monotone in the variable $y$ (see [5, Theorem 2.3]). Moreover, G. Sinnamon [11] extended the Sawyer principle of duality for measures. We apply this extension to characterize (1.3) in case $1 < p, q < \infty$ (Section 4) combining with the recent results by D.V. Prokhorov [6] for the inequality (1.3) on $f \in \mathcal{M}^{+}$ with $1 < p < \infty$ and $0 < q < \infty$ extended by the same author for the Hardy operator with Oinarov kernel [7].

We use the following notations and conventions. $A \ll B$ means that $A \leq cB$ with $c$ depending only on $p$ and $q$, $A \approx B$ is equivalent to $A \ll B \ll A$. Uncertainties of the form $0 \cdot \infty$ are taken to be zero. We also use the notation := for introducing new quantities.

2. Preliminary remarks

Denote

$$\Lambda_f(x) := \int_{[0,x]} f \, d\lambda, \quad \text{and} \quad \bar{\Lambda}_f(x) := \int_{[x,\infty)} f \, d\lambda. \quad (2.1)$$

We need the following statements.

**Lemma 2.1.** ([6], Lemma 1) If $\gamma > 0$, then

$$\frac{\Lambda_f(\infty)^{\gamma+1}}{\max\{1,\gamma+1\}} \leq \int_{[0,\infty)} f(x) \Lambda_f(x)^{\gamma} \, d\lambda(x) \leq \frac{\Lambda_f(\infty)^{\gamma+1}}{\min\{1,\gamma+1\}} \quad (2.2)$$

holds. If $\gamma \in (-1,0)$ and $\Lambda_f(\infty) < +\infty$, then (2.2) holds.

**Lemma 2.2.** ([6], Lemma 2) If $\gamma > 0$, then

$$\frac{\bar{\Lambda}_f(0)^{\gamma+1}}{\max\{1,\gamma+1\}} \leq \int_{[0,\infty)} f(x) \bar{\Lambda}_f(x)^{\gamma} \, d\lambda(x) \leq \frac{\bar{\Lambda}_f(0)^{\gamma+1}}{\min\{1,\gamma+1\}} \quad (2.3)$$

holds. If $\gamma \in (-1,0)$ and $\bar{\Lambda}_f(0) < +\infty$, then (2.3) holds.
The following two statements can be obtained from [[10], Lemma 1.2] (see also [[11], Proposition 1.5]).

**Lemma 2.3.** Let \( f \in \mathcal{M} \uparrow \) with \( f(0) = 0 \) and let \( \eta \) be a Borel measure on \([0, \infty)\). Then there exist \( f_0 \in \mathcal{M} \uparrow \) and the sequence \( \{h_n\}_{n \geq 1} \subset \mathcal{M}^+ \) such that

1. \( f_0(x) \leq f(x) \) for all \( x \in [0, \infty) \).
2. \( f_0(x) = f(x) \) for \( \eta \)-a.e. \( x \in [0, \infty) \).
3. \( f_n(x) := \int_{[0,x]} h_n d\eta \leq f_0(x) \) for all \( x \in [0, \infty) \).
4. For all \( x \in [0, \infty) \) the sequence \( \{f_n(x)\}_{n \geq 1} \) is nondecreasing in \( n \) and
   \[ f_0(x) = \lim_{n \to \infty} f_n(x) \] \( \eta \)-a.e. \( x \in [0, \infty) \).

**Lemma 2.4.** Let \( f \in \mathcal{M} \downarrow \) with \( f(+\infty) = 0 \) and let \( \eta \) be a Borel measure on \([0, \infty)\). Then there exist \( f_0 \in \mathcal{M} \downarrow \) and the sequence \( \{h_n\}_{n \geq 1} \subset \mathcal{M}^+ \) such that

1. \( f_0(x) \leq f(x) \) for all \( x \in [0, \infty) \).
2. \( f_0(x) = f(x) \) for \( \eta \)-a.e. \( x \in [0, \infty) \).
3. \( f_n(x) := \int_{[x,\infty)} h_n d\eta \leq f_0(x) \) for all \( x \in [0, \infty) \).
4. For all \( x \in [0, \infty) \) the sequence \( \{f_n(x)\}_{n \geq 1} \) is nondecreasing in \( n \) and
   \[ f_0(x) = \lim_{n \to \infty} f_n(x) \] \( \eta \)-a.e. \( x \in [0, \infty) \).

**Remark 2.5.** Two similar lemmas are valid for the approximation from above.

The following statements are taken from [7] and concern the weighted \( L_p^\mu[0, \infty) - L_q^\mu[0, \infty) \) inequality with the operator of the form

\[
(K_{\alpha}f)(x) = \int_{[0,x]} k(x,y) u(y) f(y) d\lambda(y).
\]

Here the kernel \( k(x,y) \geq 0 \) is \( \mu \times \lambda \) - measurable on \([0, \infty) \times [0, \infty) \) and satisfies the following Oinarov condition. There is a constant \( D \geq 1 \) such that

\[
D^{-1}k(x,y) \leq k(x,z) + k(z,y) \leq D k(x,y), \quad 0 \leq y \leq z \leq x.
\]  

\((2.4)\)

**Theorem 2.6.** Let \( 1 < p \leq q < \infty \). Then the inequality

\[
\left( \int_{[0, \infty)} (K_{\alpha}f)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}}
\]

holds for all \( f \in \mathcal{M}^+ \) if and only if

\[
\mathbb{A} := \max(\mathbb{A}_{0,1}, \mathbb{A}_{0,2}) < \infty,
\]

where

\[
\mathbb{A}_{0,1} := \sup_{r \in [0, \infty)} \left( \int_{[r, \infty)} v(x) k(x,t)^{\frac{q}{p}} d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[0,r]} u^p d\lambda(x) \right)^{\frac{1}{p}},
\]

\[
\mathbb{A}_{0,2} := \sup_{r \in [0, \infty)} \left( \int_{[r, \infty)} v d\mu \right)^{\frac{1}{q}} \left( \int_{[0,r]} k(t,y)^{\frac{p}{q}} u(y)^p d\lambda(y) \right)^{\frac{1}{p}}.
\]
If $1 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then inequality (2.5) holds for all $f \in \mathcal{M}$ if and only if $B := \max (B_{0,1}, B_{0,2}) < \infty$, where

$$B_{0,1} := \left( \int_{[0,\infty)} \left( \int_{[t,\infty)} v(x) k(t,x)^q \, d\mu(x) \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \left( \int_{[0,\infty)} u^p \, \, d\lambda(t) \right)^{\frac{1}{p}},$$

$$B_{0,2} := \left( \int_{[0,\infty)} \left( \int_{[0,\infty)} v d\mu \right)^{\frac{r}{p}} \right)^{\frac{1}{p}} \left( \int_{[0,\infty)} k(t,y)^p u(y)^p \, d\lambda(y) \right)^{\frac{1}{p}}.$$

The next statement is an analog of the previous theorem for the operator $K_u^*$ of the dual form

$$(K_u^* f)(x) = \int_{[x,\infty)} k(y,x) u(y)f(y) \, d\lambda (y)$$

with a kernel satisfying Oinarov’s condition (2.4).

