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Complementarity and Uncertainty in Mach-Zehnder Interferometry and beyond

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Abstract

A coherent account of the connections and contrasts between the principles of complementarity and uncertainty is developed starting from a survey of the various formalizations of these principles. The conceptual analysis is illustrated by means of a set of experimental schemes based on Mach-Zehnder interferometry. In particular, path detection via entanglement with a probe system and (quantitative) quantum erasure are exhibited to constitute instances of joint unsharp measurements of complementary pairs of physical quantities, path and interference observables. The analysis uses the representation of observables as positive-operator-valued measures (POVMs). The reconciliation of complementary experimental options in the sense of simultaneous unsharp preparations and measurements is expressed in terms of uncertainty relations of different kinds. The feature of complementarity, manifest in the present examples in the mutual exclusivity of path detection and interference observation, is recovered as a limit case from the appropriate uncertainty relation. It is noted that the complementarity and uncertainty principles are neither completely logically independent nor logical consequences of one another. Since entanglement is an instance of the uncertainty of quantum properties (of compound systems), it is moot to play out uncertainty and entanglement against each other as possible mechanisms enforcing complementarity.

Key words: Complementarity principle, uncertainty principle, joint measurement, Mach-Zehnder interferometer, path marking, erasure
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Contents

1 Introduction

Soon after the inception of quantum mechanics, the notions of complementarity and uncertainty were introduced to highlight features of the new theory, unknown to classical physics, which amounted to limitations of the preparation and measurement of atomic systems.

Complementarity and uncertainty continue to attract the attention of researchers, inspiring novel experimental tests and demonstrations. One such experiment was proposed by Scully, Englert and Walther (SEW) [1]. These authors described an atom interferometer in which entanglement is utilized to store information about the path of an atom: we will use the term "path marking" to refer to this process of path information storage in a probe system by way of establishing correlations between the atom and the probe. SEW's claim is that in their scheme, the standard explanation for the loss of interference upon path marking in terms of classical random momentum kicks, and hence the uncertainty relation, is not applicable.

The fact that in the SEW experiment the interference pattern was wiped out not by classical kicks, which supposedly could be associated with an indeterminate momentum, but by entanglement led to the suggestion that the principle of uncertainty is less significant than complementarity: "The principle of complementarity is manifest although the position-momentum uncertainty relation plays no role" ([1], p. 111); and in reply to a critique [2], SEW go further, stating that "the principle of complementarity is much deeper than the uncertainty relation although it is frequently enforced by $\delta x \delta p \geq \hbar/2$" [3].

A path-marking experiment similar to that proposed by SEW was experimentally realized in 1998 by Dürr, Nonn and Rempe [4] and confirmed the conclusion of SEW, that neither mechanical disturbance nor the position-momentum uncertainty relation could explain the loss of interference. This, in turn, caused a controversy and led to grossly misleading articles in popular science journals, announcing "An end to uncertainty - Wave goodbye to the Uncertainty Principle, you don't need it anymore." (New Scientist 1999) [5]). Numerous papers appeared with conflicting conclusions as to whether some forms of random disturbance or uncertainty relations could always be invoked to explain the loss of interference when path marking was effected via entanglement, or whether entanglement was the more fundamental mechanism for this form of complementarity (e.g. [6–12]).

The main aim of this paper is to study the roles and relative significance of the uncertainty principle (in an appropriate understanding), "disturbance" in measurement, and entanglement in the explanation of complementary quantum phenomena, such as the mutual exclusivity of path marking and interference detection.

It seems to us that the continuing lack of consensus over these questions is largely due to persistent differences of general outlook on the principles of complementarity and uncertainty that are manifest in the development just reviewed, and due to the fact that these differences are usually not reflected upon. The two notions are often considered as logically related, and the uncertainty relation is frequently presented as a quantitative expression of complementarity. By contrast, the work of SEW raised controversy over the question whether the uncertainty relation is at all relevant to complementarity. In effect, the considerations of SEW and the conclusions drawn by Dürr *et al.*

from their experiment suggest that complementarity and uncertainty refer to aspects of quantum physics that can be discriminated experimentally.

We therefore find it necessary to develop a coherent account of the notions and principles of complementarity and uncertainty before we illustrate the manifestations of these principles in the analysis of a set of simple interferometric experiments.

To this end, we will briefly analyze the existing contrasting views on complementarity and uncertainty, and review the development of various formalizations of these so-called principles (Section 2). Separating the broader "ideological" issues from the technical aspects will help to elucidate the interesting specific physical and conceptual questions indicated above, which can then be effectively addressed in terms of suitably defined notions of complementarity and uncertainty.

The analysis of the experimental illustrations of these concepts will be carried out using the general descriptions of observables in terms of positive-operator-valued measures (POVM). Projection-valued measures (PVMs) are special cases of POVM and are called sharp observables. POVMs that are not PVMs are called unsharp observables. Some of the experimental schemes presented here are instances of joint measurements of an unsharp path and an unsharp interference observable. The measurement theoretic concepts and tools required for our analysis will be briefly reviewed in Section 3.

In Section 4, we will discuss a range of atom and Mach-Zehnder interferometric experiments. We will briefly review the type of which-path and erasure experiments proposed by SEW and carried out by Dürr *et al.* (subsection 4.1). This will prepare the definition and study of analogous Mach-Zehnder interferometric setups in subsection 4.2-4.7. The following experiments are presented within one common setting and analyzed in detail: use of the interferometer for path detection; an interference detection setup; introduction of path marking via entanglement with a probe system, which precedes the interferometer; quantum erasure; and quantitative quantum erasure.

The language of POVMs proves useful in section 5 for the analysis of the relationships between specific versions of complementarity and uncertainty as they manifest themselves in the present group of experiments.

General conclusions are drawn in Section 6. In the presentation of complementarity and uncertainty developed here, these features are seen to be logically related within quantum mechanics, although they have their separate identities and roles. In a nutshell, *uncertainty* is a direct consequence of the linear structure of the Hilbert space formalism in that every vector state can be expressed as a linear combination of eigenvectors of some observable, whose values are therefore uncertain (or, more precisely, indefinite) in that state. This fact may be considered as the broadest formulation of the *uncertainty principle*, which is then usually expressed as a trade-off relation between pairs of noncommuting observables. The *complementarity principle* highlights an extreme form of that relation, stating that there are pairs of quantities that are such that complete determination of one quantity implies maximal uncertainty about the other (and vice versa). Thus, the formulation of complementarity presupposes the notion of uncertainty, but the principles of complementarity and uncertainty cannot be reduced to one another.

Similarly, it is pointed out that *entanglement* is an instance of uncertainty. It is therefore pointless to attempt to separate entanglement as a "mechanism" enforcing complementarity in addition to, and independently of, the uncertainty principle.

We hope that this study helps not only to settle some long-standing foundational issues of quantum mechanics, but also to promote the idea that the time has come to introduce the concept and application of POVMs in the undergraduate teaching of quantum mechanics. POVMs are already being taught as a standard tool in courses on quantum mechanics, quantum computation and quantum information in a small but growing number of institutions around the world. The concepts of the underlying modern quantum theory of measurements are slowly being brought to the attention of teachers of undergraduate quantum physics. For example, simple implementations of POVMs for quantum optical experiments were presented by H. Brandt in the American Journal of Physics [13]. There are a number of books giving elementary demonstrations of the use of POVMs in the analysis of conceptual problems and quantum experiments, the first being the fine text monograph by the late Asher Peres [14]. We dedicate this paper to the memory of Asher Peres, friend and mentor to one of us (PB), and mediator between quantum foundations and quantum applications.

2 The notions of complementarity and uncertainty

2.1 Bohr, Heisenberg, and the consequences

The principles of complementarity and uncertainty were introduced by Niels Bohr [15] and Werner Heisenberg [16] some eighty years ago in their efforts to develop an intuitive understanding of quantum physics. Their concern was to explain the dramatic deviations of this new theory from classical physics, which manifested themselves in wave-particle duality and the impossibility of defining and observing sharp particle trajectories.

From the beginning there was disagreement over the relative significance and merits of the notions of complementarity and uncertainty [17]. Bohr considered complementarity as fundamental for his interpretation of quantum mechanics, which was to be based on classical concepts and intuitions, rather than mathematical formalism. He thought of the uncertainty relation (primarily) as a formal, quantitative expression of complementarity. By contrast, Heisenberg seemed to have had little use for the notion of complementarity; he sought to develop an intuitive understanding of quantum mechanics that derived from the formalism itself. In his reminiscences, Heisenberg summarized the key that led him to the discovery of the uncertainty relation in the famous sentence that he ascribed to Einstein: "it is the theory which decides what can be observed" [18]. It made him realize that there was room in quantum mechanics for simultaneous approximate values of position and momentum, and thus for unsharply defined trajectories as they are observed in cloud chambers.

Bohr and Heisenberg eventually reached a compromise on their divergent views in the terminology of what came to be known as the Copenhagen interpretation of quantum mechanics. However, the fundamental differences in their philosophical outlooks on quantum mechanics were never resolved, as is manifest from their writings and recorded interviews and discussed, for example, in the book of Jammer [17]. The appearance of unity maintained nevertheless by the Copenhagen pioneers has made it difficult to obtain a coherent account of the so-called Copenhagen interpretation, and there is a vast body of *historico-philosophical* literature on this subject with rather divergent conclusions.

Here we are concerned with the reception of the notion of complementarity by the *physics* community. A survey of the textbook literature exhibits that three points of view have evolved concerning the relationship and interplay between the *principles* of complementarity and uncertainty. However, it seems to us that a fairly unambiguous, systematic account of the possible formalizations of the *notion* of complementarity within quantum mechanics has been obtained. We briefly recall the three distinct interpretational stances on the status of the principle of complementarity, and then review the possible rigorous formalizations of the notion of complementarity.

There is one group of authors who reiterate the account that has become part and parcel of the so-called Copenhagen interpretation: According to this view, the uncertainty relation constitutes the quantitative expression of the principle of complementarity (e.g., [19]), and quantum mechanics has been said to be best understood as *the* theory of complementarity [15,19–21]. Within this group, some describe the uncertainty principle as *enforcing* complementarity, while others deny the uncertainty relation the status of principle, playing down its significance in favor of complementarity [20]. (Asher Peres had his own, characteristically humorous way of doing this: in his textbook [14], the only occurrence of the term "uncertainty principle" is in the index, with the page reference indicating this very index page. On the other hand, the principle of complementarity is also described only briefly, while there are numerous occurrences of variants of uncertainty relations.)

A second group of authors (including, e.g., Dirac, von Neumann, Feynman) avoids carefully any mention of the term complementarity, apparently in accordance with a widespread perception that Bohr's presentations of this concept had remained rather obscure. These authors invoke the uncertainty principle as the reason for the impossibility of making simultaneous path and interference observations with arbitrary precision. This approach is consistent with Heisenberg's point of view, according to which it is possible but not necessary to refer to complementarity for an intuitive interpretation of quantum mechanics.

Finally, today there is a widespread sense that complementarity and uncertainty are best regarded as consequences of quantum mechanics which highlight characteristic features of that theory but need not be held up as independent principles.

We will reassess these positions after the following brief review of the evolution of current formulations of the complementarity and uncertainty principles.

2.2 The notion of complementarity

2.2.1 Complementarity in Bohr's writings

Bohr introduced the term *complementarity* in his 1927 lecture in Como: "The very nature of the quantum theory thus forces us to regard the space-time co-ordination and the claim of causality, the union of which characterizes the classical theories, as complementary but exclusive features of the description, symbolizing the idealization of observation and definition respectively." ([15], p. 580).

Complementarity was thus originally conceived as a relationship between pairs of descriptions, or phenomena, which are mutually exclusive but nevertheless both required for a complete account of the physical system under consideration. Bohr considered complementarity as a "rational generalization" of the classical notion of a causal spacetime description of physical phenomena. He argued that in quantum physics, causality (represented by conservation laws and deterministic equations of motion) and spacetime description fall apart, as a consequence of what he called the quantum postulate, according to which every observation (i.e., measurement) induces an uncontrollable, unavoidable, and non-negligible change of the phenomenon (i.e., the observed system). In fact, as Bohr put it, spacetime coordination requires observation and hence uncontrollable state changes, whereas a causal account of phenomena requires the possibility of defining a state and hence precludes interaction, and thus measurement.

For Bohr, the definition of a state and its observation constituted idealizations which were simultaneously applicable to any desired degree of accuracy in classical physics. According to the quantum postulate, the finite magnitude of Planck's quantum of action introduces a limitation to the simultaneous applicability of these idealizations: the influence of interactions required for an observation is no longer negligible for atomic systems where the characteristic quantities of the dimension of an action are comparable in magnitude to Planck's constant. According to Bohr, such measurement interactions leave the object and measuring apparatus in a situation that does not allow an independent description of either system: "Now, the quantum postulate implies that any observation of atomic phenomena will involve an interaction with the agency of observation not to be neglected. Accordingly, an independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation" [15]. We agree with D. Howard [22] that this passage should be read as expressing the entanglement of object and apparatus after the measurement interaction.

Following Bohr's own practice, it has become customary to interpret his informal descriptions of complementarity in terms of position and momentum observables as follows: these observables are complementary in that for a deterministic description of the trajectory of a particle, the values of both observables are required, but according to the quantum mechanical limitations of preparing and measuring the values of these observables, sharp values cannot be assigned to them simultaneously. This impossibility of defining sharp trajectories then gives room for the existence and explanation of interference phenomena. A lucid account of complementarity along these lines was given by W. Pauli as early as 1933 [19]. In his reply of 1935 to the famous challenge of Einstein, Podolsky and Rosen [23], Bohr defines the complementarity of position and momentum [24] by reference to "the mutual exclusion of any two experimental procedures, permitting the unambiguous definition of complementary physical quantities." Complementarity thus describes, according to Bohr, the limited way in which classical concepts can be applied in the description of quantum experiments. Such a limitation is imposed by quantum mechanics (the quantum postulate), in accordance with the failure of classical physics to explain atomic phenomena. The importance of complementarity derives, for Bohr, from the necessity of expressing all physical experience in classical physical language.

It should be noted that in contrast to the classical exclusivity of wave and particle theories that gave rise to the quantum puzzle of wave-particle duality, the concepts of position and momentum *are* simultaneously applicable in classical physics but *not* (without qualification) in quantum mechanics. This implies that wave-particle duality cannot simply be considered as an instance of complementarity, as is often suggested. There are pairs of phenomena (and of related observables) that the same type of system (e.g., electrons) can display and that are associated with intuitive ideas relating to either particles or waves. But there are also instances of particle and wave behaviors which do manifest themselves in the same experimental setup, such as the Compton scattering of a photon with an electron observed through a γ-ray microscope, followed by the subsequent wave-like propagation of the light through the microscope, which accounts for the finite optical resolution. In this and similar examples, Bohr freely used wave and particle aspects in the analysis of one and the same phenomenon. Another simple example is the interference observation in a double slit experiment where the interference is explained by wave superposition while at the same time the pattern is built up by the successive recordings of the photons or electrons through their inelastic collisions with the molecules in the photographic plate. According to Bohr, the apparent contradiction that lies in the wave-particle duality is resolved through the realization of the limited simultaneous applicability of classical concepts that is complementarity.

