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Classification of Simple Modules of the Ore Extension $K[X][Y; f \frac{d}{dX}]$

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Abstract For the algebras Λ in the title of the paper, a classification of simple modules is given, an explicit description of the prime and completely prime spectra is obtained, the global and the Krull dimensions of Λ are computed.

Keywords A skew polynomial ring · A prime ideal · A completely prime ideal · A simple module · The global dimension · The Krull dimension · A normal element

Mathematics Subject Classification 16D60 · 13N10 · 16S32

1 Introduction

Let D be a ring and $A = D[x; \sigma, \delta]$ be a skew polynomial ring where σ is an automorphism of D and δ is a σ -derivation of D (for all $a, b \in D$, $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$). The ring A is generated by D and x subject to the defining relations $xa = \sigma(a)x + \delta(a)$ for all elements $a \in D$. When D is a Dedekind domain, a classification of simple A -modules is given in [4]. This is a large class of rings. A machinery is developed in [4] to cover all possible situations (non-commutative valuations, etc).

The algebra

$$\Lambda = K[X] \left[Y; \delta := f \frac{d}{dX} \right] = \bigoplus_{i \geq 0} K[X] Y^i$$

is a particular example of the ring A where $\sigma = \text{id}$ is the identity automorphism of the polynomial ring $K[X]$, $f \in K[X]$ and $\delta = f \frac{d}{dX}$ is a K -derivation of $K[X]$ ($\delta(X) = f$). If $f = 1$ (or, more generally, $f \in K^\times \setminus \{0\}$) then the algebra $\Lambda(1)$ is the *Weyl algebra*

$$A_1 = K \langle X, \partial \mid \partial X - X\partial = 1 \rangle \simeq K[X] \left[Y; \frac{d}{dX} \right].$$

In 1981, a classification of simple A_1 -modules was obtained by Block (over the field of complex numbers) in [9] (see also [2, 3] for an alternative approach via generalized Weyl algebras in a more general situation).

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Recently, classifications of simple *weight* modules are obtained for some classical algebras (the *Euclidean* algebra, the *Schrödinger* algebra, the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$), see [5–7]. In these classifications, classifications of *all simple* modules over certain subalgebras of the Weyl algebra A_1 that contain the polynomial algebra $K[X]$ (the, so-called, *polyonic* algebras) play a crucial role. The polyonic algebras are investigated in [8]. Each polyonic algebra contains the algebra $\Lambda = \Lambda(f)$ for some non-scalar polynomial $f \in K[X]$ which play an important role in studying of its properties. This is the main reason why we decided to collect main properties of the algebras Λ in this paper. In particular, a classification of simple Λ -modules is given in Sect. 2 (Lemma 2.1 and Theorem 2.10). This classification can be derived from [4] but we give different and simpler proofs which are based on generalized Weyl algebras rather than skew polynomial rings.

An ideal $\mathfrak{p}$ of a ring R is called a *completely prime ideal* if the factor ring $R/\mathfrak{p}$ is a domain. A completely prime ideal is a prime ideal. The sets of prime and completely prime ideals of the ring R are denoted by $\text{Spec}(R)$ and $\text{Spec}_c(R)$, respectively.

In Theorem 1.1, a classification of prime and completely prime ideals of the algebra Λ is given, the Krull and global dimensions of the algebra Λ are found. The algebra Λ is a Noetherian domain of Gelfand-Kirillov dimension 2.

Theorem 1.1 *Let K be a field of characteristic zero, $\Lambda = K[X][Y; \delta := f \frac{d}{dX}]$ where $f \in K[X] \setminus K$. Let $f = p_1^{n_1} \cdots p_s^{n_s}$ be a unique (up to permutation) product of irreducible polynomials of $K[X]$. Then*

