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On a purported local extension of the quantum formalism

Joseph Melia
Department of Philosophy, University of York, York, Y01 5DD, United Kingdom

Michael Redhead
34 Coniger Road, London SW6 3TA, United Kingdom

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It is widely believed that Bell has proved there can be no consistent local extension of the quantum formalism. Against this, Angelidis has presented a hidden variable theory which, he claims, makes precisely the same predictions as the quantum formalism and which also satisfies locality. In this note, we argue that Angelidis' theory does not live up to its inventor's claims. © *1999 American Institute of Physics.* [S0022-2488(99)02607-9]

I. INTRODUCTION

Since the early days of quantum mechanics, a number of physicists have doubted whether quantum mechanics was a complete theory and wondered whether it was possible to extend the quantum formalism by adjoining hidden variables.[1] In 1952, Bohm answered this question in the affirmative[2] and in doing so refuted von Neumann's influential yet flawed proof that no such extension was possible.[3] However, Bohm's hidden variable theory has not won wide support partly because the theory is *nonlocal*: there is instantaneous action at a distance. Since there is an obvious problem reconciling such nonlocal theories with Relativity, hidden variable theories would look much more promising if they also satisfied locality. Accordingly, the question as to whether or not *local* hidden variable theories are possible assumes great significance. In 1964 Bell appeared to prove that this question had a negative answer:[4] He showed that any local hidden variables theory is incompatible with certain quantum mechanical predictions. Since these predictions have been borne out by the experiments of Aspect and others[5] the prospects for hidden variable theories have looked grim.

Angelidis disagrees.[6] He claims to have done to Bell what Bohm did to von Neumann: He has found a theory which is local and which generates a family of probability functions converging uniformly to the probability function generated by quantum mechanics. If this were true, then Angelidis' theory would be a counterexample to Bell's theorem and a promising path would once again be open to hidden variable theorists.

Unfortunately, Angelidis' theory fails to live up to his claims: As formulated, the theory does not make the same predictions as quantum mechanics, and while there is a natural extension of his theory which does make the same predictions, the extension is *not* local. Bell's Theorem stands.

II. ANGELIDIS' THEORY

The disagreement between Angelidis and Bell can most easily be understood by considering the following thought experiment, due originally to Einstein, Podolsky, and Rosen and later simplified by Bohm.[7] In this experiment, photons γ_1 and γ_2, created by the spontaneous annihilation decay of the nonfactorizable singlet state $|\gamma_1\gamma_2\rangle$, are emitted in opposite directions and arrive at polarizers P_1 and P_2, respectively. Behind each polarizer lies a photon detector. If α and β represent the angles of polarization of P_1 and P_2 then, according to quantum mechanics, the probability that both detectors register a photon is $1/2\cos^2(\alpha-\beta)$. Could a local hidden variable theory assign the same probabilities to this experiment as quantum mechanics?

To answer this, we need to know just what locality entails. First, let us fix our terminology. Let QF stand for the classical quantum formalism. Let $p^T{}_{12}(\alpha,\beta)$ be the probability that a theory

T assigns to both detectors registering a photon given that the angles of polarization are α and β. Let λ represent our hidden variable and Λ the set of values the hidden variable could take. Let $p^*{}_1(\lambda,\alpha)(p_2^*(\lambda,\beta))$ be the chance that the photon passes through $P_1(P_2)$ given that the system is in state λ and the angle of polarization is $\alpha(\beta)$. Finally, let $\rho(\lambda)$ be a weight function which represents the chance that the hidden variable takes the value λ.

Bell and Angelidis agree that any local theory should meet the following constraints:

$$(\mathrm{L1})\ p^T{}_{12}(\alpha,\beta)=\int_\Lambda \rho(\lambda)p^*{}_1(\lambda,\alpha)p^*{}_2(\lambda,\beta),$$

where the function $p^*{}_1$ must not depend upon the variable β and the function $p^*{}_2$ must not depend upon the variable α.