A dissolution, within quantum theory, of the problem of wave-particle duality was carried out somewhat differently and more formally by Heisenberg in his 1929 Chicago lectures [25]. He identifies particle and wave pictures with the quantized theories of particles and fields, and sees the consistency of the two pictures established by the equivalence of the quantum field theory, restricted to the subspace of N quanta, with the quantum mechanics of N particles. On this account, there is no need for a simultaneous application of wave and particle ideas since both formalisms can be used to give equivalent accounts of the experiments in question.

These different assessments of the significance of wave-particle duality and of its resolution in quantum mechanics are one example of the persistent discrepancies between Bohr and Heisenberg. A detailed analysis of Heisenberg's account of wave-particle duality, with similar conclusions to ours, has recently been given by Camilleri [26].

2.2.3 Modern formulations of complementarity

The complementarity idea was thus gradually transformed into the notion of complementarity of pairs of physical quantities, with an emphasis on the *negative* aspect of *mutual exclusiveness* of value assignments or experimental setups (such as path or interference detection). As will become evident from the discussion of the uncertainty principle below, the *positive* aspect of *mutual completion*, or *complementation*, that comprises the original meaning of the term complementarity, has been delegated to the uncertainty principle.

The statement of mutual exclusiveness of experimental setups can be interpreted as constituting three distinct forms of complementarity; one that refers to the possibilities of state preparation, and two that refer to the possibilities of joint and sequential measurements, respectively. This distinction allows one to take into account the freedom in placing the Heisenberg cut between preparation (object system) and measurement (probe system, measuring apparatus), as well as the fact that often the state-preparing effects of measurements are utilized. *Preparation complementarity* is the impossibility of preparing states in which the two observables in question are simultaneously assigned sharp values. *Measurement complementarity* is the impossibility of performing simultaneous sharp measurements of these observables, or the impossibility of performing their measurements in sequence without mutual disturbance. In these general forms, complementarity applies to practically all pairs of non-commuting quantities (exceptions are pairs with some common eigenstates).

Complementarity is usually understood to be a more specific relationship that singles out certain important pairs of observables, including, but not restricted to, canonically conjugate pairs such as position and momentum, or components of angular momentum and their associated angles. A comprehensive discussion of definitions of preparation and measurement complementarity together with examples of complementary pairs of observables are given in the monograph [27] (Sec. IV.2) and the review [28]. We will restrict ourselves here to a brief summary with sufficient detail for the purposes of our subsequent analyses.

2.2.4 Preparation complementarity

The most widely adoped form of *preparation complementarity* (e.g., [20,29,1,12]) is probably the following one that we will refer to as *value complementarity* (following [27]): two observables are value complementary if whenever one has a definite value, the values of the other are maximally uncertain. A value of observable A is definite if it occurs with probability equal to one in a measurement of A, and the values of observable B are maximally uncertain if they occur with equal probabilities in a measurement of B.

Formally, the value complementarity of two observables A, B with discrete nondegenerate spectra and associated eigenbasis systems $\{\psi_k : k = 1, \ldots, n\}$, $\{\phi_\ell : \ell = 1, \ldots, n\}$ amounts to the statement that any two eigenstates have constant overlap, that is, the numbers $|\langle\psi_k|\phi_\ell\rangle|$ are independent of k, ℓ. Pairs of orthonormal basis systems with this property are called *mutually unbiased*. It is well known that any observable A with nondegenerate eigenstates $\psi_1, \ldots, \psi_n$ has at least one partner B with which it forms a value complementary pair; in fact there are infinitely many such partners: first one can define any B with orthonormal system of eigenstates $\phi_\ell = \frac{1}{\sqrt{n}} \sum_k e^{2\pi i k\ell/n}\psi_k$, $\ell = 1, \ldots, n$. It is easily verified that A, B are value complementary. Note that the eigenvalues of the observables do not enter the definition of value complementarity. Further value-complementary partners B_U of A are obtained by taking $B_U := UBU^*$, where U is any unitary operator that commutes with A.

One must note that the notion of value complementarity does not easily extend to continuous quantities or those with unbounded spectra. For example, in order to consider position and momentum as value complementary, one must use their improper eigenstates; one could try to capture the idea of nearly sharp position values by means of sequences of normalizable states whose position distributions approach a Dirac distribution. In that sequence, the momentum distributions become arbitrarily widely spread out, but it is not clear how to express the idea that these momentum distributions approach a uniform distribution, which does not exist on the basis of normalizable states.

Similarly, in the case of number and conjugate phase, it is true that for a number eigenstate, the phase is distributed uniformly, but there are no states with definite, sharp values of the phase, nor are there states with uniform number distributions [30]. To maintain the idea of value complementarity, one would have to allow it to become a nonsymmetric relation.

These limitations do not arise if one adopts a slightly weaker form of preparation complementarity, known as probabilistic complementarity. Observables A, B are *probabilistically complementary* if and only if for their spectral projections $E^A(X)$, $E^B(Y)$ associated with bounded intervals X, Y (such that the projections $E^A(X)$, $E^B(Y)$ are different from the null operator O and the unit operator I) it is true that whenever the probability $\mathrm{tr}[\rho E^A(X)] = 1$ then $0 \neq \mathrm{tr}[\rho E^B(Y)] \neq 1$, and vice versa. This is to say that if the value of A is definitely in interval X, the value of B is never certain to be in any interval Y (or its complement). This notion applies without difficulty to the position-momentum and number-phase pairs; but it also allows any two spin components of a spin-$\frac{1}{2}$ system to be complementary.

Probabilistic complementarity is known to be equivalent to the statement that for all bounded intervals X, Y (such that $E^A(X), E^B(Y) \neq O, I$), the associated spectral projections of A and B satisfy

$$E^A(X) \wedge E^B(Y) = O, \tag{1}$$
$$E^A(X) \wedge E^B(\mathbb{R} \setminus Y) = O, \tag{2}$$
$$E^A(\mathrm{R} \setminus X) \wedge E^B(Y) = O. \tag{3}$$

Here, $P \wedge Q$ denotes the intersection of the projections P and Q, that is, the projection onto the intersection of their ranges; $R \setminus X$ denotes the complement of the set X. The equation $P \wedge Q = O$ is equivalent to the statement that P and Q have no common eigenvectors associated with their eigenvalue 1.

2.2.5 Measurement complementarity

Measurement complementarity can be specified in a similar vein to refer to a pair of observables A, B for which a sharp measurement of one of them makes any attempt at measuring the other one simultaneously or in immediate succession completely obsolete. The first form of measurement complementarity, the impossibility of joint measurements, is a special instance of von Neumann's theorem [31], according to which two observables are jointly measurable if, and only if, they commute. In a more specific, stronger form, the measurement complementarity of two observables A, B can be expressed as an exclusion relation for the quantum operations describing the state changes due to the measurements of A and B. This characterization was again found to be equivalent to the relation (1) (see [27, Sec. IV.2.3]), thus demonstrating the match, stipulated already by Bohr, between the possibilities of preparation and the possibilities of measurement.

Measurement complementarity in the case of sequential measurements will be taken to mean that due to the effect of the A measurement, the B measurement will not recover any information whatsoever about B in the (input) state immediately prior to the A measurement. In extreme cases it may happen that, although the second setup is devised to measure observable B, the statistics obtained in the presence of the A measurement is independent of the input state; the observable effectively measured in the second measurement provides no information about the system state prior to the A measurement. This is again a special instance of a general theorem in the quantum theory of measurement (discussed, e.g., in [27, pp. 42-44]), according to which a sequence of a measurement of observable A followed immediately by a measurement set up to measure observable B constitutes a joint measurement of A and some unsharp observable B' (represented by a POVM) that commutes with A. If A and B are value complementary observables with mutually unbiased eigenbases and if A is measured first by a von Neumann measurement, then the observable B' actually measured by a subsequent B measurement scheme is in fact trivial, in that its statistics carries no information about the system state prior to the A measurement.

Measurement complementarity is thus seen to reflect the fact that in quantum mechanics, every nontrivial measurement must alter the system's state, at least in the case of some input states. There can thus be no information gain without "disturbance" in quantum mechanics. As will be discussed in the next subsection, the related issue of a measurement of one variable disturbing the distributions of other, noncommuting variables has been highlighted by Heisenberg by means of his famous thought experiments illustrating the uncertainty relation. This feature of quantum mechanics—the fact that measurements necessarily alter the system under investigation, hence the impossibility to "see" the system in its undisturbed form—has impressed the scientifically interested public as being of such fundamental importance that it is now widely known under the name "Heisenberg effect" or also "observer effect" (as is quickly confirmed by an internet search). In fact, these terms are used to denote loosely analogous phenomena in a variety of disciplines, ranging from sociology, political science or market research over anthropology and population ecology to computer science (where certain forms of programming errors are called Heisenbugs).

We will see value complementarity and the two versions of measurement complementarity (for joint or sequential measurements) at work in the discussion of the Mach-Zehnder interferometry experiments; if the path observable is represented by σ_z, any of its associated interference observables, represented by $\cos\xi\,\sigma_x + \sin\xi\,\sigma_y$, is a complementary partner.

2.2.6 Complementarity principle

The *complementarity principle* is the statement that there are pairs of observables which stand in the relationship of complementarity. As we saw above, this is satisfied in quantum mechanics for observables the eigenvector basis systems of which are mutually unbiased. We conclude that the "principle" of complementarity, as formalized here, is a consequence of the quantum mechanical formalism. There seems to be no need to speak of a complementarity *principle*, unless one sets out to use such a principle in a more general framework to deduce quantum mechanics. Here we follow the common practice of speaking of the complementarity principle as a description of a remarkable feature of quantum mechanics.

2.3 The notion of uncertainty and the uncertainty principle

2.3.1 Origin of the uncertainty principle

The uncertainty principle was introduced in Heisenberg's seminal paper of 1927 [16]; although he did not speak of a principle, he made it very clear that he considered the uncertainty relation as fundamental for an intuitive understanding of quantum mechanics. He saw it as an expression of the physical content of the canonical commutation relations for conjugate pairs of quantities, and considered that these algebraic relations should form the basis for a derivation of the formalism of quantum mechanics.

Heisenberg introduced quantum mechanical uncertainty with the following words: "It is shown that canonically conjugate quantities can be determined simultaneously only with a characteristic inaccuracy," and "...the more precisely the position is determined the less precisely the momentum is known and conversely."

Often the uncertainty principle is reduced to the idea, referred to above as the Heisenberg effect, that any measurement "disturbs" the system in question. In the case of the canonically conjugate position and momentum observables of a particle, this "disturbance" is often identified with a momentum kick imparted on the particle during the measurement. If taken as generally valid, the explanation of the uncertainty principle in terms of momentum kicks is an incorrect conflation whose origin must be seen in Heisenberg's discussion of a position measurement with a γ-ray microscope. Heisenberg pointed out that the higher the accuracy of the position determination was, the shorter the wavelength of the photon, and therefore the larger the momentum exchange with the observed particle. In a "note added" at the end of his 1927 paper, he credits Bohr for the correction that the magnitude of the momentum transfer did not constitute a cause for the particle's momentum to become indeterminate. Bohr's explanation of the necessary momentum uncertainty was based on the dual nature of the light used in the observation: the particle aspect of the photon accounts for the momentum exchange resulting from Compton scattering, while the wave aspect gives rise to an uncertainty in the momentum inference due to the finite aperture of the microscope. Thus one could say that according to Bohr it is the quantum nature of the probe system utilized in a measurement that enforces the necessary uncertainty trade-off.

2.3.2 Three varieties of uncertainty relations

A careful reading of Heisenberg's 1927 paper and his 1930 Chicago lecture notes [25] shows that he has in fact distinguished three variants of uncertainty relations. It is evident that Heisenberg considered measurements to produce (approximate) eigenstates of the measured observable, corresponding to the measured value. Thus he describes the outcome of an attempted joint measurement of position and momentum in terms of the standard deviations of position and momentum observables in a Gaussian wave function centered at the measured values. The position and momentum uncertainties in this conditional final state are then taken to represent the inaccuracies of the joint measurement.

Here Heisenberg brings together two versions of uncertainty relations: the uncertainty relation

$$\Delta(Q,\psi)\cdot\Delta(P,\psi)\geq\frac{\hbar}{2} \tag{4}$$

for state preparations, according to which separate measurements of position and momentum in a (vector) state ψ have distributions with widths (standard deviations) satisfying this uncertainty relation; and a trade-off relation

$$\delta q\cdot\delta p\geq\frac{\hbar}{2} \tag{5}$$

for the inaccuracies of joint measurements of these noncommuting observables.

Heisenberg did not have at his disposal a precise quantum-mechanical notion of joint measurement of noncommuting observables. Such a notion was developed only several decades later, after POVMs had been introduced and made available to describe unsharp or approximate measurements. Heisenberg does grapple with the notion of joint unsharp measurement and comes close to a solution by considering sequences of measurements. For example, he considers the diffraction of a matter wave at a slit and shows that if the particle's momentum was initially sharp, this precision in the definition of momentum becomes degraded during the passage through the slit which effects an approximate localization of the particle. Considered as a sequence of a sharp momentum measurement followed immediately by an approximate position measurement, the outcome of the sharp momentum determination is thus seen to be modified into an unsharp momentum determination, due to the "disturbing" influence of the approximate position determination. The resulting inaccuracies in the definitions of position and momentum are shown to satisfy an uncertainty relation. Indeed, the variance $\Delta(P,\psi)$ of momentum after passage through the slit can be taken to represent both the accuracy δp of the momentum determination in the combined measurement scheme and a measure of the disturbance Dp of the (initially sharp) momentum through the unsharp position determination: $\Delta(P,\psi)=\delta p=Dp$. Likewise, the position uncertainty $\Delta(Q,\psi)$ upon passage through the slit reflects the accuracy δq of the position measurement. We thus have:

$$\delta q\cdot Dp\geq\frac{\hbar}{2}. \tag{6}$$

This version of uncertainty relation constitutes an accuracy-disturbance trade-off relation for sequences of measurements. It has been carefully discussed in the context of interference experiments by Pauli in his 1933 review [19]. In the form described here, the disturbance of the distribution of an observable B through a measurement of A is measured in terms of the variance of B in the state immediately after the selective A measurement operation, which is to be compared to the (near) zero variance of B in an initial (near) eigenstate. In this formulation, the accuracy-disturbance relation follows from the preparation uncertainty relation. In a more general approach, the disturbance of the distribution of B during a measurement of A should be described by some measure of the difference between the distributions of B before and after the nonselective A measurement, and it should depend on the accuracy of the A measurement. Interestingly, rigorous and general formulations of such disturbance uncertainty relations have been investigated only rather recently (e.g., [29,32,33]). A review of rigorous formulations of all three types of uncertainty relations for position and momentum is given in [34].