**Theorem 2.7.** Let $1 < p \leq q < \infty$. Then the inequality

$$\left( \int_{[0,\infty)} (K_u^* f)^q \, v \, d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p \, d\lambda \right)^{\frac{1}{p}}$$

(2.6)

holds for all $f \in \mathcal{M}$ if and only if $A := \max (A_{0,1}, A_{0,2}) < \infty$, where

$$A_{0,1} := \sup_{x \in [0,\infty)} \left( \int_{[0,x)} v(x) k(t,x)^q \, d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} u^p \, \, d\lambda(t) \right)^{\frac{1}{p}},$$

$$A_{0,2} := \sup_{x \in [0,\infty)} \left( \int_{[0,x]} v \, d\mu \right)^{\frac{1}{p}} \left( \int_{[0,\infty)} k(t,y)^p u(y)^p \, d\lambda(y) \right)^{\frac{1}{p}}.$$

If $1 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then inequality (2.6) holds for all $f \in \mathcal{M}$ if and only if $B := \max (B_{0,1}, B_{0,2}) < \infty$, where

$$B_{0,1} := \left( \int_{[0,\infty)} \left( \int_{[0,x)} v(x) k(t,x)^q \, d\mu(x) \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} \left( \int_{[0,\infty)} u^p \, \, d\lambda(t) \right)^{\frac{1}{p}},$$

$$B_{0,2} := \left( \int_{[0,\infty)} \left( \int_{[0,x]} v \, d\mu \right)^{\frac{r}{p}} \right)^{\frac{1}{p}} \left( \int_{[0,\infty)} k(t,y)^p u(y)^p \, d\lambda(y) \right)^{\frac{1}{p}}.$$

In the following theorems we collect weight versions of the results obtained by G. Sinnamon in [11] for embeddings the cones of monotone functions. Put

$$W(t) := \int_{[0,t]} wdv, \quad \text{and} \quad W(x) := \int_{[x,\infty)} wdv.$$
THEOREM 2.8. If $0 < p \leq q < \infty$, then
\[
\sup_{F \in \mathfrak{M}} \left( \int_{[0,\infty)} F^q v d\mu \right)^{\frac{1}{q}} \leq \sup_{x \in [0,\infty)} \left( \int_{[0,x]} v d\mu \right)^{\frac{1}{q}}.
\] (2.8)

THEOREM 2.9. If $0 < q < p < \infty$, and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ then
\[
\sup_{F \in \mathfrak{M}} \left( \int_{[0,\infty)} F^q v d\mu \right)^{\frac{1}{q}} \approx \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1} v d\mu \right)^{\frac{q}{p'}} dv(y) \right)^{\frac{1}{q}}.
\] (2.9)

Analogous results take place for $F \in \mathfrak{M} \uparrow$.

THEOREM 2.10. If $0 < p \leq q < \infty$, then
\[
\sup_{F \in \mathfrak{M}} \left( \int_{[0,\infty)} F^q v d\mu \right)^{\frac{1}{q}} = \sup_{x \in [0,\infty)} \left( \int_{[x,\infty)} v d\mu \right)^{\frac{1}{q}}.
\] (2.10)

THEOREM 2.11. If $0 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then
\[
\sup_{F \in \mathfrak{M}} \left( \int_{[0,\infty)} F^q v d\mu \right)^{\frac{1}{q}} \approx \left( \int_{[0,\infty)} w(y) \left( \int_{[0,y]} \tilde{W}^{-1} v d\mu \right)^{\frac{q}{p'}} dv(y) \right)^{\frac{1}{q}}.
\] (2.11)

Note that Theorems 2.9 and 2.11 with $q = 1$ give analogs of Sawyer’s principle of duality with general Borel measures.

3. The case $0 < p \leq 1$

We need the following extension of ([12], Theorem 3.3) from the weighted case to the case of measures.

THEOREM 3.1. Let $0 < q < 1$, $v = v_a + v_s$, where $d\nu_a = \frac{d\nu_a}{d\lambda} d\lambda$ and $v_s \perp \lambda$.

Then
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} f u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} f w d\nu
\] (3.1)

holds for all $f \in \mathfrak{M}^+$ if and only if
\[
\mathcal{B} := \left( \int_{[0,\infty)} \left( \int_{[0,y]} \frac{v(z) d\mu(z)}{\tilde{W}(z)} \right)^{\frac{q}{1-q}} v(y) d\mu(y) \right)^{\frac{1-q}{q}} < \infty,
\]
where
\[ \tilde{w} := \frac{w}{u} \, d\nu_a \quad \text{and} \quad \tilde{w}(x) := \inf_{t \in [0, x]} \tilde{w}(t). \] (3.2)

Moreover, \( C \approx \mathcal{B} \).