1. *The Krull dimension of Λ is $\text{Kdim}(\Lambda) = 2$.*
2. *The global dimension of Λ is $\text{gldim}(\Lambda) = 2$.*
3. *The elements $p_1, \dots, p_s$ are regular normal elements of the algebra Λ (i.e. p_i is a non-zero-divisor of Λ and $p_i \Lambda = \Lambda p_i$).*
4. *$\text{Spec}(\Lambda) = \text{Spec}_c(\Lambda) = \{0, \Lambda p_i, (p_i, q_i) \mid i = 1, \dots, s; q_i \in \text{Irr}_m(F_i[Y])\}$ where $F_i := K[X]/(p_i)$ is a field and $\text{Irr}_m(F_i[Y])$ is the set of monic irreducible polynomials of the polynomial algebra $F_i[Y]$ over the field F_i in the variable Y . If, in addition, the field K is an algebraically closed and $\lambda_1, \dots, \lambda_s$ are the roots of the polynomial f then $\text{Spec}(\Lambda) = \{0, \Lambda(X - \lambda_i), (X - \lambda_i, Y - \mu) \mid i = 1, \dots, s; \mu \in K\}$.*

The proof of Theorem 1.1 is given in Sect. 3.

2 Classification of Simple Λ -Modules

In this section, ‘module’ means left module, K is an algebraically closed field of characteristic zero, $\Lambda = K[X][Y, \delta = f \frac{d}{dX}]$ where $f \in K[X] \setminus K$. The algebra Λ is a Noetherian domain. The aim of the section is to give a classification of simple Λ -modules (Lemma 2.1 and Theorem 2.10).

The element f is a regular normal element of Λ . It follows from

$$fY = Yf - f'f = (Y - f')f, \quad \text{where } f' = \frac{df}{dX},$$

that the element f is a *normal* element of Λ (i.e. $\Lambda f = f \Lambda$). It determines a K -automorphism ω_f of the algebra Λ :

$$fu = \omega_f(u)f, \quad u \in \Lambda,$$

$$\omega_f : X \mapsto X, \quad Y \mapsto Y - f'.$$

The algebra Λ can be identified with a subalgebra of the first Weyl algebra A_1 by the map

$$\Lambda \rightarrow A_1, \quad X \mapsto X, \quad Y \mapsto f\partial. \tag{1}$$

The Weyl algebra A_1 is a generalized Weyl algebra. The Weyl algebra A_1 is a simple Noetherian domain with *restricted minimum condition*, i.e. any proper left or right factor module of A_1 has finite length, [10].

Definition, [1,2]. Let D be a ring, σ be an automorphisms of D and a be a central element of D . A **generalized Weyl algebra** (GWA) $A = D(\sigma, a)$ of degree 1, is the ring generated by D and by two indeterminates X and Y subject to the relations [1,2]: For all $\alpha \in D$,

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \quad YX = a \text{ and } XY = \sigma(a).$$

The algebra

$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$

is a $\mathbb{Z}$ -graded algebra, where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), $v_0 = 1$.

The Weyl algebra A_1 is a GWA,

$$A_1 = D(\sigma, a = H), \quad X \leftrightarrow X, \quad \partial \leftrightarrow Y, \quad \partial X \leftrightarrow H, \quad D = K[H],$$

with coefficients from a polynomial ring $K[H]$ where $\sigma \in \text{Aut}_K K[H]$ and $\sigma : H \rightarrow H - 1$.

We denote by Λ_f (resp., $A_{1,f}$) the localization of the ring Λ (resp., A_1) at the powers of the element f , i.e.

$$\Lambda_f = S_f^{-1}\Lambda, \quad (\text{resp., } A_{1,f} = S_f^{-1}A_1) \text{ where } S_f = \{f^i, i \geq 0\}.$$

By (1), Λ_f is a subalgebra of $A_{1,f}$ such that

$$\Lambda_f = A_{1,f}. \quad (2)$$

The algebras Λ and A_1 can be considered as subalgebras of $A_{1,f}$,

$$\Lambda \subseteq A_1 \subseteq \Lambda_f = A_{1,f}. \quad (3)$$

The algebra $A_{1,f}$ is a simple Noetherian domain with restricted minimum condition.

