Article:

Ruderman, M.S. (2002) DNLS equation for large-amplitude solitons propagating in an arbitrary direction in a high-[beta] Hall plasma. Journal of Plasma Physics, 67 (4). pp. 271-276. ISSN 0022-3778

http://dx.doi.org/10.1017/S002237780200168X

J. Plasma Physics (2002), *vol.* 67, *part* 4, *pp.* 271–276.
DOI: 10.1017/S002237780200168X Printed in the United Kingdom

DNLS equation for large-amplitude solitons propagating in an arbitrary direction in a high-β Hall plasma

MICHAEL S. RUDERMAN

Department of Applied Mathematics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield S3 7RH, UK

(Received 14 February 2002)

Abstract. The one-dimensional oblique propagation of magnetohydrodynamic waves with arbitrary amplitudes in a Hall plasma with isotropic pressure is studied under assumption that the plasma β is large. It is shown that the wave evolution is described by the derivative nonlinear Schrödinger equation (DNLS).

1. Introduction

It has been known for almost three decades that the evolution of small-amplitude nonlinear Alfvén waves propagating quasiparallel with respect to the background magnetic field is governed by the derivative nonlinear Schrödinger equation (DNLS). This equation was first derived by Rogister (1971) starting with a Vlasov kinetic description for the particle species. Later it was derived by Mjølhus (1976) and Mio et al. (1976) on the basis of Hall magnetohydrodynamics (MHD) for cold plasmas, and by Spangler and Sheerin (1982) and Sakai and Sonnerup (1983) from warm-fluid models. An excellent review of theory of quasiparallel small-amplitude nonlinear MHD waves based on the use of the DNLS equation and its generalizations has been given by Mjølhus and Hada (1997). Application of this theory to observation of nonlinear MHD waves at the Earth's bow shock has been discussed by Spangler (1997).

This study was motivated by a recent result by Baumgärtel (1999). He numerically solved the non-stationary system of Hall MHD equations using a dark DNLS soliton as an initial condition. In this numerical study, the propagation angle was 80° and the plasma β was 5. The numerical solution has shown that the soliton practically does not evolve with time. This implies that the dark DNLS solitons propagating at large angles with respect to the background magnetic field are stationary (or, at least, quasistationary) solutions to the system of Hall MHD equations. This result supports the suggestion by Kennel et al. (1988) that the DNLS equation might decribe nonlinear wave propagation in an arbitrary direction with respect to the background magnetic field in high-β plasmas.

The aim of this paper is to show that the numerical result obtained by Baumgärtel (1999) is not a coincidence, and that the suggestion by Kennel et al. (1988) is perfectly correct. The paper is organized as follows. In the next section, we present the governing equations and discuss main assumptions. In Sec. 3, we derive the

DNLS equation for large-amplitude waves propagate in an arbitrary direction in a high-β plasma. We discuss our results in Sec. 4.

2. Governing equations

The starting point of our analysis is the system of Hall MHD equations describing adiabatic motions of infinitely conducting plasmas with isotropic pressure. This system can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.1a}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2.1b}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + l \left(\frac{\rho_0}{\mu} \right)^{1/2} \nabla \times \frac{1}{\rho}(\mathbf{B} \times \nabla \times \mathbf{B}), \tag{2.1c}$$

$$\frac{p}{\rho^\gamma} = \text{const.} \tag{2.1d}$$

Here p is the pressure, ρ the density, $\mathbf{v}$ the velocity, $\mathbf{B}$ the magnetic field, γ the adiabatic exponent, and μ the magnetic permeability of empty space, and the subscript '0' indicates an unperturbed quantity. The ion inertia length l is given by

$$l = \left(\frac{m_i^2}{\mu e^2 \rho_0} \right)^{1/2}, \tag{2.2}$$

where m_i is the ion mass and e the elemental electric charge.