2.3.3 Uncertainty principle

We propose that the term *uncertainty principle* refers to the broad statement that there are pairs of observables for which there is a trade-off relationship in the degrees of sharpness of the preparation or measurement of their values, such that a simultaneous or sequential determination of the values requires a nonzero amount of unsharpness (latitude, inaccuracy, disturbance). This comprises the above three versions of uncertainty relations. There are a variety of measures of uncertainty, inaccuracy, and disturbance with which such trade-off relations can be formulated, usually in the form of inequalities.

The term “principle” refers here to the fact that the uncertainty relations highlight an important nonclassical feature of quantum mechanics. They are a formal consequence of the noncommutativity of the observables in question. Being inequalities, the uncertainty relations can hardly be considered adequate as postulates from which to derive quantum mechanics; however, they have been used to rule out the field of real numbers in favor of the complex numbers as the underlying field for the Hilbert space formulation of quantum mechanics [35].

2.3.4 Uncertainty or indeterminacy?

A discussion of the uncertainty principle would not be complete without a comment on the nature of quantum mechanical uncertainty. The fact that many observables do not have definite values even in pure states, which represent maximal available information about the physical system under consideration, stands in strong contrast to the situation in classical physics. If in a pure state ψ the probability for a property represented by a projection P is neither one nor zero, the value of any physical quantity represented by that projection must be considered to be *objectively indeterminate*, or *indefinite*, rather than only *subjectively unknown*, or *uncertain*. According to the Kochen-Specker theorem (discovered by Kochen and Specker [36] and independently by Bell [37]), any attempt at hypothetically assigning definite values to sets of non-commuting observables leads to contradictions even in the case of rather small, finite sets of such observables. (Asher Peres was one of the champions in finding smaller and smaller sets of observables with Kochen-Specker contradictions; see his book [14].) The assignment of definite values to the position and momentum of a particle in the Bohm interpretation is no exception to this conclusion since these value attributions are contextual, that is, they are different for different measurement setups. The conclusion is that the term *indeterminacy principle* appears to be more appropriate. However, "*uncertainty* principle" has become the standard name and shall be used here, with the proviso that one should beware the misleading connotation of subjective ignorance that it carries.

2.4 Complementarity vs. uncertainty?

The above review shows that the concepts of complementarity and uncertainty highlight two aspects of one and the same feature of quantum mechanics:

(I) the *impossibility* of assigning simultaneously sharp values to certain pairs of noncommuting observables, be it by preparation or measurement;
(P) the *possibility* of simultaneously assigning unsharp values to such observables by preparation of measurement.

One may say that in contrast to Bohr, who emphasized the *negative* aspect of complementarity in the sense of (I), Heisenberg moved further to make a *positive* statement in the form of uncertainty relations which, if satisfied, enabled the option (P). We believe that this account is endorsed by Bohr who occasionally refers to the uncertainty relation as expressing complementarity, in the sense of (I), as well as (P). This seems evident from the following passage in the published version of the Como lecture which we quote in full length:

> In the language of the relativity theory, the content of the relations (2) [the uncertainty relations] may be summarized in the statement that according to the quantum theory a general reciprocal relation exists between the maximum sharpness of definition of the space-time and energy-momentum vectors associated with the individuals. This circumstance may be regarded as a simple symbolical expression for the complementary nature of the space-time description and claims of causality. At the same time, however, the general character of this relation makes it possible to a certain extent to

> reconcile the conservation laws with the space-time co-ordination of observations, the idea of a coincidence of well-defined events in a space-time point being replaced by that of unsharply defined individuals within finite space-time regions. (Bohr 1928 [15, Sec. 2])

For Bohr, the "mutual completion" part of complementarity refers to the necessity of making use of the exclusive descriptions (or observables) in different experimental contexts that cannot be created simultaneously. Incidentally, the above quote seems to constitute the first occurrence of the term "unsharp" in connection with the question of simultaneous preparation or measurement of position and momentum, which today is formalized in terms of unsharp observables, that is, POVMs.

It seems largely a matter of terminological or interpretational preference which aspects of the two statements (I) and (P) are to be subsumed under the complementarity principle or the uncertainty principle. However, the formalizations of the features (I) and (P) reviewed in the preceding subsections, which are those most commonly used in the recent research literature, have clearly identified (I) as an expression of the idea of complementarity and (P) as the essence of the uncertainty principle. This constitutes a break with two older traditions which gave preference either to the complementarity principle or the uncertainty principle. It appears to us that with this terminological shift, a more balanced assessment has been achieved: compared to the view that emphasized complementarity over uncertainty, the positive role of the uncertainty relations as enabling joint determinations and joint measurements is now highlighted more prominently; and even though it is true (as we show later) that the uncertainty statement (P) entails (I) in a suitable limit sense, it is still appropriate to point out the strict mutual exclusivity of sharp value assignments which, after all, is the reason for the quest for an approximate reconciliation in the form of simultaneous but unsharp value assignments.

Irrespective of the particular terminological or interpretational preference, formalizing the respective statements (I) and (P) has opened up new and interesting questions: (I) and (P) have become claims that can or cannot be proven as consequences of the theory, and it becomes possible to study the logical relationships between these statements. Questions like these will be studied in the remaining part of this paper with respect to Mach-Zehnder interferometric or, more generally, qubit observables.

3 Interlude: measurement theoretic concepts and tools

In this section we review the representation of observables as positive operator valued measures (POVMs), their use in the definition of joint unsharp measurement of noncommuting observables, and the measurement theoretic implementation of a POVM. We restrict ourselves to observables with finitely many values and some examples in the context of qubit observables, as they will be used in the following sections.

3.1 Effects and POVMs

The standard representation of observables in quantum mechanics uses self-adjoint operators, or equivalently, the associated spectral measures, acting in the Hilbert space $\mathcal{H}$ associated with the physical system under consideration. For example, a standard observable A with discrete spectrum $\{a_1, a_2, \ldots, a_n\}$ has a spectral representation of the form $A = \sum_{k=1}^{n} a_k P_k$. The eigenvalues a_k and the associated mutually orthogonal spectral projections P_k define the spectral measure

$$P : \, a_k \mapsto P_k, \tag{7}$$

which satisfies the normalization condition

$$\sum_k P_k = I. \tag{8}$$

A projection-valued map of the form (7) with the property (8) is also called a *projection-valued measure* (PVM). The probabilities for the outcomes (eigenvalues) a_m in a measurement of A on state ψ are given by

$$\langle \psi | P_m | \psi \rangle =: \langle P_m \rangle_\psi = \langle P_m \rangle. \tag{9}$$

The normalization of probabilities for all unit vectors ψ is ensured by the normalization condition (8). Standard observable A and PVM $P : \, a_k \mapsto P_k$ will also be referred to as *sharp observable.*

In generalizing the formalism to include positive-operator-valued measures (POVMs) as representations of imperfect or inaccurate measurements, it is noted that probabilities can generally be represented by expectation values of positive operators that are not necessarily projection operators nor necessarily commuting. Such operators are called effects. Hence, a measurement with outcomes $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is represented by a POVM

$$E : \, \lambda_\ell \mapsto E_\ell, \tag{10}$$

with a unique collection of effects $E_1, E_2, \ldots, E_m$. The normalization condition

$$\sum_{\ell} E_\ell = I \tag{11}$$

is required to ensure the normalization of the probability distributions $\lambda_\ell \mapsto \langle\psi|E_\ell|\psi\rangle$ for each vector state ψ. Often we will simply denote a POVM E in terms of the set of its effects $\{E_1, \ldots, E_m\}$. A POVM which is not a PVM will be called *unsharp observable.*

A POVM $\lambda_\ell \mapsto E_\ell$ is taken to represent a coarse grained, or smeared version of a sharp observable (7) if there is a stochastic matrix $(w_{\ell k})$ ($w_{\ell k} \geq 0$, $\sum_\ell w_{\ell k} = 1$) such that $E_\ell := \sum_k w_{\ell k} P_k$. Such a POVM, which is always commutative, represents an approximate measurement of the sharp observable.

For example, the spectral representation of the Pauli spin-1/2 operator σ_x in $\mathcal{H} = \mathbb{C}^2$ has the form,

$$\sigma_x = a_1 P_1 + a_2 P_2 = \tfrac{1}{2}(I + \sigma_x) - \tfrac{1}{2}(I - \sigma_x), \tag{12}$$

with eigenvalues $a_1 = 1,\ a_2 = -1$ and spectral projections $P_1 = \frac{1}{2}(I + \sigma_x)$, $P_2 = \frac{1}{2}(I - \sigma_x)$.

A smeared version of this spectral measure is obtained by applying a stochastic matrix

$$(f_{jk}) := \frac{1}{2}\begin{pmatrix} 1+f & 1-f \\ 1-f & 1+f \end{pmatrix} \qquad -1 \leq f \leq 1. \tag{13}$$

The smeared version of the spectral measure of σ_x is $F = \{F_1, F_2\}$ where $F_\ell := \sum\limits_{k=1}^{2} f_{\ell k} P_k$. This gives,

$$F_1 = \tfrac{1}{2}(I + f\sigma_x), \qquad F_2 = \tfrac{1}{2}(I - f\sigma_x) \tag{14}$$

Similarly, a smeared version of σ_z, $G = \{G_1, G_2\}$ can be defined as,

$$G_1 = \tfrac{1}{2}(I + g\sigma_z), \qquad G_2 = \tfrac{1}{2}(I - g\sigma_z) \tag{15}$$

where $-1 \leq g \leq 1$.

In this example, we see that the class of POVMs is wide enough to include *trivial* POVMs: these are composed of effects which are all multiples of the unit operator I. The above POVM F becomes trivial if we put $f = 0$. Trivial POVMs give probabilities that do not depend on the state; they provide no information at all. We have already found it convenient to make reference to observables represented as trivial POVMs in the preceding section.

3.2 Joint measurability

In a joint measurement of two observables F and G, one sets out to infer the values of these observables from the output readings. Thus for every pair of values of F and G there has to be a pointer value and the statistics of these pointer values should reproduce the probabilities for the values of F and G in the object's input state. Thus, there should be a POVM, E, whose probabilities are the joint probabilities for the outcomes of F and G. This means that the probability distributions of F and G should be obtained as marginal distributions of the probability distributions of E.

Such a POVM, E, is called a *joint observable* of F and G; observables F and G are the *marginals* of E.

According to a theorem of von Neumann [31, Sec. III.3] two sharp observables have a (sharp) joint observable exactly when they commute. Thus two non-commuting observables cannot be sharply measured together. However, it has been found that smeared versions of two *noncommuting* sharp observables may have a joint observable. The pair $F = \{\frac{1}{2}(I \pm f\sigma_z)\}$, $G = \{\frac{1}{2}(I \pm g\sigma_z)\}$ are known [38] to have a joint observable exactly when,

$$f^2 + g^2 \le 1. \tag{16}$$

Thus for two POVMs to be jointly measurable their degrees of unsharpness $|f|, |g|$ must be limited by this trade-off inequality. In this case it is straightforward to give an example of a joint observable E, assuming for simplicity $0 \le f, g \le 1$,

$$E_{11} = \tfrac{1}{4}(I + f\sigma_x + g\sigma_z) \qquad E_{21} = \tfrac{1}{4}(I - f\sigma_x + g\sigma_z) \tag{17}$$

$$E_{12} = \tfrac{1}{4}(I + f\sigma_x - g\sigma_z) \qquad E_{22} = \tfrac{1}{4}(I - f\sigma_x - g\sigma_z). \tag{18}$$

Each operator $E_{k\ell}$ is positive because the eigenvalues are $\frac{1}{4}(1 \pm |(f, g)|) = \frac{1}{4}(1 \pm \sqrt{f^2 + g^2}) \ge 0$ due to (16). Moreover, the marginality relations are fulfilled:

$$E_{11} + E_{12} = F_1 \qquad E_{21} + E_{22} = F_2 \tag{19}$$

$$E_{11} + E_{21} = G_1 \qquad E_{12} + E_{22} = G_2. \tag{20}$$

In sections 4.5 to 4.7 we will give measurement implementations of similar joint observables.

3.3 Measurement implementation of a POVM

Next we give a brief explanation of how a POVM can be implemented in a suitable measurement scheme.

In a measurement a probe is coupled with the object system by a unitary evolution after which the probe pointer is read. In addition, if the object system is still available after the measurement interaction has ceased, one may also perform another measurement on it.

The following situation is encountered in sections 4.5 to 4.7: a photon gets entangled with a probe system and then passes through a Mach-Zehnder interferometer. Finally, a joint measurement is made of a probe observable and a detector observable. The purpose of this joint measurement is to obtain information about the photon state immediately prior to the interaction with the probe and subsequent passage through the interferometer. Such information is available in the form of the output probabilities if these can be expressed in terms of the photon's input state.

Given that the initial state of the probe and the interferometer settings are fixed in each run, it follows that the output probabilities are indeed the expectation values of a POVM for the photon input state.

Let $\psi_i = \alpha|1\rangle + \beta|2\rangle$ ($|\alpha|^2 + \beta|^2 = 1$) denote the input state of the photon, $|p_0\rangle$ the initial probe state, U the unitary evolution operator representing the probe and the passage through the interferometer. Then the final state of the combined system is $\Psi_f = U\psi_i \otimes |p_0\rangle$. On this the sharp output observable with projections $M_{k\ell} = |k\rangle\langle k| \otimes |r_\ell\rangle\langle r_\ell|$ is measured. Here $|k\rangle$, $k = 1, 2$, are the eigenstates of an object observable measured after the interaction with the probe, and $|r_\ell\rangle$, $\ell = 1, 2$, are eigenstates of a pointer or output observable of the probe. (In sections 4.6 and 4.7, different choices will be made for the $|r_\ell\rangle$.) The output probabilities are then

$$\begin{aligned}\langle\Psi_f|M_{k\ell}|\Psi_f\rangle &= \langle p_0|\langle\psi_i|U^*M_{k\ell}U|\psi_i\rangle|p_0\rangle \\ &= |\alpha|^2\langle p_0|\langle 1|U^*M_{k\ell}U|1\rangle|p_0\rangle + \alpha^*\beta\langle p_0|\langle 1|U^*M_{k\ell}U|2\rangle|p_0\rangle \\ &\quad + \beta^*\alpha\langle p_0|\langle 2|U^*M_{k\ell}U|1\rangle|p_0\rangle + |\beta|^2\langle p_0|\langle 2|U^*M_{k\ell}U|2\rangle|p_0\rangle.\end{aligned} \tag{21}$$

These can be written as

$$\langle\Psi_f|M_{k\ell}|\Psi_f\rangle = |\alpha|^2 E_{k\ell}^{11} + \alpha^*\beta E_{k\ell}^{12} + \beta^*\alpha E_{k\ell}^{21} + |\beta|^2 E_{k\ell}^{22}. \tag{22}$$

Being probabilities, these numbers are non-negative and hence the expression (22) is a quadratic form for the variables α and β. That is to say, for each k, ℓ, the matrix $(E_{k\ell}^{ij})_{i,j=1,2}$ is positive semi-definite; thus. it represents a positive operator $E_{k\ell}$ defined in the Hilbert space of the photon. Hence,

$$\langle\Psi_f|M_{k\ell}|\Psi_f\rangle = \langle\psi_i|E_{k\ell}|\psi_i\rangle \tag{23}$$

for all ψ_i. Normalization of the output probability entails $\sum_{k\ell} E_{k\ell} = 1$. Thus $(k,\ell) \mapsto E_{k\ell}$ is a POVM, which represents the (input) observable of the object system measured by the measurement scheme using the output observable $(k,\ell) \mapsto M_{k\ell}$. Formula (23) is the basis for the analysis of all the measurement schemes discussed in the coming sections.