$\hat{\Lambda}(f - \text{torsion})$. The sets of isoclasses of Λ -modules $\hat{\Lambda}$ and of A_1 -modules $\hat{A}_1$ are disjoint unions of f -torsion ($M_f = 0$) and f -torsionfree ($M_f \neq 0$) simple Λ -modules and A_1 -modules, respectively,

$$\hat{\Lambda} = \hat{\Lambda}(f - \text{torsion}) \coprod \hat{\Lambda}(f - \text{torsionfree}), \quad (4)$$

$$\hat{A}_1 = \hat{A}_1(f - \text{torsion}) \coprod \hat{A}_1(f - \text{torsionfree}). \quad (5)$$

Lemma 2.1 is a classification of simple f -torsion Λ -modules.

Lemma 2.1 Let $\lambda_1, \dots, \lambda_s$ be the roots of the polynomial f . Then

$$\hat{\Lambda}(f - \text{torsion}) = \{[\Lambda/\Lambda(X - \lambda_i, Y - \mu)] \mid i = 1, \dots, s; \mu \in K\}.$$

All these Λ -modules are 1-dimensional and they are the only simple finite dimensional Λ -modules (by Theorem 2.10).

Proof Each simple f -torsion Λ -module M is annihilated by the (normal) element f ($fM = 0$). So, in fact, the Λ -module M is a simple module over the factor algebra

$$\Lambda/(f) = K[X, Y]/(f)$$

which is isomorphic to the factor algebra of the polynomial ring $K[X, Y]$ in two variables at the ideal (f) generated by f , and the equality in the lemma follows.

Let N be a simple Λ -module. Then the map

$$f_N : N \rightarrow N, \quad n \mapsto fn$$

is either 0 or a bijection (f is normal in Λ). In the second case N is, in fact, a simple $(\Lambda_f \equiv A_{1,f})$ -module, so $\dim_K N = \infty$, since $A_{1,f}$ is a simple infinite dimensional algebra. If the module N is finite dimensional, then $fN = 0$, i.e. $[N] \in \hat{\Lambda}(f - \text{torsion})$. $\square$

$\hat{\Lambda}(f - \text{torsionfree})$. The sum of all simple submodules of a Λ -module M is called the *socle* of M which is denoted by $\text{soc}_\Lambda M$. It is the largest semisimple submodule of M . A Λ -module N is called Λ -*socle* (or, *socle*, for short) provided $\text{soc}_\Lambda N \neq 0$. Denote by $\hat{A}_1(\Lambda\text{-socle})$ the set of isoclasses of simple Λ -socle A_1 -modules. The proof of the following lemma is evident (see [2, Lemma 3.4] for details).

Lemma 2.2 1. *The canonical map*

$$(\cdot)_f : \hat{\Lambda}(f - \text{torsionfree}) \rightarrow \hat{A}_{1,f}(\Lambda - \text{socle}), [M] \mapsto [M_f := A_{1,f} \otimes_\Lambda M]$$

is a bijection with inverse $[N] \rightarrow [\text{soc}_\Lambda(N)]$.

2. *Each simple f -torsionfree Λ -module has the form*

$$M_{\mathfrak{m}} := \Lambda / \Lambda \cap \mathfrak{m} \tag{6}$$

for some maximal left ideal $\mathfrak{m}$ of the ring $A_{1,f}$. Two such modules are isomorphic, $M_{\mathfrak{m}} \simeq M_{\mathfrak{n}}$, iff the $A_{1,f}$ -modules $A_{1,f}/\mathfrak{m}$ and $A_{1,f}/\mathfrak{n}$ are isomorphic. $\square$

Lemma 2.3 *Let $\lambda_1, \dots, \lambda_s$ be the roots of the polynomial f . Then*

$$\hat{A}_1(f - \text{torsion}) = \{[M_i := A_1 / A_1(X - \lambda_i)] \mid i = 1, \dots, s\}.$$

Proof As a vector space the module M_i can be identified with the polynomial ring $K[y]$ in a variable $y = \partial + A_1(X - \lambda_i)$ and

$$\partial y^j = y^{j+1}, \quad X y^j = \lambda_i y^j + \dots \quad \text{for } j \geq 0$$

where by three dots we denote the sum of elements of smaller degree in the variable y . Thus the linear operator

$$X - \mu : M_i \rightarrow M_i, \quad m \mapsto (X - \mu)m, \quad m \in M_i,$$

is nilpotent if and only if $\mu = \lambda_i$; otherwise, $X - \mu$ is an isomorphism of the vector space M_i . From this fact it follows that the A_1 -modules $\{M_i\}$ are simple and non-isomorphic.