In what follows, we consider one-dimensional perturbations that only depend on the x coordinate in the Cartesian coordinates (x, y, z). Let us introduce the square of the sound speed, the square of the Alfvén speed in the x direction, and the plasma β as

$$c_S^2 = \frac{\gamma p_0}{\rho_0}, \qquad V^2 = \frac{B_x^2}{\mu \rho_0}, \qquad \beta = \frac{c_S^2}{V^2}, \tag{2.3}$$

where B_x = const is the x components of the magnetic field. We assume that $\beta \gg 1$ and use $\epsilon = \beta^{-1}$ as a small parameter. In accordance with this, we introduce the scaled sound speed, $\tilde{c}_S = \epsilon c_S$, and the scaled equilibrium pressure $\tilde{p}_0 = \epsilon p_0$.

3. Derivation of DNLS equation

To derive the DNLS equation, we use the reductive perturbation method (e.g. Taniuti and Wei 1968; Kakutani et al. 1968; Engelbreht et al. 1988). We introduce the new variables $\xi = \epsilon(x - Vt)$ and $\tau = \epsilon^2 t$. In these variables, (2.1) are rewritten as

$$\epsilon \frac{\partial \rho}{\partial \tau} - V \frac{\partial \rho}{\partial \xi} + \frac{\partial (\rho u)}{\partial \xi} = 0, \tag{3.1a}$$

$$\rho \left(\epsilon \frac{\partial u}{\partial \tau} - V \frac{\partial u}{\partial \xi} + u \frac{\partial u}{\partial \xi} \right) = -\frac{\partial}{\partial \xi} \left(p + \frac{|\mathbf{B}_\perp|^2}{2\mu} \right), \tag{3.1b}$$

$$\rho \left(\epsilon \frac{\partial \mathbf{v}_\perp}{\partial \tau} - V \frac{\partial \mathbf{v}_\perp}{\partial \xi} + u \frac{\partial \mathbf{v}_\perp}{\partial \xi} \right) = \frac{B_x}{\mu} \frac{\partial \mathbf{B}_\perp}{\partial \xi}, \tag{3.1c}$$

$$\epsilon\frac{\partial \mathbf{B}_\perp}{\partial \tau} - V\frac{\partial \mathbf{B}_\perp}{\partial \xi} = B_x\frac{\partial \mathbf{v}_\perp}{\partial \xi} - \frac{\partial (u\mathbf{B}_\perp)}{\partial \xi} + \epsilon\chi V l\,\hat{\mathbf{e}}\times\frac{\partial}{\partial \xi}\left(\frac{\rho_0}{\rho}\frac{\partial \mathbf{B}_\perp}{\partial \xi}\right), \tag{3.1d}$$

$$\frac{p}{\rho^\gamma} = \text{const}, \tag{3.1e}$$

where $\chi = \mathrm{sign}(B_x)$, $\hat{\mathbf{e}}$ is the unit vector in the x direction, u is the x component of the velocity, and $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$ are the components of the velocity and the magnetic field perpendicular to the x direction. In what follows, we only consider perturbations vanishing at infinity and assume that $\rho \to \rho_0$, $p \to p_0$, $u \to 0$, $\mathbf{v}_\perp \to 0$, and $\mathbf{B}_\perp \to \mathbf{B}_{\perp 0}$ as $|\xi| \to \infty$. We can consider the magnetic field at infinity, $\mathbf{B}_0 = B_x\hat{\mathbf{e}}_x + \mathbf{B}_{\perp 0}$, as the equilibrium magnetic field, so that perturbations propagate at an angle $\theta = \arctan(|\mathbf{B}_{\perp 0}|/|B_x|)$ with respect to the equilibrium magnetic field.