(L2) The specified range Λ of the variable λ must depend upon neither the variable α nor the variable β.

(L3) The function ρ must depend upon neither the variable α nor the variable β.[8]

Bell's claim is that no hidden variable theory which meets constraints (L1)–(L3) can yield the same statistical predictions as QF. According to Bell, the QF probability function $p^{\mathrm{QF}}{}_{12}$ cannot be represented, either precisely or arbitrarily closely in the form

$$\forall\alpha,\beta\left[1/2\cos^2(\alpha-\beta)=\int_\Lambda \rho(\lambda)p^*{}_1(\lambda,\alpha)p^*{}_2(\lambda,\beta)d\lambda.\right.$$

According to Angelidis, you can. Consider the theory T which consists of the following four postulates:

$$(\Pi_1)\quad p^*{}_1(\lambda,\alpha)=\cos^2(\lambda-\alpha),$$

$$(\Pi_2)\quad p^*{}_2(\lambda,\beta)=\cos^2(\lambda-\beta),$$

$$(\Pi_3)\quad \rho(\lambda,\mu):=1/2\left[\delta(\lambda-\mu)+\delta\left(\lambda-\mu+\frac{\pi}{2}\right)\right],$$

$$(\Pi_4)\quad \Lambda:=\{\lambda|-\infty<\lambda<+\infty\}.$$

One can think of the hidden variable λ as a common plane of polarization of the two photons emitted when the atom decays. The functions $p^*{}_i(\lambda,\gamma)$ represent the probabilities that a photon will be detected at wing i ($i=1$ or $i=2$) given that the photons are plane polarized in the λ direction *or* in the $\lambda-\frac{1}{2}\pi$ direction, and the polarizer P_i is set in the γ direction.

The third postulate is the ''conditional probability distribution for the spherically symmetric singlet state $|\gamma_1,\gamma_2\rangle$ to spontaneously disintegrate into two back to back photons plane-polarized in a *specific* but randomly chosen direction, given by a variable μ, out of *all* the equally likely choices of directions...''[9] δ is simply the Dirac delta function and the final postulate does nothing more than specify the range of λ.

T generates a family of functions $p^\mu{}_{12}$ such that

$$p^\mu{}_{12}(\alpha,\beta)=\int_\Lambda \rho(\lambda,\mu)p^*{}_1(\lambda,\alpha)p^*{}_2(\lambda,\beta)d\lambda=1/4[1+\cos 2(\mu-\alpha)\cos 2(\mu-\beta)],$$

and families of functions $p^\mu{}_1$ and $p^\mu{}_2$ such that

$$p^\mu{}_1=\int_\Lambda \rho(\lambda,\mu)p^*{}_1(\lambda,\alpha)d\lambda=\tfrac{1}{2},$$

$$p^{\mu}{}_2 = \int_{\Lambda} \rho(\lambda,\mu) p^*{}_2(\lambda,\beta) d\lambda = \tfrac{1}{2}.$$

Finally, theory T entails the following important sentence (Σ):

$$(\Sigma) \quad (\forall \epsilon>0)(\exists \eta>0)(\forall \mu \in M)(\forall \alpha,\beta \in D)[(|\mu-\alpha| < \eta) \vee (|\mu-\beta| < \eta) \rightarrow |p^{\mu}{}_{12}(\alpha,\beta) - p_{12}(\alpha,\beta)| < \epsilon].$$

A logically equivalent way of writing this sentence is

$$(\Sigma) \quad (\forall \epsilon>0)(\exists \eta>0)(\forall \mu \in M)(\forall \alpha,\beta \in D)[(\mu \in S_\alpha \cup S_\beta) \rightarrow |p^{\mu}{}_{12}(\alpha,\beta) - p_{12}(\alpha,\beta)| < \epsilon]$$

where $S_\alpha = \{\mu | -\eta+\alpha < \mu < \alpha+\eta\}$ and $S_\beta = \{\mu | -\eta+\beta < \mu < \beta+\eta\}$.

