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UNIVERSALITY OF THE LIMIT SHAPE OF CONVEX LATTICE POLYGONAL LINES

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Let $\mathit{\Pi}_n$ be the set of convex polygonal lines $\mathit{\Gamma}$ with vertices on $\mathbb{Z}_+^2$ and fixed endpoints $0=(0,0)$ and $n=(n_1,n_2)$. We are concerned with the *limit shape*, as $n\to\infty$, of "typical" $\mathit{\Gamma}\in\mathit{\Pi}_n$ with respect to a parametric family of probability measures $\{P_n^r,\, 0<r<\infty\}$ on $\mathit{\Pi}_n$, including the uniform distribution ($r=1$) for which the limit shape was found in the early 1990s independently by A. M. Vershik, I. Bárány and Ya. G. Sinai. We show that, in fact, the limit shape is *universal* in the class $\{P_n^r\}$, even though P_n^r ($r\ne 1$) and P_n^1 are asymptotically singular. Measures P_n^r are constructed, following Sinai's approach, as conditional distributions $Q_z^r(\cdot\,|\,\mathit{\Pi}_n)$, where Q_z^r are suitable product measures on the space $\mathit{\Pi}=\cup_n\mathit{\Pi}_n$, depending on an auxiliary "free" parameter $z=(z_1,z_2)$. The transition from $(\mathit{\Pi},Q_z^r)$ to $(\mathit{\Pi}_n,P_n^r)$ is based on the asymptotics of the probability $Q_z^r(\mathit{\Pi}_n)$, furnished by a certain two-dimensional local limit theorem. The proofs involve subtle analytical tools including the Möbius inversion formula and properties of zeroes of the Riemann zeta function.

1. Introduction.

1.1. *Background*: *the limit shape.* In this paper, a *convex lattice polygonal line* $\mathit{\Gamma}$ is a piecewise linear path on the plane, starting at the origin $0=(0,0)$, with vertices on the integer lattice $\mathbb{Z}_+^2:=\{(i,j)\in\mathbb{Z}^2: i,j\ge 0\}$, and such that the inclination of its consecutive edges strictly increases staying between 0 and $\pi/2$. Let $\mathit{\Pi}$ be the set of all convex lattice polygonal lines with finitely many edges, and denote by $\mathit{\Pi}_n\subset\mathit{\Pi}$ the subset of polygonal lines $\mathit{\Gamma}\in\mathit{\Pi}$ whose right endpoint $\xi=\xi_{\mathit{\Gamma}}$ is fixed at $n=(n_1,n_2)\in\mathbb{Z}_+^2$.

We are concerned with the problem of *limit shape* of "typical" $\mathit{\Gamma}\in\mathit{\Pi}_n$, as $n\to\infty$, with respect to some probability measure P_n on $\mathit{\Pi}_n$. Here the "limit shape" is understood as a planar curve γ^* such that, with overwhelming P_n-probability for large enough n, properly scaled polygonal lines $\tilde{\mathit{\Gamma}}_n=S_n(\mathit{\Gamma})$

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lie within an arbitrarily small neighborhood of γ^*. More precisely, for any $\varepsilon > 0$ it should hold that

$$\lim_{n\to\infty} P_n\{d(\tilde{\Gamma}_n, \gamma^*) \le \varepsilon\} = 1, \tag{1.1}$$

where $d(\cdot,\cdot)$ is some metric on the path space — for instance, induced by the Hausdorff distance between compact sets (in $\mathbb{R}^2$),

$$d_{\mathcal{H}}(A, B) := \max\left\{\max_{x\in A}\min_{y\in B}|x-y|,\ \max_{y\in B}\min_{x\in A}|x-y|\right\}, \tag{1.2}$$

where $|\cdot|$ is the Euclidean vector norm.

Of course, the limit shape and its very existence may depend on the probability law P_n. With respect to the *uniform distribution* on Π_n, the problem was solved independently by Vershik [32], Bárány [3] and Sinai [29], who showed that, under the scaling $S_n : (x_1, x_2) \mapsto (x_1/n_1, x_2/n_2)$, the limit shape γ^* is given by a parabola arc defined by the Cartesian equation

$$\sqrt{1-x_1} + \sqrt{x_2} = 1, \qquad 0 \le x_1, x_2 \le 1. \tag{1.3}$$

More precisely [cf. (1.1)], if $n = (n_1, n_2) \to \infty$ so that $n_2/n_1 \to c \in (0,\infty)$ then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\#\{\Gamma \in \Pi_n : d_{\mathcal{H}}(\tilde{\Gamma}_n, \gamma^*) \le \varepsilon\}}{\#(\Pi_n)} = 1. \tag{1.4}$$

(Here and in what follows, $\#(\cdot)$ denotes the number of elements in a set.)

The proofs in papers [32, 3] involved a blend of combinatorial, variational and geometric arguments and were based on a direct analysis of the corresponding generating function via a multivariate saddle-point method for a Cauchy integral [32] or a suitable Tauberian theorem [3]. Extending some of these ideas and using large deviations techniques, Vershik and Zeitouni [37] developed a systematic approach to the limit shape problem for the uniform measure on more general ensembles of convex lattice polygonal lines with various geometric restrictions.

Sinai [29] proposed an alternative, probabilistic method essentially based on randomization of the right endpoint of the polygonal line $\Gamma \in \Pi_n$; we will comment more on this approach in Section 1.3. Let us point out that the paper [29] contained the basic ideas but only sketches of the proofs. Some of these techniques were subsequently elaborated by Bogachev and Zarbaliev [6, 7] and also by Zarbaliev in his Ph.D. thesis [40]; however, a complete proof has not been published as yet.

REMARK 1.1. A polygonal line $\mathit{\Gamma} \in \mathit{\Pi}_n$ can be viewed as a vector sum of its consecutive edges, resulting in a given integer vector $n = (n_1, n_2)$; due to the convexity property, the order of parts in the sum is uniquely determined. Hence, any such $\mathit{\Gamma}$ represents an integer *vector partition* of $n \in \mathbb{Z}_+^2$ or, more precisely, a *strict* vector partition (i.e., without proportional parts; see [32]). This observation incorporates the topic of convex lattice polygonal lines in a general theory of integer partitions. For ordinary, one-dimensional partitions, the problem of limit shape can also be set up, but for a special geometric object associated with partitions, called *Young diagrams* [33, 34].

1.2. *Main result.* Vershik [32], page 20, pointed out that it would be interesting to study asymptotic properties of convex lattice polygonal lines under other probability measures P_n on $\mathit{\Pi}_n$, and conjectured that the limit shape might be universal for some classes of measures. Independently, a similar hypothesis was put forward by Prokhorov [27].

In the present paper, we prove Vershik–Prokhorov's universality conjecture for a parametric family of probability measures P_n^r $(0 < r < \infty)$ on $\mathit{\Pi}_n$ defined by

$$P_n^r(\mathit{\Gamma}) := \frac{b^r(\mathit{\Gamma})}{B_n^r}, \qquad \mathit{\Gamma} \in \mathit{\Pi}_n, \tag{1.5}$$

with

$$b^r(\mathit{\Gamma}) := \prod_{e_i \in \mathit{\Gamma}} b_{k_i}^r, \qquad B_n^r := \sum_{\mathit{\Gamma} \in \mathit{\Pi}_n} b^r(\mathit{\Gamma}), \tag{1.6}$$

where the product is taken over all edges e_i of $\mathit{\Gamma} \in \mathit{\Pi}_n$, k_i is the number of lattice points on the edge e_i except its left endpoint and

$$b_k^r := \binom{r+k-1}{k} = \frac{r(r+1)\cdots(r+k-1)}{k!}, \qquad k = 0, 1, 2, \ldots \tag{1.7}$$

Note that for $r = 1$ the measure (1.5) is reduced to the uniform distribution on $\mathit{\Pi}_n$. Qualitatively, formulas (1.6), (1.7) introduce certain probability weights for random edges on $\mathit{\Gamma}$ by encouraging $(r > 1)$ or discouraging $(r < 1)$ lattice points on each edge as compared to the reference case $r = 1$.

Assume that $0 < c_1 \le n_2/n_1 \le c_2 < \infty$, and consider the standard scaling transformation $S_n(x) = (x_1/n_1, x_2/n_2)$, $x = (x_1, x_2) \in \mathbb{R}^2$. It is convenient to work with a sup-distance between the scaled polygonal lines $\tilde{\mathit{\Gamma}}_n := S_n(\mathit{\Gamma})$ $(\mathit{\Gamma} \in \mathit{\Pi}_n)$ and the limit curve γ^*, based on the *tangential parameterization* of convex paths (see the Appendix, Section A.1). More specifically, for $t \in [0, \infty]$ denote by $\tilde{\xi}_n(t)$ the right endpoint of that part of

$\tilde{\mathit{\Gamma}}_n$ where the tangent slope (wherever it exists) does not exceed t. Similarly, the tangential parameterization of the parabola arc γ^* [see (1.3)] is given by the vector function

$$g^*(t) = \left(\frac{t^2+2t}{(1+t)^2}, \frac{t^2}{(1+t)^2}\right), \qquad 0 \le t \le \infty. \tag{1.8}$$

The *tangential distance* between $\tilde{\mathit{\Gamma}}_n$ and γ^* is then defined as

$$d_{\mathcal{T}}(\tilde{\mathit{\Gamma}}_n, \gamma^*) := \sup_{0\le t\le\infty} \bigl|\tilde{\xi}_n(t) - g^*(t)\bigr|, \tag{1.9}$$

where, as before, $|\cdot|$ is the Euclidean vector norm in $\mathbb{R}^2$ (cf. general definition (A.3) of the metric $d_{\mathcal{T}}(\cdot,\cdot)$ in the Appendix, Section A.1).

We can now state our main result about the universality of the limit shape γ^* under the measures P_n^r (cf. Theorem 8.2).

Theorem 1.1. *For each $r \in (0,\infty)$ and any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P_n^r\{d_{\mathcal{T}}(\tilde{\mathit{\Gamma}}_n, \gamma^*) \le \varepsilon\} = 1.$$

It can be shown (see the Appendix, Section A.1) that the Hausdorff distance $d_{\mathcal{H}}$ [see (1.2)] is dominated by the tangential distance $d_{\mathcal{T}}$ defined in (A.3) (however, these metrics are not equivalent). In particular, Theorem 1.1 with $r = 1$ recovers the limit shape result (1.4) for the uniform distribution on $\mathit{\Pi}_n$. As was mentioned above, in the original paper by Sinai [29] the proof of the limit shape result was only sketched, so even in the uniform case our proof seems to be the first complete implementation of Sinai's probabilistic method (which, as we will try to explain below, is far from straightforward).

Let us also point out that Theorem 1.1 is a *non-trivial* extension of (1.4) since the measures P_n^r ($r \ne 1$) are *not close* to the uniform distribution P_n^1 in total variation distance (denoted by $\|\cdot\|_{TV}$), and in fact $\|P_n^r - P_n^1\|_{TV} \to 1$ as $n \to \infty$ (see Theorem A.4 in the Appendix).

The result of Theorem 1.1 for "pure" measures P_n^r readily extends to mixed measures.

Theorem 1.2. *Let ρ be a probability measure on $(0,\infty)$, and set*

$$P_n^\rho(\mathit{\Gamma}) := \int_0^\infty P_n^r(\mathit{\Gamma})\,\rho(\mathrm{d}r), \qquad \mathit{\Gamma} \in \mathit{\Pi}_n. \tag{1.10}$$

Then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n^\rho\{d_{\mathcal{T}}(\tilde{\mathit{\Gamma}}_n, \gamma^*) \le \varepsilon\} = 1.$$

PROOF. Follows from equation (1.10) and Theorem 1.1 by Lebesgue's dominated convergence theorem. □

Theorem 1.2 shows that the limit shape result holds true (with the same limit γ^*) when the parameter r specifying the distribution P_n^r is chosen at random. Using the terminology designed for settings with random environments, Theorems 1.1 and 1.2 may be interpreted as "quenched" and "annealed" statements, respectively.

REMARK 1.2. The universality of the limit shape γ^*, established in Theorem 1.1, is not a general rule but rather an exception, holding for some, but not all, probability measures on the polygonal space $\varPi_n$. In fact, as was shown by Bogachev and Zarbaliev [7, 8], *any* C^3-smooth, strictly convex curve γ started at the origin may appear as the limit shape with respect to *some* probability measure P_n^γ on $\varPi_n$, as $n \to \infty$.

REMARK 1.3. The main results of the present paper have been recently reported (without proofs) in a brief note [9].

1.3. *Methods.* Our proof of Theorem 1.1 employs an elegant probabilistic approach first applied to convex lattice polygonal lines by Sinai [29]. This method is based on randomization of the (fixed) right endpoint $\xi = n$ of polygonal lines $\varGamma \in \varPi_n$, leading to the interpretation of a given (e.g., uniform) measure P_n on $\varPi_n$ as the conditional distribution induced by a suitable probability measure Q_z [depending on an auxiliary "free" parameter $z = (z_1, z_2)$] defined on the "global" space $\varPi = \cup_n \varPi_n$ of *all* convex lattice polygonal lines (with finitely many edges). To make the measure Q_z closer to P_n on the subspace $\varPi_n \subset \varPi$ specified by the condition $\xi = n$, it is natural to pick the parameter z from the asymptotic equation $E_z(\xi) = n\,(1 + o(1))$ $(n \to \infty)$. Then, in principle, asymptotic properties of polygonal lines $\varGamma$ (e.g., the limit shape) can be established first for $(\varPi, Q_z)$ and then transferred to $(\varPi_n, P_n)$ via conditioning with respect to $\varPi_n$ and using an appropriate local limit theorem for the probability $Q_z\{\xi = n\}$. A great advantage of working with the measure Q_z is that it may be chosen as a "multiplicative statistic" [34, 35] (i.e., a direct product of one-dimensional probability measures), thus corresponding to the distribution of a sequence of independent random variables, which immediately brings in insights and well-developed analytical tools of probability theory.

Sinai's approach in [29] was motivated by a heuristic analogy with statistical mechanics, where similar ideas are well known in the context of asymptotic equivalence, in the thermodynamic limit, of various statistical ensembles (i.e., *microcanonical*, *canonical*, and *grand canonical*) that may be associated with a given physical system (e.g., gas) by optional fixing of the total energy and/or the number of particles (see Ruelle [28]). In particular, Khinchin [22, 23] has pioneered a systematic use of local limit theorems of probability theory in problems of statistical mechanics. Deep connections between statistical properties of quantum systems (where discrete random structures naturally arise due to quantization) and asymptotic theory of random integer partitions are discussed in a series of papers by Vershik [34, 35] (see also the recent work by Comtet et al. [11] and further references therein). Note also that a general idea of randomization has proved instrumental in a large variety of combinatorial problems (see, e.g., [2, 1, 13, 15, 16, 24, 26, 33] and the vast bibliography therein).

The probabilistic method is very insightful and efficient, as it makes the arguments heuristically transparent and natural. However, the practical implementation of this approach requires substantial work, especially in the two-dimensional context of convex lattice polygonal lines as compared to the one-dimensional case exemplified by integer partitions and the corresponding Young diagrams [33, 34]. To begin with, evaluation of expected values and some higher-order statistical moments of random polygonal lines leads one to deal with various sums over the set $\mathcal{X}$ of points $x=(x_1,x_2)\in\mathbb{Z}_+^2$ with co-prime coordinates (see Section 2.1). Sinai [29] was able to obtain the limit of some basic sums of such a kind by appealing to the known asymptotic density of the set $\mathcal{X}$ in $\mathbb{Z}_+^2$ (given by $6/\pi^2$); however, this argumentation is insufficient for more refined asymptotics. In the present paper, we handle this technical problem by using the Möbius inversion formula (see Section 3), which enables one to reduce sums over $\mathcal{X}$ to more regular sums.

As already mentioned, another crucial ingredient required for the probabilistic method is a suitable local limit theorem that furnishes a "bridge" between the global distribution Q_z and the conditional one, P_n. Analytical difficulties encountered in the proof of such a result are already significant in the case of ordinary integer partitions (for more details and concrete examples, see [1, 13, 14, 15, 16] and further references therein). The case of convex lattice polygonal lines, corresponding to two-dimensional strict vector partitions (see Remark 1.1), is notoriously tedious, even though the standard method of characteristic functions is still applicable. To the best of our knowledge, after the original paper by Sinai [29] where the result was just stated [with a minor error in the determinant of the covariance matrix ([29], page 111)], full details have not been worked out in the literature (however, see [40]). We prove the following result in this direction (cf. Theorem 7.1).

THEOREM 1.3. *Suppose that the parameter z is chosen so that $a_z := E_z^r(\xi) = n\,(1+o(1))$. Then, as $n\to\infty$,*

$$Q_z^r\{\xi=n\}\sim\frac{1}{2\pi\,(\det K_z)^{1/2}}\,\exp\Bigl(-\tfrac{1}{2}\bigl|(n-a_z)K_z^{-1/2}\bigr|^2\Bigr),\tag{1.11}$$

where $K_z:=\mathrm{Cov}(\xi,\xi)$ is the covariance matrix of the random vector ξ (with respect to the probability measure Q_z^r).

REMARK 1.4. The quantities a_z and K_z, obtained via the measure Q_z^r, depend in general on the parameter r as well. For the sake of notational convenience the latter is omitted, which should cause no confusion since r is always fixed, unless stated explicitly otherwise.

One can show that the covariance matrix K_z is of the order of $|n|^{4/3}$, and in particular $\det K_z\sim\mathrm{const}\,(n_1n_2)^{4/3}$ and $\bigl\|K_z^{-1/2}\bigr\|=O\bigl(|n|^{-2/3}\bigr)$. From the right-hand side of (1.11), it is then clear that one needs to refine the error term in the asymptotic relation $E_z^r(\xi)=n\,(1+o(1))$ by estimating the deviation $E_z^r(\xi)-n$ to at least the order of $|n|^{2/3}$. We have been able to obtain the following estimate (cf. Theorem 5.1).