The above consideration illustrates a general theorem that states that every measurement scheme defines a POVM for the object system. The converse statement is also true: every POVM admits an implementation in terms of a measurement scheme. For a general introduction to these results and for original references, the reader may wish to consult [39]. To our knowledge, the first proposals of realistic models of joint measurements of unsharp qubit observables were developed in [40]. Further detailed examples of measurement implementations of POVMs are given, for instance, in the monograph [27] and in the book of de Muynck [41].

4 Path marking and erasure in an atom and Mach-Zehnder interferometry

4.1 The atom-interferometric experiment of SEW

In a two slit atom interferometer [1] each atom is prepared in a superposition

$$\psi_0(\mathbf{r}) = \tfrac{1}{\sqrt{2}}[\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})] \tag{24}$$

of path states $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$ which represent the passage through the two slits. On the far capture screen (position coordinate $\mathbf{R}$) an interference pattern will be observed according to

$$P_0(\mathbf{R}) = |\psi_0(\mathbf{R})|^2 = \tfrac{1}{2}[|\psi_1|^2 + |\psi_2|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1]. \tag{25}$$

SEW consider the situation in which atoms, prepared in an excited internal state $|a\rangle$ and propagating in a superposition of states corresponding to two collimated beam paths, arrive singly at micro-maser cavities preceding each of the double slits [1, Fig. 3]. Once in the cavity, the atoms will make a transition $|a\rangle \to |b\rangle$, spontaneously emitting a microwave photon. The state of atom plus field changes from

$$\Psi_0(\mathbf{r}) = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})]\,|a\rangle\,|0_1 0_2\rangle \tag{26}$$

to the entangled state

$$\Psi(\mathbf{r}) = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{r})\,|b\rangle\,|1_1 0_2\rangle + \psi_2(\mathbf{r})\,|b\rangle\,|0_1 1_2\rangle], \tag{27}$$

where $|1_1 0_2\rangle$ and $|0_1 1_2\rangle$ represent the field states corresponding to one photon in cavity 1 and none in cavity 2 and vice versa. Thus, the micro-maser cavities act as *which-way* detectors only if a photon left in the cavity changes the electromagnetic field in a detectable way.

The probability density on the capture screen is given by

$$\begin{aligned} P(\mathbf{R}) &= \tfrac{1}{2}[|\psi_1|^2 + |\psi_2|^2 + \psi_1^*\psi_2\langle 1_1 0_2|0_1 1_2\rangle + \psi_2^*\psi_1\langle 0_1 1_2|1_1 0_2\rangle] \\ &= \tfrac{1}{2}[|\psi_1|^2 + |\psi_2|^2]. \end{aligned} \tag{28}$$

The interference (cross) terms vanish because the field states $|1_1 0_2\rangle$ and $|0_1 1_2\rangle$ are orthogonal.

SEW also consider the possibility of recovering coherence and thus the interference pattern by deleting or *"erasing"* the path information left in the microwave cavity detectors [1, Refs. 26-31].

To model this they consider a new arrangement whereby the two cavities are separated by a shutter-detector combination [1, Fig. 5a]. This allows for the radiation either to be confined to the upper or lower cavity when the shutters are closed or for the radiation to be absorbed by a detector behind each shutter when it is opened. In the latter case the path marking information can be said to be erased. This will be explained presently.

In the erasure experiment one makes use of the fact that the state (27) has the equivalent form

$$\Psi(\mathbf{r}) = \frac{1}{\sqrt{2}}[\psi_+(\mathbf{r})\,|b\rangle\,|+\rangle + \psi_-(\mathbf{r})\,|b\rangle\,|-\rangle], \tag{29}$$

where

$$\psi_\pm = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{r}) \pm \psi_2(\mathbf{r})], \quad |\pm\rangle = \frac{1}{\sqrt{2}}[|1_1 0_2\rangle \pm |0_1 1_2\rangle]. \tag{30}$$

To display the effects of the erasure, the experimental procedure is described as follows: after an atom arrives on the far screen, the shutters are opened and the state of the detector behind the shutters, which may have changed from its initial state $|d\rangle$ to a new state $|e\rangle$ orthogonal to $|d\rangle$, is recorded. The possible transitions are as follows, reflecting the sensitivity of the detector to the field state $|+\rangle$ rather than $|-\rangle$:

$$|+\rangle\,|d\rangle \to |00\rangle\,|e\rangle, \quad |-\rangle\,|d\rangle \to |-\rangle\,|d\rangle. \tag{31}$$

In half the observations the detector will be found in an excited state indicating that there had been a photon in one of the cavities which has been absorbed. In the remaining cases there is no detection. Thus, the total state makes the following transition:

$$\frac{1}{\sqrt{2}}[\psi_+\,|b\rangle\,|+\rangle + \psi_-\,|b\rangle\,|-\rangle]|d\rangle \to \frac{1}{\sqrt{2}}[\psi_+\,|b\rangle\,|0_1 0_2\rangle\,|e\rangle + \psi_-\,|b\rangle\,|-\rangle\,|d\rangle]. \tag{32}$$

As seen in Eq. (29) the symmetric atom state ψ_+ is coupled with the symmetric cavity field state; thus the state of the atom arriving at the screen selected if the detector is found in state $|e\rangle$ is ψ_+. The probability density of those atoms will show the maxima and minima (fringes) of an interference pattern, $P_+(\mathbf{R}) = |\psi_+(\mathbf{R})|^2 = P_0(\mathbf{R})$ (Eq. (25)).

Atoms arriving at the screen for which there is no corresponding signal from the erasure detector (i.e., the detector is found in $|d\rangle$) will display "anti-fringes", $P_-(\mathbf{R}) = |\psi_-(\mathbf{R})|^2$, corresponding to the selected state ψ_-.

If all the events are counted, irrespective of the erasure detector state, the distribution is,

$$\frac{1}{2}P_+(\mathbf{R}) + \frac{1}{2}P_-(\mathbf{R}) = \frac{1}{2}[|\psi_1|^2 + |\psi_2|^2] = P(\mathbf{R}). \tag{33}$$

The maxima of one pattern coincide with the minima of the other one, washing out the fringes.

This consideration shows that in the entangled state, $\Psi(\mathbf{r})$ (Eq. (27)), the information about path as well as interference is fully available. Choosing to measure the path marking basis states, $|1_1 0_2\rangle, |0_1 1_2\rangle$, of the probe yields which way information. Measuring instead the field states $|+\rangle, |-\rangle$ allows the recovery of interference fringes or anti-fringes, respectively. The two options are mutually exclusive; in the first case it is the interference information which is lost whereas in the second case it is path information which is lost.

It should be noted that this situation is related to the Einstein, Podolsky, Rosen (EPR) experiment (in Bohm's version for spin 1/2 pairs) which also makes use of an entangled state with more than one biorthogonal decomposition as in Eqs. (27) and (29).

4.2 Mach-Zehnder interferometer: basic setup

Consider a special case of the Mach-Zehnder interferometer in Fig. 1 with no path marking and no phaseshifter. The two possible input states from I_1, I_2 will be represented by orthogonal unit vectors $|1\rangle, |2\rangle$, of a two dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^2$. When a photon entering via I_1 (represented by a "path" state $|1\rangle$) arrives at the beam splitter BS_1 its state is changed to an equally weighted superposition, with appropriate phase shift by $\pi/2$ upon reflection; and similarly for an input state $|2\rangle$:

$$|1\rangle \to \tfrac{1}{\sqrt{2}}[|1\rangle + i|2\rangle], \quad |2\rangle \to \tfrac{1}{\sqrt{2}}[i|1\rangle + |2\rangle]. \tag{34}$$

Arriving at detector D_1 will be a component via the path $I_1BS_1M_1$ reflected by BS_2 carrying a total phase shift of $\pi/2$ from M_1 plus $\pi/2$ from BS_2, and a component via the path $I_1BS_1M_2$ transmitted by BS_2 also carrying a total phase shift of $\pi/2$ from BS_1 plus $\pi/2$ from M_2. Hence, detector D_1 will register an output as these two components are in phase and interfere constructively.

Arriving at detector D_2 will be a component via the path $I_1BS_1M_1$ transmitted by BS_2 carrying a total phase shift of $\pi/2$ from M_1, and a component via the path $I_1BS_1M_2$ reflected by BS_2 carrying a total phase shift of $\pi/2$ from BS_1 plus $\pi/2$ from M_2 plus $\pi/2$ from BS_2. Hence, detector D_2 will register no output as these two components are out of phase by π and interfere destructively.

So, if $I_1BS_1M_1BS_2D_1$ is the path represented by $|1\rangle$, any measurement of the output of D_1 has associated with it projector $|1\rangle\langle 1|$ representing one of the eigenstates of the measured input observable according to Eq. (23) (which defines the measured POVM). We identify this with the spectral projection of the Pauli operator σ_z associated with the eigenvalue 1, $|1\rangle\langle 1| = \frac{1}{2}(I + \sigma_z)$.

Similarly, $I_2BS_1M_2BS_2D_2$ corresponds to $|2\rangle$ and any measurement of the output of D_2 has associated with it the input projector $|2\rangle\langle 2| = \frac{1}{2}(I - \sigma_z)$.

We are now in a position to consider a Mach-Zehnder interferometer with path marking before the beam splitter BS_1. This will be implemented by

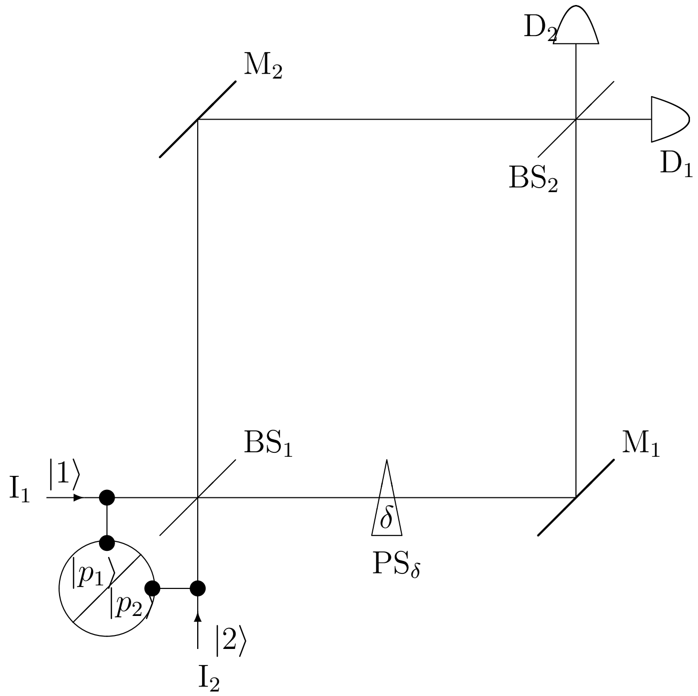

Fig. 1. A Mach-Zehnder interferometer with path marking and phase shifter.

introducing a probe system which interacts with the photon. The probe is represented by a two dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^2$, with path marking ("pointer") states $|p_1\rangle$ and $|p_2\rangle$. The interaction between the photon and the probe is to be arranged in such a way that $|p_1\rangle$ becomes correlated with $|1\rangle$ and $|p_2\rangle$ with $|2\rangle$, so that registration of these pointer states allows one to infer that the photon was in the corresponding path eigenstate (see Eq. (38) below). A phase shifter, δ in one path after BS_1 completes the analogy with the SEW experiment.

A general input from I_1, I_2 without the path marking interaction switched on (object in input state ψ_i and probe remaining in a neutral state $|p_0\rangle$) can be represented by

$$\psi_i \otimes |p_0\rangle = (\alpha|1\rangle + \beta|2\rangle) \otimes |p_0\rangle \tag{35}$$

Taking into account the phase shift in path 1 (see Fig. 1), the photon input states $|1\rangle, |2\rangle$ undergo the following evolution upon passage through the interferometer and before entering one of the detectors D_1, D_2:

$$|1\rangle \to \tfrac{1}{2}[(-e^{i\delta} - 1)|1\rangle + i(e^{i\delta} - 1)|2\rangle], \tag{36}$$
$$|2\rangle \to \tfrac{1}{2}[i(-e^{i\delta} + 1)|1\rangle - (1 + e^{i\delta})|2\rangle]. \tag{37}$$

When the path marking is turned on the total state after the path marking interaction has ceased and before the photon enters BS_1 is

$$\Psi_e = \alpha|1\rangle \otimes |p_1\rangle + \beta|2\rangle \otimes |p_2\rangle \tag{38}$$

The photon states $|1\rangle, |2\rangle$ evolve according to (36); this leads to the total final (output) state after the photon passes through beam splitter BS_2 as

$$\Psi_f^\delta = \tfrac{1}{2}\alpha[(-e^{i\delta} - 1)|1\rangle + i(e^{i\delta} - 1)|2\rangle] \otimes |p_1\rangle \tag{39}$$
$$+\tfrac{1}{2}\beta[i(-e^{i\delta} + 1)|1\rangle - (1 + e^{i\delta})|2\rangle] \otimes |p_2\rangle\,. \tag{40}$$

We are now ready to discuss a variety of possible experiments.

4.3 Path detection in outputs D_1, D_2

The simplest case of this Mach-Zehnder interferometer is with no path marking, $|p_1\rangle = |p_2\rangle = |p_0\rangle$ and no phase shift, $\delta = 0$, analogous to a double slit interferometer (SEW) with no path marking and the far field detector placed centrally; the output state is,

$$\Psi_f^o = -(\alpha|1\rangle + \beta|2\rangle) \otimes |p_0\rangle \tag{41}$$

Observing the output of detectors D_1, D_2 with no path marking is represented by the projections $M_1 = |1\rangle\langle 1| \otimes I$, $M_2 = |2\rangle\langle 2| \otimes I$.