According to Angelidis, (Σ) ''expresses the *formal definition* of the uniform convergence of the family of functions $\{p^{\mu}{}_{12} | \mu \in M\}$ to the function $p^{\mathrm{QF}}{}_{12}$.''

Angelidis bases his physical interpretation of this theory around (Σ): ''For *any* chosen values of α and β, whenever a value of μ, characterising the random direction of the common plane of polarization of a single pair of back to back photons, *happens by pure chance* to belong to subset S_α or S_β, this single pair of back to back photons gets through polarisers P_1 and P_2 *and causes* a coincidence count with probability given by a *value* of the QF probability function $p^{\mathrm{QF}}{}_{12}$.'' [10]

So if μ is close to either α or β, then the chance of a coincidence count is close to the chance predicted by QF. But what if μ is not close to α or β? Well, in that case, (Σ) is still true just because the antecedant is false. However, we cannot infer that the ''single pair of back to back photons with $\mu_1 \in M$ causes a coincidence count with probability $\frac{1}{2}\cos^2(\alpha_1-\beta_1)$. But the single pair of back to back photons with $\mu_1 \in M$ may fall inside another subset, say, $S_{\alpha 4}$ or $S_{\beta 4}$ of the set M... so that it causes a coincidence count with a different probability $\frac{1}{2}\cos^2(\alpha_4-\beta_4)$.'' Angelidis concludes that ''The universal quantifiers $(\forall \mu \in M)$ and $(\forall \alpha,\beta \in D)$ occurring in the prefix of the sentence Σ take into account the *whole* array of such possibilities... so that the detectors accordingly register coincidence (and single) counts with the same probabilities as those given by QF for *each* and *every* pair of back to back photons emitted by the source.'' [11]

This ends the summary of Angelidis' theory. I shall now argue that the paper contains two flaws: (1) Angelidis' family of functions does *not* converge uniformly to the QF probability function; (2) Angelidis' theory does not predict the same probability count as those given by QF for each and every pair of back to back photons emitted by the source.

III. UNIFORM CONVERGENCE

Let us examine a little more closely Angelidis' notion of uniform convergence.

We know when a countable sequence of functions $\{q^n | n \in N\}$ defined on some domain D uniformly converges to q: they converge uniformly if, for any small number ϵ we please, there is an n such that any $q^{n'}$ (with n' larger than n) is within an ϵ of q for *any* value of q and $q^{n'}$. More formally:

$$(\forall \epsilon>0)(\exists n \in N)(\forall n' \in N)(n'>n \rightarrow \forall \alpha\beta\{|q^n(\alpha,\beta) - q(\alpha,\beta)| < \epsilon\}).$$

However, since Angelidis' theory deals with the uniform convergence of an *uncountable* family of functions, the definition must be extended to cover this case. So when does the set $\{f^{\mu} | \mu \in M\}$, with M uncountable, converge to g?

Angelidis extends the definition of uniform convergence by introducing the notion of a direction: N is a *direction* in X precisely when (a) N is a set of subsets of X partially ordered by reverse inclusion; (b) for any $x,y \in N$ there is a $z \in N$ with $z \subseteq x$ and $z \subseteq y$. Example: if X is the set of real numbers, then the set of basic neighborhoods containing the number 0 is a direction in X.[12]

Then Angelidis' definition of *uniform convergence* is as follows: Let D^2 be a subset of R^2, and let N be a direction in M. The family of functions $\{f^{\mu} | \mu \in M\}$ is said to converge uniformly to g on D^2 if for every $\epsilon > 0$ there exists an $\eta > 0$ (with η depending only on ϵ) corresponding to a basic neighborhood N_η in N such that for any μ in M and any x in D^2 *whenever* the values of μ are in N_η then $|f^\mu(x) - g(x)| < \epsilon$ holds. In symbols this becomes,

$$(\forall \epsilon > 0)(\exists \eta > 0)(\forall \mu \in M)(\forall x \in D^2)(\mu \in N_\eta \rightarrow |p^\mu{}_{12}(\alpha,\beta) - p_{12}(\alpha,\beta)| < \epsilon].$$