THEOREM 1.4. *Under the conditions of Theorem 1.3,*

$$E_z^r(\xi)=n+o\bigl(|n|^{2/3}\bigr),\qquad n\to\infty.\tag{1.12}$$

The proof of this result is quite involved. The main idea is to apply the Mellin transform and use the inversion formula to obtain a suitable integral representation for the difference E_z^r-n of the form $(j=1,2)$

$$E_z^r(\xi_j)-n_j=\frac{r}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\widetilde{F}_j(s)\zeta(s+1)}{(-\ln z_j)^{s+1}\zeta(s)}\,\mathrm{d}s\qquad(1<c<2),\tag{1.13}$$

where $\ln z_j$ happens to be of the order of $|n|^{-1/3}$ (according to the "optimal" choice of z as explained at the beginning of Section 1.3; cf. Theorem 3.1), $\widetilde{F}_j(s)$ is an explicit function analytic in the strip $1<\Re s<2$ and $\zeta(s)$ is the Riemann zeta function. As usual, to obtain a better estimate of the integral one has to shift the integration contour in (1.13) as far to the left as possible, and it turns out that to get an estimate of order $o\bigl(|n|^{2/3}\bigr)$ one needs to enter the critical strip $0<\Re s<1$, which requires information about zeroes of the zeta function in view of the denominator $\zeta(s)$ in (1.13).

Layout. The rest of the paper is organized as follows. In Section 2, we explain the basics of the probability method in the polygonal context and define the parametric families of measures Q_z^r and P_n^r $(0 < r < \infty)$. In Section 3, we choose suitable values of the parameter $z = (z_1, z_2)$ (Theorem 3.1), which implies convergence of "expected" polygonal lines to the limit curve γ^* (Section 4, Theorems 4.1 and 4.2). The refined error estimate (1.12) is proved in Section 5 (Theorem 5.1). Higher-order moment sums are analyzed in Section 6; in particular, the asymptotics of the covariance matrix K_z is obtained in Theorem 6.1. Section 7 is devoted to the proof of the local central limit theorem (Theorem 7.1). Finally, the limit shape result, with respect to both Q_z^r and P_n^r, is proved in Section 8 (Theorems 8.1 and 8.2). The Appendix includes necessary details about the tangential parameterization and the tangential metric $d_{\mathcal{T}}$ on the space of convex paths (Section A.1), as well as a discussion of the total variation distance between the measures P_n^r $(r \neq 1)$ and the uniform distribution P_n^1 (Section A.2, Theorems A.2 and A.4).

Notation. Let us fix some general notations frequently used in the paper. For a row-vector $x = (x_1, x_2) \in \mathbb{R}^2$, its Euclidean norm (length) is denoted by $|x| := (x_1^2 + x_2^2)^{1/2}$, and $\langle x, y\rangle := x\,y^{\top} = x_1y_1 + x_2y_2$ is the corresponding inner product of vectors $x, y \in \mathbb{R}^2$. We denote $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k \geq 0\}$, $\mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$, and similarly $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$.

2. Probability measures on spaces of convex polygonal lines.

2.1. *Encoding.* As was observed by Sinai [29], one can encode convex lattice polygonal lines via suitable integer-valued functions. More specifically, consider the set $\mathcal{X}$ of all pairs of co-prime non-negative integers,

$$\mathcal{X} := \{x = (x_1, x_2) \in \mathbb{Z}_+^2 : \gcd(x_1, x_2) = 1\}, \tag{2.1}$$

where $\gcd(\cdot,\cdot)$ stands for the greatest common divisor of two integers. (In particular, the pairs $(0, 1)$ and $(1, 0)$ are included in this set, while $(0, 0)$ is not.) Let $\varPhi := (\mathbb{Z}_+)^{\mathcal{X}}$ be the space of functions on $\mathcal{X}$ with non-negative integer values, and consider the subspace of functions with *finite support*,

$$\varPhi_0 := \{\nu \in \varPhi : \#(\operatorname{supp} \nu) < \infty\},$$

where $\operatorname{supp} \nu := \{x \in \mathcal{X} : \nu(x) > 0\}$. It is easy to see that the space $\varPhi_0$ is in one-to-one correspondence with the space $\varPi = \cup_{n\in\mathbb{Z}_+^2} \varPi_n$ of all (finite) convex lattice polygonal lines

$$\varPhi_0 \ni \nu \;\longleftrightarrow\; \varGamma \in \varPi.$$

Indeed, let us interpret points $x \in \mathcal{X}$ as radius-vectors (pointing from the origin to x). Now, for any $\nu \in \varPhi_0$, a finite collection of nonzero vectors $\{x\nu(x),\, x \in \operatorname{supp}\nu\}$, arranged in the order of increase of their slope $x_2/x_1 \in [0,\infty]$, determines consecutive edges of some convex lattice polygonal line $\varGamma \in \varPi$. Conversely, vector edges of a lattice polygonal line $\varGamma \in \varPi$ can be uniquely represented in the form xk, with $x \in \mathcal{X}$ and integer $k > 0$; setting $\nu(x) := k$ for such x and zero otherwise, we obtain a function $\nu \in \varPhi_0$. (The special case where $\nu(x) \equiv 0$ for all $x \in \mathcal{X}$ corresponds to the "trivial" polygonal line $\varGamma_0$ with coinciding endpoints.)

That is to say, each $x \in \mathcal{X}$ determines the *direction* of a potential edge, only utilized if $x \in \operatorname{supp}\nu$, in which case the value $\nu(x) > 0$ specifies the *scaling factor*, altogether yielding a vector edge $x\nu(x)$; finally, assembling all such edges into a polygonal line is uniquely determined by the fixation of the starting point (at the origin) and the convexity property.

Note that, according to the above construction, $\nu(x)$ has the meaning of the number of lattice points on the edge $x\nu(x)$ (except its left endpoint). The right endpoint $\xi = \xi_\varGamma$ of the polygonal line $\varGamma \in \varPi$ associated with a configuration $\nu \in \varPhi_0$ is expressed by the formula

$$\xi = \sum_{x\in\mathcal{X}} x\nu(x). \tag{2.2}$$

In what follows, we shall identify the spaces $\varPi$ and $\varPhi_0$. In particular, any probability measure on $\varPi$ can be treated as the distribution of a $\mathbb{Z}_+$-valued random field $\nu(\cdot)$ on $\mathcal{X}$ with almost surely (a.s.) finite support.

2.2. *Global measure Q_z and conditional measure P_n.* Let $b_0, b_1, b_2, \dots$ be a sequence of non-negative numbers such that $b_0 > 0$ (without loss of generality, we put $b_0 = 1$) and not all b_k vanish for $k \ge 1$, and assume that the generating function

$$\beta(s) := \sum_{k=0}^{\infty} b_k s^k \tag{2.3}$$

is finite for $|s| < 1$. Let $z = (z_1, z_2)$ be a two-dimensional parameter, with $z_1, z_2 \in (0,1)$. Throughout the paper, we shall use the multi-index notation

$$z^x := z_1^{x_1} z_2^{x_2}, \qquad x = (x_1, x_2) \in \mathbb{Z}_+^2.$$

We now define the "global" probability measure Q_z on the space $\varPhi = (\mathbb{Z}_+)^{\mathcal{X}}$ as the distribution of a random field $\nu = \{\nu(x),\ x \in \mathcal{X}\}$ with mutually independent values and marginal distributions of the form

$$Q_z\{\nu(x) = k\} = \frac{b_k z^{kx}}{\beta(z^x)}, \qquad k \in \mathbb{Z}_+ \qquad (x \in \mathcal{X}). \tag{2.4}$$

PROPOSITION 2.1. *For each $z \in (0,1)^2$, the condition*

$$\tilde{\beta}(z) := \prod_{x\in\mathcal{X}} \beta(z^x) < \infty \tag{2.5}$$

is necessary and sufficient in order that $Q_z(\varPhi_0) = 1$.

PROOF. According to (2.4), $Q_z\{\nu(x) > 0\} = 1 - \beta(z^x)^{-1}$ ($x \in \mathcal{X}$). Since the random variables $\nu(x)$ are mutually independent for different $x \in \mathcal{X}$, Borel–Cantelli's lemma implies that $Q_z\{\nu \in \varPhi_0\} = 1$ if and only if

$$\sum_{x\in\mathcal{X}} \left(1 - \frac{1}{\beta(z^x)}\right) < \infty.$$

In turn, the latter inequality is equivalent to (2.5). □

That is to say, under condition (2.5) a sample configuration of the random field $\nu(\cdot)$ belongs (Q_z-a.s.) to the space $\varPhi_0$ and therefore determines a (random) finite polygonal line $\varGamma \in \varPi$. By the mutual independence of the values $\nu(x)$, the corresponding Q_z-probability is given by

$$Q_z(\varGamma) = \prod_{x\in\mathcal{X}} \frac{b_{\nu(x)}\, z^{x\nu(x)}}{\beta(z^x)} = \frac{b(\varGamma)\, z^{\xi}}{\tilde{\beta}(z)}, \qquad \varGamma \in \varPi, \tag{2.6}$$

where $\xi = \sum_{x\in\mathcal{X}} x\nu(x)$ is the right endpoint of $\varGamma$ [see (2.2)], and

$$b(\varGamma) := \prod_{x\in\mathcal{X}} b_{\nu(x)} < \infty, \qquad \varGamma \in \varPi. \tag{2.7}$$

Note that the infinite product in (2.7) contains only finitely many terms different from 1, since for $x \notin \operatorname{supp}\nu$ we have $b_{\nu(x)} = b_0 = 1$. Hence, expression (2.7) can be rewritten in a more intrinsic form [cf. (1.6)]

$$b(\varGamma) := \prod_{e_i\in\varGamma} b_{k_i}, \tag{2.8}$$

where the product is taken over all edges e_i of $\varGamma \in \varPi_n$ and k_i is the number of lattice points on the edge e_i except its left endpoint (see Section 2.1).

In particular, for the trivial polygonal line $\varGamma_0 \leftrightarrow \nu \equiv 0$ formula (2.6) yields

$$Q_z(\varGamma_0) = \tilde{\beta}(z)^{-1} > 0.$$

Note, however, that $Q_z(\varGamma_0) < 1$ since, due to our assumptions, (2.3) implies $\beta(s) > \beta(0) = 1$ for $s > 0$ and hence, according to (2.5), $\tilde{\beta}(z) > 1$.

On the subspace $\mathit{\Pi}_n \subset \mathit{\Pi}$ of polygonal lines with the right endpoint fixed at $n = (n_1, n_2)$, the measure Q_z induces the conditional distribution

$$P_n(\mathit{\Gamma}) := Q_z(\mathit{\Gamma} | \mathit{\Pi}_n) = \frac{Q_z(\mathit{\Gamma})}{Q_z(\mathit{\Pi}_n)}, \qquad \mathit{\Gamma} \in \mathit{\Pi}_n, \tag{2.9}$$

provided, of course, that $Q_z(\mathit{\Pi}_n) > 0$ [i.e., there is at least one $\mathit{\Gamma} \in \mathit{\Pi}_n$ with $b(\mathit{\Gamma}) > 0$, cf. (2.6)]. The parameter z may be dropped from the notation for P_n due to the following fact.

PROPOSITION 2.2. *The measure P_n in (2.9) does not depend on z.*

PROOF. If $\mathit{\Pi}_n \ni \mathit{\Gamma} \leftrightarrow \nu \in \mathit{\Phi}_0$ then $\xi = n$ and hence formula (2.6) is reduced to

$$Q_z(\mathit{\Gamma}) = \frac{b(\mathit{\Gamma}) z^n}{\tilde{\beta}(z)}, \qquad \mathit{\Gamma} \in \mathit{\Pi}_n.$$

Accordingly, using (2.5) and (2.9) we get the expression

$$P_n(\mathit{\Gamma}) = \frac{b(\mathit{\Gamma})}{\sum_{\mathit{\Gamma}' \in \mathit{\Pi}_n} b(\mathit{\Gamma}')}, \qquad \mathit{\Gamma} \in \mathit{\Pi}_n, \tag{2.10}$$

which is z-free. □

2.3. *Parametric families $\{Q_z^r\}$ and $\{P_n^r\}$.* Let us consider a special parametric family of measures $\{Q_z^r,\ 0 < r < \infty\}$, determined by formula (2.4) with the coefficients b_k of the form

$$b_k^r := \binom{r+k-1}{k} = \frac{r(r+1)\cdots(r+k-1)}{k!}, \qquad k \in \mathbb{Z}_+ \tag{2.11}$$

(note that $b_0^r = \binom{r-1}{0} = 1$, in accordance with our convention in Section 2.2).

By the binomial expansion formula, the generating function (2.3) of the sequence (2.11) is given by

$$\beta^r(s) = (1-s)^{-r}, \qquad |s| < 1, \tag{2.12}$$

and from (2.4) it follows that under the law Q_z^r the random variable $\nu(x)$ has the probability generating function

$$E_z^r\bigl(s^{\nu(x)}\bigr) = \frac{\beta^r(sz^x)}{\beta^r(z^x)} = \frac{(1-z^x)^r}{(1-sz^x)^r}, \qquad 0 \le s \le 1. \tag{2.13}$$

Consequently, formula (2.4) specializes to

$$Q_z^r\{\nu(x) = k\} = \binom{r+k-1}{k} z^{kx}(1-z^x)^r, \qquad k \in \mathbb{Z}_+ \qquad (x \in \mathcal{X}). \tag{2.14}$$

That is to say, with respect to the measure Q_z^r the random variable $\nu(x)$ has a negative binomial distribution with parameters r and $p = 1 - z^x$ ([12], Section VI.8, page 165]); in particular, its expected value and variance are given by (see [12], Section XI.2, page 269)

$$E_z^r[\nu(x)] = \frac{r z^x}{1 - z^x}, \qquad \mathrm{Var}[\nu(x)] = \frac{r z^x}{(1 - z^x)^2} \qquad (x \in \mathcal{X}). \tag{2.15}$$

According to formulas (2.9) and (2.10), the corresponding conditional measure $P_n^r(\cdot) := Q_z^r(\cdot\,|\varPi_n)$ is expressed as

$$P_n^r(\varGamma) = \frac{Q_z^r(\varGamma)}{Q_z^r(\varPi_n)} = \frac{b^r(\varGamma)}{\sum_{\varGamma' \in \varPi_n} b^r(\varGamma')}, \qquad \varGamma \in \varPi_n, \tag{2.16}$$

where $b^r(\varGamma)$ is given by the general formula (2.8) specialized to the coefficients b_k^r defined in (2.11).

In the special case $r = 1$, we have $b_k^1 = \binom{k}{k} \equiv 1$ so that (2.14) is reduced to the geometric distribution (with parameter $p = 1 - z^x$)

$$Q_z^1\{\nu(x) = k\} = z^{kx}(1 - z^x), \qquad k \in \mathbb{Z}_+ \qquad (x \in \mathcal{X}),$$

whereas the conditional measure (2.16) specifies the uniform distribution on $\varPi_n$ (cf. [29])

$$P_n^1(\varGamma) = \frac{1}{\#(\varPi_n)}, \qquad \varGamma \in \varPi_n.$$

REMARK 2.1. Since $b_{k+1}^r / b_k^r = (r + k)/(k + 1)$, the sequence $\{b_k^r\}$ is strictly increasing or decreasing in k according as $r > 1$ or $r < 1$, respectively. That is to say, the measures Q_z^r and P_n^r encourage ($r > 1$) or discourage ($r < 1$) lattice points on edges, as compared to the reference case $r = 1$.

It is easy to see that condition (2.5) is satisfied and, by Proposition 2.1,

$$Q_z^r(\varPhi_0) = 1, \qquad 0 < r < \infty.$$

Indeed, using (2.12) we have

$$\tilde{\beta}^r(z) = \prod_{x \in \mathcal{X}} (1 - z^x)^{-r} = \exp\left(-r \sum_{x \in \mathcal{X}} \ln(1 - z^x)\right) < \infty$$

whenever $\sum_{x \in \mathcal{X}} \ln(1 - z^x) > -\infty$, and the latter condition is fulfilled since

$$\sum_{x \in \mathcal{X}} z^x \le \sum_{x \in \mathbb{Z}_+^2} z^x = \sum_{x_1=0}^{\infty} z_1^{x_1} \sum_{x_2=0}^{\infty} z_2^{x_2} = \frac{1}{(1 - z_1)(1 - z_2)} < \infty.$$

3. Calibration of the parameter z. In what follows, the asymptotic notation of the form $a_n \asymp b_n$ [where $n = (n_1, n_2)$] means that

$$0 < \liminf_{n_1,n_2\to\infty} \frac{a_n}{b_n} \le \limsup_{n_1,n_2\to\infty} \frac{a_n}{b_n} < \infty.$$

We also use the standard notation $a_n \sim b_n$ for $a_n/b_n \to 1$ as $n_1, n_2 \to \infty$.

Throughout the paper, we shall work under the following convention about the limit $n = (n_1, n_2) \to \infty$.

ASSUMPTION 3.1. The notation $n \to \infty$ signifies that $n_1, n_2 \to \infty$ in such a way that $n_1 \asymp n_2$. In particular, this implies that $|n| = (n_1^2+n_2^2)^{1/2} \to \infty$ as $n \to \infty$, and $n_1 \asymp |n|$, $n_2 \asymp |n|$.

The goal of this section is to use the freedom of the conditional distribution $P_n^r(\cdot) = Q_z^r(\cdot\,|\mathit{\Pi}_n)$ from the parameter z (see Proposition 2.2) in order to better adapt the measure Q_z^r to the subspace $\mathit{\Pi}_n \subset \mathit{\Pi}$ determined by the condition $\xi = n$ [where $\xi = (\xi_1, \xi_2)$ is defined in (2.2)]. To this end, it is natural to require that the latter condition be satisfied (at least asymptotically) for the *expected* value of ξ (cf. [29, 6]). More precisely, we will seek $z = (z_1, z_2)$ as a solution to the following asymptotic equations:

$$E_z^r(\xi_1) \sim n_1, \qquad E_z^r(\xi_2) \sim n_2 \qquad (n \to \infty), \tag{3.1}$$

where E_z^r denotes expectation with respect to the distribution Q_z^r.