The probabilities for an output at D_1 and D_2 are,

$$\langle\Psi_f^o|M_1|\Psi_f^o\rangle = \langle\psi_i|1\rangle\langle 1|\psi_i\rangle = |\alpha|^2 \tag{42}$$
$$\langle\Psi_f^o|M_2|\Psi_f^o\rangle = \langle\psi_i|2\rangle\langle 2|\psi_i\rangle = |\beta|^2. \tag{43}$$

The input observable measured by this experiment is the POVM $E^0 = \{E_1^0, E_2^0\}$ defined by

$$\langle\Psi_f^o|M_k|\Psi_f^o\rangle = \langle\psi_i|E_k^o|\psi_i\rangle \tag{44}$$

for all ψ_i and $k = 1, 2$ It follows that E^0 is a PVM with projections

$$E_1^0 = \tfrac{1}{2}(I + \sigma_z), \quad E_2^0 = \tfrac{1}{2}(I - \sigma_z). \tag{45}$$

This reproduces the discussion of path detection connected with Fig. 1: If $\psi_i = |1\rangle$, then the probabilities of a detection at D_1 and D_2 are $\langle 1|E_1^0|1\rangle = 1$

and $\langle 1|E_2^0|1\rangle = 0$, respectively. A similar consideration applies to an input state $\psi_i = |2\rangle$. Thus, the measured observable is the path observable σ_z.

4.4 Interference detection at D_1, D_2

We now consider the use of the Mach-Zehnder interferometer for an interference measurement. In a double slit interferometer both slits would be open and a detector placed at the first minimum. Here we choose $\delta = -\pi/2$; then the output state is,

$$\Psi_f^{-\pi/2} = \tfrac{-(1-i)}{\sqrt{2}} \left[\alpha \tfrac{1}{\sqrt{2}}(|1\rangle - |2\rangle) + \beta \tfrac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\right] \otimes |p_0\rangle. \tag{46}$$

If a measurement of $M_k = |k\rangle\langle k| \otimes I$, $k = 1,2$ is now applied by observing the outputs of D_k the associated probabilities are

$$\langle \Psi_f^{-\pi/2}|M_1|\Psi_f^{-\pi/2}\rangle = \tfrac{1}{2}(|\alpha+\beta|^2) = \langle\psi_i|E_1^{-\pi/2}|\psi_i\rangle \tag{47}$$
$$\langle \Psi_f^{-\pi/2}|M_2|\Psi_f^{-\pi/2}\rangle = \tfrac{1}{2}(|\alpha-\beta|^2) = \langle\psi_i|E_2^{-\pi/2}|\psi_i\rangle, \tag{48}$$

from which $E_1^{-\pi/2}$ and $E_1^{-\pi/2}$ are extracted:

$$E_1^{-\pi/2} = \tfrac{1}{2}(I + \sigma_x), \quad E_2^{-\pi/2} = \tfrac{1}{2}(I - \sigma_x). \tag{49}$$

Following customary practice we consider an interference observable one with the form $\cos\delta\,\sigma_x + \sin\delta\,\sigma_y$, $0 \le \delta < \pi$, given that the path is represented by σ_z. Interference observables are singled out by the condition that the interference contrast can assume its maximum possible value. In this case their eigenstates give equal probabilities of 1/2 to the path projections $|1\rangle\langle 1|, |2\rangle\langle 2|$.

In the present experiment, the measured input observable is defined by the projections of Eq. (49); these are the spectral projections of the operator σ_x, which is indeed an interference observable.

4.5 The path-marking setup

Now consider the case where $|p_1\rangle$ and $|p_2\rangle$ are mutually orthogonal , $\langle p_1|p_2\rangle = 0$. This is an analog of SEW's path-marking experiment. We can find the influence of the path marking on the outputs of the detectors using each of the four measurement projections of path $|k\rangle$ jointly with marker $|p_\ell\rangle$, $M'_{k\ell} = |k\rangle\langle k| \otimes |p_\ell\rangle\langle p_\ell|$, $k, \ell = 1, 2$, e.g.

$$\langle \Psi_f^\delta | M'_{11} | \Psi_f^\delta \rangle = |\tfrac{1}{2}\alpha(e^{i\delta} + 1)|^2 = \tfrac{1}{4}|\alpha|^2(1 + \cos\delta). \tag{50}$$

The input POVM, the measured observable is again defined by Eq. (23),

$$\langle \Psi_f^\delta | M'_{k\ell} | \Psi_f^\delta \rangle = \langle \psi_i | E'_{k\ell} | \psi_i \rangle, \tag{51}$$

and we obtain

$$E'_{11} = \tfrac{1}{2}(I + \sigma_z)\cos^2\tfrac{\delta}{2} \qquad E'_{21} = \tfrac{1}{2}(I + \sigma_z)\sin^2\tfrac{\delta}{2} \tag{52}$$
$$E'_{12} = \tfrac{1}{2}(I - \sigma_z)\sin^2\tfrac{\delta}{2} \qquad E'_{22} = \tfrac{1}{2}(I - \sigma_z)\cos^2\tfrac{\delta}{2}\,. \tag{53}$$

These effects are all fractions of a path projection.

We now give the marginal input POVM associated with the detectors D_1, D_2,

$$F'_1 = E'_{11} + E'_{12} = \tfrac{1}{2}(I + \cos\delta\,\sigma_z), \tag{54}$$
$$F'_2 = E'_{22} + E'_{21} = \tfrac{1}{2}(I - \cos\delta\,\sigma_z). \tag{55}$$

This POVM represents a path observable smeared by $\cos\delta$. The unsharpness inherent in the detector marginal is reflected in the non-zero probability of the marker indicating path 1 but detector D_2 firing even if the input state is a path eigenstate $|1\rangle$. Here we see the effect of the perfect path-marking interaction: irrespective of the phase parameter value, the Mach-Zehnder interferometer does not detect any interference effects. When $\delta = 0$, the POVM $\{F'_1, F'_2\}$ becomes a sharp path observable and when the interferometer is set to observe interference, $\delta = \frac{\pi}{2}$, this POVM is reduced to being trivial, $F'_1 = \frac{1}{2}I = F'_2$, giving no path nor interference information. This is in line with the prediction of SEW: path marking results in the interference pattern being washed out. After the path marking interaction, all the detectors are able to "see" is a "shadow" of the path information provided by the path marker: indeed, the marginal POVM measured by the path marker is given by the effects

$$G'_1 = E'_{11} + E'_{21} = \tfrac{1}{2}(I + \sigma_z), \tag{56}$$
$$G'_2 = E'_{22} + E'_{12} = \tfrac{1}{2}(I - \sigma_z). \tag{57}$$

These represent a sharp path observable irrespective of the value of δ.

It is possible to define a third 'marginal',

$$H'_1 = E'_{11} + E'_{22} = \cos^2 \tfrac{\delta}{2}\, I, \tag{58}$$
$$H'_2 = E'_{12} + E'_{21} = \sin^2 \tfrac{\delta}{2}\, I, \tag{59}$$

which in the present experiment also turns out to be trivial.

4.6 Quantum erasure

In the previous experiment, each path was correlated with one of two orthogonal marker states. We can now consider a new set of pointer states, which are superpositions of the two orthogonal path-marker states,

$$|q_1\rangle = \tfrac{1}{\sqrt{2}}(|p_1\rangle + e^{i\gamma}|p_2\rangle), \quad |q_2\rangle = \tfrac{1}{\sqrt{2}}(|p_1\rangle - e^{i\gamma}|p_2\rangle)\,. \tag{60}$$

Observing these symmetric states involves outputs for which both $|p_1\rangle$ and $|p_2\rangle$ are equally likely, so no information about the path is recorded.

The final state (39) in terms of the new pointer states is

$$\begin{aligned}\Psi_f^{\delta,\gamma} \;&= \tfrac{1}{2\sqrt{2}}\Big[\Big(-\alpha(1+e^{i\delta}) + ie^{-i\gamma}\beta(1-e^{i\delta})\Big)\,|1\rangle \\ &+ \Big(-i\alpha(1-e^{i\delta}) - e^{-i\gamma}\beta(1+e^{i\delta})\Big)\,|2\rangle\Big] \otimes |q_1\rangle \\ &+ \tfrac{1}{2\sqrt{2}}\Big[\Big(-\alpha(1+e^{i\delta}) - ie^{-i\gamma}\beta(1-e^{i\delta})\Big)\,|1\rangle \\ &+ \Big(-i\alpha(1-e^{i\delta}) + e^{-i\gamma}\beta(1+e^{i\delta})\Big)\,|2\rangle\Big] \otimes |q_2\rangle.\end{aligned} \tag{61}$$

As before we can find the four joint probabilities for the marker and detector outputs, defined as the expectations of the projections $M''_{k\ell} = |k\rangle\langle k| \otimes |q_\ell\rangle\langle q_\ell|$, $k, \ell = 1, 2$.

The associated input POVM E'' is determined via the relation

$$\langle \Psi_f^{\delta,\gamma}|M''_{k\ell}|\Psi_f^{\delta,\gamma}\rangle = \langle \psi_i|E''_{k\ell}|\psi_i\rangle. \tag{62}$$

We obtain:

$$E''_{11} = \tfrac{1}{4}(I - \sin\delta\cos\gamma\,\sigma_x - \sin\delta\sin\gamma\,\sigma_y + \cos\delta\,\sigma_z) = \tfrac{1}{4}(I - \mathbf{n}''\cdot\sigma)\,, \tag{63}$$
$$E''_{21} = \tfrac{1}{4}(I + \sin\delta\cos\gamma\,\sigma_x + \sin\delta\sin\gamma\,\sigma_y - \cos\delta\,\sigma_z) = \tfrac{1}{4}(I + \mathbf{n}''\cdot\sigma)\,, \tag{64}$$
$$E''_{12} = \tfrac{1}{4}(I + \sin\delta\cos\gamma\,\sigma_x + \sin\delta\sin\gamma\,\sigma_y + \cos\delta\,\sigma_z) = \tfrac{1}{4}(I + \mathbf{m}''\cdot\sigma)\,, \tag{65}$$
$$E''_{22} = \tfrac{1}{4}(I - \sin\delta\cos\gamma\,\sigma_x - \sin\delta\sin\gamma\,\sigma_y - \cos\delta\,\sigma_z) = \tfrac{1}{4}(I - \mathbf{m}''\cdot\sigma)\,. \tag{66}$$

Here we have introduced the unit vectors

$$\mathbf{n}'' = (\sin\delta\cos\gamma, \sin\delta\sin\gamma, -\cos\delta), \;\; \mathbf{m}'' = (\sin\delta\cos\gamma, \sin\delta\sin\gamma, \cos\delta). \tag{67}$$

The marginal POVM associated with the detector outputs is obtained by summing over both probe outputs:

$$F_1'' = E_{11}'' + E_{12}'' = \tfrac{1}{2}(I + \cos\delta\,\sigma_z), \;\; F_2'' = E_{21}'' + E_{22}'' = \tfrac{1}{2}(I - \cos\delta\,\sigma_z). \tag{68}$$

This is a smeared path observable. The marginal POVM associated with the probe outputs is obtained by summing over both detection outputs:

$$G_1'' = E_{11}'' + E_{21}'' = \tfrac{1}{2}\,I, \quad G_2'' = E_{12}'' + E_{22}'' = \tfrac{1}{2}\,I. \tag{69}$$

This is a trivial observable, it provides no information about the input state ψ_i.

The fact that the detector POVM is a smeared path observable and the probe POVM is trivial can be understood as follows. The entanglement between probe and photon is devised to establish a strict correlation between the path states $|1\rangle$, $|2\rangle$ and the pointer states $|p_1\rangle$, $|p_2\rangle$, for any photon input state ψ_i. This correlation information is not accessible by measuring a probe observable with the eigenstates $|q_1\rangle$, $|q_2\rangle$ because these are equal weight superpositions of the path marker states. Further, the reduced state of the photon after the coupling has been established is a mixture of the path states, so that any phase relation between these states has been erased. Accordingly, the detector outputs cannot detect any interference indicative of coherence between the path input states, and the only information left about the input is path information.

A third 'marginal' input POVM is defined as follows:

$$\begin{aligned} H_1'' &= E_{11}'' + E_{22}'' = \tfrac{1}{2}\left(I - \sin\delta(\cos\gamma\,\sigma_x - \sin\gamma\,\sigma_y)\right) \\ H_2'' &= E_{12}'' + E_{21}'' = \tfrac{1}{2}\left(I + \sin\delta(\cos\gamma\sigma_x + \sin\gamma\,\sigma_y)\right). \end{aligned} \tag{70}$$

This is a smeared interference observable, the unsharpness being determined by the term $\sin\delta$ and the direction of the associated Poincaré sphere vector being given by $\pm(\cos\gamma, \sin\gamma, 0)$. By varying γ from 0 to 2π, all possible interference observables can be realized. The erasure scheme presented here constitutes a joint unsharp measurement of path and interference observables as represented by the marginal POVMs $\{F_1'', F_2''\}$ and $\{H_1'', H_2''\}$.

We also see that for $\delta = -\pi/2$, all four effects $E_{k\ell}$ are fractions of spectral projections of a sharp interference observable; the marginal $\{H_1'', H_2''\}$ becomes a sharp interference observable and the marginal $\{F_1'', F_2''\}$ becomes a trivial observable. Here we have recovered the observation of SEW, that the detector statistics conditional on the probe output readings display perfect interference with perfect visibility. In fact, we have found somewhat more: independently of the photon input state, the conditional probabilities for detections at D_1, D_2 given a probe recording of $|q_1\rangle$ (say) are

$$prob(D_1|q_1) = \frac{\langle\psi_i|E_{11}''\psi_i\rangle}{\langle\psi_i|G_1''\psi_i\rangle} = \langle\psi_i|\tfrac{1}{2}(I + \cos\gamma\,\sigma_x + \sin\gamma\,\sigma_y)\psi_i\rangle, \qquad (71)$$

$$prob(D_2|q_1) = \frac{\langle\psi_i|E_{21}''\psi_i\rangle}{\langle\psi_i|G_1''\psi_i\rangle} = \langle\psi_i|\tfrac{1}{2}(I - \cos\gamma\,\sigma_x - \sin\gamma\,\sigma_y)\psi_i\rangle. \qquad (72)$$

For $\gamma = 0$ and the input state $\psi_i = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, this gives $prob(D_1|q_1) = 1$ and $prob(D_2|q_1) = 0$. This corresponds to the case of perfect interference fringes. Similarly, for the detector probabilities conditional on $|q_2\rangle$ and the above input eigenstate of σ_x, we obtain probabilities 0 and 1 for D_1 and D_2, respectively, which are characteristic of interference antifringes.

This situation is a consequence of the fact that for the above input and $\delta = -\pi/2$, $\gamma = 0$, the state Ψ_e and also the total output state $\Psi_f^{-\pi/2}$ is an EPR state, analogous to the state described in the SEW erasure setup of subsection 4.1:

$$\begin{aligned}\Psi_f^{-\pi/2} &= \tfrac{-(1-i)}{\sqrt{2}}\left[\tfrac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\otimes|p_1\rangle + \tfrac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\otimes|p_2\rangle\right] \\ &= \tfrac{-(1-i)}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\left[|1\rangle\otimes|q_1\rangle + |2\rangle\otimes|q_2\rangle\right]. \end{aligned} \qquad (73)$$

4.7 Quantitative erasure

The two possible experimental options discussed in the preceding subsections, of path marking and erasure are mutually exclusive in that they require settings and operations that cannot be performed at the same time: for path determination, the probe eigenstates $|p_1\rangle, |p_2\rangle$ must be read out, while for the recovery of interference it is necessary to record the detector outputs conditional on the probe output states $|q_1\rangle, |q_2\rangle$. Erasure was achieved by choosing $\delta = \frac{\pi}{2}$, which led to the POVM $\{E_{k\ell}''\}$ being constituted of (fractions of) spectral projections of an interference observable. Accordingly, the only non-trivial marginal is the sharp interference observable $\{H_1'', H_2''\}$.