Now we look for the solution to (3.1) in the form of expansions in power series with respect to ϵ. We write these expansions in the form

$$p = \epsilon^{-1}\tilde{p}_0 + p_1 + \epsilon p_2 + \ldots \tag{3.2}$$

for the pressure, and in the form

$$f = f_1 + \epsilon f_2 + \ldots \tag{3.3}$$

for all other variables. In the first-order approximation, we collect terms of the order of unity in (3.1) to obtain the system of equations

$$V\frac{\partial \rho_1}{\partial \xi} - \frac{\partial(\rho_1 u_1)}{\partial \xi} = 0, \tag{3.4a}$$

$$\rho_1\left(V\frac{\partial u_1}{\partial \xi} - u_1\frac{\partial u_1}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(p_1 + \frac{|\mathbf{B}_{\perp 1}|^2}{2\mu}\right), \tag{3.4b}$$

$$\rho_1\left(V\frac{\partial \mathbf{v}_{\perp 1}}{\partial \xi} - u_1\frac{\partial \mathbf{v}_{\perp 1}}{\partial \xi}\right) = -\frac{B_x}{\mu}\frac{\partial \mathbf{B}_{\perp 1}}{\partial \xi}, \tag{3.4c}$$

$$V\frac{\partial \mathbf{B}_{\perp 1}}{\partial \xi} = -B_x\frac{\partial \mathbf{v}_{\perp 1}}{\partial \xi} + \frac{\partial(u_1\mathbf{B}_{\perp 1})}{\partial \xi}, \tag{3.4d}$$

$$\frac{p_0}{\rho_1^\gamma} = \frac{p_0}{\rho_0^\gamma}. \tag{3.4e}$$

In deriving (3.4e), we have used the boundary conditions at infinity.

It follows from (3.4e) that $\rho_1 = \rho_0$. Then, using the boundary conditions at infinity, we immediately find from (3.4) that

$$u_1 = 0, \tag{3.5a}$$

$$\mathbf{v}_{\perp 1} = -\frac{V}{B_x}(\mathbf{B}_{\perp 1} - \mathbf{B}_{\perp 0}), \tag{3.5b}$$

$$p_1 = -\frac{1}{2\mu}(|\mathbf{B}_{\perp 1}|^2 - |\mathbf{B}_{\perp 0}|^2). \tag{3.5c}$$

We see that only the perturbations of $\mathbf{v}_\perp$, $\mathbf{B}_\perp$ and p are of the order of unity, while the perturbations of ρ and u are of the order of ϵ.

In the second-order approximation, we collect terms of the order of ϵ in (3.1). As a result, we obtain the system of equations

$$V\frac{\partial \rho_2}{\partial \xi} - \rho_0\frac{\partial u_2}{\partial \xi} = 0, \tag{3.6a}$$

$$\rho_0 V\frac{\partial u_2}{\partial \xi} = \frac{\partial}{\partial \xi}\left(p_2 + \frac{\mathbf{B}_{\perp 1}\cdot\mathbf{B}_{\perp 2}}{\mu}\right), \tag{3.6b}$$

$$\rho_0\left(\frac{\partial \mathbf{v}_{\perp 1}}{\partial \tau} - V\frac{\partial \mathbf{v}_{\perp 2}}{\partial \xi}\right) = \frac{B_x}{\mu}\frac{\partial \mathbf{B}_{\perp 2}}{\partial \xi}, \tag{3.6c}$$

$$\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} - V\frac{\partial \mathbf{B}_{\perp 2}}{\partial \xi} = \frac{\partial}{\partial \xi}(B_x\mathbf{v}_{\perp 2} - u_2\mathbf{B}_{\perp 1}) + \chi V l\,\hat{\mathbf{e}}\times\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \xi^2}, \tag{3.6d}$$

$$p_1 = \tilde{c}_S^2\rho_2. \tag{3.6e}$$

In deriving (3.6c) we have used (3.6a). Using (3.5c), (3.6a), and (3.6e), we obtain

$$u_2 = -\frac{V}{2\mu\rho_0\tilde{c}_S^2}(|\mathbf{B}_{\perp 1}|^2 - |\mathbf{B}_{\perp 0}|^2). \tag{3.7}$$