From (2.2), using the first formula in (2.15), we obtain

$$E_z^r(\xi) = \sum_{x\in\mathcal{X}} x\, \frac{r z^x}{1 - z^x} = r \sum_{k=1}^{\infty} \sum_{x\in\mathcal{X}} x\, z^{kx}. \tag{3.2}$$

Let us represent the parameters z_1, z_2 in the form

$$z_j = \mathrm{e}^{-\alpha_j}, \qquad \alpha_j = \delta_j\, n_j^{-1/3} \qquad (j = 1, 2), \tag{3.3}$$

where the quantities $\delta_1, \delta_2 > 0$ (possibly depending on the ratio n_2/n_1) are presumed to be bounded from above and separated from zero. Hence, (3.2) takes the form

$$E_z^r(\xi) = r \sum_{k=1}^{\infty} \sum_{x\in\mathcal{X}} x\, \mathrm{e}^{-k\langle\alpha,x\rangle}. \tag{3.4}$$

THEOREM 3.1. *Conditions* (3.1) *are satisfied if* δ_1, δ_2 *in* (3.3) *are chosen to be*

$$\delta_1 = \kappa r^{1/3}(n_2/n_1)^{1/3}, \qquad \delta_2 = \kappa r^{1/3}(n_1/n_2)^{1/3}, \tag{3.5}$$

where $\kappa := \bigl(\zeta(3)/\zeta(2)\bigr)^{1/3}$ *and* $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ *is the Riemann zeta function.*

PROOF. Let us prove the first of the asymptotic relations (3.1). Set

$$f(x) := r x_1 \mathrm{e}^{-\langle \alpha, x\rangle}, \qquad x \in \mathbb{R}_+^2, \tag{3.6}$$

and

$$F^\sharp(h) := \sum_{x\in\mathcal{X}} f(hx), \qquad h > 0.$$

Then we can rewrite (3.4) in projection to the first coordinate as

$$E_z^r(\xi_1) = \sum_{k=1}^{\infty} \sum_{x\in\mathcal{X}} \frac{f(kx)}{k} = \sum_{k=1}^{\infty} \frac{F^\sharp(k)}{k}. \tag{3.7}$$

Let us also consider the function

$$F(h) := \sum_{m=1}^{\infty} F^\sharp(hm) = \sum_{m=0}^{\infty} \sum_{x\in\mathcal{X}} f(hmx), \qquad h > 0 \tag{3.8}$$

(adding terms with $m = 0$ does not affect the sum, since $f(\cdot)$ vanishes at the origin). Recalling the definition of the set $\mathcal{X}$ [see (2.1)], we note that $\mathbb{Z}_+^2$ can be decomposed as a *disjoint* union of multiples of $\mathcal{X}$: $\mathbb{Z}_+^2 = \bigsqcup_{m=0}^{\infty} m\mathcal{X}$. Hence, the double sum in (3.8) is reduced to

$$\begin{aligned} F(h) &= \sum_{x\in\mathbb{Z}_+^2} f(hx) = rh \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-h\alpha_1 x_1} \sum_{x_2=0}^{\infty} \mathrm{e}^{-h\alpha_2 x_2} \\ &= \frac{rh\mathrm{e}^{-h\alpha_1}}{(1-\mathrm{e}^{-h\alpha_1})^2(1-\mathrm{e}^{-h\alpha_2})}. \end{aligned} \tag{3.9}$$

By the Möbius inversion formula (see [18], Theorem 270, page 237)

$$F(h) = \sum_{m=1}^{\infty} F^\sharp(hm) \iff F^\sharp(h) = \sum_{m=1}^{\infty} \mu(m) F(hm), \tag{3.10}$$

where $\mu(m)$ ($m \in \mathbb{N}$) is the *Möbius function* defined as follows: $\mu(1) = 1$, $\mu(m) = (-1)^d$ if m is a product of d different primes and $\mu(m) = 0$ if m

has a squared factor ([18], Section 16.3, page 234); in particular, $|\mu(\cdot)| \le 1$. A sufficient condition for (3.10) is that the double series $\sum_{k,m} |F^\sharp(hkm)|$ should be convergent, which is easily verified in our case: $F^\sharp(\cdot) \ge 0$ and, according to (3.8) and (3.9),

$$\sum_{k,m=1}^{\infty} F^\sharp(kmh) = \sum_{k=1}^{\infty} F(kh) = rh \sum_{k=1}^{\infty} \frac{k\,\mathrm{e}^{-hk\alpha_1}}{(1-\mathrm{e}^{-hk\alpha_1})^2(1-\mathrm{e}^{-hk\alpha_2})} < \infty.$$

Using (3.9) and (3.10), we can rewrite (3.7) as

$$E_z^r(\xi_1) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^{\infty} \mu(m) F(km) = \sum_{k,m=1}^{\infty} \frac{rm\mu(m)\,\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2(1-\mathrm{e}^{-km\alpha_2})}. \tag{3.11}$$

Note that (3.3) and (3.5) imply

$$\alpha_1^2\alpha_2 = \frac{r\kappa^3}{n_1}, \qquad \alpha_1\alpha_2^2 = \frac{r\kappa^3}{n_2}, \qquad \alpha_2 n_2 = \alpha_1 n_1, \tag{3.12}$$

where κ is defined in Theorem 3.1. Hence, we can rewrite (3.11) in the form

$$n_1^{-1} E_z^r(\xi_1) = \frac{1}{\kappa^3} \sum_{k,m=1}^{\infty} \frac{m\mu(m)\alpha_1^2\alpha_2\,\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2(1-\mathrm{e}^{-km\alpha_2})}. \tag{3.13}$$

We now need an elementary estimate, which will also be instrumental later on.

LEMMA 3.2. *For any $k > 0$, $\theta > 0$, there exists $C = C(k,\theta) > 0$ such that, for all $t > 0$,*

$$\frac{\mathrm{e}^{-\theta t}}{(1-\mathrm{e}^{-t})^k} \le \frac{C\,\mathrm{e}^{-\theta t/2}}{t^k}. \tag{3.14}$$

PROOF OF LEMMA 3.2. Set $g(t) := t^k \mathrm{e}^{-\theta t/2}(1-\mathrm{e}^{-t})^{-k}$ and note that

$$\lim_{t\to 0+} g(t) = 1, \qquad \lim_{t\to\infty} g(t) = 0.$$

By continuity, the function $g(t)$ is bounded on $(0,\infty)$, and (3.14) follows. □

By Lemma 3.2, the general term of the series (3.13) is estimated, uniformly in k and m, by $O\big(k^{-3}m^{-2}\big)$. Hence, by Lebesgue's dominated convergence theorem one can pass to the limit in (3.13) termwise

$$\lim_{n\to\infty} n_1^{-1} E_z^r(\xi_1) = \frac{1}{\kappa^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{1}{\kappa^3} \cdot \frac{\zeta(3)}{\zeta(2)} = 1. \tag{3.15}$$

Here the expression for the second sum (over m) is obtained using the Möbius inversion formula (3.10) with $F^{\sharp}(h) = h^{-2}$, $F(h) = \sum_{m=1}^{\infty}(hm)^{-2} = h^{-2}\zeta(2)$ (cf. [18], Theorem 287, page 250).

Similarly, we can check that, as $n \to \infty$,

$$n_2^{-1} E_z^r(\xi_2) = \frac{1}{\kappa^3} \sum_{k,m=1}^{\infty} \frac{m\mu(m)\,\alpha_1\alpha_2^2\,\mathrm{e}^{-km\alpha_2}}{(1-\mathrm{e}^{-km\alpha_1})(1-\mathrm{e}^{-km\alpha_2})^2} \to 1.$$

The theorem is proved. □

Remark 3.1. The term $1/\zeta(2) = 6/\pi^2$ appearing in formula (3.15) and the like, equals the asymptotic density of co-prime pairs $x = (x_1, x_2) \in \mathcal{X}$ among all integer points on $\mathbb{Z}_+^2$ (see [18], Theorem 459, page 409).

Assumption 3.2. Throughout the rest of the paper, we assume that the parameters z_1, z_2 are chosen according to formulas (3.3), (3.5). In particular, the measure Q_z^r becomes dependent on $n = (n_1, n_2)$, as well as all Q_z^r-probabilities and mean values.

4. Asymptotics of "expected" polygonal lines. For $\varGamma \in \varPi$, denote by $\varGamma(t)$ ($t \in [0,\infty]$) the part of $\varGamma$ where the slope does not exceed tn_2/n_1. Hence, the path $\tilde{\varGamma}_n(t) = S_n(\varGamma(t))$ serves as a tangential parameterization of the scaled polygonal line $\tilde{\varGamma}_n = S_n(\varGamma)$, where $S_n(x_1,x_2) = (x_1/n_1, x_2/n_2)$ (see Section 1.2, and also Section A.1 below). Consider the set

$$\mathcal{X}(t) := \{x \in \mathcal{X} : \, x_2/x_1 \le tn_2/n_1\}, \qquad t \in [0,\infty]. \tag{4.1}$$

According to the association $\varPi \ni \varGamma \leftrightarrow \nu \in \varPhi_0$ described in Section 2.1, for each $t \in [0,\infty]$ the polygonal line $\varGamma(t)$ is determined by a truncated configuration $\{\nu(x),\, x \in \mathcal{X}(t)\}$, hence its right endpoint $\xi(t) = (\xi_1(t), \xi_2(t))$ is given by

$$\xi(t) = \sum_{x \in \mathcal{X}(t)} x\,\nu(x), \qquad t \in [0,\infty]. \tag{4.2}$$

In particular, $\mathcal{X}(\infty) = \mathcal{X}$, $\xi(\infty) = \xi$ [cf. (2.2)]. Similarly to (3.2) and (3.4),

$$E_z^r[\xi(t)] = r \sum_{k=1}^{\infty} \sum_{x \in \mathcal{X}(t)} x\,\mathrm{e}^{-k\langle \alpha, x\rangle}, \qquad t \in [0,\infty]. \tag{4.3}$$

Let us also set [cf. (1.8)]

$$g_1^*(t) := \frac{t^2+2t}{(1+t)^2}, \qquad g_2^*(t) := \frac{t^2}{(1+t)^2}, \qquad t \in [0,\infty]. \tag{4.4}$$

As will be verified in the Appendix (see Section A.1), the vector-function $g^*(t) = \bigl(g_1^*(t), g_2^*(t)\bigr)$ gives a tangential parameterization of the parabola γ^* defined in (1.3).

The goal of this section is to establish the convergence of the (scaled) expectation $E_z^r[\xi(t)]$ to the limit $g^*(t)$, first for each $t \in [0,\infty]$ (Section 4.1) and then uniformly in $t \in [0,\infty]$ (Section 4.2).

4.1. *Pointwise convergence.*

THEOREM 4.1. *For each* $t \in [0,\infty]$,

$$\lim_{n\to\infty} n_j^{-1} E_z^r[\xi_j(t)] = g_j^*(t) \qquad (j=1,2). \tag{4.5}$$

PROOF. Theorem 3.1 implies that (4.5) holds for $t=\infty$. Assume that $t<\infty$ and let $j=1$ (the case $j=2$ is considered in a similar manner). Setting for brevity $c_n := n_2/n_1$ and arguing as in the proof of Theorem 3.1 [see (3.4), (3.7) and (3.13)], from (4.3) we obtain

$$\begin{aligned} E_z^r[\xi_1(t)] &= r \sum_{k,m=1}^{\infty} m\mu(m) \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1} \sum_{x_2=0}^{\hat{x}_2} \mathrm{e}^{-km\alpha_2 x_2} \\ &= r \sum_{k,m=1}^{\infty} \frac{m\mu(m)}{1-\mathrm{e}^{-km\alpha_2}} \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1} \Bigl(1-\mathrm{e}^{-km\alpha_2(\hat{x}_2+1)}\Bigr), \end{aligned} \tag{4.6}$$

where $\hat{x}_2 = \hat{x}_2(t)$ denotes the integer part of $tc_n x_1$, so that

$$0 \le tc_n x_1 - \hat{x}_2 < 1. \tag{4.7}$$

Aiming to replace $\hat{x}_2 + 1$ by $tc_n x_1$ in (4.6), we recall (3.12) and rewrite the sum over x_1 as

$$\sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1} \bigl(1-\mathrm{e}^{-km\alpha_1 t x_1}\bigr) + \Delta_{k,m}(t,\alpha), \tag{4.8}$$

where

$$\Delta_{k,m}(t,\alpha) := \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1(1+t)} \Bigl(1-\mathrm{e}^{-km\alpha_2(\hat{x}_2+1-tc_n x_1)}\Bigr).$$

Using that $0 < \hat{x}_2 + 1 - tc_n x_1 \le 1$ [see (4.7)] and applying Lemma 3.2, we obtain, uniformly in $k,m \ge 1$ and $t \in [0,\infty]$,

$$0 < \frac{\Delta_{k,m}(t,\alpha)}{1-\mathrm{e}^{-km\alpha_2}} \le \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1} = \frac{\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2} = O(1)\,\frac{\mathrm{e}^{-m\alpha_1/2}}{(km\alpha_1)^2}.$$

Substituting this estimate into (4.6), we see that the error resulting from the replacement of $\hat{x}_2 + 1$ by $tc_n x_1$ is dominated by

$$O\big(\alpha_1^{-2}\big) \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-m\alpha_1/2}}{m} = O\big(\alpha_1^{-2}\big) \ln\big(1 - \mathrm{e}^{-\alpha_1/2}\big) = O\big(\alpha_1^{-2} \ln \alpha_1\big).$$

Returning to representation (4.6) and computing the sum in (4.8), we find

$$E_z^r[\xi_1(t)] = r \sum_{k,m=1}^{\infty} \frac{m\mu(m)}{1 - \mathrm{e}^{-km\alpha_2}} \cdot \frac{\mathrm{e}^{-km\alpha_1 y}}{(1 - \mathrm{e}^{-km\alpha_1 y})^2} \bigg|_{y=1+t}^{y=1} + O\big(\alpha_1^{-2} \ln \alpha_1\big). \tag{4.9}$$

Passing to the limit by Lebesgue's dominated convergence theorem, similarly to the proof of Theorem 3.1 [cf. (3.15)] we get, as $n \to \infty$,

$$n_1^{-1} E_z^r[\xi_1(t)] \to \frac{1}{\kappa^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} \left(1 - \frac{1}{(1+t)^2}\right) = \frac{t^2 + 2t}{(1+t)^2},$$

which coincides with $g_1^*(t)$, as claimed. □

4.2. *Uniform convergence.* There is a stronger version of Theorem 4.1.

THEOREM 4.2. *Convergence in* (4.5) *is uniform in* $t \in [0, \infty]$, *that is,*

$$\lim_{n\to\infty} \sup_{0 \le t \le \infty} \big| n_j^{-1} E_z^r[\xi_j(t)] - g_j^*(t) \big| = 0 \qquad (j = 1, 2).$$

For the proof, we need the following general lemma.

LEMMA 4.3. *Let* $\{f_n(t)\}$ *be a sequence of nondecreasing functions on a finite interval* $[a, b]$, *such that, for each* $t \in [a, b]$, $\lim_{n\to\infty} f_n(t) = f(t)$, *where* $f(t)$ *is a continuous* (*nondecreasing*) *function on* $[a, b]$. *Then the convergence* $f_n(t) \to f(t)$ *as* $n \to \infty$ *is uniform on* $[a, b]$.

PROOF OF LEMMA 4.3. Since f is continuous on a closed interval $[a, b]$, it is uniformly continuous. Therefore, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t') - f(t)| < \varepsilon$ whenever $|t' - t| < \delta$. Let $a = t_0 < t_1 < \cdots < t_N = b$ be a partition such that $\max_{1 \le i \le N}(t_i - t_{i-1}) < \delta$. Since $\lim_{n\to\infty} f_n(t_i) = f(t_i)$ for each $i = 0, 1, \ldots, N$, there exists n^* such that $\max_{0 \le i \le N} |f_n(t_i) - f(t_i)| < \varepsilon$ for all $n \ge n^*$. By monotonicity of f_n and f, this implies that for any $t \in [t_i, t_{i+1}]$ and all $n \ge n^*$

$$f_n(t) - f(t) \le f_n(t_{i+1}) - f(t_i) \le f_n(t_{i+1}) - f(t_{i+1}) + \varepsilon \le 2\varepsilon.$$

Similarly, $f_n(t) - f(t) \ge -2\varepsilon$. Therefore, $\sup_{t\in[a,b]} |f_n(t) - f(t)| \le 2\varepsilon$, and the uniform convergence follows. □

PROOF OF THEOREM 4.2. Suppose that $j = 1$ (the case $j = 2$ is handled similarly). Note that for each n the function

$$f_n(t) := n_1^{-1} E_z^r[\xi_1(t)] = \frac{1}{n_1} \sum_{x\in\mathcal{X}(t)} x_1 E_z^r[\nu(x)]$$

is nondecreasing in t. Therefore, by Lemma 4.3 the convergence (4.5) is uniform on any interval $[0, t^*]$. Furthermore, since $n_1^{-1}E_z^r[\xi_1(\infty)] \to g_1^*(\infty)$ and the function $g_1^*(t)$ is continuous at infinity [see (4.4)], for the proof of the uniform convergence on a suitable interval $[t^*, \infty]$ it suffices to show that for any $\varepsilon > 0$ one can choose t^* such that, for all large enough n_1, n_2 and all $t \ge t^*$,

$$n_1^{-1} E_z^r |\xi_1(\infty) - \xi_1(t)| \le \varepsilon. \tag{4.10}$$

On account of (4.9) we have

$$E_z^r[\xi_1(\infty) - \xi_1(t)] = \sum_{k,m=1}^{\infty} \frac{rm\mu(m)}{1 - \mathrm{e}^{-km\alpha_2}} \cdot \frac{\mathrm{e}^{-km\alpha_1(1+t)}}{\bigl(1 - \mathrm{e}^{-km\alpha_1(1+t)}\bigr)^2} + O\bigl(\alpha_1^{-2} \ln \alpha_1\bigr). \tag{4.11}$$

Note that by Lemma 3.2, uniformly in $k, m \ge 1$,

$$\frac{\mathrm{e}^{-km\alpha_2}}{1 - \mathrm{e}^{-km\alpha_2}} \cdot \frac{\mathrm{e}^{-km\alpha_1(1+t)}}{(1 - \mathrm{e}^{-km\alpha_1(1+t)})^2} = \frac{O(1)}{\alpha_1^2\alpha_2(km)^3(1 + t)^2}.$$

Returning to (4.11), we obtain, uniformly in $t \ge t^*$,

$$\alpha_1^2\alpha_2 E_z^r[\xi_1(\infty) - \xi_1(t)] = \frac{O(1)}{(1 + t)^2} \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{O(1)}{(1 + t^*)^2},$$

whence by (3.3) we get (4.10). □

5. Further refinement. For future applications, we need to refine the asymptotic formulas (3.1) by estimating the error term. The following theorem is one of the main technical ingredients of our work.