If, however, the interferometric parameter δ is varied between 0 and $\frac{\pi}{2}$, then the POVM $\{E_{k\ell}''\}$ is a joint observable for an unsharp path and an unsharp interference observable. In this case the experiment provides simultaneous information about these noncommuting quantities. In the limit of $\delta = 0$, the interference marginal $\{H_1'', H_2''\}$ becomes trivial and the path marginal $\{F_1'', F_2''\}$ becomes sharp.

The possibility of obtaining some joint information about both observables, path and interference, can also be achieved by modifying the path marking coupling in such a way that the correlation between the paths and the probe indicator observable is not perfect. This has been described as *quantitative erasure* (e.g., [42]). We show here that quantitative erasure is again an instance of a joint unsharp measurement.

We take the path-marking interaction to be of the same form as before, Eq. (39), but now we specify the marker states $|p_1\rangle, |p_2\rangle$ to be nonorthogonal. Their associated Poincaré sphere vectors will be chosen to be tilted by an angle θ away from the $\pm z$ axis towards the positive x axis. As pointer states we choose $|q_1\rangle, |q_2\rangle$ equal to the up and down eigenstates of σ_z. Thus we define

$$|p_1\rangle = \cos\tfrac{\theta}{2}|q_1\rangle + \sin\tfrac{\theta}{2}|q_2\rangle,\ \ |p_2\rangle = \sin\tfrac{\theta}{2}|q_1\rangle + \cos\tfrac{\theta}{2}|q_2\rangle. \tag{74}$$

The final state after the path-marking interaction is

$$\begin{aligned}\Psi_f^{\delta,\theta} \;=\;& \Big[\Big(-\tfrac{\alpha}{2}\cos\tfrac{\theta}{2}(1+e^{i\delta}) + i\tfrac{\beta}{2}\sin\tfrac{\theta}{2}(1-e^{i\delta})\Big)\ |1\rangle \\ &+ \Big(-i\tfrac{\alpha}{2}\cos\tfrac{\theta}{2}(1-e^{i\delta}) - \tfrac{\beta}{2}\sin\tfrac{\theta}{2}(1+e^{i\delta})\Big)\ |2\rangle\Big]\otimes|q_1\rangle \\ &+ \Big[\Big(-\tfrac{\alpha}{2}\sin\tfrac{\theta}{2}(1+e^{i\delta}) + i\tfrac{\beta}{2}\cos\tfrac{\theta}{2}(1-e^{i\delta})\Big)\ |1\rangle \\ &+ \Big(-i\tfrac{\alpha}{2}\sin\tfrac{\theta}{2}(1-e^{i\delta}) - \tfrac{\beta}{2}\cos\tfrac{\theta}{2}(1+e^{i\delta})\Big)\ |2\rangle\Big]\otimes|q_2\rangle.\end{aligned} \tag{75}$$

Now we determine the input effects $E_{k\ell}'''$ associated with the output projections $M_{k\ell}''' = |k\rangle\langle k|\otimes|q_\ell\rangle\langle q_\ell|$ via $\langle\Psi_f^{\delta,\theta}|M_{k\ell}'''|\Psi_f^{\delta,\theta}\rangle = \langle\psi_i|E_{k\ell}'''|\psi_i\rangle$,

$$\begin{aligned}E_{11}''' &= \tfrac14[I(1+\cos\theta\cos\delta) - \sin\delta\sin\theta\,\sigma_x + (\cos\delta+\cos\theta)\,\sigma_z] \\ &= \tfrac14[I(1+\cos\theta\cos\delta) + \mathbf{m}'''\cdot\sigma] \\ E_{21}''' &= \tfrac14[I(1-\cos\theta\cos\delta) + \sin\delta\sin\theta\,\sigma_x - (\cos\delta-\cos\theta)\,\sigma_z] \\ &= \tfrac14[I(1-\cos\theta\cos\delta) - \mathbf{n}'''\cdot\sigma] \\ E_{12}''' &= \tfrac14[I(1-\cos\theta\cos\delta) - \sin\delta\sin\theta\,\sigma_x + (\cos\delta-\cos\theta)\,\sigma_z] \\ &= \tfrac14[I(-\cos\theta\cos\delta) + \mathbf{n}'''\cdot\sigma] \\ E_{22}''' &= \tfrac14[I(1+\cos\theta\cos\delta) + \sin\delta\sin\theta\,\sigma_x - (\cos\delta+\cos\theta)\,\sigma_z] \\ &= \tfrac14[I(1+\cos\theta\cos\delta) - \mathbf{m}'''\cdot\sigma],\end{aligned} \tag{76}$$

where

$$
\begin{aligned}
\mathbf{m}''' &= (-\sin\delta\sin\theta, 0, (\cos\delta + \cos\theta)), \\
\mathbf{n}''' &= (-\sin\delta\sin\theta, 0, (\cos\delta - \cos\theta)).
\end{aligned} \tag{77}
$$

The three marginal POVMs are determined as before:

$$
\begin{aligned}
F_1''' &= E_{11}''' + E_{12}''' = \tfrac{1}{2}(I - \sin\delta\sin\theta\,\sigma_x + \cos\delta\,\sigma_z) \\
F_2''' &= E_{22}''' + E_{21}''' = \tfrac{1}{2}(I + \sin\delta\sin\theta\,\sigma_x - \cos\delta\,\sigma_z) \\
G_1''' &= E_{11}''' + E_{21}''' = \tfrac{1}{2}(I + \cos\theta\,\sigma_z) \\
G_2''' &= E_{22}''' + E_{12}''' = \tfrac{1}{2}(I - \cos\theta\,\sigma_z) \\
H_1''' &= E_{11}''' + E_{22}''' = \tfrac{1}{2}I(1 + \cos\theta\cos\delta) \\
H_2''' &= E_{12}''' + E_{21}''' = \tfrac{1}{2}I(1 - \cos\theta\cos\delta).
\end{aligned} \tag{78}
$$

For $\delta = -\pi/2$, the first marginal POVM (corresponding to the detector statistics) becomes an unsharp interference observable, while the second marginal POVM (corresponding to the probe output statistics) is an unsharp path observable. In both cases the unsharpness is determined by the parameter θ.

We note for later reference that instead of the choice of pointer states $|q_1\rangle, |q_2\rangle$, one could have measured any pair of orthogonal probe states $|r_1\rangle, |r_2\rangle$. It is straightforward to show that the marker marginal is always an unsharp path observable.

5 Complementarity and uncertainty in Mach-Zehnder interferometry

5.1 Manifestations of complementarity in Mach-Zehnder interferometry

The sequence of experiments in section 4 is a demonstration of complementarity in different guises. In the first two experiments path detection (4.3) and interference detection (4.4) are mutually exclusive because this requires settings of the parameter δ which cannot be realized in the same experiment, namely, $\delta = 0$ for path (σ_z) measurement and $\delta = -\pi/2$ for interference (σ_x) measurement, respectively. Here we have an instance of the complementarity of measurement setups or measurement complementarity: these two noncommuting sharp observables do not admit any joint measurement.

These experiments can also be used to confirm preparation complementarity. We recall that sending a path eigenstate $|1\rangle$ or $|2\rangle$ into the Mach-Zehnder interferometric setup to observe interference leads to the probability $\langle 1|E_1^{-\pi/2}|1\rangle = \langle 1|E_2^{-\pi/2}|1\rangle = 1/2$, interference is completely uncertain. And, if we feed an interference eigenstate, $|\pm x\rangle$, into the interferometer to measure path ($\delta = 0$), the path observable is uncertain $\langle \pm x|E_1^0|\pm x\rangle = \langle \pm x|E_2^0|\pm x\rangle = 1/2$.

Value complementarity can indeed be used to explain the disappearance of interference fringes resulting from path marking. Once perfect correlation between the path states and the marker states is established in the entangled state (39), the reduced state of the photon is a mixture of the path eigenstates $|1\rangle$ and $|2\rangle$. In each of these, the path is definite, and therefore, in accordance with value complementarity, the outcomes of a subsequent interference measurement are equally probable. No interference fringes show up. Indeed this remains true for any mixture of path eigenstates.

This account in terms of preparation complementarity views the path marking interaction as part of a preparation process. An alternative explanation is possible in terms of measurement complementarity as follows.

In the experiment of section 4.5, where sharp path marking is followed by the interference setup with $\delta = -\pi/2$, it was found that the path measurement interaction leads to a complete loss of interference information detectable in D_1, D_2. All that the detectors can "see" is path information, irrespective of the value of δ (Eq. (54)).

If the sharp path marking is relaxed into unsharp path marking, section 4.7, setting the interferometer with $\delta = -\pi/2$ defines an unsharp interference observable, which is jointly measured with the path that can be recorded at the path marker.

It is found that the less accurate the path marking is set by making $\cos\theta$ in $G'''_{1,2} = \frac{1}{2}(I \pm \cos\theta\,\sigma_z)$ smaller, the sharper will be the interference measurement as $\sin\theta$ in $F'''_{1,2} = \frac{1}{2}(I \mp \sin\theta\,\sigma_x)$ becomes larger.

We see here that measurement complementarity follows in the limits of making the path marginal or the interference marginal perfectly sharp, rendering the other trivial.

A similar analysis applies to the erasure setup (Sec. 4.6): if $0 < \delta < \pi/2$, this setup realizes a joint measurement of the POVMs $\{F''_1, F''_2\}$ and $\{H''_1, H''_2\}$ which are unsharp path and interference observables.

Consider the case of $\delta = \pi/2$, where $F''_{1,2} = \frac{1}{2}I$ and $H''_{1,2} = \frac{1}{2}(I \mp \cos\gamma\sigma_x \mp \sin\gamma\sigma_y)$. Here we have a sharp interference measurement and no path measurement. With $\delta = 0$, $F''_{1,2} = \frac{1}{2}(I \pm \sigma_z)$ and $H''_1 = \frac{1}{2}I$, so that we have a sharp path measurement and no interference. These two limit cases of a joint measurement scheme illustrate once more measurement complementarity.

We recall that a pair of value complementary observables A, B is characterized by the condition that in each of the eigenstates of A, all eigenvalues of B are equally likely to occur as outcomes of a B measurement; and vice versa. We now show that for qubit observables such as those occurring in the Mach-Zehnder interferometry measurements discussed here, the value complementarity property can always be obtained as a consequence of some suitable uncertainty relation for the observables in question.

In what follows we allow general states represented as density operators $\rho = \frac{1}{2}(I + \mathbf{r}\cdot\sigma)$, $|\mathbf{r}| \le 1$. For the observables represented by σ_x, σ_y and σ_z, we then have

$$\langle\sigma_k\rangle = r_k^2, \quad k = 1, 2, 3, \tag{79}$$

and the variances, defined as $\mathrm{Var}(A) \equiv \mathrm{Var}(A, \rho) = \langle A^2\rangle - \langle A\rangle^2$ (where $\langle A\rangle = \mathrm{tr}[A\rho]$, etc., $\mathrm{tr}[\cdot]$ denoting the trace operation), are

$$\mathrm{Var}(\sigma_k) = 1 - \langle\sigma_k\rangle^2 = 1 - r_k^2, \quad k = 1, 2, 3. \tag{80}$$

With these expressions it is straightforward to confirm the general uncertainty relation for variances, with the commutator and covariance terms contributing the the lower bound:

$$\mathrm{Var}(\sigma_x)\mathrm{Var}(\sigma_z) \;\ge\; \tfrac{1}{4}|\langle[\sigma_x, \sigma_z]\rangle|^2 + \tfrac{1}{4}[\langle\sigma_x\sigma_z + \sigma_z\sigma_x\rangle - 2\langle\sigma_x\rangle\langle\sigma_z\rangle]^2. \tag{81}$$

It is easy to verify that the pair σ_x, σ_z is value complementary: in an eigenstate of σ_z, we have $r_1^2 = 1$ and so $r_2 = 0$. So, probability 1 for a value of σ_z goes along with probability 1/2 for the values of σ_x; and vice versa.

However, when it comes to deciding whether the statement of value complementarity can be inferred from the uncertainty principle, one should look at the uncertainty relation (81) alone, rather than using the explicit values of its terms. But using solely the above variance inequality one cannot recover value complementarity without further information on the terms of the left hand side. Still it suffices to use the algebraic and spectral properties of the Pauli operators (we imagine that we are given only this information but not the explicit expression of the probabilities or expectations): then one finds the right hand side of Eq. (81) to be equal to $\langle\sigma_y\rangle^2 + \langle\sigma_x\rangle^2\langle\sigma_z\rangle^2$, and one can argue as follows: if the path is definite, that is, if $\rho = |\psi\rangle\langle\psi|$ with ψ an eigenstate of σ_z, the left hand side of the uncertainty relation (81) is zero, and therefore the terms on the right hand side must vanish, too. Thus as $\langle\sigma_z\rangle = 1$, then

$\langle\sigma_x\rangle = \langle\sigma_y\rangle = 0$. Since the eigenvalues of these quantities are ±1, it follows that these observables are uniformly distributed in ψ. By symmetry, σ_z is uniformly distributed if σ_x has a definite value. (Note that we did not have to use the full explicit expressions for the expectation values, which would of course make this consideration entirely trivial.)

There are other forms of uncertainty relations which yield the statement of value complementarity without recourse to specific properties of the Pauli operators. Here we only mention the entropic uncertainty for two observables A, B with spectral representations $A = \sum_{i=1}^2 a_i P_i$, $B = \sum_{k=1}^2 b_k Q_k$. The (Shannon) entropy of A in a state ψ is defined as

$$H(A,\psi) = -\sum_{i=1}^{2} \langle\psi|P_i|\psi\rangle \log_2 \langle\psi|P_i|\psi\rangle. \tag{82}$$

This quantity is a measure of uncertainty concerning the value of A as encoded in the probability distribution of A in the given state: note that $0 \le H(A,\psi) \le \log_2 2 = 1$, where the lower bound is assumed for any eigenstate of A and the upper bound arises for any state which assigns equal probability $1/2$ to all eigenvalues. The following *additive* trade-off relation then holds [43]:

$$H(A,\psi) + H(B,\psi) \ge -2\log_2\left(\max_{i,k} \frac{|\langle\psi|P_i Q_k|\psi\rangle|}{\|P_i\psi\|\,\|Q_k\psi\|}\right). \tag{83}$$

If A, B are a pair of value complementary observables, so that any pair of eigenstates ψ_i of A and ϕ_k of B have overlap given by $|\langle\psi_i|\phi_k\rangle| = 1/\sqrt{2}$, it follows that the lower bound on the right hand side is $\log_2 2$. Thus, for $A = \sigma_z$, $B = \sigma_x$, one obtains [44]:

$$H(\sigma_z,\psi) + H(\sigma_x,\psi) \ge 1. \tag{84}$$

The combined lack of information about σ_z and σ_x is never less than one bit. Now, it is easily seen from this inequality alone that if one observable has a definite value, e.g., $H(\sigma_z,\psi) = 1$ (which happens in the eigenstates of σ_z), then the other observable is maximally uncertain, it is uniformly distributed since $H(\sigma_x,\psi) = 1$.