Now, it isn't at all clear what the N_η are supposed to be here. Angelidis tells us that they are basic neighborhoods (unlike Angelidis' N_x) and it is natural to think that they are basic neighborhoods of η. But then, why quantify over the variable η? And indeed, it would be perfectly all right to say that $\{f^\mu | \mu \in M\}$ uniformly converges to f^η iff, for any ϵ there is some basic neighborhood of η such that any μ in N_η, $|f^\mu(x) - f^\eta(x)| < \epsilon$. But here η is a *name* for an element of R—it is not a free variable which can be quantified over; nor is there any reason why η has to be greater than zero.

The ambiguity of the N_η allows Angelidis to make a serious mistake in his formal definition of uniform convergence. Angelidis claims that sentence (Σ) expresses the formal definition of uniform convergence. Recall that this sentence is

$$(\Sigma) \quad (\forall \epsilon > 0)(\exists \eta > 0)(\forall \mu \in M)(\forall \alpha,\beta \in D)[(|\mu - \alpha| < \eta) \vee (|\mu - \beta| < \eta) \rightarrow |p^\mu{}_{12}(\alpha,\beta) - p_{12}(\alpha,\beta)| < \epsilon].$$

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In this case $N_\eta = \{\mu | \alpha - \eta < \mu < \alpha + \eta\}$. Again, this significantly differs from Angelidis' own definition of S_α on p. 1645, where $S_\alpha = \{\mu | \alpha - 2\epsilon < \mu < \alpha + 2\epsilon\}$. For N_η the subscript is an index of the distance from α that the μ in N_η are allowed to be. For S_α the subscript tells us which value of D the μ in S_α are close to.

Worse still, (Σ) does *not* express the notion of uniform convergence. For (Σ) says that if μ is close to α or is close to β then $p^\mu{}_{12}$ is close to p_{12} *at* (α,β). We require something more of uniform convergence—we require that if μ be close to α or β then $p^\mu{}_{12}$ be close to p_{12} for *all* values of these functions. To see how short of uniform convergence Angelidis' definition falls, consider the family of functions $\{q^\mu(\alpha) := \alpha - \mu\}$. Let $q(\alpha)$ be the zero function (so $q(\alpha) = 0$ for all α). Now, by letting $\eta = \epsilon$ it is easy to see that

$$(\forall \epsilon > 0)(\exists \eta > 0)(\forall \mu \in R)(\forall \alpha \in R)(|\mu - \alpha| < \eta \rightarrow |q^\mu(\alpha) - q(\alpha)| < \epsilon).$$

So, if μ is close to α then q^μ is close to q *at* α. But there is no reasonable sense of *uniform convergence* on which the family of functions can be said to converge to the zero function. True, for any μ and for any x, if μ is sufficiently close to x then the function q^μ is sufficiently close to the function q *at the point* α—but this is a far cry from implying that the function q^μ is close to q for all values of α.

It is clear that a family of functions f^μ will *not* uniformly converge to the function g if there is some ϵ such that, for every μ there is some α,β with $|f^\mu(\alpha,\beta) - g(\alpha,\beta)| \geqslant \epsilon$. For in such a case, the family is always at least an ϵ away from g at some point $\langle \alpha,\beta \rangle$. In Angelidis' theory, we can find an ϵ such that ϵ equals 1/4. For, for any μ let $\alpha = \mu + 45$ and let $\beta = \mu - 45$. Now,

$$p^{\mathrm{QF}}{}_{12}(\alpha,\beta) = 1/2 \cos^2(\alpha - \beta) = 1/2 \cos^2(90) = 0$$

while

$$p^\mu{}_{12}(\alpha,\beta) = 1/4[1 + \cos 2(\mu - \alpha)\cos 2(\mu - \beta)] = 1/4[1 + \cos 2(-45)\cos 2(45)] = 1/4.$$

Since every one of Angelidis' functions is at least 1/4 away from the QF function at some point $\langle \alpha,\beta \rangle$, the set does not uniformly converge to the QF function.