Now, let $[M] \in \hat{A}_1(f - \text{torsion})$. Then there exists i such that M is an epimorphic image of M_i , hence $M \simeq M_i$. $\square$

Theorem 2.4 *The map*

$$\hat{A}_1(f - \text{torsionfree}) = \hat{A}_1 \setminus \{[M_1], \dots, [M_s]\} \rightarrow \hat{A}_{1,f}, \quad [M] \mapsto [M_f]$$

is bijective.

Proof The map above is well defined and injective.

Let $[N] \in \hat{A}_{1,f}$. Then $N \simeq A_{1,f}/J$ for some nonzero maximal left ideal J of $A_{1,f}$. Then $I = J \cap A_1 \neq 0$ and A_1/I is a A_1 -submodule of N . The A_1 -module A_1/I has finite length [10], thus it contains a simple A_1 -submodule, say M . Then $N \simeq M_f$ which means that the map above is surjective. $\square$

Recall that $D = K[H]$. The localization $B = S^{-1}A_1$ of the Weyl algebra A_1 at the Ore set $S = D \setminus \{0\}$ is a *skew Laurent polynomial ring*

$$B = K(H)[X, X^{-1}; \sigma], \quad \sigma(H) = H - 1,$$

with coefficients from the field $K(H)$ of rational functions. The algebra B is a right and left *Euclidean* domain with respect to the ‘length’ function

$$l : B \setminus \{0\} \rightarrow \mathbb{N} := \{0, 1, 2, \dots\}, \quad l(\alpha X^m + \beta X^{m+1} + \dots + \gamma X^n) = n - m, \quad \alpha \neq 0, \gamma \neq 0 \in K(H),$$

hence, it is a right and left principal ideal domain.

We have

$$\hat{A}_1 = \hat{A}_1(D - \text{torsion}) \coprod \hat{A}_1(D - \text{torsionfree})$$

where a simple A_1 -module M belongs to the first (resp., second) set if $S^{-1}M = 0$ (resp., $S^{-1}M \neq 0$).

For $\lambda \in K$ set $\mathcal{O}(\lambda) := \lambda + \mathbb{Z}$. We say that scalars λ and μ are *equivalent*, $\lambda \sim \mu$, if either $\mathcal{O}(\lambda) = \mathcal{O}(\mu) \neq \mathbb{Z}$ or both λ and μ belong either to $(-\infty, 0] := \{i \in \mathbb{Z} \mid i \leq 0\}$ or to $[1, \infty) := \{i \in \mathbb{Z} \mid i \geq 1\}$. Then $\sim$ is an equivalence relation on K . Let $K/\sim$ be the set of equivalence classes of K under $\sim$. So, the elements of the set $K/\sim$ are distinct sets $\lambda + \mathbb{Z}$ where $\lambda \notin \mathbb{Z}$ and the two sets $(-\infty, 0]$ and $[1, \infty)$. Notice that $\mathbb{Z} = (-\infty, 0] \coprod [1, \infty)$.

Proposition 2.5 ([2, Theorem 3.1]) *The map*

$$K/\sim \rightarrow \hat{A}_1(D - \text{torsion}), \quad [\Gamma] \mapsto [L(\Gamma)],$$

is a bijection, where

1. If $\Gamma = \mathcal{O}(\lambda) \neq \mathbb{Z}$, then $L(\Gamma) = A_1/A_1(H - \lambda)$.
2. If $\Gamma = (-\infty, 0]$, then $L(\Gamma) = A_1/A_1X$.
3. If $\Gamma = [1, \infty)$, then $L(\Gamma) = A_1/A_1(H - 1, Y)$. □

Corollary 2.6

$$\hat{A}_1(D - \text{torsion}, f - \text{torsion}) = \begin{cases} \{[L((-\infty, 0]) = A_1/A_1X]\} & \text{if } 0 \text{ is a root of } f(X), \\ \emptyset & \text{if } 0 \text{ is not a root of } f(X). \end{cases}$$