We note that the explicit expressions for the probabilities have gone into the derivation of this entropic inequality. But all that is needed to recover the property of value complementarity is contained in this inequality.

The inequality (84) is indeed easily verified by using the expressions $\frac{1}{2}(1 \pm r_z)$, $\frac{1}{2}(1 \pm r_x)$ for the probabilities and applying calculus to determine the minima.

It is similarly straightforward to verify the following *triple* uncertainty relation:

$$H(\sigma_x, \psi) + H(\sigma_y, \psi) + H(\sigma_z, \psi) \geq 2. \tag{85}$$

5.3 Quantitative duality relations are uncertainty relations

In the debates of the 1990s over complementarity in the context of interferometry and which-path experiments, the meaning of the term wave particle duality has gradually shifted away from a relation of strict exclusion of path determination and interference observation (in the same setup) to the broader idea of a continuous trade-off between approximate path determination and approximate interference determination. These discussions were eventually linked with earlier theoretical and experimental work of the 1980s on simultaneous but imperfect path determination and interference observation (e.g., [45,46,40,47]), as reviewed in [11]. The original intuitive ideas of the pioneers on an approximate reconciliation of complementary operational options (cf. Sec. 2) have thus been turned into precise trade-off relations which are being tested experimentally.

Trade-off relations of the form $P^2+V^2 \leq 1$ were derived as characterizations of the duality between path predictability and interference visibility [48–50]. (In [50], a stronger result, $P^2+V^2 = 1$, was shown to hold for certain experimental situation.) Soon afterwards, similar relations were formulated for quantum erasure (for a lucid and comprehensive review, see [42]). Vivid debates took place over the question whether the associated quantities are connected with uncertainties, and it has been shown that the respective trade-off relations are related in various ways to some forms of uncertainty relations. In the context of the Mach-Zehnder interferometer, we refer, in particular, to the work of Björk *et al.* [9], Dürr and Rempe [11], and Luis [12].

Using familiar measures of uncertainty, we give a simple demonstration that in the context of Mach-Zehnder interferometry experiments, quantitative duality relations are indeed equivalent to the uncertainty relation for an appropriate pair of associated observables.

Any state ρ can be represented by a matrix of the following form in the basis of eigenvectors of σ_z:

$$\rho = \begin{pmatrix} w_+ & re^{-i\theta} \\ re^{i\theta} & w_- \end{pmatrix}, \quad \text{where} \begin{cases} w_\pm \geq 0,\ w_+ + w_- = 1, \\ 0 \leq r \leq \sqrt{w_+ w_-},\ 0 \leq \theta < 2\pi. \end{cases} \tag{86}$$

We define the *path contrast* of ρ as

$$C_P = C_P(\rho) := |prob(\sigma_z = +1, \rho) - prob(\sigma_z = -1, \rho)| = |w_+ - w_-|. \quad (87)$$

This is identical to the predictability P entering the duality relation $P^2+V^2 \leq 1$. Similarly we define the interference contrast of ρ as

$$C_I = C_I(\rho) = |prob(\sigma_x = +1, \rho) - prob(\sigma_x = -1, \rho)| = 2r\cos\theta. \quad (88)$$

With the specification $\theta = 0$, or with an alternative choice of interference observable, this reduces to the visibility $V := (I_{max} - I_{min})/(I_{max} + I_{min}) = 2r$ (where I_{max}, I_{min} denote the maximal and minimal intensities of the measured interference pattern, obtained by variation of the interference observables). Using $r^2 \leq w_+w_-$, the following duality relation is easily verified:

$$C_P^2 + C_I^2 = w_+^2 + w_-^2 - 2w_+w_- + 4r^2\cos^2\theta \leq 1. \quad (89)$$

Now we observe that

$$C_P^2 = \langle\sigma_z\rangle^2 = 1 - \mathrm{Var}(\sigma_z), \quad C_I^2 = \langle\sigma_x\rangle^2 = 1 - \mathrm{Var}(\sigma_x). \quad (90)$$

Therefore, the above duality inequality can be equivalently expressed as

$$\mathrm{Var}(\sigma_z) + \mathrm{Var}(\sigma_x) = 2 - (C_P^2 + C_I^2) \geq 1. \quad (91)$$

Thus, our duality relation is equivalent to a form of uncertainty trade-off relation. As before, value complementarity is again entailed as a limit case.

We now show that this last inequality is actually a direct consequence of the uncertainty relation (81). Using Eq. (79), that relation is readily found to be equivalent to

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 \leq 1, \quad (92)$$

which expresses the positivity of the state ρ. Using Eq. (80), we thus see that the uncertainty relation (81) is indeed equivalent to the following inequality:

$$\mathrm{Var}(\sigma_x) + \mathrm{Var}(\sigma_y) + \mathrm{Var}(\sigma_z) \geq 2. \quad (93)$$

We note that besides σ_x, the operator σ_y also constitutes an interference observable with respect to the path σ_z. Thus, substituting $\mathrm{Var}(\sigma_z) = 1 - C_P^2$, $\mathrm{Var}(\sigma_x) = 1 - C_I^2 \equiv 1 - C_{I,x}^2$, and a similar term$\mathrm{Var}(\sigma_y) = 1 - C_{I,y}^2$, we obtain a generalized and sharpened duality relation which involves one path and two complementary interference observables:

$$C_P^2 + C_{I,x}^2 + C_{I,y}^2 \leq 1. \tag{94}$$

Thus the full uncertainty relation for σ_x, σ_z, including the commutator and covariance terms, is equivalent to the additive triple trade-off relation for the variances of $\sigma_x, \sigma_y, \sigma_z$ as well as this new trade-off relation for three mutually complementary observables.

5.4 Measurement complementarity from measurement inaccuracy relations

The measurement schemes of subsections 4.6 and 4.7 were found to constitute joint measurements of unsharp path and interference observables of the form $F = \{F_{1,2} = \frac{1}{2}(I \pm f\sigma_x\}$ and $G = \{G_{1,2} = \frac{1}{2}(I \pm g\sigma_z)$. For instance, in equation (78), setting $\delta = -\pi/2$, we have $f = \sin\theta$ and $g = \cos\theta$, so that we have $f^2 + g^2 = 1$. This is an instance of the criterion (16) which ensures the joint measurability of the POVMs F, G.

For a general state $\rho = \frac{1}{2}(I + \mathbf{r}\cdot\sigma)$, $|\mathbf{r}| \leq 1$, we define the contrasts of the distributions of F and G,

$$\begin{aligned} C_F(\rho) &= |\mathrm{tr}[\rho F_1] - \mathrm{tr}[\rho F_2])| = |f r_1|, \\ C_G(\rho) &= |\mathrm{tr}[\rho G_1] - \mathrm{tr}[\rho G_2]| = |g r_3|. \end{aligned} \tag{95}$$

The contrasts of the POVMs F, G are the respective maximal contrasts over all states ρ:

$$C_F = |f|, \quad C_G = |g|. \tag{96}$$

These quantities measure the degrees of unsharpness,

$$U_F := 1 - C_F^2 = 1 - f^2, \quad U_G := 1 - C_G^2 = 1 - g^2, \tag{97}$$

in the POVMs F, G. The unsharpness of F can also be defined as the minimum variance of the distribution of F for all states ρ. Indeed, it is not hard to verify that

$$\begin{aligned} \mathrm{Var}_\rho(F) = 1 - f^2 r_1^2 &= 1 - f^2 + f^2(1 - r_1^2) \\ &= U_F + (1 - U_F)\mathrm{Var}_\rho(\sigma_x) \geq U_F. \end{aligned} \tag{98}$$

Taking the minimum over all ρ gives

$$\mathrm{Var}_{min}(F) = U_F. \tag{99}$$

The above joint measurability criterion can be written in terms of the degrees of unsharpness:

$$U_F + U_G \geq 1. \tag{100}$$

This inequality is an uncertainty trade-off relation which must be satisfied if the two noncommuting unsharp path and interference observables F and G are to be jointly measurable. We have here an instance of Heisenberg's uncertainty principle for the inaccuracies which are necessarily present in joint measurements. As far as we are aware, this is one [38] of two cases in which an inaccuracy relation has been proven as a necessary condition for joint measurability. The other example is the case of position and momentum [51,33], and the corresponding uncertainty relation for joint measurements is reviewed in [34].

Finally we note that the variances of the marginals F, G in a joint measurement satisfy the uncertainty relation

$$\mathrm{Var}_\rho(F) + \mathrm{Var}_\rho(G) \geq U_F + U_G \geq 1. \tag{101}$$

Measurement complementarity is obtained as a limiting case for a pair F, G which are jointly measurable: if it is stipulated that one marginal, say F, becomes sharp, $U_F = 0$, or $|f| = 1$, then the other marginal, G, becomes a trivial POVM, $g = 0$, $G_{1,2} = \frac{1}{2}I$. Thus, if the path F is measured sharply, any attempt at obtaining information on interference will fail as the only unsharp interference observable G that can be measured jointly with F is trivial.

5.5 Disturbance versus accuracy

The setup discussed in Subsection 4.7 corresponds to a sequence of measurements where first path marking and registration can be achieved and then an interferometric measurement is carried out. As was observed in Subsection 5.1, if the path marking correlation is perfect, all the interferometric detectors can "see" is unsharp path information. If the setting is $\delta = \pi/2$, then the input observable indicated by the detectors is trivial: no interference information whatsoever about the input state is observed. If the path marking is unsharp, due to imperfect correlations or nonorthogonal path marker states $|p_1\rangle, |p_2\rangle$, then *some* interference information about the input state can pass through the path marking interaction. Here we will give a quantitative expression of this trade-off between the "disturbance" of the interference information through (imperfect) path marking and the accuracy of this path-marking process.

The path-marking interaction transforms the initial state $(\alpha|1\rangle+\beta|2\rangle)|p_0\rangle$ into

$$\Psi_e := \alpha|1\rangle|p_1\rangle + \beta|2\rangle|p_2\rangle. \tag{102}$$

This state has reduced photon-state operator

$$\rho_e = |\alpha|^2|1\rangle\langle 1| + |\beta|^2|2\rangle\langle 2| \ + \alpha\beta^*\langle p_2|p_1\rangle|1\rangle\langle 2| + \alpha^*\beta\langle p_1|p_2\rangle|2\rangle\langle 1|. \tag{103}$$

In what follows we want to express the necessary "disturbance" of the interference detection due to the path-marking entanglement. We start with a situation where $|p_1\rangle, |p_2\rangle$ are not necessarily orthogonal, but arranged as described in subsection 4.7. We also allow a general set of orthogonal pointer states $|r_1\rangle, |r_2\rangle$.

The quality of the path marking can be determined by following the entangling interaction with a path detection at D_1, D_2, where $\delta = 0$. If a reading r_1 (r_2) is taken to infer the path to be path 1 (path 2), then the subsequent path detection in the interferometer can be used to verify this inference. Thus, a joint measurement is made of the PVM with projections $M_{k\ell} = |k\rangle\langle k| \otimes |\ell\rangle\langle\ell|$, with probabilities

$$p_{k\ell} = \langle\Psi_f^0|M_{k\ell}|\Psi_f^0\rangle = |\langle k|\langle\ell|\Psi_e\rangle|^2. \tag{104}$$

The probability $prob(corr)$ of a correct inference is given as the sum of the probabilities of the corresponding coincident outputs:

$$prob(corr) = p_{11} + p_{22} = \langle\psi_i|H_1^0|\psi_i\rangle \ = |\alpha|^2|\langle r_1|p_1\rangle|^2 + |\beta|^2|\langle r_2|p_2\rangle|^2. \tag{105}$$

Similarly, the probability $prob(err)$ of error is given by the non-coincident combinations:

$$prob(err) = p_{12} + p_{21} = \langle\psi_i|H_2^0|\psi_i\rangle \ = |\alpha|^2|\langle r_2|p_1\rangle|^2 + |\beta|^2|\langle r_1|p_2\rangle|^2. \tag{106}$$

Here we have introduced the input marginal $H^0 = \{H_1^0, H_2^0\}$ which represents the coincidence events in this joint measurement:

$$\begin{aligned} H_1^0 &= |1\rangle\langle 1||\langle r_1|p_1\rangle|^2 + |2\rangle\langle 2||\langle r_2|p_2\rangle|^2, \\ H_2^0 &= |1\rangle\langle 1||\langle r_2|p_1\rangle|^2 + |2\rangle\langle 2||\langle r_1|p_2\rangle|^2. \end{aligned} \tag{107}$$

On writing $|r_m\rangle\langle r_m| = \frac{1}{2}(I+\mathbf{r}_m\cdot\sigma)$, $\mathbf{r}_1 = \mathbf{r}$, $\mathbf{r}_2 = -\mathbf{r}$, and $|p_\ell\rangle\langle p_\ell| = \frac{1}{2}(I+\mathbf{p}_\ell\cdot\sigma)$, we obtain the following forms for H^0_1, H^0_2:

$$H^0_1 = \tfrac{1}{2}\left(I(1+\tfrac{1}{2}\mathbf{r}\cdot(\mathbf{p}_1-\mathbf{p}_2)) + \tfrac{1}{2}\mathbf{r}\cdot(\mathbf{p}_1+\mathbf{p}_2)\sigma_z\right), \tag{108}$$

$$H^0_2 = \tfrac{1}{2}\left(I(1-\tfrac{1}{2}\mathbf{r}\cdot(\mathbf{p}_1-\mathbf{p}_2)) - \tfrac{1}{2}\mathbf{r}\cdot(\mathbf{p}_1+\mathbf{p}_2)\sigma_z\right). \tag{109}$$

in this notation the probability $prob(corr)$ becomes

$$prob(corr) = \tfrac{1}{2}\left(1+\mathbf{r}\cdot(|\alpha|^2\mathbf{p}_1 - |\beta|^2\mathbf{p}_2)\right). \tag{110}$$

In order to determine the maximum quality path determination available by a suitable choice of the path marker output observable (with eigenstates $|r_1\rangle, |r_2\rangle$), we maximize the probability $prob(corr)$ of correct inferences on the path from the registrations of r_1, r_2. The maximum $prob_{max}(corr)$ is obained at

$$\mathbf{r} = \mathbf{r}^0 = \frac{|\alpha|^2\mathbf{p}_1 - |\beta|^2\mathbf{p}_2}{||\alpha|^2\mathbf{p}_1 - |\beta|^2\mathbf{p}_2|}, \tag{111}$$

and its value is

$$prob_{max}(corr) = \tfrac{1}{2}(1+||\alpha|^2\mathbf{p}_1 - |\beta|^2\mathbf{p}_2|) =: L. \tag{112}$$