THEOREM 5.1. *Suppose that the parameter z is chosen according to formulas (3.3), (3.5), so that $E_z^r(\xi) = n\,(1 + o(1))$ (see Theorem 3.1). Then $E_z^r(\xi) = n + o\bigl(|n|^{2/3}\bigr)$ as $n \to \infty$.*

For the proof of this theorem, some preparations are needed.

5.1. *Approximation of sums by integrals.* Let a function $f\colon \mathbb{R}_+^2 \to \mathbb{R}$ be continuous and absolutely integrable on $\mathbb{R}_+^2$, together with its partial derivatives up to the second order. Set

$$F(h) := \sum_{x\in\mathbb{Z}_+^2} f(hx), \qquad h>0 \tag{5.1}$$

(as verified below, the series in (5.1) is absolutely convergent for all $h>0$), and assume that for some $\beta>2$

$$F(h) = O\bigl(h^{-\beta}\bigr), \qquad h\to\infty. \tag{5.2}$$

Consider the Mellin transform of $F(h)$ (see, e.g., [38], Chapter VI, Section 9),

$$\widehat{F}(s) := \int_0^\infty h^{s-1} F(h)\,\mathrm{d}h. \tag{5.3}$$

LEMMA 5.2. *Under the above conditions, the function $\widehat{F}(s)$ is meromorphic in the strip $1<\Re s<\beta$, with a single (simple) pole at $s=2$. Moreover, $\widehat{F}(s)$ satisfies the identity*

$$\widehat{F}(s) = \int_0^\infty h^{s-1}\Delta_f(h)\,\mathrm{d}h, \qquad 1<\Re s<2, \tag{5.4}$$

where

$$\Delta_f(h) := F(h) - \frac{1}{h^2}\int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x, \qquad h>0. \tag{5.5}$$

REMARK 5.1. Identity (5.4) is a two-dimensional analogue of Müntz's formula for univariate functions (see [31], Section 2.11, pages 28 and 29).

PROOF OF LEMMA 5.2. Let a function $\phi:\mathbb{R}_+\to\mathbb{R}$ be continuous and continuously differentiable, and suppose that both ϕ and ϕ' are absolutely integrable on $\mathbb{R}_+$. It follows that $\lim_{x\to\infty}\phi(x)=0$; indeed, note that

$$\int_0^\infty \phi'(x)\,\mathrm{d}x = \lim_{x\to\infty}\int_0^x \phi'(y)\,\mathrm{d}y = \lim_{x\to\infty}\phi(x)-\phi(0),$$

hence $\lim_{x\to\infty}\phi(x)$ exists and, since ϕ is integrable, the limit must equal zero. Then the well-known Euler–Maclaurin summation formula states that

$$\sum_{j=0}^\infty \phi(hj) = \frac{1}{h}\int_0^\infty \phi(x)\,\mathrm{d}x + \int_0^\infty \tilde{B}_1\Bigl(\frac{x}{h}\Bigr)\,\phi'(x)\,\mathrm{d}x, \tag{5.6}$$

where $\tilde{B}_1(x) := x - [x] - 1$ (cf. [5], Section A.4, page 254).

Applying formula (5.6) twice to the double series (5.1), we obtain

(5.7)
$$F(h) = \frac{1}{h^2}\int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x + \frac{1}{h}\int_{\mathbb{R}_+^2}\left(\tilde{B}_1\Big(\frac{x_1}{h}\Big)\frac{\partial f(x)}{\partial x_1} + \tilde{B}_1\Big(\frac{x_2}{h}\Big)\frac{\partial f(x)}{\partial x_2}\right)\mathrm{d}x + \int_{\mathbb{R}_+^2}\tilde{B}_1\Big(\frac{x_1}{h}\Big)\tilde{B}_1\Big(\frac{x_2}{h}\Big)\frac{\partial^2 f(x)}{\partial x_1\partial x_2}\,\mathrm{d}x.$$

Since $|\tilde{B}_1(\cdot)| \le 1$, the above conditions on the function f imply that all integrals in (5.7) exist, hence $F(h)$ is well defined for all $h > 0$. Moreover, from (5.7) it follows that

$$F(h) = O(h^{-2}), \qquad \Delta_f(h) = O(h^{-1}) \qquad (h \to 0). \tag{5.8}$$

Estimates (5.2) and (5.8) imply that $\widehat{F}(s)$ as defined in (5.3) is a regular function for $2 < \Re s < \beta$. Let us now note that for such s we can rewrite (5.3) as

$$\begin{aligned}\widehat{F}(s) &= \int_1^\infty h^{s-1}F(h)\,\mathrm{d}h + \int_0^1 h^{s-1}F(h)\,\mathrm{d}h\\ &= \int_1^\infty h^{s-1}F(h)\,\mathrm{d}h + \int_0^1 h^{s-3}\,\mathrm{d}h\int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x + \int_0^1 h^{s-1}\Delta_f(h)\,\mathrm{d}h\\ &= \int_1^\infty h^{s-1}F(h)\,\mathrm{d}h + \frac{1}{s-2}\int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x + \int_0^1 h^{s-1}\Delta_f(h)\,\mathrm{d}h. \qquad (5.9)\end{aligned}$$

According to condition (5.2), the first term on the right-hand side of (5.9), as a function of s, is regular for $\Re s < \beta$, whereas the last term is regular for $\Re s > 1$ by (5.8). Hence, formula (5.9) furnishes an analytic continuation of the function $\widehat{F}(s)$ into the strip $1 < \Re s < \beta$, where it is meromorphic and, moreover, has a single (simple) pole at point $s = 2$. Finally, observing that

$$\frac{1}{s-2} = -\int_1^\infty h^{s-3}\,\mathrm{d}h, \qquad \Re s < 2,$$

and rearranging the terms in (5.9) using (5.5), we obtain (5.4). ◻

Lemma 5.3. *Under the conditions of Lemma 5.2,*

$$\Delta_f(h) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^{-s}\widehat{F}(s)\,\mathrm{d}s, \qquad 1 < c < 2. \tag{5.10}$$

PROOF. From (5.5), (5.7) we have $\Delta_f(h) = O(h^{-2})$ as $h \to \infty$. Combined with estimate (5.8) established in the proof of Lemma 5.2, this implies that the integral (5.4) converges absolutely in the strip $1 < \Re s < 2$. Representation (5.10) then follows from (5.4) by the inversion formula for the Mellin transform (see [38], Theorem 9a, pages 246 and 247). $\square$

5.2. *Proof of Theorem* 5.1. Let us consider the first coordinate, ξ_1 (for ξ_2 the proof is similar). The proof consists of several steps.

Step 1. According to (3.11) we have

$$E_z^r(\xi_1) = \sum_{k,m=1}^{\infty} \frac{\mu(m)}{k} F(km), \tag{5.11}$$

where [see (3.6), (3.9)]

$$F(h) = \sum_{x \in \mathbb{Z}_+^2} f(hx) = \frac{rh\,\mathrm{e}^{-\alpha_1 h}}{(1-\mathrm{e}^{-\alpha_1 h})^2(1-\mathrm{e}^{-\alpha_2 h})}, \qquad h > 0,$$

$$f(x) = r x_1 \mathrm{e}^{-\langle \alpha, x\rangle}, \qquad x \in \mathbb{R}_+^2.$$

Note that

$$\int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x = r \int_0^\infty x_1 \mathrm{e}^{-\alpha_1 x_1}\,\mathrm{d}x_1 \int_0^\infty \mathrm{e}^{-\alpha_2 x_2}\,\mathrm{d}x_2 = \frac{r}{\alpha_1^2 \alpha_2}.$$

Moreover, using (3.12) we have

$$\sum_{k,m=1}^{\infty} \frac{\mu(m)}{k} \cdot \frac{r}{(km)^2 \alpha_1^2 \alpha_2} = \frac{n_1}{\kappa^3} \sum_{k=1}^{\infty} \frac{1}{k^3} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = n_1 \tag{5.12}$$

[cf. (3.15)]. Subtracting (5.12) from (5.11), we obtain the representation

$$E_z^r(\xi_1) - n_1 = \sum_{k,m=1}^{\infty} m\mu(m)\,\Delta_f(km), \tag{5.13}$$

where $\Delta_f(h)$ is defined in (5.5). Clearly, the functions f and F satisfy the hypotheses of Lemma 5.2 (with $\beta = \infty$). Setting $c_n := n_2/n_1$ and using (3.12), the Mellin transform of $F(h)$ defined by (5.3) can be represented as

$$\widehat{F}(s) = r\alpha_1^{-(s+1)} \widetilde{F}(s),$$

where

$$\widetilde{F}(s) := \int_0^\infty \frac{y^s \mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2\,(1-\mathrm{e}^{-y/c_n})}\,\mathrm{d}y, \qquad \Re s > 2. \tag{5.14}$$

As a result, applying Lemma 5.3 we can rewrite (5.13) as

$$E_z^r(\xi_1) - n_1 = \frac{r}{2\pi i}\sum_{k,m=1}^\infty m\mu(m)\int_{c-i\infty}^{c+i\infty} \frac{\widetilde{F}(s)}{\alpha_1^{s+1}(km)^s}\,\mathrm{d}s \qquad (1<c<2). \tag{5.15}$$

Step 2. It is not difficult to find explicitly the analytic continuation of the function $\widetilde{F}(s)$ into the domain $1 < \Re s < 2$. To this end, let us represent the integral (5.14) as

$$\widetilde{F}(s) = J(s) + c_n\int_0^\infty \frac{y^{s-1}\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y + \frac{1}{2}\int_0^\infty \frac{y^s\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y, \tag{5.16}$$

where

$$J(s) := \int_0^\infty \frac{y^s\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\left(\frac{1}{1-\mathrm{e}^{-y/c_n}} - \frac{c_n}{y} - \frac{1}{2}\right)\mathrm{d}y. \tag{5.17}$$

The last two integrals in (5.16) are easily evaluated:

$$\begin{aligned}\int_0^\infty \frac{y^{s-1}\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y &= \int_0^\infty y^{s-1}\sum_{k=1}^\infty k\mathrm{e}^{-ky}\,\mathrm{d}y = \sum_{k=1}^\infty k\int_0^\infty y^{s-1}\mathrm{e}^{-ky}\,\mathrm{d}y\\ &= \sum_{k=1}^\infty \frac{1}{k^{s-1}}\int_0^\infty u^{s-1}\mathrm{e}^{-u}\,\mathrm{d}u = \zeta(s-1)\,\Gamma(s),\end{aligned} \tag{5.18}$$

where $\Gamma(s) = \int_0^\infty u^{s-1}\mathrm{e}^{-u}\,\mathrm{d}u$ is the gamma function, and similarly

$$\int_0^\infty \frac{y^s\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y = \zeta(s)\,\Gamma(s+1). \tag{5.19}$$

Substituting expressions (5.18) and (5.19) into (5.16), we obtain

$$\widetilde{F}(s) = J(s) + c_n\zeta(s-1)\,\Gamma(s) + \frac{1}{2}\,\zeta(s)\,\Gamma(s+1). \tag{5.20}$$

Since the expression in the parentheses in (5.17) is $O(y)$ as $y \to 0$ and $O(1)$ as $y \to \infty$, the integral (5.17) is absolutely convergent (and therefore the function $J(s)$ is regular) for $\Re s > 0$. Furthermore, it is well known that $\Gamma(s)$ is analytic for $\Re s > 0$ ([30], Section 4.41, page 148), while $\zeta(s)$ has a single pole at point $s = 1$ ([30], Section 4.43, page 152). Hence, the right-hand side of (5.20) is meromorphic in the semi-plane $\Re s > 0$ with poles at $s = 1$ and $s = 2$.

Step 3. Let us estimate the function $\widetilde{F}(c+it)$ as $t \to \infty$. First, by integration by parts in (5.17) it is easy to show that, uniformly in a strip $0 < c_1 \le \sigma \le c_2 < \infty$,

$$J(\sigma+it) = O\bigl(|t|^{-2}\bigr), \qquad t \to \infty. \tag{5.21}$$

The gamma function in such a strip is known to satisfy a uniform estimate

$$\Gamma(\sigma+it) = O(1)\,|t|^{\sigma-(1/2)}\,\mathrm{e}^{-\pi|t|/2}, \qquad t \to \infty \tag{5.22}$$

(see [30], Section 4.42, page 151). Furthermore, the zeta function is obviously bounded in any semi-plane $\sigma \ge c_1 > 1$

$$|\zeta(\sigma+it)| \le \sum_{n=1}^{\infty} \frac{1}{|n^{\sigma+it}|} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \le \sum_{n=1}^{\infty} \frac{1}{n^{c_1}} = O(1). \tag{5.23}$$

We also have the following bounds, uniform in σ, on the growth of the zeta function as $t \to \infty$ (see [20], Theorem 1.9, page 25):

$$\zeta(\sigma+it) = \begin{cases} O(\ln|t|), & 1 \le \sigma \le 2, \\ O\bigl(t^{(1-\sigma)/2}\ln|t|\bigr), & 0 \le \sigma \le 1, \\ O\bigl(t^{1/2-\sigma}\ln|t|\bigr), & \sigma \le 0. \end{cases} \tag{5.24}$$

As a result, by (5.22), (5.23) and (5.24) the second and third summands on the right-hand side of (5.20) give only exponentially small contributions as compared to (5.21), so that

$$\widetilde{F}(c+it) = O\bigl(|t|^{-2}\bigr), \qquad t \to \infty \qquad (1 < c < 2). \tag{5.25}$$

Step 4. In view of (5.25), for $1 < c < 2$ there is an absolute convergence on the right-hand side of (5.15),

$$\begin{aligned} &\sum_{k,m=1}^{\infty} m|\mu(m)| \int_{c-i\infty}^{c+i\infty} \frac{|\widetilde{F}(s)|}{|\alpha_1^{s+1}(km)^{s+1}|}\,|\mathrm{d}s| \\ &\qquad\qquad \le \frac{1}{\alpha_1^{c+1}} \sum_{k=1}^{\infty} \frac{1}{k^{c+1}} \sum_{m=1}^{\infty} \frac{1}{m^{c}} \int_{-\infty}^{\infty} |\widetilde{F}(c+it)|\,\mathrm{d}t < \infty. \end{aligned}$$

Hence, the summation and integration in (5.15) can be interchanged to yield

$$\begin{aligned} E_z^r(\xi_1) - n_1 &= \frac{r}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\widetilde{F}(s)}{\alpha_1^{s+1}} \sum_{k=1}^{\infty} \frac{1}{k^{s+1}} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}\,\mathrm{d}s \\ &= \frac{r}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\widetilde{F}(s)\zeta(s+1)}{\alpha_1^{s+1}\zeta(s)}\,\mathrm{d}s. \end{aligned} \tag{5.26}$$

While evaluating the sum over m here, we used the Möbius inversion formula (3.10) with $F^{\sharp}(h) = h^{-s}$, $F(h) = \sum_m (hm)^{-s} = h^{-s}\zeta(s)$ (cf. (3.15); see also [18], Theorem 287, page 250). Substituting (5.20) into (5.26), we finally obtain

$$E_z^r(\xi_1) - n_1 = \frac{r}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varPsi(s)\,\mathrm{d}s \qquad (1<c<2), \tag{5.27}$$

where

$$\varPsi(s) := \frac{\zeta(s+1)}{\alpha_1^{s+1}}\left[\frac{J(s)+c_n\zeta(s-1)\Gamma(s)}{\zeta(s)} + \frac{1}{2}\,\Gamma(s+1)\right] \tag{5.28}$$

and the function $J(s)$ is given by (5.17).

Step 5. By the La Vallée Poussin theorem (see [21], Section 4.2, Theorem 5, page 69), there exists a constant $A>0$ such that $\zeta(\sigma+it)\neq 0$ in the domain

$$\sigma \ge 1 - \frac{A}{\ln(|t|+2)} =: \eta(t), \qquad t\in\mathbb{R}. \tag{5.29}$$

Moreover, it is known (see [31], equation (3.11.8), page 60) that in the domain (5.29) the following uniform estimate holds:

$$\frac{1}{\zeta(\sigma+it)} = O(\ln|t|), \qquad t\to\infty. \tag{5.30}$$

Without loss of generality, one can assume $A<\ln 2$, so that [see (5.29)]

$$\eta(t)\ge\eta(0) = 1-\frac{A}{\ln 2} > 0, \qquad t\in\mathbb{R}.$$

Therefore, $\varPsi(s)$ [see (5.28)] is regular for all $s=\sigma+it$ such that $2>\sigma\ge\eta(t)$ $(t\in\mathbb{R})$.

Let us show that the integration contour $\Re s = c$ in (5.27) can be replaced by the curve $\sigma=\eta(t)$ $(t\in\mathbb{R})$. By the Cauchy theorem, it suffices to check that

$$\lim_{T\to\pm\infty}\int_{\eta(T)+iT}^{c+iT}\varPsi(s)\,\mathrm{d}s = 0.$$

We have

$$\left|\int_{\eta(T)+iT}^{c+iT}\varPsi(s)\,\mathrm{d}s\right| \le \int_{\eta(T)}^{c}|\varPsi(\sigma+iT)|\,\mathrm{d}\sigma \le \int_{\eta(0)}^{c}|\varPsi(\sigma+iT)|\,\mathrm{d}\sigma. \tag{5.31}$$

In view of the remark after formula (5.30), we have $\eta(0) > 0$, hence application of estimate (5.23) gives, for $s = \sigma + iT,\ \eta(T) \le \sigma \le c$,

$$\left|\frac{\zeta(s+1)}{\alpha_1^{s+1}}\right| \le \frac{\zeta(\sigma+1)}{\alpha_1^{\sigma+1}} \le \frac{\zeta(\eta(0)+1)}{\alpha_1^{c+1}}$$

(since $\alpha_1 \to 0$, we may assume that $\alpha_1 < 1$).