Proof Straightforward. □

Corollary 2.7 *Let $[M] \in \hat{A}_1(D - \text{torsion}, f - \text{torsionfree})$.*

1. *If $M = A_1/A_1X$ (i.e. 0 is not a root of f , by Corollary 2.6) then M is a simple f -torsionfree Λ -module with $M = M_f$.*
2. *If $M \neq A_1/A_1X$ then $\text{soc}_\Lambda M = \text{soc}_\Lambda M_f = 0$. The set $\hat{\Lambda}(D - \text{torsion}, f - \text{torsionfree})$ is equal to $\{A_1/A_1/X\}$ if 0 is not a root of f and $\emptyset$, otherwise.*

Proof 1. As a vector space the module $M = A_1/A_1X$ has the basis $\{y^i = \partial^i + A_1X, i \geq 0\}$, and

$$\partial y^i = y^{i+1}, \quad X y^i = -i y^{i-1} \quad \text{and} \quad Y y^i = f(0) y^{i+1} + \sum_{0 \leq j \leq i} \mu_j y^j,$$

for some scalars $\mu_j \in K$. Now, it is obvious that the Λ -module M is a simple f -torsionfree Λ -module ($f(0) \neq 0$). Moreover, the linear map $f_M : M \rightarrow M, m \mapsto fm$ is a bijection, hence, $M = M_f$.

2. Since $\Lambda_f = A_{1,f}$, $\text{soc}_\Lambda M = \text{soc}_\Lambda M_f$. Let M belongs to the first (resp., third) class of modules from Proposition 2.5, i.e.

$$M = L(\Gamma), \quad \Gamma = \mathcal{O}(\lambda) \neq \mathbb{Z} \quad (\text{resp., } \Gamma = [1, \infty)).$$

The element $\bar{1} = 1 + A_1(H - \lambda)$ (resp., $\bar{1} = 1 + A_1(H - 1, Y)$) is a canonical generator of the A_1 -module M . In both cases, for $i \geq 0$, set $x^i = X^i \bar{1}$. In the first case, for $i < 0$, set $x^i = \mu_i \partial^{-i} \bar{1}$, $\mu_i \in K$. The scalars μ_i can be chosen in such a way that (in both cases) $X x^i = x^{i+1}$ for all possible i . Degree argument shows that the module M contains a strictly descending chain of Λ -submodules

$$M \supset fM \supset \cdots \supset f^n M \supset \cdots \quad \text{with} \quad \bigcap_{n \geq 0} f^n M = 0.$$

Suppose that $N := \text{soc}_\Lambda M \neq 0$, then, in a view of Lemma 2.2 and Theorem 2.4, N is an essential simple Λ -submodule of both M_f and M , hence $0 \neq N \subseteq \bigcap_{n \geq 0} f^n M = 0$, a contradiction. □

An element of a ring is called *regular* if it is not a zero divisor. Given a ring A and a multiplicatively closed subset S of A which consists of regular normal elements. Let $B = S^{-1}A$ be the localization of A at S .

Theorem 2.8 *Let A , B , and S be as above and let $\mathfrak{m}$ be a maximal left ideal of B . The following are equivalent.*

1. The A -module $M_{\mathfrak{m}} := A/A \cap \mathfrak{m}$ is simple.
2. The socle $\text{soc}_A(M_{\mathfrak{m}}) \neq 0$.
3. $A = As + A \cap \mathfrak{m}$ for all $s \in S$.

□

Remark. If $S = \{f^n, n \geq 0\}$ for some regular normal element f of A , then the last condition of this lemma is equivalent to $A = Af + A \cap \mathfrak{m}$. We shall use this fact in what follows. In general situation, it suffices to check whether the third condition holds only for generators of the monoid S .

Proof The implications $(1 \Rightarrow 2)$ and $(1 \Rightarrow 3)$ are obvious.