IV. ON THE STATISTICAL PREDICTIONS OF ANGELIDIS' THEORY

In this section we argue that Angelidis' theory does not make the same statistical predictions for the EPRB experiment as the quantum formalism.

Suppose we fix an α and a β and repeat the EPRB experiment many times. Then what proportion of coincidence counts does Angelidis' theory say we should expect? There has been a suspicious change of notation in Angelidis' paper which makes this question surprisingly difficult to answer. ${p^{\mathrm{QF}}}_{12}(\alpha,\beta)$, is the chance that both detectors fire given the polarizers are set at angles α and β, respectively, according to QF. We would expect any rival theory to QF to yield a similar probability function. But Angelidis' theory actually yields a set of probability functions ${p^{\mu}}_{12}(\alpha,\beta)$. Moreover, the superscript μ no longer represents a *theory* (as it does in ''${p^{\mathrm{QF}}}_{12}(\alpha,\beta)$''). Rather, it has come to represent the direction of polarization of the two photons.

This is odd. We expected any competitor of QF to produce a function ${p^{T}}_{12}(\alpha,\beta)$ as close to ${p^{\mathrm{QF}}}_{12}(\alpha,\beta)$ as is compatible with experimental error. But ${p^{\mu}}_{12}(\alpha,\beta)$ tells us only the chance of a coincidence *given* that the common plane of polarization of the two photons is μ. In order to work out the chance of a coincidence full stop, we need a weight function $\rho^*(\mu)$ which tells us how likely it is that the atom will decay into two photons plane polarized in the μ direction. The chance of a coincidence will then be equal to $\int_{\mu}\rho^*(\mu){p^{\mu}}_{12}(\alpha,\beta)d\mu$. But Angelidis never tells us what this weight function is. Accordingly, it is hard to see how his theory manages to make any statistical predictions at all for the EPRB experiment he is attempting to model.

Angelidis seems to think that there is no need for him to specify this weight function. He seems to think that sentence (Σ) contains all the information we need to know. Recall that (Σ) says that, when hidden variable μ happens by pure chance to be close to α or β, then the two photons get through their respective polarizers with a probability close to ${p^{\mathrm{QF}}}_{12}(\alpha,\beta)$. But, as Angelidis admits, the conditional sentence (Σ) tells us nothing about what happens when μ is *not* close to either α or β. However, ''the single pair of back-to-back photons with $\mu\in M$ may fall inside another subset, say, $S_{\alpha 4}$ or $S_{\beta 4}$ of the set M, that is, $\mu\in S_{\alpha 4}$ OR $\mu\in S_{\beta 4}$, so that it causes a coincidence count with a different probability $\frac{1}{2}\cos^2(\alpha_4-\beta_4)$, determined by the consequent in Σ deduced from Σ (by *modus ponens*) under another value assignment.'' [13] He goes on to add ''The universal quantifiers $(\forall\mu\in M)$ and $(\forall\alpha,\beta\in D)$ occurring in the prefix of the sentence Σ take into account the *whole* array of such possibilities so that the detectors accordingly register coincidence (and single counts) with the same probabilities as those given by QF for *each* and *every* pair of back to back photons emitted by the source.''

This is not so. Angelidis' explanation of how to interpret the physical significance of Σ is not complete. Standard quantum mechanics tells us that if a particular α and β are chosen so that α and β are at right angles then we will never, no matter how many times we repeat the experiment, register photons at both polarizers. Now, it is true that, on those particular occasions when the back to back photons are emitted so that their common plane of polarization μ is very close to either α or β, then the chance of a correlation will be very small. But what happens on those occasions where μ is not close to the settings of either of the polarizers? It is true, as Angelidis says, that there *exists* an α^* such that α^* is close to $\mu\alpha\nu\delta$ that, *had been the case* that the polarizer had been placed at angle α^* then the probabilities ascribed by T to a coincidence count are the same as that ascribed by quantum mechanics. But this does not tell us what we wanted to know! The situations where polarizer 1 is set at angle α^* is a *different* physical situation from the one that was under consideration. We need to know what happens when polarizers are at the particular settings α and β and the hidden variable μ is not close to either. Angelidis' advice that we choose an α^* close to μ simply dodges the question. In effect, Angelidis is only considering experiments where μ *is* close to one of the two polarizer settings. This information is not sufficient to tell us what proportion of coincidences we should expect if the polarizers are set at α and β and the experiment repeated many times.