To estimate the expression in the square brackets in (5.28), we use the estimates (5.21), (5.22), (5.24) and (5.30). As a result, we obtain

$$\Psi(\sigma + iT) = O\bigl(|T|^{-2} \ln |T|\bigr), \qquad T \to \pm\infty, \tag{5.32}$$

which implies that the right-hand side of (5.31) tends to zero as $T \to \pm\infty$, as required. Therefore, the integral in (5.27) can be rewritten in the form

$$D_n := \int_{-\infty}^{\infty} \Psi(\eta(t) + it)\, \mathrm{d}(\eta(t) + it). \tag{5.33}$$

Step 6. It remains to estimate the integral (5.33) as $n \to \infty$. Let us set

$$\begin{aligned} \Psi_0(s) &:= \alpha_1^{s+1} \Psi(s) \\ &= \zeta(s+1) \left[\frac{J(s) + c_n \zeta(s-1)\Gamma(s)}{\zeta(s)} + \frac{1}{2}\,\Gamma(s+1)\right] \end{aligned} \tag{5.34}$$

[see (5.28)], then equation (5.33) is rewritten as

$$D_n = \alpha_1^{-2} \int_{-\infty}^{\infty} \alpha_1^{1-\eta(t)-it}\, \Psi_0(\eta(t) + it)\, (\eta'(t) + i)\, \mathrm{d}t.$$

Using that $\alpha_1 = \delta_1 / n^{1/3}$ [see (3.3)], we get

$$\begin{aligned} |D_n| &= O\bigl(n_1^{2/3}\bigr) \int_{-\infty}^{\infty} \alpha_1^{1-\eta(t)}\, |\Psi_0(\eta(t) + it)|\, (|\eta'(t)| + 1)\, \mathrm{d}t \\ &= O\bigl(n_1^{2/3}\bigr) \int_{-\infty}^{\infty} \alpha_1^{1-\eta(t)}\, |\Psi_0(\eta(t) + it)|\, \mathrm{d}t, \end{aligned} \tag{5.35}$$

since by (5.29)

$$|\eta'(t)| = \frac{A}{(|t|+2)\ln^2(|t|+2)} \le \frac{A}{2\ln^2 2} = O(1).$$

Let us now note that, as $n \to \infty$, the integrand function in (5.35) tends to zero for each t, because $\alpha_1 \to 0$ and $1 - \eta(t) > 0$ [see (5.29)]. Finally, eligibility of passing to the limit under the integral sign follows from Lebesgue's dominated convergence theorem. Indeed, the integrand function in (5.35) is bounded by $|\varPsi_0(\eta(t)+it)|$, and integrability of the latter is easily checked by applying the estimates (5.21), (5.22), (5.24) and (5.30) to the expression (5.34), which yields [cf. (5.32)]

$$|\varPsi_0(\eta(t)+it)| = O\big(|t|^{-2}\ln|t|\big), \qquad t\to\pm\infty.$$

Thus, we have shown that the integral in (5.35) is $o(1)$ as $n\to\infty$, hence $D_n = o\big(|n|^{2/3}\big)$. Substituting this estimate into (5.27), we obtain the statement of Theorem 5.1. The proof is complete.

6. Asymptotics of higher-order moments.

6.1. *The variance.* According to the second formula in (2.15), we have

$$\operatorname{Var}[\nu(x)] = \frac{rz^x}{(1-z^x)^2} = r\sum_{k=1}^{\infty} kz^{kx}. \tag{6.1}$$

Let $K_z := \operatorname{Cov}(\xi,\xi)$ be the covariance matrix (with respect to the measure Q_z^r) of the random vector $\xi = \sum_{x\in\mathcal{X}} x\nu(x)$. Recalling that the random variables $\nu(x)$ are independent for different $x\in\mathcal{X}$ and using (6.1), we see that the elements $K_z(i,j) = \operatorname{Cov}(\xi_i,\xi_j)$ of the matrix K_z are given by

$$K_z(i,j) = \sum_{x\in\mathcal{X}} x_ix_j \operatorname{Var}[\nu(x)] = r\sum_{k=1}^{\infty}\sum_{x\in\mathcal{X}} kx_ix_jz^{kx}, \qquad i,j\in\{1,2\}. \tag{6.2}$$

THEOREM 6.1. *As $n\to\infty$,*

$$K_z(i,j) \sim \frac{(n_1n_2)^{2/3}}{r^{1/3}\kappa}\, B_{ij}, \qquad i,j\in\{1,2\}, \tag{6.3}$$

where κ is defined in Theorem 3.1 and the matrix $B := (B_{ij})$ is given by

$$B = \begin{pmatrix} 2n_1/n_2 & 1 \\ 1 & 2n_2/n_1 \end{pmatrix}. \tag{6.4}$$

PROOF. Let us consider $K_z(1,1)$ (the other elements of K_z are analyzed in a similar manner). Substituting (3.3) into (6.2), we obtain

$$K_z(1,1) = r\sum_{k=1}^{\infty}\sum_{x\in\mathcal{X}} kx_1^2 \mathrm{e}^{-k\langle\alpha,x\rangle}. \tag{6.5}$$

Using the Möbius inversion formula (3.10), similarly to (3.13) expression (6.5) can be rewritten in the form

$$
\begin{aligned}
K_z(1,1) &= r \sum_{k,m=1}^{\infty} km^2\mu(m) \sum_{x\in\mathbb{Z}_+^2} x_1^2 \mathrm{e}^{-km\langle\alpha,x\rangle} \\
&= r \sum_{k,m=1}^{\infty} km^2\mu(m) \sum_{x_1=1}^{\infty} x_1^2 \mathrm{e}^{-km\alpha_1 x_1} \sum_{x_2=0}^{\infty} \mathrm{e}^{-km\alpha_2 x_2} \\
&= r \sum_{k,m=1}^{\infty} \frac{km^2\mu(m)}{1-\mathrm{e}^{-km\alpha_2}} \sum_{x_1=1}^{\infty} x_1^2 \mathrm{e}^{-km\alpha_1 x_1}. \qquad (6.6)
\end{aligned}
$$

Note also that

$$
\sum_{x_1=1}^{\infty} x_1^2 \mathrm{e}^{-km\alpha_1 x_1} = \frac{\mathrm{e}^{-km\alpha_1}(1+\mathrm{e}^{-km\alpha_1})}{(1-\mathrm{e}^{-km\alpha_1})^3} = \frac{O(1)}{k^3m^3\alpha_1^3}. \qquad (6.7)
$$

Returning to representation (6.6) and using (6.7), we obtain

$$
\alpha_1^3\alpha_2 K_z(1,1) = r \sum_{k,m=1}^{\infty} km^2\mu(m) \frac{\alpha_1^3\alpha_2\, \mathrm{e}^{-km\alpha_1}(1+\mathrm{e}^{-km\alpha_1})}{(1-\mathrm{e}^{-km\alpha_1})^3(1-\mathrm{e}^{-km\alpha_2})}. \qquad (6.8)
$$

By Lemma 3.2, the general term in the series (6.8) admits a uniform estimate $O\bigl(k^{-3}m^{-2}\bigr)$. Hence, by Lebesgue's dominated convergence theorem one can pass to the limit in (6.8) to obtain

$$
\alpha_1^3\alpha_2 K_z(1,1) \to \frac{2r\zeta(3)}{\zeta(2)} = 2r\kappa^3, \qquad \alpha_1,\alpha_2\to 0.
$$

Using (3.3) and (3.5), this yields

$$
K_z(1,1) \sim \frac{2\, n_1/n_2}{r^{1/3}\kappa}\, (n_1n_2)^{2/3}, \qquad n\to\infty,
$$

as required [cf. (6.3), (6.4)]. □

6.2. *Statistical moments of* $\nu(x)$. Denote

$$
\nu_0(x) := \nu(x) - E_z^r[\nu(x)], \qquad x\in\mathcal{X}, \qquad (6.9)
$$

and for $k\in\mathbb{N}$ set

$$
m_k(x) := E_z^r\bigl[\nu(x)^k\bigr], \qquad \mu_k(x) := E_z^r\bigl|\nu_0(x)^k\bigr| \qquad (6.10)
$$

(for notational simplicity, we suppress the dependence on r and z).

LEMMA 6.2. *For each $k \in \mathbb{N}$ and all $x \in \mathcal{X}$,*

$$\mu_k(x) \le 2^k m_k(x). \tag{6.11}$$

PROOF. Omitting for brevity the argument x, by Newton's binomial formula and Lyapunov's inequality we obtain

$$\begin{aligned}\mu_k \le E_z^r\big[(\nu+m_1)^k\big] &= \sum_{i=0}^{k}\binom{k}{i} m_i\, m_1^{k-i}\\ &\le \sum_{i=0}^{k}\binom{k}{i} m_k^{i/k}\, m_k^{(k-i)/k} = 2^k m_k,\end{aligned}$$

and (6.11) follows. □

LEMMA 6.3. *For each $k \in \mathbb{N}$, there exist positive constants $c_k = c_k(r)$ and $C_k = C_k(r)$ such that, for all $x \in \mathcal{X}$,*

$$\frac{c_k\, z^{kx}}{(1-z^x)^k} \le m_k(x) \le \frac{C_k z^x}{(1-z^x)^k}. \tag{6.12}$$

PROOF. Fix $x \in \mathcal{X}$ and let $\varphi(s) \equiv \varphi_{\nu(x)}(s) := E_z^r\big[\mathrm{e}^{is\nu(x)}\big]$ be the characteristic function of the random variable $\nu(x)$ with respect to the measure Q_z^r. From (2.13) it follows that

$$\varphi(s) = \frac{\beta^r(z^x \mathrm{e}^{is})}{\beta^r(z^x)} = \frac{(1-z^x)^r}{(1-z^x\mathrm{e}^{is})^r}. \tag{6.13}$$

Let us first prove that for any $k \in \mathbb{N}$

$$(1-z^x)^{-r}\,\frac{\mathrm{d}^k\varphi(s)}{\mathrm{d}s^k} = i^k \sum_{j=1}^{k} c_{j,k}\,\frac{(z^x\mathrm{e}^{is})^j}{(1-z^x\mathrm{e}^{is})^{r+j}}, \tag{6.14}$$

where $c_{j,k} \equiv c_{j,k}(r) > 0$. Indeed, if $k=1$ then differentiation of (6.13) yields

$$(1-z^x)^{-r}\,\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = \frac{irz^x\mathrm{e}^{is}}{(1-z^x\mathrm{e}^{is})^{r+1}},$$

which is in accordance with (6.14) if we put $c_{1,1} := r$. Assume now that

(6.14) is valid for some k. Differentiating (6.14) once more, we obtain

$$\begin{aligned}(1-z^x)^{-r}\,\frac{\mathrm{d}^{k+1}\varphi(s)}{\mathrm{d}s^{k+1}} &= i^{k+1}\sum_{j=1}^{k} c_{j,k}\,\frac{j(z^x\mathrm{e}^{is})^j}{(1-z^x\mathrm{e}^{is})^{r+j}}\\ &\qquad + i^{k+1}\sum_{j=1}^{k} c_{j,k}\,\frac{(r+j)(z^x\mathrm{e}^{is})^{j+1}}{(1-z^x\mathrm{e}^{is})^{r+j+1}}\\ &= i^{k+1}\sum_{j=1}^{k+1} c_{j,k+1}\frac{(z^x\mathrm{e}^{is})^j}{(1-z^x\mathrm{e}^{is})^{r+j}}\,,\end{aligned}$$

where we have set

$$c_{j,k+1} := \begin{cases} c_{1,k}\,, & j=1,\\ j\,c_{j,k}+(r+j-1)c_{j-1,k}\,, & 2\le j\le k,\\ (r+k)c_{k,k}\,, & j=k+1.\end{cases}$$

Hence, by induction, formula (6.14) is valid for all k.

Now, by (6.14) we have

$$m_k(x) = i^{-k}\,\frac{\mathrm{d}^k\varphi(s)}{\mathrm{d}s^k}\bigg|_{s=0} = \sum_{j=1}^{k} c_{j,k}\,\frac{z^{jx}}{(1-z^x)^j} \le \frac{z^x}{(1-z^x)^k}\sum_{j=1}^{k} c_{j,k}\,,$$

since $0<z^x<1$. Hence, inequalities (6.12) hold with $c_k=c_{k,k}$, $C_k=\sum_{j=1}^{k} c_{j,k}$. □

6.3. *Asymptotics of the moment sums.*

LEMMA 6.4. *For any $k\in\mathbb{N}$ and $\theta>0$,*

$$\sum_{x\in\mathcal{X}} |x|^k\frac{z^{\theta x}}{(1-z^x)^k} \asymp |n|^{(k+2)/3}, \qquad n\to\infty. \tag{6.15}$$

PROOF. Using (3.3), by Lemma 3.2 we have

$$\frac{z^{\theta x}}{(1-z^x)^k} = \frac{\mathrm{e}^{-\theta\langle\alpha,x\rangle}}{(1-\mathrm{e}^{-\langle\alpha,x\rangle})^k} \le \frac{C\,\mathrm{e}^{-(\theta/2)\langle\alpha,x\rangle}}{\langle\alpha,x\rangle^k} \le \frac{C\,\mathrm{e}^{-(\theta/2)\langle\alpha,x\rangle}}{\alpha_0^k\,|x|^k}, \tag{6.16}$$

where $\alpha_0:=\min\{\alpha_1,\alpha_2\}$. On the other hand,

$$\frac{z^{\theta x}}{(1-z^x)^k} = \frac{\mathrm{e}^{-\theta\langle\alpha,x\rangle}}{(1-\mathrm{e}^{-\langle\alpha,x\rangle})^k} \ge \frac{\mathrm{e}^{-\theta\langle\alpha,x\rangle}}{\langle\alpha,x\rangle^k} \ge \frac{\mathrm{e}^{-\theta\langle\alpha,x\rangle}}{|\alpha|^k\,|x|^k}. \tag{6.17}$$

Since $\alpha_0 \asymp |n|^{-1/3}$ and $|\alpha| \asymp |n|^{-1/3}$, from (6.16) and (6.17) we see that for the proof of (6.15) it remains to show

$$\sum_{x\in\mathcal{X}} \mathrm{e}^{-\langle\alpha,x\rangle} \asymp |n|^{2/3}, \qquad n\to\infty. \tag{6.18}$$

Using the Möbius inversion formula (3.10), similarly as in Sections 3 and 4 we obtain

$$\begin{aligned}
\sum_{x\in\mathcal{X}} \mathrm{e}^{-\langle\alpha,x\rangle} &= \sum_{m=1}^{\infty}\mu(m)\sum_{x\in\mathbb{Z}_+^2\setminus\{0\}} \mathrm{e}^{-m\langle\alpha,x\rangle}\\
&= \sum_{m=1}^{\infty}\mu(m)\left(\frac{1}{(1-\mathrm{e}^{-m\alpha_1})(1-\mathrm{e}^{-m\alpha_2})}-1\right)\\
&= \sum_{m=1}^{\infty}\mu(m)\,\frac{\mathrm{e}^{-m\alpha_1}+\mathrm{e}^{-m\alpha_2}-\mathrm{e}^{-m(\alpha_1+\alpha_2)}}{(1-\mathrm{e}^{-m\alpha_1})(1-\mathrm{e}^{-m\alpha_2})}. \qquad (6.19)
\end{aligned}$$

By Lemma 3.2, the general term of the series (6.19) is $O\big(\alpha_1^{-1}\alpha_2^{-1}m^{-2}\big)$ (uniformly in m). Hence, by Lebesgue's dominated convergence theorem, the right-hand side of (6.19) is asymptotically equivalent to

$$\frac{1}{\alpha_1\alpha_2}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} = \frac{1}{\alpha_1\alpha_2\zeta(2)} \asymp |n|^{2/3},$$

and (6.18) follows. □

Lemma 6.5. *For any $k\in\mathbb{N}$,*

$$\sum_{x\in\mathcal{X}} |x|^k m_k(x) \asymp |n|^{(k+2)/3}, \qquad n\to\infty.$$

Proof. Readily follows from the estimates (6.12) and Lemma 6.4. □

Lemma 6.6. *For any integer $k\ge 2$,*

$$\sum_{x\in\mathcal{X}} |x|^k \mu_k(x) \asymp |n|^{(k+2)/3}, \qquad n\to\infty.$$

Proof. An upper bound follows (for all $k\ge 1$) from inequality (6.11) and Lemma 6.5. On the other hand, by Lyapunov's inequality and formula (6.1), for any $k\ge 2$ we have

$$\mu_k(x) \ge \mu_2(x)^{k/2} = \big(\mathrm{Var}[\nu(x)]\big)^{k/2} = \frac{r^{k/2}z^{kx/2}}{(1-z^x)^k},$$

and a lower bound follows by Lemma 6.4. □

LEMMA 6.7. *For each $k \in \mathbb{N}$ and $j = 1, 2$*

$$E_z^r\bigl[\xi_j - E_z^r(\xi_j)\bigr]^{2k} = O\bigl(|n|^{4k/3}\bigr), \qquad n \to \infty.$$

PROOF. Let $j = 1$ (the case $j = 2$ is considered similarly). Using the notation (6.9), we obtain, by the multinomial expansion,

$$E_z^r\bigl[\xi_1 - E_z^r(\xi_1)\bigr]^{2k} = E_z^r\Biggl(\sum_{x\in\mathcal{X}} x_1 \nu_0(x)\Biggr)^{2k}$$

$$= \sum_{\ell=1}^{2k} \sum_{\substack{k_1,\ldots,k_\ell \ge 1,\\ k_1+\cdots+k_\ell = 2k}} C_{k_1,\ldots,k_\ell} \sum_{\{x^1,\ldots,x^\ell\}\subset\mathcal{X}} \prod_{i=1}^{\ell} (x_1^i)^{k_i} E_z^r\bigl[\nu_0(x^i)^{k_i}\bigr], \tag{6.20}$$

where $C_{k_1,\ldots,k_\ell}$ are combinatorial coefficients accounting for the number of identical terms in the expansion. Using that $E_z^r[\nu_0(x)] = 0$, we can assume that $k_i \ge 2$ for all $i = 1, \ldots, \ell$. Since $k_1 + \cdots + k_\ell = 2k$, this implies that $\ell \le k$. Hence, recalling the notation (6.10) and using Lemma 6.6, we see that the internal sum in (6.20) (over $\{x^1, \ldots, x^\ell\} \subset \mathcal{X}$) is bounded by

$$\begin{aligned}\sum_{\{x^1,\ldots,x^\ell\}\subset\mathcal{X}} \prod_{i=1}^{\ell} |x^i|^{k_i} \mu_{k_i}(x^i) &\le \prod_{i=1}^{\ell} \sum_{x\in\mathcal{X}} |x|^{k_i} \mu_{k_i}(x)\\ &= O(1) \prod_{i=1}^{\ell} |n|^{(k_i+2)/3}\\ &= O(1)\cdot |n|^{2(k+\ell)/3} = O\bigl(|n|^{4k/3}\bigr),\end{aligned}$$

and the lemma is proved. ☐

7. Local limit theorem. As was explained in the Introduction (see Section 1.3), the role of a local limit theorem in our approach is to yield the asymptotics of the probability $Q_z^r\{\xi = n\} \equiv Q_z^r(\varPi_n)$ appearing in the representation of the measure P_n^r as a conditional distribution, $P_n^r(A) = Q_z^r(A\,|\,\varPi_n) = Q_z^r(A)/Q_z^r(\varPi_n)$, $A \subset \varPi_n$ [see (2.16)].