With the aid of this result and (3.5b), we rewrite (3.6c, d) as

$$V\frac{\partial \mathbf{B}_{\perp 2}}{\partial \xi} + B_x\frac{\partial \mathbf{v}_{\perp 2}}{\partial \xi} = -\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau}, \tag{3.8a}$$

$$\begin{aligned} V\frac{\partial \mathbf{B}_{\perp 2}}{\partial \xi} + B_x\frac{\partial \mathbf{v}_{\perp 2}}{\partial \xi} = {} & \frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} + \chi V l\,\hat{\mathbf{e}}\times\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \xi^2} \\ & - \frac{V}{2\mu\rho_0\tilde{c}_S^2}\frac{\partial}{\partial \xi}[\mathbf{B}_{\perp 1}(|\mathbf{B}_{\perp 1}|^2 - |\mathbf{B}_{\perp 0}|^2)]. \end{aligned} \tag{3.8b}$$

The compatibility condition for this system of two equations for $\mathbf{B}_{\perp 2}$ and $\mathbf{v}_{\perp 2}$ is that their right-hand sides have to be equal. As a result, we obtain

$$\frac{\partial \mathbf{B}_{\perp 1}}{\partial \tau} - \frac{V}{4\mu\rho_0\tilde{c}_S^2}\frac{\partial}{\partial \xi}[\mathbf{B}_{\perp 1}(|\mathbf{B}_{\perp 1}|^2 - |\mathbf{B}_{\perp 0}|^2)] + \tfrac{1}{2}\chi V l\,\hat{\mathbf{e}}\times\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \xi^2} = 0. \tag{3.9}$$

Returning to the original independent variables and introducing $\mathbf{b} = \mathbf{B}_{y1} + i\mathbf{B}_{z1}$, we eventually arrive at the DNLS equation

$$\frac{\partial b}{\partial t} + V\frac{\partial b}{\partial x} - \frac{V}{4\beta B_x^2}\frac{\partial}{\partial x}\,[b(|b|^2 - |b_0|^2)] + \frac{i}{2}\chi V l\frac{\partial^2 b}{\partial x^2} = 0, \tag{3.10}$$

where $b_0 = \lim_{|x|\to\infty} b$. Hence we have shown that, in high-β Hall plasmas, the DNLS equation does describe oblique propagation of MHD waves with arbitrary amplitudes.

4. Discussion

The results of the previous section clearly show that, in contrast to plasmas with low and moderate β, in high-β plasmas, the DNLS equation describes nonlinear waves with arbitrary amplitude of the magnetic field perturbation and propagating in an arbitrary direction.

However, one has to be very cautious when applying this result to real problems in plasma physics (e.g. to problems related to space plasmas). The point of concern is that the system of Hall MHD equations is often used for the description of motions of collisionless plasmas. This is only justified when the plasma β is small. In plasmas with moderate and large β, kinetic effects become important (e.g. Mjølhus and Willer 1986, 1988; Mjølhus 1988; Mjølhus and Hada 1997). Hence, both the system of Hall MHD equations and the DNLS equation (3.10) can be used for the description of high-β plasmas only if these plasmas are collisional.

A plasma can be considered as collisional if the characteristic spatial scale, which is βl, is much larger than the mean free path of the charged particles. This condition can be easily reduced to $\Omega_i \tau \ll \beta$, where $\Omega_i = eB_0/m_i$ is the ion gyrofrequency and τ if the mean collision time. In order that we can neglect the plasma resistivity and consider it as infinitely conducting, the condition $\Omega_e \tau \gg 1$ must be satisfied, where $\Omega_e = eB_0/m_e$ is the electron gyrofrequency, and m_e is the electron mass.