$(2 \Rightarrow 1)$ If $\text{soc}_A(M_{\mathfrak{m}}) \neq 0$ then it is a simple A -module which for some $s \in S$ is equal to

$$(As + A \cap \mathfrak{m})/A \cap \mathfrak{m} \simeq As/As \cap \mathfrak{m} \simeq A/A \cap \mathfrak{m}s^{-1} = \omega_s(A)/\omega_s(A \cap \mathfrak{m}) \simeq \omega_s^{-1}M_{\mathfrak{m}},$$

where $\omega_s^{-1}M_{\mathfrak{m}}$ is the twisted A -module $M_{\mathfrak{m}}$ by the automorphism ω_s^{-1} of A (the element s is regular and normal). Since the A -module $\omega_s^{-1}M_{\mathfrak{m}}$ is simple, so is $M_{\mathfrak{m}}$.

$(3 \Rightarrow 1)$ If J is a left ideal of A which contains $A \cap \mathfrak{m}$ but does not coincide with it, then, by the maximality of $\mathfrak{m}$, $S^{-1}J = B$. Therefore $J \cap S \neq \emptyset$. Let $s \in J \cap S$. Then $J \supseteq As + A \cap \mathfrak{m} = A$, that is $M_{\mathfrak{m}}$ is a simple A -module. □

$\hat{A}_1(D - \text{torsionfree})$. Let us recall a description of $\hat{A}_1(D - \text{torsionfree})$ from [2]. In the set $S = K[H] \setminus \{0\}$ consider the relation $<: \alpha < \beta$ if there are no roots λ and μ of the polynomial α and β respectively and such that $\lambda - \mu$ is non-negative integer.

Definition, [2]. An element $b = Y^m \beta_{-m} + \dots + \beta_0 \in A_1, m > 0$, all $\beta_i \in D$, is called ***l-normal*** if $\beta_0 < \beta_{-m}$ and $\beta_0 < H$, (i.e. the polynomial β_0 has no root from $\{0, 1, 2, \dots\}$ and there are no roots λ and μ of the polynomials β_0 and β_m respectively with $\lambda - \mu \in \{0, 1, 2, \dots\}$).

Theorem 2.9 ([2, Theorem 3.8]) *Let $b = Y^m \beta_{-m} + \dots + \beta_0 \in A_1, m > 0$, all $\beta_i \in D$, be an *l-normal* and irreducible element in B . Then*

$$\mathcal{M}_b := A_1/A_1 \cap Bb$$

is a simple D -torsionfree A_1 -module. Two such A_1 -modules are isomorphic, $\mathcal{M}_b \simeq \mathcal{M}_c$, iff $B/Bb \simeq B/Bc$ as B -modules. Each simple D -torsionfree A_1 -module is isomorphic to some $\mathcal{M}_b$. □

Set

$$B_f := S_f^{-1}B = A_{1,f} \otimes_{\Lambda} B = \Lambda_f \otimes_{\Lambda} B$$

for the *localization of the (left) Λ -module B at S_f* . Then the algebra $A_{1,f} = \Lambda_f$ can be considered as a $(A_{1,f} = \Lambda_f)$ -submodules of B_f . For any nonzero $b \in B$, $(Bb)_f = B_f b$.

Theorem 2.10 is a classification of simple f -torsionfree Λ -modules.

Theorem 2.10 *Let $b = Y^m \beta_{-m} + \dots + \beta_0 \in A_1, m > 0$, all $\beta_i \in D$, be an *l-normal* and irreducible element in B such that*

1. $\Lambda = \Lambda f + \Lambda \cap B_f b (= \Lambda f + \Lambda \cap Bb)$, and
2. the simple B -module B/Bb is not isomorphic to any of modules $B/B(X - \lambda)$ where λ runs through the nonzero roots of f .

Then

$$\mathcal{M}_b := \Lambda / \Lambda \cap Bb \quad (= \Lambda / \Lambda \cap B_f b)$$

is a simple f -torsionfree Λ -module. Two such Λ -modules are isomorphic, $\mathcal{M}_b \simeq \mathcal{M}_c$, iff $B/Bb \simeq B/Bc$ as B -modules.