**Proof.** Let us start with proving that (3.1) is equivalent to the following inequality
\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} f \, ud\lambda \right)^q \, v(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f \frac{d\nu_a}{d\lambda}. \tag{3.3}
\]

Obviously, (3.3) implies (3.1). Let (3.1) hold and \( f \in \mathcal{M}^+ \). If \( \nu_s \perp \lambda \), then there exists \( A \subset [0, \infty) \) such that \( \lambda(A) = 0 \), \( \text{supp} \nu_s = A \) and \( \text{supp} \nu_a = [0, \infty) \setminus A \). Let \( \tilde{f} = f \chi_{[0, \infty) \setminus A} \). Then
\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} f \, ud\lambda \right)^q \, v(x) \, d\mu(x) \right)^{\frac{1}{q}} = \left( \int_{[0, \infty)} \left( \int_{[0, x]} \tilde{f} \, ud\lambda \right)^q \, v(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} \tilde{f} \, w \, d\nu = C \left( \int_{[0, \infty)} \tilde{f} \, w \, d\nu_a + \int_{[0, \infty)} \tilde{f} \, w \, d\nu_s \right) = C \int_{[0, \infty)} \tilde{f} \, w \, d\nu_a.
\]

Now if we use (3.2), then (3.3) is equivalent to
\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} f \, ud\lambda \right)^q \, v(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f \, \tilde{w} \, d\lambda. \tag{3.4}
\]

Then, by [10, Theorem 3.1] and changing \( f u \) to \( f \), we get that (3.4) is equivalent to
\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} f \, d\lambda \right)^q \, v(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f \, \tilde{w} \, d\lambda. \tag{3.5}
\]

Now we follow the proof of [12, Theorem 3.3]. First let \( \tilde{w}(x) = \int_{[x, \infty)} b \, d\lambda \) for \( \lambda \)-a.e. \( x \in [0, \infty) \), \( \int_{[0, \infty)} b \, d\lambda = \infty \) and \( \int_{[x, \infty)} b \, d\lambda < \infty \). Then by changing order of integration the right hand side of (3.5) is equal to
\[
C \int_{[0, \infty)} \left( \int_{[0, x]} f \, d\lambda \right) b(x) \, d\lambda(x)
\]
and so (3.5) is equivalent to
\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} f \, d\lambda \right)^q \, v(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} \left( \int_{[0, x]} f \, d\lambda \right) b(x) \, d\lambda(x). \tag{3.6}
\]
Since \( \int_{[0, x]} f \, d\lambda \) is increasing we can replace it with \( F \) and so (3.6) is equivalent to
\[
\left( \int_{[0, \infty)} F^q \, v \, d\mu \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} F \, b \, d\lambda \text{ with } F \in \mathcal{M} \uparrow \tag{3.7}.\]
By [11, Theorem 2.5] and using Lemma 2.2 we get

\[
C \approx \left( \int_{[0, \infty)} \left( \int_{[0, x]} \frac{v(y) d\mu(y)}{\tilde{w}_1(y)} \right)^{\frac{1}{1-q}} b(x) d\lambda(x) \right)^{\frac{q}{1-q}}
\]

\[
\approx \left( \int_{[0, \infty)} \int_{[0, x]} \frac{v(y) d\mu(y)}{\tilde{w}_1(y)} \left( \int_{[0, y]} \frac{v(z) d\mu(z)}{\tilde{w}_1(z)} \right)^{\frac{q}{1-q}} b(x) d\lambda(x) \right)^{\frac{1-q}{q}}
\]

\[
= \left( \int_{[0, \infty)} \left( \int_{[0, y]} \frac{v(z) d\mu(z)}{\tilde{w}_1(z)} \right)^{\frac{q}{1-q}} v(y) d\mu(y) \right)^{\frac{1-q}{q}}.
\]

For a general \( \tilde{w}_1 \) we may and shall suppose that \( \tilde{w}_1(x) < \infty \) for all \( x > 0 \). Let \( N \in \mathbb{N} \) and

\[
w_N(x) := \chi_{[0,N]}(x) \tilde{w}_1(x).
\]

Then \( w_N(+\infty) = 0 \) and similar to Lemma 2.4 we find \( w_N^{(0)} \in M_- \) and \( h_n \in M_+ (n \in \mathbb{N}) \) such that

1. \( w_N(x) \leq w_N^{(0)}(x) \) for all \( x \in [0, \infty) \).
2. \( w_N(x) = w_N^{(0)}(x) \) for \( \lambda \)-a.e. \( x \in [0, \infty) \).
3. \( w_{N,k}(x) := \int_{[x, \infty)} h_d\lambda \geq w_N^{(0)}(x) \) for all \( x \in [0, \infty) \).
4. The sequence \( \{w_{N,k}(x)\}_{k \geq 1} \) is nonincreasing in \( k \) for all \( x \in [0, \infty) \) and \( w_N^{(0)}(x) = \lim_{k \to \infty} w_{N,k}(x) \) \( \lambda \)-a.e. \( x \in [0, \infty) \). Then by the previous part of the proof for any \( f \in M_+ \) we have

\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} f d\lambda \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1}{q}} \leq \left( \int_{[0, \infty)} \left( \int_{[0, x]} \frac{v(z) d\mu(z)}{w_{N,k}(z)} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1}{q}} \int_{[0, \infty)} f w_{N,k} d\lambda.
\]

By [6, Lemma 5] this is equivalent to

\[
\left( \int_{[0, \infty)} \left( \int_{[0, x]} \frac{f}{w_{N,k}} d\lambda \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1}{q}} \leq \left( \int_{[0, \infty)} \left( \int_{[0, x]} \frac{v(z) d\mu(z)}{w_{N,k}(z)} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1}{q}} \int_{[0, \infty)} f d\lambda.
\]

By (3) and (1) we have \( \frac{1}{w_{N,k}(x)} \leq \frac{1}{w_N(x)} \) and by (4), (2) and Monotone Convergence Theorem

\[
\lim_{k \to \infty} \int_{[0, x]} \frac{f}{w_{N,k}} d\lambda = \int_{[0, x]} \frac{f}{w_N^{(0)}} d\lambda = \int_{[0, x]} \frac{f}{w_N} d\lambda.
\]
Making the reverse change \( \frac{f}{w_N} \rightarrow f \) we find
\[
\left( \int_{[0,N]} \left( \int_{[0,x]} f \, d\lambda \right)^q v(x) \, d\mu(x) \right)^{\frac{1}{q}}
\ll \left( \int_{[0,N]} \left( \int_{[0,x]} \frac{v(z) \, d\mu(z)}{w_N(z)} \right)^{\frac{1-q}{q}} v(x) \, d\mu(x) \right)^{\frac{1}{1-q}} \int_{[0,N]} f \, w_N \, d\lambda
\]
\[
= \left( \int_{[0,N]} \left( \int_{[0,x]} \frac{v(z) \, d\mu(z)}{\tilde{w}_1(z)} \right)^{\frac{1-q}{q}} v(x) \, d\mu(x) \right)^{\frac{1}{1-q}} \int_{[0,N]} f \, \tilde{w}_1 \, d\lambda
\]
\[
\leq \mathcal{B} \int_{[0,\infty)} f \, \tilde{w}_1 \, d\lambda.
\]
Letting \( N \rightarrow \infty \) we arrive at \( C \ll \mathcal{B} \). To show the reverse inequality we again approximate \( \tilde{w}_1 \) from above by a monotone sequence of functions \( w_k(x) := \int_{[x,\infty)} b_k \, d\lambda \downarrow \tilde{w}_1 \). Then applying (3.6), (3.7) and [11, Theorem 2.5] we find
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} v(z) \, d\mu(z) \right)^{\frac{1-q}{q}} v(y) \, d\mu(y) \right)^{\frac{1}{1-q}} \ll C
\]
and since \( w_k^{-1} \uparrow \tilde{w}_1^{-1} \) the result follows. \( \square \)