An example of a plasma where both inequalities are satisfied is the solar photospheric plasma. In this plasma, we can use the approximation $\tau \approx \tau_{ei}$, where τ_{ei} is the electron–ion collision time (Priest 1982). If we take $B_0 \approx 0.1\,\mathrm{T}\,(= 1000\,\mathrm{G})$, then $\Omega_e \tau_{ei}$ varies from 2 in the lower part of the photosphere to 2×10^3 in the upper part of the photosphere (Ruderman et al. 1997). Hence, we can neglect the resistive term in Ohm's equation in comparison with the Hall term in the middle and upper part of the photosphere. Since $\Omega_i \approx 0.5 \times 10^{-3} \Omega_e$, $\Omega_i \tau_{ei} \leqslant 1 \ll \beta$ in the solar photosphere. Therefore, (3.10) can be used for the description of nonlinear MHD waves (e.g. solitons) in the middle and upper part of the solar photosphere. The characteristic length of these solitons is $\beta l \approx 0.01\beta\,\mathrm{cm}$.

Acknowledgement

The author gratefully acknowledges financial support from PPARC.

References

Baumgärtel, K. 1999 Soliton approach to magnetic holes. *J. Geophys. Res.* **104**, 28295–28308.

Engelbreht, J. K., Fridman, V. E. and Pelinovskii, E. N. 1988 *Nonlinear Evolution Equations.* London: Longman.

Kakutani, T., Ono, H., Taniuti, T. and Wei, C.-C. 1968 Reductive perturbation method in nonlinear wave propagation. II. Application to hydromagnetic waves in cold plasmas. *J. Phys. Soc. Jpn* **24**, 1159–1166.

Kennel, C. F., Buti, B., Hada, T. and Pellat, R. 1988 Nonlinear, dispersive, elliptically polarized Alfvén waves. *Phys. Fluids* **31**, 1949–1961.

Mio, K., Ogino, T., Minami, K. and Takeda, S. 1976 Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *J. Phys. Soc. Jpn* **41**, 265–271.

Mjølhus, E. 1976 On the modulational instability of hydromagnetic waves parallel to the magnetic field. *J. Plasma Phys.* **16**, 321–334.

Mjølhus, E. 1988 The onset of Alfvénic turbulence – Comment. *Phys. Fluids* **31**, 2723–2724.

Mjølhus, E. and Hada, T. 1997 Soliton theory of quasi-parallel MHD waves. In: *Nonlinear Waves and Chaos in Space Plasmas* (ed. T. Hada and H. Matsumoto). Tokyo: TERRA-PUB, pp. 121–169.

Mjølhus, E. and Willer, J. 1986 Alfvén solitons. *Physica Scripta* **33**, 442–451.

Mjølhus, E. and Willer, J. 1988 Nonlinear Alfvén waves in a finite-beta plasma. *J. Plasma Phys.* **40**, 299–318.

Priest, E. R. 1982 *Solar Magnetohydrodynamics.* Dordrecht: Reidel.

Rogister, A. 1971 Parallel propagation of nonlinear low-frequency waves in high β plasma. *Phys. Fluids* **14**, 2733–2743.

Ruderman, M. S., Hollweg, J. V. and Goossens, M. 1997 Nonlinear theory of resonant slow waves in dissipative layers. *Phys. Plasmas* **4**, 75–90.

Sakai, J. I. and Sonnerup, B. U. Ö. 1983 Modulational instability of finite-amplitude dispersive Alfvén waves. *J. Geophys. Res.* **88**, 9069–9079.

Spangler, S. R. 1997 Nonlinear evolution of MHD waves at the Earth's bow shock: Opinions on the confrontation between theory, simulations, and measurements. In: *Nonlinear Waves and Chaos in Space Plasmas* (ed. T. Hada and H. Matsumoto). Tokyo: TERRAPUB, pp. 171–223.

Spangler, S. R. and Sheerin, J. P. 1982 Properties of Alfvén solitons in a finite-beta plasma. *J. Plasma Phys.* **27**, 193–198.

Taniuti, T. and Wei, C.-C. 1968 Reductive perturbation method in nonlinear wave propagation. I. *J. Phys. Soc. Jpn* **24**, 941–946.