Perhaps, though, Angelidis could augment his theory so that μ always *is* close to one of the two polarizer settings. Should Angelidis accept the postulate the value of the hidden variable μ is always close to the angle of one of the two polarizers, then his theory would both ascribe a probability to a coincidence count and, since ${p^{\mu}}_{12}(\alpha,\beta)$ approaches arbitrarily close to

$p^{QF}{}_{12}(\alpha,\beta)$, this probability can be made arbitrarily close to the probabilities ascribed by QF. The trouble with this proposal is that it straightforwardly violates the postulates of locality. In particular, it violates (L3), which effectively forbids that the angle of polarization of the two back to back photons be a function of the settings of the polarizers themselves.

V. CONCLUSION

The conclusion of this paper is clear: Angelidis has failed to provide us with a theory which is both local and which makes the same predictions as the standard quantum formalism. As such, Angelidis' theory simply leaves Bell's theorems untouched and the prospects for a local extension of the quantum formalism look as slim as ever.

[1] The most famous attack on the completeness of quantum mechanics comes from A. Einstein, B. Podolsky, and N. Rosen, ''Can quantum-mechanical description of physical reality be considered complete?'' Phys. Rev. **47**, 777–780 (1935).

[2] D. Bohm, ''A suggested interpretation of the quantum theory in terms of 'hidden' variables. I,'' Phys. Rev. **85**, 166–179 (1952).

[3] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).

[4] J. S. Bell, ''On the Einstein–Podolsky–Rosen paradox'' Physics (Long Island City, NY) **1**, 195–200 (1964).

[5] See A. Aspect, P. Grangier, and G. Roger, ''Experimental tests of realistic local theories via Bell's Theorem,'' Phys. Rev. Lett. **47**, 460–463 (1981); ''Experimental realisation of Einstein–Podolsky–Rosen–Bohm Gedankenexperiment: A new violation of Bell's inequalities,'' **49**, 91–94 (1982).

[6] T. D. Angelidis, ''A local extension of the quantum formalism,'' J. Math. Phys. **34**, 1635–1653 (1993).

[7] D. Bohm, *Quantum Theory*. (Prentice–Hall, Englewood Cliffs, NJ, 1951).

[8] After all, if a theory did make ρ depend upon either of these variables, then the theory is effectively saying that the setting of the polarizers influences the way in which the singlet state decays. That these three conditions are entailed by locality is widely accepted. See, for example, J. P. Jarrett, ''On the physical significance of the locality conditions in the Bell arguments,'' Nous **18**, 569–589 (1984).

[9] T. D. Angelidis, ''A local extension of the quantum formalism,'' J. Math. Phys. **34**, 1642 (1993).

[10] T. D. Angelidis, ''A local extension of the quantum formalism,'' J. Math. Phys. **34**, 1646 (1993).

[11] T. D. Angelidis, ''A local extension of the quantum formalism,'' J. Math. Phys. **34**, 1646 (1993).

[12] Angelidis makes a mess of the definition of N. He lets each N_x be the *set* of basic neighborhoods containing x, and then defines direction N as $\{N_x | x \in X\}$ thus making a direction a set of sets of subsets of X. This conflicts with his own use of the symbol N_x in the very next paragraph. Moreover, it is not true that for any two N_x and N_y there is an N_z which is contained in both. N is not even a directed set.

[13] T. D. Angelidis, ''A local extension of the quantum formalism,'' J. Math. Phys. **34**, 1646 (1993).