**Definition 3.2.** Let \( w \in \mathcal{M} \downarrow \) and be continuous on the left. It is known ([8, Chapter 12, §3]), that there exists a Borel measure, say \( \eta_w \), such that \( w(x) = \int_{[x,\infty)} d\eta_w + w(+) \). We say that \( w \in \mathcal{S}_2(0) \) if there exist a constant \( C \geq 1 \) such that
\[
\frac{1}{w(x)} - \frac{1}{w(0)} \leq C \int_{[0,x]} \frac{d\eta_w}{w^2}, \quad x > 0.
\]

**Corollary 3.3.** Let \( 0 < q < 1 \), \( w \in \mathcal{M} \downarrow \) and \( w \in \mathcal{S}_2(0) \). Then
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} h \, d\lambda \right)^q v(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} h \, w \, d\lambda
\]
holds for all \( h \in \mathcal{M}^+ \) if and only if
\[
\mathcal{B} := \left( \int_{[0,\infty)} \left( \int_{[0,x]} \frac{v \, d\mu}{w} \right)^{\frac{q}{1-q}} v(x) \, d\mu(x) \right)^{\frac{1-q}{q}} < \infty.
\]
Moreover, \( C \approx \mathcal{B} \approx \mathcal{B}_0 + \mathcal{B}_1 \), where
\[
\mathcal{B}_0 := \left( \int_{[0,\infty)} v \, d\mu \right)^{\frac{1}{q}} w(0)^{-\frac{1}{p}},
\]
$$\mathbb{B}_1 := \left( \int_{[0, \infty)} w(x)^{-\frac{q}{1-q}} \left( \int_{[x, \infty)} v d\mu(x) \right)^\frac{q}{1-q} v(x) d\mu(x) \right)^\frac{1-q}{q}. $$

**Proof.** It follows from Theorem 3.1, Lemma 2.2 and [11, Theorem 2.6]. □

Denote

$$\Lambda(t) := \Lambda_u(t) = \int_{[0,t]} u d\lambda$$

and observe that by the change $f^p \to f$ in the inequality (1.3) we get the following equivalent inequality

$$\left( \int_{[0, \infty)} \left( Hf^p \right)^q v d\mu \right)^\frac{q}{p} \leq C^p \left( \int_{[0, \infty)} f w d\nu \right), \quad f \in \mathcal{M}.$$  (3.9)

**Theorem 3.4.** (a) Let $0 < p \leq q < \infty$ and $0 < p \leq 1$. Then (1.3) holds for all $f \in \mathcal{M}$ if and only if

$$A_0 := \sup_{t \in [0, \infty)} \left( \int_{[0, t]} w d\nu \right)^{-\frac{1}{p}} \left( \int_{[0, t]} \Lambda^q v d\mu \right)^\frac{1}{q} < \infty,$$

$$\mathcal{A}_1 := \sup_{t \in [0, \infty)} \Lambda(t) \left( \int_{[0, t]} w d\nu \right)^{-\frac{1}{p}} \left( \int_{[t, \infty)} v d\mu \right)^\frac{1}{q} < \infty$$

and $C \approx A_0 + \mathcal{A}_1$.

(b) Let $0 < q < 1 = p$. Then (1.3) holds for all $f \in \mathcal{M}$ if and only if

$$\mathbb{B}_0 := \left( \int_{[0, \infty)} w(y) \left( \int_{[y, \infty)} W^{-1} \Lambda^q v d\mu \right) dv(y) \right)^\frac{1-q}{p} \leq \infty,$$

$$\mathbb{B}_1 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} \text{ess sup}_{s \in [0, x]} \frac{\Lambda(s)}{W(s)} v(t) d\mu(t) \right)^\frac{q}{1-q} v(x) d\mu(x) \right)^\frac{1-q}{q} < \infty$$

and $C \approx \mathbb{B}_0 + \mathbb{B}_1$.

(c) Let $0 < q < p < 1$, $\mathcal{V}_p(t) := \text{ess sup}_{s \in [0, t]} \frac{\Lambda^p(s)}{W(s)}$. Then (1.3) holds for all $f \in \mathcal{M}$ if

$$\mathcal{B}_0 := \left( \int_{[0, \infty)} w(y) \left( \int_{[y, \infty)} W^{-1} \Lambda^q v d\mu \right)^\frac{p}{p-q} \right)^\frac{p-q}{pq} < \infty,$$
\[ B_1 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} \mathcal{V}_p(t) v(x) d\mu(x) \right)^{\frac{q}{p-q}} v(t) d\mu(t) \right)^{\frac{p-q}{pq}} < \infty \]

and only if \( B_0 + B_1 < \infty \), provided \( \mathcal{V}_p(t) \) is continuous on \((0, \infty)\) and \( \frac{1}{\mathcal{F}(t)} \in \mathcal{I}_2(0) \). Then \( C \approx B_0 + B_1 \).

**Proof.** (a) Since \( f \in \mathcal{M} \downarrow \), then \((H_{uf})(x) \geq f(x) \Lambda(x)\) and (1.3) implies

\[
\left( \int_{[0, \infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p v d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{M} \downarrow.
\]

It is known (see Theorem 2.8) that \( C = A_0 \) for \( 0 < p \leq q < \infty \).

Now, if \( f_t = \chi_{[0, t]} \) in (1.3) then

\[
C \left( \int_{[0, t]} v d\mu \right)^{\frac{1}{q}} \geq \left( \int_{[t, \infty)} (H_{uf_t})^q v d\mu \right)^{\frac{1}{q}} = \Lambda(t) \left( \int_{[t, \infty)} v d\mu \right)^{\frac{1}{q}},
\]

which implies that \( C \geq A_1 \). Consequently, \( A_0 + A_1 \leq 2C \).

