

Symmetrization of exterior parabolic problems and probabilistic interpretation

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Abstract We prove a comparison theorem for the spatial mass of the solutions of two exterior parabolic problems, one of them having symmetrized geometry, using approximation of the Schwarz symmetrization by polarizations, as it was introduced in Brock and Solynin (Trans Am Math Soc 352(4):1759–1796, 2000). This comparison provides an alternative proof, based on PDEs, of the isoperimetric inequality for the Wiener sausage, which was proved in Peres and Sousi (Geom Funct Anal 22(4):1000–1014, 2012).

Keywords Schwarz symmetrization · Polarization · Parabolic equations · Wiener sausage

Mathematics Subject Classification 35K20 · 35B51

1 Introduction

In the present article we prove a comparison theorem for the spatial mass, at any time t , for the solutions of two parabolic exterior problems, the second being the “symmetrization” of the first one. In order to do so, we show that the spatial mass of the solution decreases under polarization, and since the Schwarz symmetrization is the limit of compositions of polarizations, we carry the comparison to the limit. This technique was introduced in [4].

Our result is motivated by a problem in probability theory. Namely, the isoperimetric inequality for the Wiener sausage, which was proved in [15]. The problem is the

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following. If $(w_t)_{t \geq 0}$ is a Wiener process in \mathbb{R}^d , one wants to minimize the expected volume of the set $\cup_{t \leq T} (w_t + A)$, for $T \geq 0$, over “all” subsets A of \mathbb{R}^d of a given measure. It was proved in [15] that the minimizer is the ball (the result was for a more general setting, see Sect. 2 below). This was proved by obtaining a similar result for random walks by using rearrangement inequalities of Brascamp-Lieb-Luttinger type on the sphere, which were proved in [6], and then by Donsker’s theorem, the authors obtain the result for the Wiener process. It is known that the expected volume of the Wiener sausage up to time t , can be expressed as the integral over $x \in \mathbb{R}^d$ of the probability that a Wiener process starting from $x \in \mathbb{R}^d$ hits the set A by time t . It is also known that this collection of probabilities, as a function of (t, x) , satisfies a parabolic equation on $(0, T) \times \mathbb{R}^d \setminus A$. For properties of these hitting times and applications to the Wiener sausage we refer the reader to [3] and references therein, and for the case of Riemannian manifolds, we refer to [11]. Therefore, we provide an alternative proof of the isoperimetric inequality for the Wiener sausage, based on PDE techniques.

Comparison results between solutions of partial differential equations and solutions of their symmetrized counterparts, were first proved in [16]. Since then, much work has been done in this area, for elliptic and parabolic equations, and we refer the reader to [2, 4, 13, 14] and references therein. The equations under consideration at these works, are on a bounded domain, with Dirichlet or Neumann boundary conditions. Our approach is based on the techniques introduced in [4].

Let us now introduce some notation that will be frequently used throughout the paper. We denote by \mathbb{R}^d the Euclidean space of dimension $1 \leq d < \infty$. For A, B subsets of \mathbb{R}^d , we write

$$A + B := \left\{ z \in \mathbb{R}^d \mid z = x + y, x \in A, y \in B \right\},$$

and for $x \in \mathbb{R}^d$ we write $x + A := \{x\} + A$. The open centered ball of radius $\rho > 0$ in \mathbb{R}^d will be denoted by B_ρ . Let $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$ and let H be a closed half-space. If A is measurable, $|A|$ will stand for the Lebesgue measure of A . We will write $\sigma_H(x)$ and A_H for the reflections of x and A respectively, with respect to the shifted hyperplane ∂H . We will write \bar{A} and \underline{A} for the closure and the interior of A respectively. We will use the notation $P_H A$ for the polarization of A with respect to H , that is

$$P_H A := \left((A \cup A_H) \cap H \right) \cup \left(A \cap A_H \right).$$

For a non-negative function u on \mathbb{R}^d we will write $P_H u$ for the polarization of u with respect to H , that is

$$P_H u(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{if } x \in H \\ \min\{u(x), u(\sigma_H(x))\}, & \text{if } x \in H^c. \end{cases}$$

We will denote by \mathcal{H} the set of all half-spaces H such that $0 \in H$. For positive functions f and g on \mathbb{R}^d and for $H \in \mathcal{H}$, we will write $f \triangleleft_H g$, if $f(x) + f(\sigma_H(x)) \leq$

$g(x) + g(\sigma_H(x))$ for a.e. $x \in H$. For a bounded set $V \subset \mathbb{R}^d$, we will denote by V^* the closed, centered ball of volume $|V|$. For a positive function u on \mathbb{R}^d such that $|\{u > r\}| < \infty$ for all $r > 0$, we denote by u^* its symmetric decreasing rearrangement. For an open set $D \subset \mathbb{R}^d$ we denote by $H^1(D)$ the space of all functions in $u \in L_2(D)$ whose distributional derivatives $\partial_i u := \frac{\partial}{\partial x_i} u, i = 1, \dots, d$, lie in $L_2(D)$, equipped with the norm

$$\|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \sum_{i=1}^d \|\partial_i u\|_{L_2}^2.$$