7.1. *Statement of the theorem.* As before, we denote $a_z := E_z^r(\xi)$, $K_z := \operatorname{Cov}(\xi, \xi) = E_z^r(\xi - a_z)^\top(\xi - a_z)$, where the random vector $\xi = (\xi_1, \xi_2)$ is defined in (2.2). From (6.2), it is easy to see (e.g., using the Cauchy–Schwarz inequality together with the characterization of the equality case) that the matrix K_z is positive definite; in particular, $\det K_z > 0$ and hence K_z is invertible. Let $V_z = K_z^{-1/2}$ be the (unique) square root of the matrix K_z^{-1}

(see, e.g., [4], Chapter 6, Section 5, pages 93 and 94), that is, a symmetric, positive definite matrix such that $V_z^2 = K_z^{-1}$.

Denote by $f_{0,I}(\cdot)$ the density of a standard two-dimensional normal distribution $\mathcal{N}(0, I)$ (with zero mean and identity covariance matrix),

$$f_{0,I}(x) = \frac{1}{2\pi}\,\mathrm{e}^{-|x|^2/2}, \qquad x \in \mathbb{R}^2.$$

Then the density of the normal distribution $\mathcal{N}(a_z, K_z)$ (with mean a_z and covariance matrix K_z) is given by

$$f_{a_z,K_z}(x) = (\det K_z)^{-1/2} f_{0,I}\bigl((x - a_z)\,V_z\bigr), \qquad x \in \mathbb{R}^2. \tag{7.1}$$

With these notations, we can now state our local limit theorem.

THEOREM 7.1. *Uniformly in* $m \in \mathbb{Z}_+^2$,

$$Q_z^r\{\xi = m\} = f_{a_z,K_z}(m) + O\bigl(|n|^{-5/3}\bigr), \qquad n \to \infty. \tag{7.2}$$

REMARK 7.1. Theorem 7.1 is a two-dimensional local central limit theorem for the sum $\xi = \sum_{x\in\mathcal{X}} x\nu(x)$ with independent terms whose distribution depends on a large parameter $n = (n_1, n_2)$; however, the summation scheme is rather different from the classic one, since the number of non-vanishing terms is not fixed in advance, and, moreover, the summands actually involved in the sum are determined by sampling.

One implication of Theorem 7.1 will be particularly useful.

COROLLARY 7.2. *As* $n \to \infty$,

$$Q_z^r\{\xi = n\} \sim \frac{r^{1/3}\kappa}{2\sqrt{3}\,\pi}\,(n_1 n_2)^{-2/3}, \tag{7.3}$$

where $\kappa = \bigl(\zeta(3)/\zeta(2)\bigr)^{1/3}$.

Before proving the theorem, we have to make some (quite lengthy) technical preparations, collected below in Sections 7.2–7.4.

7.2. *Lemmas about the matrix norm.* The matrix norm induced by the Euclidean vector norm $|\cdot|$ is defined by $\|A\| := \sup_{|x|=1} |xA|$. It is well known that for a (real) square matrix A its norm is given by

$$\|A\| = \sqrt{\lambda(A^{\top}A)}, \tag{7.4}$$

where $\lambda(\cdot)$ is the spectral radius of a matrix, defined to be the largest modulus of its eigenvalues (see, e.g., [25], Section 6.3, pages 210 and 211).

We need some general facts about the matrix norm $\|\cdot\|$. Even though they are mostly well known, specific references are not easy to find (cf., e.g., [4, 25, 19]). For the reader's convenience, we give neat proofs of the lemmas below based on the spectral characterization (7.4).

LEMMA 7.3 (cf. [17], Section 22, Theorem 4, page 40). *If A is a real matrix then* $\|A^\top A\| = \|A\|^2$.

PROOF. The matrix $A^\top A$ is symmetric and non-negative definite, hence, using (7.4), we obtain $\|A^\top A\| = \lambda(A^\top A) = \|A\|^2$, as claimed. $\square$

LEMMA 7.4 (cf. [19], Section 5.6, Problem 23, hints (2,5) and (5,2), pages 313 and 314). *If $A = (a_{ij})$ is a real $d \times d$ matrix, then*

$$\frac{1}{d}\sum_{i,j=1}^{d} a_{ij}^2 \le \|A\|^2 \le \sum_{i,j=1}^{d} a_{ij}^2. \tag{7.5}$$

PROOF. Note that $\sum_{i,j=1}^{d} a_{ij}^2 = \operatorname{tr}(A^\top A)$, where $\operatorname{tr}(\cdot)$ denotes the trace, and furthermore

$$\lambda(A^\top A) \le \operatorname{tr}(A^\top A) \le d\cdot\lambda(A^\top A).$$

Since $\lambda(A^\top A) = \|A\|^2$ by (7.4), this implies (7.5). $\square$

The following simple fact pertaining to dimension $d = 2$ seems to be less known.

LEMMA 7.5. *Let A be a symmetric 2×2 matrix with* $\det A \ne 0$. *Then*

$$\|A^{-1}\| = \frac{\|A\|}{|\det A|}. \tag{7.6}$$

PROOF. Let λ_1 and λ_2 ($|\lambda_2| \ge |\lambda_1| > 0$) be the eigenvalues of A, then $|\det A| = |\lambda_1|\cdot|\lambda_2|$ and, according to (7.4),

$$\|A\| = \sqrt{\lambda(A^2)} = |\lambda_2|, \qquad \|A^{-1}\| = \sqrt{\lambda\big((A^{-1})^2\big)} = |\lambda_1|^{-1},$$

which makes equality (7.6) obvious. $\square$

7.3. *Estimates for the covariance matrix.* In this section, we collect some information about the asymptotic behavior of the matrix $K_z = \mathrm{Cov}(\xi, \xi)$. The next lemma is a direct consequence of Theorem 6.1.

LEMMA 7.6. *As $n \to \infty$,*

$$\det K_z \sim \frac{3(n_1 n_2)^{4/3}}{r^{2/3}\kappa^2}.$$

Let us now estimate the norms of the matrices K_z and $V_z = K_z^{-1/2}$.

LEMMA 7.7. *As $n \to \infty$, one has $\|K_z\| \asymp |n|^{4/3}$.*

PROOF. Lemma 7.4 and Theorem 6.1 imply

$$\|K_z\|^2 \asymp \sum_{i,j=1}^{2} K_z(i,j)^2 \asymp (n_1 n_2)^{4/3} \asymp |n|^{8/3} \qquad (n \to \infty),$$

and the required estimate follows. □

LEMMA 7.8. *For the matrix $V_z = K_z^{-1/2}$, one has $\|V_z\| \asymp |n|^{-2/3}$ as $n \to \infty$.*

PROOF. Using Lemmas 7.3 and 7.5 we have

$$\|V_z\|^2 = \|V_z^2\| = \|K_z^{-1}\| = \frac{\|K_z\|}{\det K_z},$$

and an application of Lemmas 7.6 and 7.7 completes the proof. □

We also need to estimate the so-called *Lyapunov coefficient*

$$L_z := \|V_z\|^3 \sum_{x \in \mathcal{X}} |x|^3 \mu_3(x), \tag{7.7}$$

where $\mu_3(x) = E_z^r |\nu_0(x)^3|$ [see (6.10)].

LEMMA 7.9. *As $n \to \infty$, one has $L_z \asymp |n|^{-1/3}$.*

PROOF. The proof follows from (7.7) using Lemmas 7.8 and 6.6 (with $k = 3$). □

7.4. *Estimates of the characteristic functions.* Recall from Section 2.1 that, with respect to the measure Q_z^r, the random variables $\{\nu(x)\}_{x\in\mathcal{X}}$ are independent and have negative binomial distribution with parameters r and $p=1-z^x$. In particular, $\nu(x)$ has the characteristic function [see (6.13)]

$$\varphi_{\nu(x)}(s) := E_z^r\bigl(\mathrm{e}^{is\nu(x)}\bigr) = \frac{(1-z^x)^r}{(1-z^x\mathrm{e}^{is})^r}\,, \qquad s\in\mathbb{R}, \tag{7.8}$$

and hence the characteristic function $\varphi_\xi(\lambda) := E_z^r\bigl(\mathrm{e}^{i\langle\lambda,\,\xi\rangle}\bigr)$ of the vector $\xi=\sum_{x\in\mathcal{X}} x\nu(x)$ is given by

$$\varphi_\xi(\lambda) = \prod_{x\in\mathcal{X}} \varphi_{\nu(x)}(\langle\lambda,x\rangle) = \prod_{x\in\mathcal{X}} \frac{(1-z^x)^r}{(1-z^x\,\mathrm{e}^{i\langle\lambda,x\rangle})^r}\,, \qquad \lambda\in\mathbb{R}^2.$$

Lemma 7.10. *Let $\varphi_{\nu_0(x)}(s)$ be the characteristic function of the random variable $\nu_0(x)=\nu(x)-E_z^r[\nu(x)]$. Then*

$$|\varphi_{\nu_0(x)}(s)| \le \exp\Bigl\{-\tfrac{1}{2}\mu_2(x)s^2 + \tfrac{1}{3}\mu_3(x)|s|^3\Bigr\}, \qquad s\in\mathbb{R}, \tag{7.9}$$

where $\mu_k(x)=E_z^r|\nu_0(x)^k|$ [see (6.10)].

Proof. Let a random variable $\nu_1(x)$ be independent of $\nu_0(x)$ and have the same distribution, and set $\tilde\nu(x):=\nu_0(x)-\nu_1(x)$. Note that $E_z^r[\tilde\nu(x)]=0$ and $\mathrm{Var}[\tilde\nu(x)]=2\,\mathrm{Var}[\nu(x)]=2\mu_2(x)$. We also have the inequality

$$E_z^r\bigl|\tilde\nu(x)^3\bigr| \le 4E_z^r\bigl|\nu_0(x)^3\bigr| = 4\mu_3(x)$$

(see [5], Lemma 8.8, pages 66 and 67). Hence, by Taylor's formula, the characteristic function of $\tilde\nu(x)$ can be represented in the form

$$\varphi_{\tilde\nu(x)}(s) = 1-\mu_2(x)\,s^2 + \tfrac{2}{3}\,\theta\,\mu_3(x)s^3, \tag{7.10}$$

where $|\theta|\le 1$. Now, using the elementary inequality $|y|\le \mathrm{e}^{(y^2-1)/2}$ and the fact that $\varphi_{\tilde\nu(x)}(s)=\bigl|\varphi_{\nu_0(x)}(s)\bigr|^2$, we get

$$|\varphi_{\nu_0(x)}(s)| \le \exp\Bigl\{\tfrac{1}{2}\bigl(|\varphi_{\nu_0(x)}(s)|^2-1\bigr)\Bigr\} = \exp\Bigl\{\tfrac{1}{2}\bigl(\varphi_{\tilde\nu(x)}(s)-1\bigr)\Bigr\},$$

and the lemma follows by (7.10). □

The characteristic function of the vector $\xi_0:=\xi-a_z=\sum_{x\in\mathcal{X}} x\nu_0(x)$ is given by

$$\varphi_{\xi_0}(\lambda) := E_z^r\bigl(\mathrm{e}^{i\langle\lambda,\,\xi_0\rangle}\bigr) = \prod_{x\in\mathcal{X}} E_z^r\bigl(\mathrm{e}^{i\langle\lambda,\,x\rangle\,\nu_0(x)}\bigr) = \prod_{x\in\mathcal{X}} \varphi_{\nu_0(x)}(\langle\lambda,x\rangle). \tag{7.11}$$

LEMMA 7.11. *If* $V_z = K_z^{-1/2}$, *then for all* $\lambda \in \mathbb{R}^2$

$$|\varphi_{\xi_0}(\lambda V_z)| \le \exp\Bigl\{-\tfrac{1}{2}|\lambda|^2 + \tfrac{1}{3}L_z|\lambda|^3\Bigr\}. \tag{7.12}$$

PROOF. Using (7.11) and (7.9), we obtain

$$|\varphi_{\xi_0}(\lambda V_z)| \le \exp\Biggl\{-\frac{1}{2}\sum_{x\in\mathcal{X}}\langle\lambda V_z, x\rangle^2\mu_2(x) + \frac{1}{3}\sum_{x\in\mathcal{X}}|\langle\lambda V_z, x\rangle|^3\mu_3(x)\Biggr\}. \tag{7.13}$$

The first sum in (7.13) is evaluated exactly as

$$\begin{aligned}\sum_{x\in\mathcal{X}}\langle\lambda V_z, x\rangle^2\mu_2(x) = \operatorname{Var}\langle\lambda V_z, \xi\rangle &= \lambda V_z \operatorname{Cov}(\xi,\xi)\,(\lambda V_z)^\top \\ &= \lambda V_z K_z V_z \lambda^\top = |\lambda|^2,\end{aligned} \tag{7.14}$$

since $\operatorname{Cov}(\xi,\xi) = K_z = V_z^{-2}$. For the second sum in (7.13), by the Cauchy–Schwarz inequality and on account of (7.7) we have

$$\sum_{x\in\mathcal{X}}|\langle\lambda V_z, x\rangle|^3\mu_3(x) \le |\lambda|^3\,\|V_z\|^3\sum_{x\in\mathcal{X}}|x|^3\mu_3(x) = |\lambda|^3 L_z. \tag{7.15}$$

Now, substituting (7.14), (7.15) into (7.13), we get (7.12). □

LEMMA 7.12. *If* $|\lambda| \le L_z^{-1}$ *then*

$$\Bigl|\varphi_{\xi_0}(\lambda V_z) - \mathrm{e}^{-|\lambda|^2/2}\Bigr| \le 16L_z|\lambda|^3\,\mathrm{e}^{-|\lambda|^2/6}. \tag{7.16}$$

PROOF. Let us first suppose that $\frac{1}{2}L_z^{-1/3} \le |\lambda| \le L_z^{-1}$. Then $\frac{1}{8} \le L_z|\lambda|^3 \le |\lambda|^2$, so (7.12) implies $|\varphi_{\xi_0}(\lambda V_z)| \le \mathrm{e}^{-|\lambda|^2/6}$. Hence,

$$\begin{aligned}\bigl|\varphi_{\xi_0}(\lambda V_z) - \mathrm{e}^{-|\lambda|^2/2}\bigr| &\le |\varphi_{\xi_0}(\lambda V_z)| + \mathrm{e}^{-|\lambda|^2/2} \\ &\le 2\,\mathrm{e}^{-|\lambda|^2/6} \le 16L_z|\lambda|^3\,\mathrm{e}^{-|\lambda|^2/6},\end{aligned}$$

in accord with (7.16).

Suppose now that $|\lambda| \le \frac{1}{2}L_z^{-1/3}$. Taylor's formula implies

$$\varphi_{\nu_0(x)}(s) - 1 = -\tfrac{1}{2}\mu_2(x)s^2 + \tfrac{1}{6}\,\theta_x\mu_3(x)s^3, \tag{7.17}$$

where $|\theta_x| \le 1$. By Lyapunov's inequality, $\mu_2(x) \le \mu_3(x)^{2/3}$, so

$$|\varphi_{\nu_0(x)}(s) - 1| \le \tfrac{1}{2}|s|^2\mu_3(x)^{2/3} + \tfrac{1}{6}|s|^3\mu_3(x). \tag{7.18}$$

For $s = \langle\lambda V_z, x\rangle$, we have from (7.15)

$$|\langle\lambda V_z, x\rangle|\,\mu_3(x)^{1/3} \le L_z^{1/3}|\lambda| \le \tfrac{1}{2}, \tag{7.19}$$

and so (7.18) yields

$$\bigl|\varphi_{\nu_0(x)}(\langle\lambda V_z, x\rangle) - 1\bigr| \le \tfrac{1}{2}\cdot\tfrac{1}{4} + \tfrac{1}{6}\cdot\tfrac{1}{8} < \tfrac{1}{2}. \tag{7.20}$$

Similarly, using the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$, from (7.18) we obtain

$$\bigl|\varphi_{\nu_0(x)}(s) - 1\bigr|^2 \le \tfrac{1}{2}\Bigl(|s|\mu_3(x)^{1/3} + \tfrac{1}{9}|s|^3\mu_3(x)\Bigr)|s|^3\mu_3(x),$$

whence, in view of (7.19) and a general bound $|\langle\lambda V_z, x\rangle| \le |\lambda|\cdot\|V_z\|\cdot|x|$, it follows that

$$\begin{aligned}\bigl|\varphi_{\nu_0(x)}(\langle\lambda V_z, x\rangle) - 1\bigr|^2 &\le \tfrac{1}{2}\Bigl(\tfrac{1}{2} + \tfrac{1}{9}\cdot\tfrac{1}{8}\Bigr)|\lambda|^3\|V_z\|^3|x|^3\mu_3(x)\\ &\le \tfrac{1}{3}|\lambda|^3\|V_z\|^3|x|^3\mu_3(x).\end{aligned} \tag{7.21}$$

Consider the function $\ln(1+y)$ of complex variable y, choosing the principal branch of the logarithm (i.e., such that $\ln 1 = 0$). Taylor's expansion implies $\ln(1+y) = y + \theta y^2$ for $|y| \le \frac{1}{2}$, where $|\theta| \le 1$. By (7.17), (7.20) and (7.21) this yields

$$\ln\varphi_{\nu_0(x)}(\langle\lambda V_z, x\rangle) = -\tfrac{1}{2}\langle\lambda V_z, x\rangle^2\mu_2(x) + \tfrac{1}{2}\tilde\theta_x|\lambda|^3\|V_z\|^3|x|^3\mu_3(x),$$

where $|\tilde\theta_x| \le 1$. Substituting this into (7.11), due to (7.14) and (7.15) we obtain

$$\ln\varphi_{\xi_0}(\lambda V_z) = \sum_{x\in\mathcal{X}}\ln\varphi_{\nu_0}(\langle\lambda V_z, x\rangle) = -\tfrac{1}{2}|\lambda|^2 + \tfrac{1}{2}\theta_1 L_z|\lambda|^3 \qquad (|\theta_1| \le 1).$$