For the sufficiency we suppose first that \( f \in \mathcal{M} \downarrow, f(x) = \int_{[x, \infty)} hud\lambda \) for \( \lambda \)-a.e. \( x \in [0, \infty) \), where \( h \in \mathcal{M}^+ \) and \( f(x) \geq \int_{[x, \infty)} hud\lambda \), for all \( x \in [0, \infty) \). Let \( 0 < p < 1 \). We have by Lemma 2.2

\[
\int_{[0, x]} \left( \int_{[s, \infty)} hud\lambda \right) u(s) d\lambda(s)
\]

\[
\approx \int_{[0, x]} \left( \int_{[s, \infty)} \left( \int_{[s, \infty)} hud\lambda \right)^{p-1} h(y)u(y)d\lambda(y) \right)^{\frac{1}{p}} u(s) d\lambda(s)
\]

\[
\leq \int_{[0, x]} \left( \int_{[s, \infty)} \left( \int_{[s, \infty)} hud\lambda \right)^{p-1} h(y)u(y)d\lambda(y) \right)^{\frac{1}{p}} u(s) d\lambda(s) + \Lambda(x)f(x)
\]

[by Minkowski inequality]

\[
\leq \left( \int_{[0, x]} \left( \int_{[s, \infty)} hud\lambda \right)^{p-1} h(y)u(y)\Lambda(y)^p d\lambda(y) \right)^{\frac{1}{p}} + \Lambda(x)f(x).
\]

Applying (3.10) we obtain

\[
\left( \int_{[0, \infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \ll \left( \int_{[0, \infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} + J,
\]

(3.11)
where

\[ J := \left( \int_{[0, \infty)} \left( \int_{[0, x]} \left( \int_{[y, \infty)} hud\lambda \right)^{p-1} h(y)u(y)\Lambda(y) d\lambda(y) \right)^{\frac{q}{p}} v(x) d\mu(x) \right)^{\frac{1}{q}}. \]

For the first term on the right hand side of (3.11) by Theorem 2.8 we have

\[ \left( \int_{[0, \infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} \leq A_0 \left( \int_{[0, \infty)} f^p w d\nu \right)^{\frac{1}{p}}. \quad (3.12) \]

For the second term on the right hand side of (3.11) by Minkowski inequality with \( \frac{q}{p} \geq 1 \) and Lemma 2.2 we find

\[
J \leq \left( \int_{[0, \infty)} \left( \int_{[y, \infty)} hud\lambda \right)^{p-1} h(y)u(y)\Lambda(y) \left( \int_{[y, \infty)} v d\mu \right)^{\frac{p}{q}} d\lambda(y) \right)^{\frac{1}{p}}
\]

\[
\leq \mathcal{A}_1 \left( \int_{[0, \infty)} \left( \int_{[y, \infty)} hud\lambda \right)^{p-1} h(y)u(y) \left( \int_{[0, y]} w d\nu \right)^{\frac{1}{p}} d\lambda(y) \right)^{\frac{1}{p}}
\]

\[
\approx \mathcal{A}_1 \left( \int_{[0, \infty)} \left( \int_{[y, \infty)} hud\lambda \right)^{p} w(s) d\nu(s) \right)^{\frac{1}{p}} \leq \mathcal{A}_1 \left( \int_{[0, \infty)} f^p d\nu \right)^{\frac{1}{p}}
\]

and the inequality

\[
\left( \int_{[0, \infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \ll (A_0 + \mathcal{A}_1) \left( \int_{[0, \infty)} f^p w d\nu \right)^{\frac{1}{p}} \quad (3.13)
\]

in this case follows. For an arbitrary \( f \in \mathcal{M} \downarrow \) without loss of generality we may suppose that \( f(\infty) = 0 \) and find by Lemma 2.4 that \( f_0 \in \mathcal{M} \downarrow \) and a sequence \( \{h_n\}_{n \geq 1} \subset \mathcal{M}^+ \) such that

1. \( f_0(x) \leq f(x) \) for all \( x \in [0, \infty) \).
2. \( f_0(x) = f(x) \) for \( \lambda \)-a.e. \( x \in [0, \infty) \).
3. \( f_n(x) := \int_{[x, \infty)} h_n u d\lambda \leq f_0(x) \) for all \( x \in [0, \infty) \).
4. For all \( x \in [0, \infty) \) the sequence \( \{f_n(x)\}_{n \geq 1} \) is nondecreasing in \( n \) and \( f_0(x) = \lim_{n \to \infty} f_n(x) \) \( \lambda \)-a.e. \( x \in [0, \infty) \). Then by the Monotone Convergence
Theorem and (3.13), it yields that
\[
\left( \int_{[0,\infty)} (Hf)^q \mathrm{d}\mu \right)^{\frac{1}{q}} \overset{(2)}{=} \left( \int_{[0,\infty)} (Hf_0)^q \mathrm{d}\mu \right)^{\frac{1}{q}}
\]
\[
\overset{(4)}{=} \lim_{n \to \infty} \left( \int_{[0,\infty)} (Hf_n)^q \mathrm{d}\mu \right)^{\frac{1}{q}} \overset{(3.13)}{\leq} (A_0 + \mathcal{A}_1) \lim_{n \to \infty} \left( \int_{[0,\infty)} f_n^p \nu \mathrm{d}v \right)^{\frac{1}{p}}
\]
\[
\leq (A_0 + \mathcal{A}_1) \left( \int_{[0,\infty)} f_0^p \nu \mathrm{d}v \right)^{\frac{1}{p}} \overset{(1)}{\leq} (A_0 + \mathcal{A}_1) \left( \int_{[0,\infty)} f^p \nu \mathrm{d}v \right)^{\frac{1}{p}}
\]
and the upper bound \( C \ll A_0 + \mathcal{A}_1 \) is proved. The case \( p = 1 \) is treated by the same method, but even simpler.