We will write $H_0^1(D)$ for the closure of $C_c^\infty(D)$ (the space of smooth, compactly supported real functions on D) in $H^1(D)$. We will write $\mathbb{H}^1(D)$, and $\mathbb{H}_0^1(D)$ for $L_2((0, T); H^1(D))$, and $L_2((0, T); H_0^1(D))$ respectively. Also we define, $\mathcal{H}^1(D) := \mathbb{H}^1(D) \cap C([0, T]; L_2(D))$ and $\mathcal{H}_0^1(D) := \mathbb{H}_0^1(D) \cap C([0, T]; L_2(D))$. The notation (\cdot, \cdot) , will be used for the inner product in $L_2(\mathbb{R}^d)$. Also, the summation convention with respect to integer valued repeated indices will be in use.

The rest of the article is organized as follows. In Sect. 2 we state our main results. In Sect. 3 we prove a version of the parabolic maximum principle, and some continuity properties of the solution map with respect to the set A . These tools are then used in Sect. 4 in order to prove the main theorems.

2 Main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a standard Wiener process $(w_t)_{t \geq 0}$ with values in \mathbb{R}^d , and let A be compact subset of \mathbb{R}^d . For $T \geq 0$ we let us consider the expected volume of the Wiener sausage generated by A , that is, the quantity $\mathbb{E} |\cup_{t \leq T} (w_t + A)|$. In [15], the following theorem is proved.

Theorem 1 *For any $T \geq 0$ we have*

$$\mathbb{E} |\cup_{t \leq T} (w_t + A^*)| \leq \mathbb{E} |\cup_{t \leq T} (w_t + A)|. \tag{1}$$

The result in [15] is stated for open sets A , and the set A is allowed to depend on time. As it was mentioned above, this was proved by obtaining a similar inequality for random walks, using rearrangement inequalities of Brascamp-Lieb-Luttinger type on the sphere, which were proved in [6], and then by using Donsker’s theorem, the authors obtain the inequality for the Wiener process.

Let us now move to our main result, and see the connection with Theorem 1. For a compact set $A \subset \mathbb{R}^d$, and for $\psi \in L_2(\mathbb{R}^d \setminus A)$, let us denote by $\Pi(A, \psi)$ the problem

$$\begin{cases} dv_t = \frac{1}{2} \Delta v_t \, dt \text{ in } (0, T) \times \mathbb{R}^d \setminus A; \\ v_t(x) = 1 \quad \text{on } [0, T] \times \partial A; \\ v_0(x) = \psi(x) \text{ in } \mathbb{R}^d \setminus A \end{cases} \tag{2}$$

Definition 1 We will say that u is a solution of the problem $\Pi(A, \psi)$ if

- (i) $u \in \mathcal{H}^1(\mathbb{R}^d \setminus A)$,
- (ii) for each $\phi \in C_c^\infty(\mathbb{R}^d \setminus A)$,

$$(u_t, \phi) = (\psi, \phi) - \int_0^t \frac{1}{2} (\partial_i u_s, \partial_i \phi) ds,$$

for all $t \in [0, T]$

- (iii) $v - \xi \in \mathbb{H}_0^1(\mathbb{R}^d \setminus A)$, for any $\xi \in H_0^1(\mathbb{R}^d)$ with $\xi = 1$ on a compact set A' , $A \subset A'$.

The following is very well known.

Theorem 2 *There exists a unique solution of the problem $\Pi(A, \psi)$.*

If $\psi \in L_2(\mathbb{R}^d)$, then by $\Pi(A, \psi)$ we obviously mean $\Pi(A, \psi|_{\mathbb{R}^d \setminus A})$. Our two main results read as follows.

Theorem 3 *Let $\psi \in L_2(\mathbb{R}^d)$ with $0 \leq \psi \leq 1$, $\psi = 1$ on A . Let u, v be the solutions of the problems $\Pi(A, \psi)$ and $\Pi(P_H A, P_H \psi)$, extended to 1 on A and $P_H A$ respectively. Then for all $t \in [0, T]$, we have $v_t \triangleleft_H u_t$.*

Theorem 4 *Let $\psi \in L_2(\mathbb{R}^d)$ with $0 \leq \psi \leq 1$, and $\psi = 1$ on A . Suppose that $|A| > 0$. Let u, v be the solutions of the problems $\Pi(A, \psi)$ and $\Pi(A^*, \psi^*)$ respectively. Then for any $t \in [0, T]$ we have*

$$\int_{\mathbb{R}^d} v_t dx \leq \int_{\mathbb{R}^d} u_t dx, \tag{3}$$

where u_t and v_t are extended to 1 on A and A^* respectively.