Using the elementary inequality $|\mathrm{e}^y - 1| \le |y|\mathrm{e}^{|y|}$, which holds for any $y \in \mathbb{C}$, we have

$$\begin{aligned}\bigl|\varphi_{\xi_0}(\lambda V_z) - \mathrm{e}^{-|\lambda|^2/2}\bigr| &= \mathrm{e}^{-|\lambda|^2/2}\bigl|\mathrm{e}^{\theta_1 L_z|\lambda|^3/2} - 1\bigr|\\ &\le \mathrm{e}^{-|\lambda|^2/2}\cdot\tfrac{1}{2}L_z|\lambda|^3\mathrm{e}^{L_z|\lambda|^3/2} \le \mathrm{e}^{-|\lambda|^2/2}L_z|\lambda|^3,\end{aligned}$$

and the proof is complete. □

Lemma 7.13. *For all* $\lambda \in \mathbb{R}^2$,

$$|\varphi_{\xi_0}(\lambda)| \le \exp\bigl\{-\tfrac{1}{4}rJ_\alpha(\lambda)\bigr\}, \tag{7.22}$$

where

$$J_\alpha(\lambda) := \sum_{x\in\mathcal{X}}\mathrm{e}^{-\langle\alpha,\, x\rangle}(1 - \cos\langle\lambda, x\rangle). \tag{7.23}$$

Proof. According to (7.11), we have

$$|\varphi_{\xi_0}(\lambda)| = |\varphi_{\xi}(\lambda)| = \exp\biggl\{\sum_{x\in\mathcal{X}} \ln\bigl|\varphi_{\nu(x)}(\langle\lambda,\, x\rangle)\bigr|\biggr\}. \tag{7.24}$$

Using (7.8), for any $s\in\mathbb{R}$ we can write

$$\begin{aligned}\ln|\varphi_{\nu(x)}(s)| = \frac{r}{2}\ln\frac{|1-z^x|^2}{|1-z^x\mathrm{e}^{is}|^2} &\le \frac{r}{2}\left(\frac{|1-z^x|^2}{|1-z^x\mathrm{e}^{is}|^2}-1\right)\\ &= -\frac{rz^x(1-\cos s)}{|1-z^x\mathrm{e}^{is}|^2} \le -\frac{rz^x(1-\cos s)}{4}.\end{aligned}$$

Utilizing this estimate under the sum in (7.24) (with $s=\langle\lambda,\, x\rangle$) and recalling the notation (3.3), we arrive at (7.22). □

7.5. *Proof of Theorem* 7.1 *and Corollary* 7.2. Let us first deduce the corollary from the theorem.

Proof of Corollary 7.2. According to Theorem 5.1, $a_z := E_z^r(\xi) = n + o\bigl(|n|^{2/3}\bigr)$. Together with Lemma 7.8 this implies

$$\begin{aligned}|(n-a_z)V_z| &\le |n-a_z|\cdot\|V_z\|\\ &= o\bigl(|n|^{2/3}\bigr)\,O\bigl(|n|^{-2/3}\bigr) = o(1).\end{aligned}$$

Hence, by Lemma 7.6 we get

$$\begin{aligned}f_{a_z,K_z}(n) &= \frac{1}{2\pi}\,(\det K_z)^{-1/2}\,\mathrm{e}^{-|(n-a_z)V_z|^2/2}\\ &= \frac{r^{1/3}\kappa}{2\sqrt{3}\,\pi}\,(n_1n_2)^{-2/3}(1+o(1)),\end{aligned}$$

and (7.3) follows from (7.2). □

Proof of Theorem 7.1. By definition, the characteristic function of the random vector $\xi_0 = \xi - a_z$ is given by the Fourier series

$$\varphi_{\xi_0}(\lambda) := E_z^r\bigl(\mathrm{e}^{i\langle\lambda,\,\xi_0\rangle}\bigr) = \sum_{m\in\mathbb{Z}_+^2} Q_z^r\{\xi=m\}\,\mathrm{e}^{i\langle\lambda,\,m-a_z\rangle},\qquad \lambda\in\mathbb{R}^2,$$

and hence the Fourier coefficients are expressed as

$$Q_z^r\{\xi=m\} = \frac{1}{4\pi^2}\int_{T^2}\mathrm{e}^{-i\langle\lambda,\,m-a_z\rangle}\,\varphi_{\xi_0}(\lambda)\,\mathrm{d}\lambda,\qquad m\in\mathbb{Z}_+^2, \tag{7.25}$$

where $T^2 := \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 : |\lambda_1| \le \pi,\, |\lambda_2| \le \pi\}$. On the other hand, the characteristic function corresponding to the normal probability density $f_{a_z,K_z}(x)$ [see (7.1)] is given by

$$\varphi_{a_z,K_z}(\lambda) = \mathrm{e}^{i\langle\lambda, a_z\rangle - |\lambda V_z^{-1}|^2/2}, \qquad \lambda \in \mathbb{R}^2,$$

so by the Fourier inversion formula

$$f_{a_z,K_z}(m) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \mathrm{e}^{-i\langle\lambda,\, m-a_z\rangle - |\lambda V_z^{-1}|^2/2}\,\mathrm{d}\lambda, \qquad m \in \mathbb{Z}_+^2. \tag{7.26}$$

Note that if $|\lambda V_z^{-1}| \le L_z^{-1}$ then, according to Lemmas 7.8 and 7.9,

$$|\lambda| \le |\lambda V_z^{-1}| \cdot \|V_z\| \le L_z^{-1}\|V_z\| = O\bigl(|n|^{-1/3}\bigr) = o(1),$$

which implies that $\lambda \in T^2$. Using this observation and subtracting (7.26) from (7.25), we get, uniformly in $m \in \mathbb{Z}_+^2$,

$$\bigl|Q_z^r\{\xi = m\} - f_{a_z,K_z}(m)\bigr| \le I_1 + I_2 + I_3, \tag{7.27}$$

where

$$\begin{aligned}
I_1 &:= \frac{1}{4\pi^2}\int_{\{\lambda:\, |\lambda V_z^{-1}| \le L_z^{-1}\}} \bigl|\varphi_{\xi_0}(\lambda) - \mathrm{e}^{-|\lambda V_z^{-1}|^2/2}\bigr|\,\mathrm{d}\lambda,\\
I_2 &:= \frac{1}{4\pi^2}\int_{\{\lambda:\, |\lambda V_z^{-1}| > L_z^{-1}\}} \mathrm{e}^{-|\lambda V_z^{-1}|^2/2}\,\mathrm{d}\lambda,\\
I_3 &:= \frac{1}{4\pi^2}\int_{T^2\cap\{\lambda:\, |\lambda V_z^{-1}| > L_z^{-1}\}} |\varphi_{\xi_0}(\lambda)|\,\mathrm{d}\lambda.
\end{aligned}$$

By the substitution $\lambda = yV_z$, the integral I_1 is reduced to

$$\begin{aligned}
I_1 &= \frac{|\det V_z|}{4\pi^2}\int_{|y|\le L_z^{-1}} \bigl|\varphi_{\xi_0}(yV_z) - \mathrm{e}^{-|y|^2/2}\bigr|\,\mathrm{d}y\\
&= O(1)\,(\det K_z)^{-1/2}\,L_z\int_{\mathbb{R}^2} |y|^3\mathrm{e}^{-|y|^2/6}\,\mathrm{d}y = O\bigl(|n|^{-5/3}\bigr),
\end{aligned}\tag{7.28}$$

on account of Lemmas 7.6, 7.9 and 7.12. Similarly, again putting $\lambda = yV_z$ and passing to the polar coordinates, we get, due to Lemmas 7.6 and 7.9,

$$\begin{aligned}
I_2 &= \frac{|\det V_z|}{2\pi}\int_{L_z^{-1}}^{\infty} |y|\,\mathrm{e}^{-|y|^2/2}\,\mathrm{d}|y|\\
&= O\bigl(|n|^{-4/3}\bigr)\,\mathrm{e}^{-L_z^{-2}/2} = o\bigl(|n|^{-5/3}\bigr).
\end{aligned}\tag{7.29}$$

Estimation of I_3 is the main part of the proof. Using Lemma 7.13, we obtain

$$I_3 = O(1) \int_{T^2 \cap \{|\lambda V_z^{-1}| > L_z^{-1}\}} \mathrm{e}^{-J_\alpha(\lambda)}\,\mathrm{d}\lambda, \tag{7.30}$$

where $J_\alpha(\lambda)$ is given by (7.23). The condition $|\lambda V_z^{-1}| > L_z^{-1}$ implies that $|\lambda| > \eta|\alpha|$ for a suitable (small enough) constant $\eta > 0$ and hence

$$\max\{|\lambda_1|/\alpha_1, |\lambda_2|/\alpha_2\} > \eta,$$

for otherwise from Lemmas 7.7, 7.9 and 7.12 it would follow

$$1 < L_z|\lambda V_z^{-1}| \le L_z\eta|\alpha| \cdot \|K_z\|^{1/2} = O(\eta) \to 0 \quad \text{as} \quad \eta \downarrow 0.$$

Hence, the estimate (7.30) is reduced to

$$I_3 = O(1) \left(\int_{|\lambda_1| > \eta\alpha_1} + \int_{|\lambda_2| > \eta\alpha_2} \right) \mathrm{e}^{-J_\alpha(\lambda)}\,\mathrm{d}\lambda. \tag{7.31}$$

To estimate the first integral in (7.31), by keeping in the sum (7.23) only pairs of the form $x = (x_1, 1)$, $x_1 \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
\mathrm{e}^{\alpha_2} J_\alpha(\lambda) &\ge \sum_{x_1=0}^{\infty} \mathrm{e}^{-\alpha_1 x_1} \Bigl(1 - \Re\, \mathrm{e}^{i(\lambda_1 x_1 + \lambda_2)}\Bigr) \\
&= \frac{1}{1 - \mathrm{e}^{-\alpha_1}} - \Re\left(\frac{\mathrm{e}^{i\lambda_2}}{1 - \mathrm{e}^{-\alpha_1 + i\lambda_1}}\right) \\
&\ge \frac{1}{1 - \mathrm{e}^{-\alpha_1}} - \frac{1}{|1 - \mathrm{e}^{-\alpha_1 + i\lambda_1}|}, \qquad (7.32)
\end{aligned}$$

because $\Re u \le |u|$ for any $u \in \mathbb{C}$. Since $\eta\alpha_1 \le |\lambda_1| \le \pi$, we have

$$\bigl|1 - \mathrm{e}^{-\alpha_1 + i\lambda_1}\bigr| \ge \bigl|1 - \mathrm{e}^{-\alpha_1 + i\eta\alpha_1}\bigr| \sim \alpha_1(1 + \eta^2)^{1/2} \qquad (\alpha_1 \to 0).$$

Substituting this estimate into (7.32), we conclude that $J_\alpha(\lambda)$ is asymptotically bounded from below by $C(\eta)\,\alpha_1^{-1} \asymp |n|^{1/3}$, uniformly in $\eta\alpha_1 \le |\lambda_1| \le \pi$. Thus, the first integral in (7.31) is bounded by

$$O(1)\exp\bigl(-\mathrm{const} \cdot |n|^{1/3}\bigr) = o\bigl(|n|^{-5/3}\bigr).$$

Similarly, the second integral in (7.31) (where $|\lambda_2| > \eta\alpha_2$) is estimated by reducing summation in (7.23) to that over $x = (1, x_2)$ only. As a result, we obtain that $I_3 = o\bigl(|n|^{-5/3}\bigr)$. Substituting this estimate together with (7.28) and (7.29) into (7.27), we get (7.2), and so the theorem is proved. $\square$

8. Proof of the limit shape results. Recall the notation [see (4.1), (4.2)] $\xi(t) = \sum_{x\in\mathcal{X}(t)} x\nu(x)$, where $\mathcal{X}(t) = \{x \in \mathcal{X} : x_2/x_1 \le t(n_2/n_1)\}$, $t \in [0,\infty]$. As stated at the beginning of Section 4, the tangential parameterization of the scaled polygonal line $\tilde{\mathit{\Gamma}}_n = S_n(\mathit{\Gamma})$ is given by

$$\tilde{\xi}_n(t) := S_n(\xi(t)) = \bigl(n_1^{-1}\xi_1(t),\, n_2^{-1}\xi_2(t)\bigr), \qquad t \in [0,\infty], \tag{8.1}$$

whereas the limit shape γ^* determined by equation (1.3) is parameterized by the vector-function $g^*(t) = (g_1^*(t), g_2^*(t))$ defined in (4.4) (see more details in the Appendix, Section A.1).

The goal of this section is to use the preparatory results obtained so far and prove the uniform convergence of random paths $\tilde{\xi}_n(\cdot)$ to the limit $g^*(\cdot)$ in probability with respect to both Q_z^r (Section 8.1) and P_n^r (Section 8.2). Let us point out that, in view of (8.1), Theorems 8.1 and 8.2 below can be easily reformulated (cf. Theorem 1.1 stated in the Introduction) using the tangential distance $d_{\mathcal{T}}(\tilde{\mathit{\Gamma}}_n, \gamma^*) = \sup_{0\le t\le\infty}\bigl|\tilde{\xi}_n(t) - g^*(t)\bigr|$ (see (1.9); cf. general definition (A.3) in Section A.1 below).

8.1. *Limit shape under Q_z^r.* Let us first establish the universality of the limit shape under the measures Q_z^r, which, in conjunction with the next Theorem 8.2, illustrates the asymptotic "equivalence" of the probability spaces $(\mathit{\Pi}, Q_z^r)$ and $(\mathit{\Pi}_n, P_n^r)$.

THEOREM 8.1. *For each $\varepsilon > 0$,*

$$\lim_{n\to\infty} Q_z^r\biggl\{\sup_{0\le t\le\infty}\bigl|n_j^{-1}\xi_j(t) - g_j^*(t)\bigr| \le \varepsilon\biggr\} = 1 \qquad (j = 1, 2).$$

PROOF. By Theorems 4.1 and 4.2, the expectation of the random process $n_j^{-1}\xi_j(t)$ uniformly converges to $g_j^*(t)$ as $n \to \infty$. Therefore, we only need to check that for each $\varepsilon > 0$

$$\lim_{n\to\infty} Q_z^r\biggl\{\sup_{0\le t\le\infty} n_j^{-1}\bigl|\xi_j(t) - E_z^r[\xi_j(t)]\bigr| > \varepsilon\biggr\} = 0.$$

Note that the random process $\xi_{0j}(t) := \xi_j(t) - E_z^r[\xi_j(t)]$ has independent increments and zero mean; hence it is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma\{\nu(x),\, x \in \mathcal{X}(t)\}$, $t \in [0,\infty]$. From the definition of $\xi_j(t)$ [see (4.2)], it is also clear that $\xi_{0j}(t)$ is a càdlàg process; that is, its paths are everywhere right-continuous and have left limits. Therefore, applying the Kolmogorov–Doob submartingale inequality (see, e.g., [39], [Corollary 2.1, page 14) and using Theorem 6.1, we obtain

$$Q_z^r\biggl\{\sup_{0\le t\le\infty}|\xi_{0j}(t)| > \varepsilon n_j\biggr\} \le \frac{\mathrm{Var}(\xi_j)}{(\varepsilon n_j)^2} = O\bigl(|n|^{-2/3}\bigr) \to 0,$$

and the theorem is proved. □

8.2. *Limit shape under P_n^r.* We are finally ready to prove our main result about the universality of the limit shape under the measures P_n^r (cf. Theorem 1.1).

THEOREM 8.2. *For any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P_n^r\left\{\sup_{0\le t\le\infty}\left|n_j^{-1}\xi_j(t) - g_j^*(t)\right| \le \varepsilon\right\} = 1 \qquad (j = 1, 2).$$

PROOF. Similarly as in the proof of Theorem 8.1, the claim of the theorem is reduced to the limit

$$\lim_{n\to\infty} P_n^r\left\{\sup_{0\le t\le\infty}\left|\xi_{0j}(t)\right| > \varepsilon n_j\right\} = 0,$$

where $\xi_{0j}(t) = \xi_j(t) - E_z^r[\xi_j(t)]$. Using (2.9) we get

$$P_n^r\left\{\sup_{0\le t\le\infty}\left|\xi_{0j}(t)\right| > \varepsilon n_j\right\} \le \frac{Q_z^r\left\{\sup_{0\le t\le\infty}\left|\xi_{0j}(t)\right| > \varepsilon n_j\right\}}{Q_z^r\{\xi = n\}}. \tag{8.2}$$

Applying the Kolmogorov–Doob submartingale inequality and using Lemma 6.7 (with $k = 3$), we obtain

$$Q_z^r\left\{\sup_{0\le t\le\infty}\left|\xi_{0j}(t)\right| > \varepsilon n_j\right\} \le \frac{E_z^r\left[\xi_j - E_z^r(\xi_j)\right]^6}{(\varepsilon n_j)^6} = O\bigl(|n|^{-2}\bigr).$$

On the other hand, by Corollary 7.2

$$Q_z^r\{\xi = n\} \asymp (n_1 n_2)^{-2/3} \asymp |n|^{-4/3}.$$

In view of these estimates, the right-hand side of (8.2) is dominated by a quantity of order of $O\bigl(|n|^{-2/3}\bigr) \to 0$, and the theorem is proved. □

APPENDIX

A.1. Tangential distance between convex paths. Let $\mathcal{G}_0$ be the space of paths in $\mathbb{R}_+^2$ starting from the origin and such that each path $\gamma \in \mathcal{G}_0$ is continuous, piecewise C^1-smooth (i.e., everywhere except a finite set), bounded and convex, and, furthermore, its tangent slope (where it exists) is non-negative, including the possible value $+\infty$. Convexity implies that the slope is non-decreasing as a function of the natural parameter (i.e., the length along the path measured from the origin).