(b) Necessity. It follows from the inequality
\[
\left( \int_{[0,\infty)} (Hf)^q \mathrm{d}\mu \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} f \nu \mathrm{d}v, \quad f \in \mathcal{M}^-, \quad (3.14)
\]
that
\[
\left( \int_{[0,\infty)} f^q \Lambda^q \mathrm{d}\mu \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} f \nu \mathrm{d}v, \quad f \in \mathcal{M}^- . \quad (3.15)
\]
The last inequality is characterized by \( \mathbb{B}_0 \) (see Theorem 2.9 with \( p = 1 \).) Hence, \( \mathbb{B}_0 \leq C \). Now, suppose \( h \in \mathcal{M}^+ \) and \( f(x) = \int_{[x,\infty)} hu \nu \mathrm{d}\lambda \). Then \( f \in \mathcal{M}^- \) and (3.14) gives
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[x,\infty)} hu \nu \mathrm{d}\lambda \right) \mathrm{d}\mu(x) \right)^q \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} \left( \int_{[0,x]} hu \nu \mathrm{d}\lambda \right) w(s) \nu \mathrm{d}s .
\]
This implies
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} h\Lambda \nu \mathrm{d}\lambda \right)^q \nu \mathrm{d}\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} hWud\lambda.
\]
Changing the variable \( h\Lambda u \to h \) we obtain
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} h\nu \mathrm{d}\lambda \right)^q \nu \mathrm{d}\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} hWd\lambda .
\]
The last inequality is characterized by Theorem 3.1. Consequently, \( \mathbb{B}_1 \ll C \).
Sufficiency. Again, suppose first, that \( f \in \mathcal{M} \downarrow \), \( f(x) = \int_{[x,\infty)} h ud\lambda \) for \( \lambda \) -a.e. \( x \in [0,\infty) \), where \( h \in \mathcal{M} \) and \( f(x) = \int_{[x,\infty)} h ud\lambda \) for all \( x \in [0,\infty) \). Then we have

\[
\left( \int_{[0,\infty)} (Hf)^q vd\mu \right)^{\frac{1}{q}} = \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[x,\infty)} h ud\lambda \right) u(s)d\lambda(s) \right)^q vd\mu \right)^{\frac{1}{q}}
\]

\[
\ll \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[x,\infty)} h ud\lambda \right) u(s)d\lambda(s) \right)^q vd\mu \right)^{\frac{1}{q}}
+ \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} h ud\lambda \right)^q \Lambda^q(x) v(x)d\mu(x) \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_{[0,\infty)} \left( \int_{[0,x]} h ud\lambda \right)^q v(x)d\mu(x) \right)^{\frac{1}{q}} + \left( \int_{[0,\infty)} f^q \Lambda^q vd\mu \right)^{\frac{1}{q}}
\]

[applying Theorem 3.1 and Theorem 2.9]

\[
\ll \mathbb{B}_1 \left( \int_{[0,\infty)} \left( \int_{[x,\infty]} h ud\lambda \right) w(x)d\nu(x) \right) + \mathbb{B}_0 \left( \int_{[0,\infty)} f wd\nu \right)
\]

\[
\leq (\mathbb{B}_0 + \mathbb{B}_1) \int_{[0,\infty)} f wd\nu.
\]

For an arbitrary \( f \in \mathcal{M} \downarrow \) we use the arguments from the end of the part (a).

(c) Sufficiency. To prove (3.9) we again, suppose first that \( f \in \mathcal{M} \downarrow \), \( f(x) = \int_{[x,\infty)} h ud\lambda \) for \( \lambda \) -a.e. \( x \in [0,\infty) \), where \( h \in \mathcal{M}^+ \) and \( f(x) = \int_{[x,\infty)} h ud\lambda \) for all \( x \in [0,\infty) \). Then, arguing as before and applying Minkowski’s inequality, we find

\[
\left( \int_{[0,\infty)} (Hf^\frac{1}{p})^q vd\mu \right)^{\frac{p}{q}}
\]

\[
= \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[x,\infty)} h ud\lambda \right)^\frac{1}{p} u(s)d\lambda(s) \right)^q v(x)d\mu(x) \right)^{\frac{p}{q}}
\]

\[
\ll \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[x,\infty)} h ud\lambda \right)^\frac{1}{p} u(s)d\lambda(s) \right)^q v(x)d\mu(x) \right)^{\frac{p}{q}}
+ \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} h ud\lambda \right)^{\frac{q}{p}} \Lambda^q(x) v(x)d\mu(x) \right)^{\frac{p}{q}}
\]

\[
\leq \left( \int_{[0,\infty)} \left( \int_{[0,x]} h \Lambda^{p} ud\lambda \right)^{\frac{q}{p}} v(x)d\mu(x) \right)^{\frac{p}{q}} + \left( \int_{[0,\infty)} f^{\frac{q}{p}} \Lambda^q vd\mu \right)^{\frac{p}{q}}
\]
applying Theorem 3.1 and Theorem 2.9
\[
\leq B_1^p \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} hu d\lambda \right) w(x) d\nu(x) \right) + B_0^p \int_{[0,\infty)} f w d\nu.
\]
For an arbitrary \( f \in \mathcal{M} \) we again use the arguments from the end of the part (a).

Necessity. The inequality \( B_0 \leq C \) follows by using similar arguments as in the proof of \( A_0 \leq C \) and \( B_0 \leq C \) in the parts (a) and (b).

For the rest it is sufficient to show that (3.9) implies the inequality \( C \gg B_1 \).

Suppose for simplicity, that \( V_p(0) = 0 \). Let
\[
g(t) := \max \left\{ 2^m, m \in \mathbb{Z}; 2^m \leq \mathcal{V}_p^\rho(t) \right\}
\]
and
\[
\tau_m := \inf \left\{ y \in [0, \infty) : 2^m \leq \mathcal{V}_p^\rho(y) \right\}.
\]
Since \( \mathcal{V}_p^\rho(t) \) is continuous, then \( \tau_m \) exists for all \( m \in \mathbb{Z}, \tau_m \uparrow \) and
\[
\frac{\Lambda(\tau_m)^r}{W(\tau_m)^p} = 2^m = \mathcal{V}_p^\rho(\tau_m) \leq \mathcal{V}_p^\rho(t) \leq 2^{m+1}, \quad t \in [\tau_m, \tau_{m+1}].
\]
\[
g(\tau_m) = 2^m, \quad g(s) \leq 2^{m-1} \text{ for all } s \in [0, \tau_m).
\]