Each simple f -torsionfree Λ -module is isomorphic either to some $\mathcal{M}_b$ or to the module $M = A_1/A_1X$ from Corollary 2.7, if 0 is not a root of f (the Λ -module M is not isomorphic to any $\mathcal{M}_b$). The condition 1 above is equivalent to the condition that $\Lambda = \Lambda(X - \lambda_i) + \Lambda \cap B_f b \quad (= \Lambda(X - \lambda_i) + \Lambda \cap Bb)$ for all roots λ_i of the polynomial f .

Each simple f -torsionfree Λ -module is infinite dimensional.

Proof By Lemma 2.2,

$$[M] \in \hat{\Lambda}(f - \text{torsionfree}) \Leftrightarrow [M_f] \in \hat{A}_{1,f}(\Lambda - \text{socle})$$

and $M = \text{soc}_\Lambda(M_f) \simeq \Lambda / \Lambda \cap \mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of $A_{1,f}$. By Corollary 2.7, either $M_f \simeq A_1/A_1X$ (0 is not a root of f) or $M_f \in \hat{A}_{1,f}(D - \text{torsionfree}, \Lambda - \text{socle})$. In the first case, $M = \text{soc}_\Lambda(M_f) = M_f = A_1/A_1X$ (Corollary 2.7).

In the second case, by Theorems 2.4 and 2.9,

$$M_f \simeq (\mathcal{M}_b)_f = A_{1,f}/A_{1,f} \cap B_f b$$

for some l -normal irreducible element b from Theorem 2.9. Note that the left ideal $\mathfrak{m} = A_{1,f} \cap B_f b$ of $A_{1,f}$ is maximal. By Lemma 2.3 and Theorem 2.9, $[\mathcal{M}_b] \in \hat{A}_1(D - \text{torsionfree}, f - \text{torsionfree})$ iff the second condition of the theorem holds. Now,

$$\text{soc}_\Lambda(M_f) = \text{soc}_\Lambda(\mathcal{M}_b)_f = \text{soc}_\Lambda(\Lambda / \Lambda \cap A_{1,f} \cap B_f b) = \text{soc}_\Lambda(\Lambda / \Lambda \cap B_f b). \quad (7)$$

By Theorem 2.8 and by the Remark after it,

$$\text{soc}_\Lambda(M_f) \neq 0 \quad \text{iff} \quad \Lambda = \Lambda f + \Lambda \cap (A_{1,f} \cap B_f b) = \Lambda f + \Lambda \cap B_f b.$$

In this case,

$$\text{soc}_\Lambda(M_f) = \Lambda / \Lambda \cap (A_{1,f} \cap B_f b) = \Lambda / \Lambda \cap B_f b.$$

Let us show that (in this case) the natural Λ -module epimorphism

$$\varphi : M_b = \Lambda / \Lambda \cap Bb \rightarrow \Lambda / \Lambda \cap B_f b, \quad \lambda + \Lambda \cap Bb \rightarrow \lambda + \Lambda \cap B_f b,$$

is an isomorphism. Note that

$$\ker \varphi = \Lambda \cap B_f b / \Lambda \cap Bb.$$

The A_1 -module $\mathcal{M}_b$ is a submodule of $(\mathcal{M}_b)_f \simeq M_f$. So,

$$\text{soc}_\Lambda(M_f) = \text{soc}_\Lambda(\mathcal{M}_b) = \text{soc}_\Lambda(\Lambda / \Lambda \cap Bb).$$

By assumption $\text{soc}_\Lambda(M_f) \neq 0$, then it is a simple essential f -torsionfree Λ -submodule of M_f . If $\ker \varphi \neq 0$, then $\text{soc}_\Lambda(M_f) \subseteq \ker \varphi$, but $\ker \varphi$ is an f -torsion Λ -module, a contradiction.