For $\gamma \in \mathcal{G}_0$, let $g_\gamma(t) = (g_1(t), g_2(t))$ denote the right endpoint of the (closure of the) part of γ where the tangent slope does not exceed $t \in [0,\infty]$. Note that the functions $x_1 = g_1(t)$, $x_2 = g_2(t)$ are càdlàg (i.e., right-continuous with left limits), and

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{g_2'(t)}{g_1'(t)} = t. \tag{A.1}$$

More precisely, equation (A.1) holds at points where the tangent exists and its slope is strictly growing; corners on γ correspond to intervals where both functions g_1 and g_2 are constant, whereas flat (straight line) pieces on γ lead to simultaneous jumps of g_1 and g_2.

The canonical limit shape curve γ^* [see (1.3)] is determined by the parametric equations [cf. (1.8), (4.4)]

$$x_1 = g_1^*(t) = \frac{t^2+2t}{(1+t)^2}\,, \qquad x_2 = g_2^*(t) = \frac{t^2}{(1+t)^2}\,, \qquad t \in [0,\infty]. \tag{A.2}$$

Indeed, it can be readily seen that the functions (A.2) satisfy the Cartesian equation (1.3) for γ^*; moreover, it is easy to check that t in equations (A.2) is the tangential parameter,

$$\frac{\mathrm{d}g_1^*(t)}{\mathrm{d}t} = \frac{2}{(1+t)^3}\,, \qquad \frac{\mathrm{d}g_2^*}{\mathrm{d}t} = \frac{2t}{(1+t)^3}\,,$$

and hence [cf. (A.1)]

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{\mathrm{d}g_2^*/\mathrm{d}t}{\mathrm{d}g_1^*/\mathrm{d}t} = t.$$

The tangential distance $d_{\mathcal{T}}$ between paths in $\mathcal{G}_0$ is defined as follows:

$$d_{\mathcal{T}}(\gamma_1,\gamma_2) := \sup_{0\le t\le\infty} |g_{\gamma_1}(t) - g_{\gamma_2}(t)|, \qquad \gamma_1,\gamma_2 \in \mathcal{G}_0. \tag{A.3}$$

LEMMA A.1. *The Hausdorff distance $d_{\mathcal{H}}$ defined in (1.2) is dominated by the tangential distance $d_{\mathcal{T}}$*

$$d_{\mathcal{H}}(\gamma_1,\gamma_2) \le d_{\mathcal{T}}(\gamma_1,\gamma_2), \qquad \gamma_1,\gamma_2 \in \mathcal{G}_0. \tag{A.4}$$

PROOF. First of all, note that any path $\gamma \in \mathcal{G}_0$ can be approximated, simultaneously in metrics $d_{\mathcal{H}}$ and $d_{\mathcal{T}}$, by polygonal lines γ^m (for instance, by inscribing polygonal lines with refined edges in the arc γ) so that

$$\lim_{m\to\infty} d_{\mathcal{H}}(\gamma,\gamma^m) = 0, \qquad \lim_{m\to\infty} d_{\mathcal{T}}(\gamma,\gamma^m) = 0.$$

This reduces inequality (A.4) to the case where γ_1, γ_2 are polygonal lines. Moreover, by symmetry it suffices to show that

$$\max_{x\in\gamma_1}\min_{y\in\gamma_2}|x-y|\le d_{\mathcal{T}}(\gamma_1,\gamma_2). \tag{A.5}$$

Note that if a point $x\in\gamma_1$ can be represented as $x=g_{\gamma_1}(t_0)$ with some t_0 (i.e., x is a vertex of γ_1), then

$$\min_{y\in\gamma_2}|x-y|=\min_{y\in\gamma_2}|g_{\gamma_1}(t_0)-y|\le|g_{\gamma_1}(t_0)-g_{\gamma_2}(t_0)|\le d_{\mathcal{T}}(\gamma_1,\gamma_2),$$

and inequality (A.5) follows.

Suppose now that $x\in\gamma_1$ lies on an edge — say of slope t^* — between two consecutive vertices $g_{\gamma_1}(t_*)$ and $g_{\gamma_1}(t^*)$, then $x=\theta g_{\gamma_1}(t_*)+(1-\theta)g_{\gamma_1}(t^*)$ with some $\theta\in(0,1)$ and

$$\begin{aligned}\min_{y\in\gamma_2}|x-y|&=\min_{y\in\gamma_2}|\theta g_{\gamma_1}(t_*)+(1-\theta)g_{\gamma_1}(t^*)-y|\\&\le|\theta g_{\gamma_1}(t_*)+(1-\theta)g_{\gamma_1}(t^*)-g_{\gamma_2}(t^*)|\\&\le\theta|g_{\gamma_1}(t_*)-g_{\gamma_2}(t^*)|+(1-\theta)|g_{\gamma_1}(t^*)-g_{\gamma_2}(t^*)|\\&\le\theta|g_{\gamma_1}(t_*)-g_{\gamma_2}(t^*)|+(1-\theta)\,d_{\mathcal{T}}(\gamma_1,\gamma_2).\end{aligned} \tag{A.6}$$

Note that for all $t\in[t_*,t^*)$ we have $g_{\gamma_1}(t)=g_{\gamma_1}(t_*)$, hence

$$|g_{\gamma_1}(t_*)-g_{\gamma_2}(t)|=|g_{\gamma_1}(t)-g_{\gamma_2}(t)|\le d_{\mathcal{T}}(\gamma_1,\gamma_2)\qquad(t_*\le t<t^*). \tag{A.7}$$

If $g_{\gamma_2}(t)$ is continuous at $t=t^*$, then inequality (A.7) extends to $t=t^*$:

$$|g_{\gamma_1}(t_*)-g_{\gamma_2}(t^*)|\le d_{\mathcal{T}}(\gamma_1,\gamma_2).$$

Substituting this inequality into the right-hand side of (A.6), we see that $\min_{y\in\gamma_2}|x-y|\le d_{\mathcal{T}}(\gamma_1,\gamma_2)$, which implies (A.5).

If $g_{\gamma_2}(t^*-0)\ne g_{\gamma_2}(t^*)$, then t^* coincides with the slope of some edge on γ_2 (with the endpoints, say, $g_{\gamma_2}(t'_*)$ and $g_{\gamma_2}(t^*)$), which is thus parallel to the edge on γ_1 where the point x lies (i.e., with the endpoints $g_{\gamma_1}(t_*)$, $g_{\gamma_1}(t^*)$). Setting $s_*:=\max\{t'_*,t_*\}<t^*$, we have $g_{\gamma_1}(t_*)=g_{\gamma_1}(s_*)$, $g_{\gamma_2}(t'_*)=g_{\gamma_2}(s_*)$.

To complete the proof, it remains to observe that the shortest distance from a point on a base of a trapezoid to the opposite base does not exceed the maximum length of the two lateral sides. Hence,

$$\begin{aligned}\min_{y\in\gamma_2}|x-y|&\le\min\bigl\{|x-y|:\,y\in[g_{\gamma_2}(s_*),g_{\gamma_2}(t^*)]\bigr\}\\&\le\max\bigl\{|g_{\gamma_1}(s_*)-g_{\gamma_2}(s_*)|,|g_{\gamma_1}(t^*)-g_{\gamma_2}(t^*)|\bigr\}\\&\le d_{\mathcal{T}}(\gamma_1,\gamma_2),\end{aligned}$$

and the bound (A.5) follows. ∎

REMARK A.1. Note, however, that the metrics $d_{\mathcal{H}}$ and $d_{\mathcal{T}}$ are *not equivalent*. For instance, if $\gamma \in \mathcal{G}_0$ is a smooth, strictly convex curve with curvature bounded below by a constant $\varkappa_0 > 0$, then for an inscribed polygonal line $\varGamma_\varepsilon$ with edges of length no more than $\varepsilon > 0$, its tangential distance from γ is of the order of ε, but the Hausdorff distance is of the order of ε^2:

$$d_{\mathcal{T}}(\varGamma_\varepsilon, \gamma) \asymp \varepsilon, \qquad d_{\mathcal{H}}(\varGamma_\varepsilon, \gamma) \asymp \varkappa_0 \varepsilon^2 \qquad (\varepsilon \to 0).$$

Moreover, in the degenerate case where the curvature may vanish, the difference between the two metrics may be even more dramatic. For instance, it is possible that two polygonal lines are close to each other in the Hausdorff distance while their tangential distance is quite large. For an example, consider two straight line segments $\varGamma_1, \varGamma_2 \subset \mathbb{R}^2_+$ of the same (large) length L, both starting from the origin and with very close slopes, so that the Euclidean distance δ between their right endpoints is small; then $d_{\mathcal{H}}(\varGamma_1, \varGamma_2) \le \delta$ whereas $d_{\mathcal{T}}(\varGamma_1, \varGamma_2) = L$.

A.2. Total variation distance between P_n^r and P_n^1. Note that if probability measures $\tilde{P}_n$ and P_n on the polygonal space $\varPi_n$ are asymptotically close to each other in *total variation* (TV), that is, $\|\tilde{P}_n - P_n\|_{TV} \to 0$ as $n \to \infty$, where

$$\|\tilde{P}_n - P_n\|_{TV} := \sup_{A \subset \varPi_n} |\tilde{P}_n(A) - P_n(A)|,$$

then the problem of universality of the limit shape is resolved in a trivial way, in that if a limit shape γ^* exists under P_n then the same curve γ^* provides the limit shape under $\tilde{P}_n$. Indeed, assuming that the event $A_\varepsilon = \{d(\tilde{\varGamma}_n, \gamma^*) > \varepsilon\}$ satisfies $P_n(A_\varepsilon) \to 0$ as $n \to \infty$, we have

$$\begin{aligned} \tilde{P}_n(A_\varepsilon) &\le P_n(A_\varepsilon) + \bigl|\tilde{P}_n(A_\varepsilon) - P_n(A_\varepsilon)\bigr| \\ &\le P_n(A_\varepsilon) + \|\tilde{P}_n - P_n\|_{TV} \to 0 \qquad (n \to \infty). \end{aligned}$$

However, the family $\{P_n^r\}$, defined by formula (2.10) with the coefficients (2.11), is *not* close to P_n^1 in total variation, at least uniformly in $r \in (0, \infty)$.

THEOREM A.2. *For every fixed n, the limiting distance in total variation between P_n^r and P_n^1, as $r \to 0$ or $r \to \infty$, is given by*

$$\lim_{r \to 0, \infty} \|P_n^r - P_n^1\|_{TV} = 1 - \frac{1}{\#(\varPi_n)}. \tag{A.8}$$

PROOF. To obtain a lower bound for $\|P_n^r - P_n^1\|_{TV}$ in the case $r \to \infty$, consider the polygonal line $\Gamma^* \in \Pi_n$ consisting of two edges, horizontal [from the origin to $(n_1, 0)$] and vertical [from $(n_1, 0)$ to $n = (n_1, n_2)$]. The corresponding configuration ν_{Γ^*} is determined by the conditions $\nu_{\Gamma^*}(1,0) = n_1$, $\nu_{\Gamma^*}(0,1) = n_2$ and $\nu_{\Gamma^*}(x) = 0$ otherwise. Note that $b_k^r \sim r^k/k!$ as $r \to \infty$ [see (2.11)], hence $b^r(\Gamma) = O\big(r^{N_\Gamma}\big)$, where $N_\Gamma := \sum_{x \in \mathcal{X}} \nu_\Gamma(x)$ is the total number of integer points on Γ (excluding the origin). We have $N_{\Gamma^*} = n_1 + n_2$, so $b^r(\Gamma^*) = b_{n_1}^r b_{n_2}^r \sim r^{n_1+n_2}/(n_1! n_2!)$ as $r \to \infty$. On the other hand, for any $\Gamma \in \Pi_n$ $(\Gamma \neq \Gamma^*)$ one has $b^r(\Gamma) = o(r^{n_1+n_2})$. Indeed, for $x \in \mathcal{X}$ we have $x_1 + x_2 \geq 1$ and, moreover, $x_1 + x_2 > 1$ unless $x = (1,0)$ or $x = (0,1)$. Hence, for any $\Gamma \in \Pi_n$ $(\Gamma \neq \Gamma^*)$,

$$N_\Gamma = \sum_{x \in \mathcal{X}} \nu_\Gamma(x) < \sum_{x \in \mathcal{X}} (x_1 + x_2)\nu_\Gamma(x) = n_1 + n_2,$$

so that $N_\Gamma < n_1 + n_2$ and $b^r(\Gamma) = O(r^{N_\Gamma}) = o(r^{n_1+n_2})$ as $r \to \infty$. Therefore, from (2.10) we get

$$P_n^r(\Gamma^*) = \frac{b^r(\Gamma^*)}{b^r(\Gamma^*) + \sum_{\Gamma \neq \Gamma^*} b^r(\Gamma)} = \frac{r^{n_1+n_2}}{r^{n_1+n_2} + o(r^{n_1+n_2})} \to 1 \qquad (r \to \infty),$$

and it follows that

$$\|P_n^r - P_n^1\|_{TV} \geq \big|P_n^r(\Gamma^*) - P_n^1(\Gamma^*)\big| \to 1 - \frac{1}{\#(\Pi_n)} \qquad (r \to \infty). \tag{A.9}$$

For the case $r \to 0$, consider the polygonal line $\Gamma_* \in \Pi_n$ consisting of one edge leading from the origin to $n = (n_1, n_2)$. That is, $\nu_{\Gamma_*}(n/k_n) = k_n$ and $\nu(x) = 0$ otherwise, where $k_n := \gcd(n_1, n_2)$. Clearly, $b^r(\Gamma_*) = b_{k_n}^r = r/k_n$, while for any other $\Gamma \in \Pi_n$ (i.e., with more than one edge), by (2.11) we have $b^r(\Gamma) = O(r^2)$ as $r \to 0$. Therefore, according to (2.10),

$$P_n^r(\Gamma_*) = \frac{b^r(\Gamma_*)}{b^r(\Gamma_*) + \sum_{\Gamma \neq \Gamma_*} b^r(\Gamma)} = \frac{r}{r + O(r^2)} \to 1 \qquad (r \to 0),$$

and similarly to (A.9) we obtain

$$\|P_n^r - P_n^1\|_{TV} \geq \big|P_n^r(\Gamma_*) - P_n^1(\Gamma_*)\big| \to 1 - \frac{1}{\#(\Pi_n)} \qquad (r \to 0). \tag{A.10}$$

The upper bound for $\|P_n^r - P_n^1\|_{TV}$ (uniform in r) follows from the known fact (see [10], page 472, and also [1], Section 3.1, pages 67 and 68) that the total variation distance can be expressed in terms of a certain Vasershtein (–Kantorovich–Rubinstein, cf. [36]) distance:

$$\|P_n^r - P_n^1\|_{TV} = \inf_{X,Y} \mathsf{E}[\varrho(X,Y)].$$

Here the infimum is taken over all pairs of random elements X and Y defined on a common probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with values in $\varPi_n$ and the marginal distributions P_n^r and P_n^1, respectively; E denotes expectation with respect to the probability measure P, and the function $\varrho(\cdot,\cdot)$ on $\varPi_n \times \varPi_n$ is such that $\varrho(\varGamma,\varGamma') = 1$ if $\varGamma \ne \varGamma'$ and $\varrho(\varGamma,\varGamma') = 0$ if $\varGamma = \varGamma'$ (therefore defining a discrete metric in $\varPi_n$). Choosing X and Y so that they are independent of each other, we obtain

$$\begin{aligned}
\|P_n^r - P_n^1\|_{TV} &\le \mathsf{E}[\varrho(X,Y)] = 1 - \mathsf{P}\{X = Y\} \\
&= 1 - \sum_{\varGamma \in \varPi_n} \mathsf{P}\{X = \varGamma, Y = \varGamma\} = 1 - \sum_{\varGamma \in \varPi_n} P_n^r(\varGamma) \cdot P_n^1(\varGamma) \\
&= 1 - \frac{1}{\#(\varPi_n)} \sum_{\varGamma \in \varPi_n} P_n^r(\varGamma) = 1 - \frac{1}{\#(\varPi_n)}.
\end{aligned}$$

Combining this estimate with (A.9) and (A.10), we obtain (A.8). ☐

In the limit $n \to \infty$, Theorem A.2 yields

$$\lim_{n\to\infty} \lim_{r\to 0,\infty} \|P_n^r - P_n^1\|_{TV} = 1.$$

That is to say, in the successive limit $r \to 0\,(\infty)$, $n \to \infty$, the measures P_n^r and P_n^1 become singular with respect to each other.

Moreover, one can show that the distance $\|P_n^r - P_n^1\|_{TV}$ is not small even for a fixed $r \ne 1$. To this end, it suffices to find a function on $\varPi_n$ possessing a limiting distribution (possibly degenerate) under each P_n^r, with the limit depending on the parameter r. Recalling Remark 2.1, it is natural to seek such a function in the form referring to integer points on $\varGamma \in \varPi_n$. Indeed, for the statistic $N_\varGamma = \sum_{x\in\mathcal{X}} \nu_\varGamma(x)$ introduced in the proof of Theorem A.2, the following law of large numbers holds (see Bogachev and Zarbaliev [6], Theorem 3, and Zarbaliev [40], Section 1.10).

Lemma A.3. *Under Assumption 3.1, for each $r \in (0,\infty)$ and any $\varepsilon > 0$,*

$$\lim_{n\to\infty} P_n^r(A_\varepsilon^r) = 1, \quad \textit{where} \quad A_\varepsilon^r := \left\{ \left| \frac{N_\varGamma}{(n_1 n_2)^{1/3}} - \frac{r^{1/3}}{\kappa^2} \right| \le \varepsilon \right\}. \tag{A.11}$$

The result (A.11) implies that, for any $r \neq 1$ and all $\varepsilon > 0$ small enough,

$$\|P_n^r - P_n^1\|_{TV} \geq \left|P_n^r(A_\varepsilon^r) - P_n^1(A_\varepsilon^r)\right| \to |1-0| = 1 \qquad (n \to \infty),$$

and we arrive at the following result.

THEOREM A.4. *For every fixed $r \neq 1$, we have*

$$\lim_{n\to\infty} \|P_n^r - P_n^1\|_{TV} = 1,$$

and hence the measures P_n^r and P_n^1 on Π_n are asymptotically singular with respect to each other as $n \to \infty$.

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