We note that
\[
g(t) = \sum_{m \in \mathbb{Z}} 2^m I_{[\tau_m, \tau_{m+1})}(t) \leq \mathcal{V}_p^\rho(t)
\]
and define
\[
f(t) := \int_{[t, \infty)} \left( \int_{[x, \infty)} v d\mu \right)^\frac{r}{q} dg(x).
\]
Then \( f \in \mathcal{M} \) and by Lemma 2.2
\[
\int_{[0,\infty)} f w d\nu = \int_{[0,\infty)} \left( \int_{[x,\infty)} v d\mu \right)^\frac{r}{q} dg(x)
\]
\[
\approx \int_{[0,\infty)} g(x) \left( \int_{[x,\infty)} v d\mu \right)^\frac{r}{q} v(x) d\mu(x)
\]
\[
\leq \int_{[0,\infty)} \mathcal{V}_p^\rho(x) \left( \int_{[x,\infty)} v d\mu \right)^\frac{r}{q} v(x) d\mu(x) := B_{2,1}^p.
\]
On the other hand
\[
\left( \int_{[0, \infty)} \left( \int_{[0, \infty)} f \right)^q (y) d\Lambda (y) \right)^{\frac{1}{q}} v (x) d\mu (x)
\]
\[
\geq \left( \sum_m \int_{[\tau_m, \tau_{m+1}]} v (x) \left( \int_{[0, \tau_m]} \int_{[y, \tau_m]} \left( \frac{f_{[y, \infty)}}{W (s)} d\mu (s) \right)^{\frac{p}{q}} d\Lambda (y) \right) d\mu (x) \right)^{\frac{1}{q}}
\]
\[
\geq \left( \sum_m \int_{[\tau_m, \tau_{m+1}]} v d\mu \left( \int_{[\tau_m, \tau_{m+1}]} v d\mu \right)^{\frac{p}{q}} \times \left( W (\tau_m)^{-\frac{1}{p}} \int_{[0, \tau_m]} (g (\tau_m) - g (y))^\frac{p}{q} d\Lambda (y) \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
\]
\[
\geq \left( \sum_m \int_{[\tau_m, \tau_{m+1}]} v d\mu \left( \int_{[\tau_m, \tau_{m+1}]} v d\mu \right)^{\frac{p}{q}} \left( \frac{2^{\frac{m}{p}} \Lambda (\tau_m)}{W (\tau_m)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}
\]
\[
\geq \left( \sum_m \frac{2^m}{\int_{[\tau_m, \tau_{m+1}]} \left( \int_{[\tau_m, \tau_{m+1}]} v d\mu \right)^{\frac{p}{q}} v (s) d\mu (s) \right)^{\frac{1}{q}} =: B_{2, 1}^q
\]

With such \( f (x) \) the inequality (3.9) implies \( C^p B_{2, 1}^q \gg B_{2, 0}^q \Rightarrow C \gg B_{2, 1}. \) Now, if we put \( f = \chi_{[0]} \) in (3.9), we find that
\[
C \geq \left( \int_{[0, \infty)} v d\mu \right)^{\frac{p}{q}} \left( \frac{W (0)}{\Lambda^p (0)} \right)^{-\frac{1}{p}} = \left( \int_{[0, \infty)} v d\mu \right)^{\frac{p}{q}} \left( \frac{1}{\Lambda^p (0)} \right)^{-\frac{1}{p}} =: B_{2, 0}. \]

It follows from Corollary 3.3, that \( B_{2, 1} + B_{2, 0} \gg B_1. \) Hence, \( C \gg B_1 \) and the proof is complete.

In conclusion of this section we give an analog of part (a) of the previous theorem for non-decreasing functions.

**Theorem 3.5.** Let \( 0 < p \leq q < \infty \) and \( 0 < p \leq 1. \) Then, (1.3) holds for all \( f \in \mathfrak{M} \uparrow \) if and only if
\[
\tilde{A}_1 := \sup_{t \in [0, \infty)} \left( \int_{[t, \infty)} \Lambda^q (x, t) v (x) d\mu (x) \right)^{\frac{1}{q}} W^{-\frac{1}{p}} (t) < \infty,
\]
where

\[ \Lambda (x, t) := \int_{[t,x]} ud\lambda, \]

and \( C \approx \tilde{\Lambda}_1. \)

**Proof.** Replacing \( f \) in (1.3) by \( f_t := \chi_{[t,\infty]} \) we find \( \tilde{\Lambda}_1 \leq C. \) For sufficiency we suppose that

\[ f (x) = \int_{[0,x]} hud\lambda, \quad h \in \mathcal{M}^+ \]

and let \( 0 < p < 1. \) Then, by Minkowski inequality and Lemma 2.1, we find

\[
\begin{align*}
\int_{[0,x]} \left( \int_{[0,s]} hud\lambda \right) u(s) d\lambda (s) &
\approx \int_{[0,x]} \left( \int_{[0,s]} \left( \int_{[0,y]} hud\lambda \right)^{p-1} h(y) u(y) d\lambda (y) \right)^{\frac{q}{p}} u(s) d\lambda (s)
\leq \left( \int_{[0,x]} \left( \int_{[0,y]} hud\lambda \right)^{p-1} h(y) u(y) \Lambda^p (x, y) d\lambda (y) \right)^\frac{1}{p}.
\end{align*}
\]

Thus, again by Minkowski inequality

\[
\left( \int_{[0,\infty)} (Hf)^q vd\mu \right)^\frac{1}{pq} \leq \left( \int_{[0,\infty)} \left( \int_{[0,y]} hud\lambda \right)^{p-1} h(y) u(y) \Lambda^p (x, y) d\lambda (y) \right)^\frac{q}{pq} v(x) d\mu (x)
\leq \left( \int_{[0,\infty)} \left( \int_{[0,y]} hud\lambda \right)^{p-1} h(y) u(y) \left( \int_{[y,\infty)} \Lambda^q (x, y) v(x) d\mu (x) \right)^\frac{q}{pq} d\lambda (y) \right)^\frac{1}{pq}
\leq \tilde{\Lambda}_1 \left( \int_{[0,\infty)} \left( \int_{[0,y]} hud\lambda \right)^{p-1} h(y) u(y) \left( \int_{[y,\infty)} wd\nu \right) d\lambda (y) \right)^\frac{1}{pq}
\approx \tilde{\Lambda}_1 \left( \int_{[0,\infty)} f^p wd\nu \right)^\frac{1}{pq}.
\]

\( \square \)

A general case \( f \in \mathcal{M} \uparrow \) follows by Lemma 2.3 similar to the proof of Theorem 3.4.
4. The case $1 < p, q < \infty$

The result of this section is based on the following statement, which follows from Theorems 2.9 and 2.11 with $q = 1$.