Let $\mathcal{M}_b$ and $\mathcal{M}_c$ be as in the theorem. By Lemma 2.2, $\mathcal{M}_b \simeq \mathcal{M}_c$ as Λ -modules $\Leftrightarrow A_{1,f} \otimes_\Lambda \mathcal{M}_b \simeq A_{1,f} \otimes_\Lambda \mathcal{M}_c$ as $A_{1,f}$ -modules. Since

$$A_{1,f} \otimes_\Lambda \mathcal{M}_b \simeq A_{1,f} / A_{1,f} \cap B_f b \simeq (\mathcal{M}_b)_f,$$

by Theorem 2.4, the above $A_{1,f}$ -modules are isomorphic iff $\mathcal{M}_b \simeq \mathcal{M}_c$ as A_1 -modules, so, by Theorem 2.9, $B/Bb \simeq B/Bc$ as B -modules.

The condition 1 of the theorem is equivalent to the condition that $\Lambda = \Lambda(X - \lambda_i) + \Lambda \cap B_f b (= \Lambda(X - \lambda_i) + \Lambda \cap Bb)$ for all roots λ_i of the polynomial f (since the elements $X - \lambda_i$ are regular normal elements of Λ and λ_i are the roots of f).

By Lemma 2.1, each simple f -torsionfree Λ -module is infinite dimensional. If 0 is not a root of f , then the modules $M = A_1/A_1X$ and $\mathcal{M}_b$ (from the theorem) are not isomorphic, since the linear map $X_M : M \rightarrow M$, $m \mapsto Xm$ is locally nilpotent but $\ker X_{\mathcal{M}_b} = 0$. $\square$

3 The Prime Ideals, the Krull and Global Dimensions of the Algebra Λ

In this section, K is a field of characteristic zero (not necessarily algebraically closed) and $f = p_1^{n_1} \cdots p_s^{n_s}$ is a nonscalar polynomial of $K[X]$ where $p_1, \dots, p_s$ are irreducible, co-prime divisors of f (i.e. $K[X]p_i + K[X]p_j = K[X]$ for all $i \neq j$). The aim of this section is to give a proof of Theorem 1.1.

Proof of Theorem 1.1 3. The elements $p_1, \dots, p_s$ are regular normal elements of the algebra Λ since

$$Yp_i = p_i(Y - p_i^{-1}f) \quad \text{and} \quad Xp_i = p_iX.$$

4. The algebra Λ is a domain, hence $0 \in \text{Spec}_c(\Lambda)$.

Since

$$\Lambda/\Lambda p_i \simeq F_i[Y] \tag{8}$$

is a polynomial algebra with coefficients in the field F_i (since $YX - XY = f \in \Lambda p_i$), the ideal Λp_i is a completely prime ideal of Λ .

By (3), $\Lambda_f = A_{1,f}$ is a simple algebra (as a localization of a simple Noetherian algebra). If $\mathfrak{p}$ is a nonzero prime ideal of the algebra Λ then $f^n \in \mathfrak{p}$ for some natural number $n \geq 1$. Hence, $p_i \in \mathfrak{p}$ for some i , by statement 3. By (8), $\mathfrak{p} = (p_i, g_i)$ for some monic irreducible polynomial g_i of the polynomial algebra $F_i[Y]$.

1. By [11, Theorem 6.5.4.(i)], $\text{Kdim}(\Lambda) \leq \text{Kdim}(K[X]) + 1 = 1 + 1 = 2$.

Since p_i is a regular normal element of the algebra Λ ,

$$\text{Kdim}(\Lambda) \geq \text{Kdim}(\Lambda/\Lambda p_i) + 1 \stackrel{(8)}{=} \text{Kdim}(F_i[Y]) + 1 = 1 + 1 = 2,$$

by [11, Theorem 6.5.9]. Therefore, $\text{Kdim}(\Lambda) = 2$.

2. By [11, Theorem 7.5.3.(i)], $\text{gldim}(\Lambda) \leq \text{gldim}(K[X]) + 1 = 1 + 1 = 2$.

By (8), $\text{gldim}(\Lambda/\Lambda p_i) = \text{gldim}(F_i[Y]) = 1 < \infty$. Now, by [11, Theorem 7.3.5.(i)],

$$\text{gldim}(\Lambda) \geq \text{gldim}(\Lambda/\Lambda p_i) + 1 \stackrel{(8)}{=} \text{gldim}(F_i[Y]) + 1 = 1 + 1 = 2.$$

Therefore, $\text{gldim}(\Lambda) = 2$. $\square$

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