The (path) distinguishability is defined as $D = 2L-1$ [49]. We obtain:

$$D = \Big||\alpha|^2\mathbf{p}_1 - |\beta|^2\mathbf{p}_2\Big| = \sqrt{1-4|\alpha|^2|\beta|^2|\langle p_1|p_2\rangle|^2}. \tag{113}$$

We note that the probabilities for correct and wrong inferences can be expressed in terms of the coincidence POVM $H^0 = \{H^0_1, H^0_2\}$: noting that $Prob_{min}(err) = 1 - prob_{max}(corr) = 1 - L$, we have

$$D = \Big[\langle\psi_i|H^0_1|\psi_i\rangle - \langle\psi_i|H^0_2|\psi_i\rangle\Big]_{\mathbf{r}=\mathbf{r}^0}. \tag{114}$$

We calculate the variance of the distribution of H^0 in the state ψ_i, using

$$\bar{t} = \int t\, d\langle\psi_i|H^0_t|\psi_i\rangle = \langle\psi_i|H^0_1|\psi_i\rangle - \langle\psi_i|H^0_2|\psi_i\rangle, \tag{115}$$

we obtain

$$\mathrm{Var}(H,\psi_i) = \int (t-\bar{t})^2\, d\langle\psi_i|H^0_t|\psi_i\rangle \;= 1 - \Big[\langle\psi_i|H^0_1|\psi_i\rangle - \langle\psi_i|H^0_2|\psi_i\rangle\Big]^2. \tag{116}$$

Thus we obtain that

$$D^2 = 1 - \mathrm{Var}(H^0, \psi_i)|_{\mathbf{r}=\mathbf{r}_0}. \tag{117}$$

The distinguishability is thus found to be directly related to the uncertainty of the coincidence observable. It gives a measure of the accuracy of the path determination, and it is dependent on the degree of entanglement between the photon and the marker system, which may be quantified by the overlap quantity $|\langle p_1|p_2\rangle|$.

We now determine the visibility V_e of an interference observation available in the reduced photon state ρ_e after the entangling interaction. Thus we consider a joint measurement of the kind studied in subsection 4.7. The usual definition of visibility in terms of the difference between maximal and minimal probability of an outcome at D_1 (or D_2) reduces to

$$V_e = V(\rho_e) = \left|\mathrm{tr}(\rho_e|+,\mathbf{n}\rangle\langle+,\mathbf{n}|) - \mathrm{tr}(\rho_e|-,\mathbf{n}\rangle\langle-,\mathbf{n}|)\right|, \tag{118}$$

where the interference observable

$$\sigma_{\mathbf{n}} = |+,\mathbf{n}\rangle\langle+,\mathbf{n}| - |-,\mathbf{n}\rangle\langle-,\mathbf{n}| = \mathbf{n}\cdot\sigma = \begin{pmatrix} 0 & e^{-i\delta} \\ e^{i\delta} & 0 \end{pmatrix}, \tag{119}$$

(with $\mathbf{n} = (\cos\delta, \sin\delta, 0)$) is chosen such that the above difference becomes maximal:

$$V_e = 2|\mathrm{Re}(\alpha\beta^*\langle p_2|p_1\rangle\, e^{i\delta})|_{max} = 2|\alpha||\beta||\langle p_2|p_1\rangle|. \tag{120}$$

Since $V_e = |\langle\sigma_{\mathbf{n}}\rangle|$, we have

$$V_e^2 = V(\rho_e) = 1 - \mathrm{Var}(\sigma_{\mathbf{n}}, \rho_e), \tag{121}$$

and finally [50]

$$V_e^2 + D^2 = 1. \tag{122}$$

This is a limiting case of a general inequality $V_e^2 + D^2 \le 1$, reviewed in [42], where it is shown that equality arises whenever the total system of photon plus marker is in a pure state as is the case in the present context.

The above quantitative erasure duality relation can be written in terms of the associated uncertainties:

$$\mathrm{Var}(H^0, \psi_i) + \mathrm{Var}(\sigma_{\mathbf{n}}, \rho_e) = 1. \tag{123}$$

This relation shows the trade-off that happens as a result of the path-marking interaction: the better the path marking is set, the more the interference term becomes attenuated. This is to be understood as a relation between the accuracy of path determination and the "disturbance" of the interference capability of the quantum state that passes through the path marking interaction. This can be seen particularly clearly if the input state ψ_i is chosen to be an interference eigenstate, $|\alpha| = |\beta| = 1/\sqrt{2}$. In this case the distinguishability becomes minimal among all input states, and the visibility becomes maximal, and both quantities depend solely on the overlap between the marker states and hence the degree of entanglement between photon and marker:

$$D = |\mathbf{p}_1 - \mathbf{p}_2| = \sqrt{1 - |\langle p_1|p_2\rangle|^2}, \;\; V_e = |\langle p_1|p_2\rangle|. \tag{124}$$

The deviation of V_e from 1, its original value for the input interference eigenstate, represents the minimal degradation of the interference capability required by the gain in path distinguishability. If the path marking is made perfect, by requiring $\langle p_1|p_2\rangle = 0$, then $D = 1$ and $V_e = 0$, that is, the disturbance of the interference observation becomes maximal, no interference can be detected.

5.6 Uncertainty, disturbance, or entanglement?

We finally turn to the question of whether entanglement provides a more fundamental or more general explanation of the loss of interference in path-marking experiments than uncertainty. We think that this question arose from the erroneous conflation of the uncertainty principle with the idea of classical random disturbance. The discussion of that conflation has led to some very interesting clarifications and distinctions between classical, random momentum transfers (or phase kicks) and quantum-mechanical momentum transfers (or phase shifts) [6,8], with the conclusion that often, if not always, one and the same experiment may admit explanations of the loss of interference both in terms of classical random disturbances and in terms of quantum disturbances.

It seems to be a desire for causal explanation that induced the search for mechanical causes (classical or quantum) enforcing the uncertainty principle. In the preceding subsection we have seen that the reverse perspective leads to a satisfactory account of the disturbance of interference through path marking: this disturbance is expressed by means of an uncertainty relation.

We have used the term “disturbance” in the operational sense of a change in the distribution of the values of the interference observable. There was no question of a (random) mechanical kick. Quite the contrary, the coupling for the path marking process was arranged so as to constitute a *non-demolition* measurement: if the input is a path eigenstate, say $|1\rangle$, the total state after path marking is $|1\rangle|p_1\rangle$; that is, the system has remained undisturbed, it is still in the original path eigenstate.

The magnitude of the loss of interference capability or coherence that arises in the transition from an interference eigenstate as input to the reduced photon state ρ_e after the path marking interaction is determined solely by the overlap $c := |\langle p_1|p_2\rangle|$, Eq. (124). The quantity c describes the degree of correlation between the path eigenstates and the marker states, as well as the degree of entanglement in the total state Ψ_e.

It is evident that in the present quantitative erasure experiment, the explanations of the degradation of coherence either in terms of uncertainties for the photon system alone (Eq. (123)) or in terms of entanglement draw on the same crucial entity — the quantity c. The deeper reason for this may be seen in the fact that entanglement can itself be explained as an instance of uncertainty. This was made manifest in the papers of Kim and Mahler [10] and Björk *et al.* [9], who used uncertainty relations for observables of the entangled object and probe system to explain the loss of interference.

A pure state of a compound system is *entangled* if it cannot be represented as a product state. Product states are also called *separable*. The (normalized) state $\Psi_e = \alpha|1\rangle|p_1\rangle + \beta|2\rangle|p_2\rangle$ is entangled exactly when $c = |\langle p_1|p_2\rangle| < 1$ and $0 < |\alpha|, |\beta| < 1$. As an entangled state, Ψ_e possesses a biorthogonal decomposition $\Psi_e = \sqrt{w}\psi_1 \otimes \phi_1 + \sqrt{1-w}\psi_2 \otimes \phi_2$, where $\langle\psi_1|\psi_2\rangle = 0 = \langle\phi_1|\phi_2\rangle$. The only values of the (non-negative) parameter w for which Ψ_e is separable are $w = 1$ or $w = 0$. Then the state Ψ_e is separable exactly when the adapted observable $S = |\psi_1\rangle\langle\psi_1| \otimes |\phi_1\rangle\langle\phi_1| - |\psi_2\rangle\langle\psi_2| \otimes |\phi_2\rangle\langle\phi_2|$ has a definite value. Conversely, the state Ψ_e is entangled if and only if the outcomes of an S measurement are uncertain. Seen in this way, entanglement is quite generally an instance of uncertainty, and like uncertainty, it is rooted in the existence of states which are superpositions of orthogonal families of states. It is therefore to be expected that whenever there is an explanation for a loss of interference due to entanglement, there is also an associated explanation in terms of an uncertainty relation. Our consideration of the previous subsection shows that such uncertainty relations can even be formulated without recourse to the probe system.

It has been argued that in the case of the experiments of SEW and Dürr *et al.*, the position-momentum uncertainty relation does not provide an explanation of the loss of interference, and that no explanation based on another uncertainty relation has been found [11]. However, the loss of interference capability in the SEW double-slit experiment, discussed in Section 4.1 above, is due to the loss of coherence in the transition from $\psi_0 = \frac{1}{\sqrt{2}}[\psi_1 + \psi_2]$ to the associated mixed state,

$$|\psi_0\rangle\langle\psi_0| \longrightarrow \tfrac{1}{2}|\psi_1\rangle\langle\psi_1| + \tfrac{1}{2}|\psi_2\rangle\langle\psi_2|, \tag{125}$$

which is completely analogous to the transition $|\psi_i\rangle\langle\psi_i| \to \rho_e$ of the photon state in the Mach-Zehnder context. All that matters for the explanation of the loss of interference is the presence or absence of a definite phase relation between the path states ψ_1 and ψ_2 (or $|1\rangle, |2\rangle$). This can be fully described by analogs of Mach-Zehnder interferometric observables in the two dimensional subspace spanned by the path states ψ_1, ψ_2. Thus if the path observable is represented as

$$\Sigma_z := |\psi_1\rangle\langle\psi_1| - |\psi_2\rangle\langle\psi_2|, \tag{126}$$

then the phase relation between the path states can be tested by a phase sensitive observable such as

$$\Sigma_x := \tfrac{1}{2}|\psi_1 + \psi_2\rangle\langle\psi_1 + \psi_2| - \tfrac{1}{2}|\psi_1 - \psi_2\rangle\langle\psi_1 - \psi_2|. \tag{127}$$

The explanations and formulations of loss of interference in terms of uncertainty relations given in the context of Mach-Zehnder interferometry in the present section can then be literally transferred to the case of double-slit interference and path-marking experiments.

The same conclusion has been drawn by Luis [12] who based his argument on a continuous phase POVM conjugate to the path observable (σ_z), instead of the phase-sensitive interference observables used in the present paper. Similarly, Björk *et al.* [9] and de Muynck and Hendrikx [52] have demonstrated that analogous atom-interferometric experiments can be interpreted in terms of joint unsharp measurements of path and interference observables, and that complementarity emerges as a limiting case of an entropic uncertainty relation for the accuracies of these joint measurements.

6 Conclusion

We have reviewed the evolution of the understanding and current formalizations of the concepts of complementarity and uncertainty. In particular, we exhibited three distinct versions of complementarity and uncertainty relations, respectively, referring to the limitations and possibilities of preparing and measuring simultaneously sharp or unsharp values, and to a necessary trade-off between accuracy and state disturbance.

It was shown that that these formulations can be usefully applied in the context of Mach-Zehnder interferometry, to explain the loss or degradation of interference due to path marking. In particular, it was found that value complementarity and measurement complementarity can be recovered from appropriate uncertainty relations for preparations, joint, and sequential measurements, respectively. Furthermore, duality relations for the trade-off between partial path determinations and reduced-visibility interference observations were shown to be expressible as uncertainty relations.

Next, we have noted that entanglement is properly understood as an instance of uncertainty in the context of the description of compound systems, rather than a separate feature. It is therefore to be expected that whenever an explanation of the loss of interference can be given in terms of entanglement, this can be accompanied with an explanation in terms of a form of uncertainty relation.

We have pointed out that "disturbance" in quantum measurements cannot be reduced to the naive notion of classical (random) kicks; the generic concept of disturbance concerns the necessary state change induced by measurements, which inevitably goes along with the build-up of entanglement between object system and probe. This applies, in particular, to the non-demolition measurement couplings employed here for path marking, which are exactly of the same kind as the coupling used in the SEW and Dürr *et al.* experiments. Such (approximately) repeatable measurements were also at the heart of Heisenberg's thought experiments illustrating the various types of uncertainty relations. The necessary state change required for the extraction of path information leads to a disturbance of the distribution of an associated interference observable.

All these notions — uncertainty, complementarity, entanglement, and disturbance — are ultimately rooted in the linear structure of quantum mechanics, with the concomitant noncommutativity of observables.

Taken all this together, it seems indeed moot to try and establish a hierarchy of principles of uncertainty, complementarity, or entanglement *within* quantum mechanics. As seen from within this theory, these features are linked with each other but cannot be claimed to be reducible to one another. They are not logically independent, nor simply consequences of each other. Such logical relations can only be analyzed within a framework more general than quantum mechanics, in which each principle can be introduced as a separate postulate. In fact, it has been shown that there are theories in which there is an uncertainty principle without a complementarity principle (suitably specified), and theories with complementarity but without a Heisenberg uncertainty relation [53]. In quantum mechanics, uncertainty and complementarity are consequences of the formalism which are neither reducible to each other nor entirely unrelated.

There is thus no need in quantum mechanics to speak of a complementarity or uncertainty "principle", unless one uses this term informally as a way of highlighting these "principal" implications of the theory, fundamental for its intuitive understanding. If, however, one sets out to use complementarity or uncertainty as *postulates*, from which to deduce quantum mechanics in Hilbert space starting from a more general framework, then it is indeed appropriate to refer to them as principles. In this spirit, Bohr [15] and Pauli [19] have referred to quantum mechanics as "the theory of complementarity", and the programme implied by this slogan has been carried out in quite different ways by P. Lahti together with the late S. Bugajski [54] (who use the convexity, or operational approach) and by J. Schwinger [20] (based on his idea of measurement algebra). Similarly compelling derivations of quantum mechanics from the uncertainty principle, as envisaged by Heisenberg [16], are still outstanding. This task seems impossible as long as the uncertainty principle is expressed only in terms of an inequality; however, if the uncertainty principle is reinterpreted in the stronger formal sense of the canonical commutation relations, which represent the characteristic shift covariance properties of localization observables, then a derivation of quantum mechanics from such an enhanced postulate becomes possible starting, for example, within a quantum logical framework [55].

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