**Corollary 4.1.** Let $(Tf)(x) = \int_{[0,\infty)} k(x,y)f(y)d\lambda(y)$, where $k(x,y)$ is a defined on $[0,\infty) \times [0,\infty)$, non-negative, $\mu \times \lambda$-measurable kernel.

(a) The inequality

$$\left( \int_{[0,\infty)} (Tf)^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\nu \right)^{\frac{1}{p}}$$

for $f \in \mathcal{M} \downarrow$, holds if and only if the inequality

$$\left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1}(T^*g) d\lambda \right)^{p'} d\nu(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} q^q d\mu \right)^{\frac{1}{q}}, \quad g \in \mathcal{M}^+,$n

holds with $(T^*g)(z) = \int_{[0,\infty)} k(z,x)g(z)v(z)d\mu(z)$.

(b) The inequality (4.1) for $f \in \mathcal{M} \uparrow$ holds if and only if the following inequality holds:

$$\left( \int_{[0,\infty)} w(y) \left( \int_{[0,y]} W^{-1}(T^*g) d\lambda \right)^{p'} d\nu(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} q^q d\mu \right)^{\frac{1}{q}}, \quad g \in \mathcal{M}^+.$$

Now let us present our result for the case $1 < p, q < \infty$.

**Theorem 4.2.** Let $k(x,y) = \int_{[y,\infty]} W^{-1} u d\lambda$ and $f \in \mathcal{M} \downarrow$. The inequality (1.3) holds for $1 < p \leq q < \infty$ if and only if $\mathcal{A} = \max \{ \mathcal{A}_{0,1} + \mathcal{A}_{0,2} \} < \infty$, where

$$\mathcal{A}_{0,1} = \sup_{r \in [0,\infty)} \left( \int_{[0,r]} w(y)k(t,y)^p d\nu(y) \right)^{\frac{1}{p}} \left( \int_{[r,\infty)} v^q d\mu \right)^{\frac{1}{q}},$$

$$\mathcal{A}_{0,2} = \sup_{r \in [0,\infty)} \left( \int_{[0,r]} w d\nu \right)^{\frac{1}{p'}} \left( \int_{[r,\infty)} v(x)k(x,t)^q d\mu(x) \right)^{\frac{1}{q}}.$$n

Moreover, if $C$ is the best constant in (1.3), then $C = \mathcal{A}$.

In the case $1 < q < p < \infty$ the inequality (1.3) holds if and only if $\mathcal{B} = \max \{ \mathcal{B}_{0,1} + \mathcal{B}_{0,2} \} < \infty$, where

$$\mathcal{B}_{0,1} = \sup_{r \in [0,\infty)} \left( \int_{[0,r]} w(y)k(t,y)^p d\nu(y) \right)^{\frac{1}{p}} \left( \int_{[r,\infty)} v^q d\mu \right)^{\frac{1}{q}},$$

$$\mathcal{B}_{0,2} = \sup_{r \in [0,\infty)} \left( \int_{[0,r]} w d\nu \right)^{\frac{1}{p'}} \left( \int_{[r,\infty)} v(x)k(x,t)^q d\mu(x) \right)^{\frac{1}{q}}.$$
\[ B_{0,1} = \left( \int_{[0,\infty)} \left( \int_{[0,1]} w(y)k(t, y)^p \, dv(y) \right)^{\frac{\nu}{p'}} \left( \int_{[1,\infty)} v(t) \, d\mu(t) \right)^{\frac{\nu}{p}} \right)^{\frac{1}{\nu}} \]

\[ B_{0,2} = \left( \int_{[0,\infty)} \left( \int_{[0,1]} w \, dv \right)^{\frac{\nu}{q'}} \left( \int_{[1,\infty)} v(x)k(x, t)^q \, d\mu(x) \right)^{\frac{\nu}{q}} w(t) \, dv(t) \right)^{\frac{1}{\nu}} \]

and \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \). Moreover, if \( C \) is the best constant in (1.3), then \( C = B \).

**Proof.** Because of Corollary 4.1 (a) the inequality (1.3) is equivalent to

\[
\left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W(x)^{-1} \left( \int_{[x,\infty)} g \, dv \right) u(x) \right) \, d\lambda(x) \right)^{\frac{\nu}{p'}} \left( \int_{[0,\infty)} v(y) \, d\mu(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} q^q \, dv \right)^{\frac{1}{q'}}.
\]

By changing the order of integration in the left hand side of (4.3) we obtain the Hardy inequality with Oinarov kernel of the form

\[
\left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} g(z)k(z, y)v(z) \, d\mu(z) \right) \, dv(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} q^q \, dv \right)^{\frac{1}{q'}}.
\]

By substitution \( f = g^q \) and according to Lemma 7 from [7] the last inequality is equivalent to

\[
\left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} f(z)k(z, y)v(z)^{1/q} \, d\mu(z) \right) \, dv(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} f^q \, d\mu \right)^{\frac{1}{q'}}.
\]

Thus the proof follows by applying Theorem 2.7. \( \square \)

Similarly we can obtain the result for non-decreasing functions as follows.

**THEOREM 4.3.** Let \( \bar{k}(y, x) = \int_{[y,\infty]} \bar{W}^{-1} \, ud\lambda \) and \( f \in \mathcal{M}^{\uparrow} \). The inequality (1.3) holds for \( 1 < p \leq q < \infty \) if and only if \( \mathcal{A} = \max \{ \mathcal{A}_{0,1} + \mathcal{A}_{0,2} \} < \infty \), where

\[
\mathcal{A}_{0,1} = \sup_{t \in [0,\infty)} \left( \int_{[1,\infty)} w(y)k(y, t)^p \, dv(y) \right)^{\frac{1}{p'}} \left( \int_{[0,1]} \, dv \right)^{\frac{1}{q'}}.
\]
Moreover, if $C$ is the best constant in (1.3), then $C = \tilde{A}$. In the case $1 < q < p < \infty$ the inequality (1.3) holds if and only if

$$\mathcal{B}_{0,1} := \left( \int_{0,\infty} \left( \int_{t,\infty} w(y) \bar{K}(y, t)^p dv(y) \right)^{\frac{\tau}{p}} \left( \int_{0,\infty} v(x) \bar{K}(x, t)q d\mu(x) \right)^{\frac{\tau}{q}} \right)^{\frac{1}{\tau}}$$

and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Moreover, if $C$ is the best constant in (1.3), then $C = \mathcal{B}^\ast$.

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