It is easy to check that

$$\mathbb{E} |\cup_{t \leq T} (w_t + A)| = \int_{\mathbb{R}^d} \mathbb{P}(\tau_A^x \leq t) dx.$$

where

$$\tau_A^x := \inf\{t \geq 0 : x + w_t \in A\}.$$

It is also known that the unique solution of the problem $\Pi(A, 0)$ is given by

$$u_t(x) = \mathbb{P}(\tau_A^x \leq t). \tag{4}$$

Consequently Theorem 1 follows by Theorem 4 by choosing $\psi = 0$ on $\mathbb{R}^d \setminus A$, if $|A| > 0$. If $|A| = 0$ then (1) trivially holds.

Remark 1 All of the arguments in the next sections can be repeated in exactly the same way, if the operator $\frac{1}{2}\Delta$ is replaced by an operator of the form $L_t u := \partial_i(a_t^{ij}\partial_j u)$, such that for $j, i \in \{1, \dots, d\}$, $a^{ij} \in L_\infty((0, T))$ (independent of $x \in \mathbb{R}^d$), and there exists a constant $\kappa > 0$ such that for almost all $t \in [0, T]$,

$$a_t^{ij} z_i z_j \geq \kappa |z|^2, \quad (5)$$

for all $z = (z_1, \dots, z_d) \in \mathbb{R}^d$. Consequently one can replace w_t in Theorem 1 by “non-degenerate” stochastic integrals of the form $y_t = \int_0^t \sigma_s dB_s$ where B_t is an m -dimensional Wiener process and σ is a measurable function from $[0, T]$ to the set of $d \times m$ matrices such that $(\sigma_t \sigma_t^\top)_{i,j=1}^d$ satisfies (5).

3 Auxiliary results

In this section we prove some tools that we will need in order to obtain the proof of our main theorems. Namely, we present a version of the parabolic maximum principle for functions that are not necessarily continuous up to the parabolic boundary. This result (Lemma 1 below) is probably well known but we provide a proof for the convenience of the reader. The maximum principle is the main tool used in order to show the comparison of the solution of the problem $\Pi(A, \psi)$ and its polarized version. The reason that we need this version of the maximum principle is that, $P_H A$ is not guaranteed to have any “good” properties, even if ∂A is of class C^∞ , and therefore one can not expect the solution of $\Pi(P_H A, P_H \psi)$ to be continuous up to the boundary. We also present certain continuity properties of the solution map with respect to the set A , so that we can then iterate Theorem 3 in order to obtain Theorem 4.

In this section we consider $a^{ij} \in L_\infty((0, T) \times \mathbb{R}^d)$ for $i, j = 1, \dots, d$, and we assume that there exists a constant $\kappa > 0$ such that for any $z = (z_1, \dots, z_d) \in \mathbb{R}^d$ we have

$$a_t^{ij}(x) z_i z_j \geq \kappa |z|^2,$$

for a.e. $(t, x) \in [0, T] \times \mathbb{R}^d$. We will denote by $K := \max_{i,j} \|a^{ij}\|_{L_\infty}$. For an open set $Q \subset \mathbb{R}^d$, let $\Psi(Q)$ be the set of functions $u \in \mathcal{H}^1(Q)$, such that for any $\phi \in C_c^\infty(Q)$

$$(u_t, \phi) = (u_0, \phi) - \int_0^t (a_s^{ij} \partial_i u_s, \partial_i \phi) ds, \quad (6)$$

for all $t \in [0, T]$. Notice that by the De Giorgi-Moser-Nash theorem, if $u \in \Psi(Q)$, then $u \in C((0, T) \times Q)$.

Let us also introduce the functions $\alpha_r(s)$, $\beta_r(s)$ and $\gamma_r(s)$ on \mathbb{R} , for $r > 0$, that will be needed in the next lemma, given by

$$\gamma_r(s) = \begin{cases} 2 & \text{if } s > r \\ \frac{2s}{r} & \text{if } 0 \leq s \leq r \\ 0 & \text{if } s < 0, \end{cases}$$

$$\beta_r(s) = \int_0^s \gamma_r(t) dt, \quad \alpha_r(s) = \int_0^s \beta(t) dt.$$

For all $s \in \mathbb{R}$ we have $\gamma_r(s) \rightarrow 2I_{s>0}$, $\beta_r(s) \rightarrow 2s_+$ and $\alpha_r(s) \rightarrow (s_+)^2$ as $r \rightarrow 0$. Also, for all $s \in \mathbb{R}$ and $r > 0$, the following inequalities hold

$$|\gamma_r(s)| \leq 2, \quad |\beta_r(s)| \leq 2|s|, \quad |\alpha_r(s)| \leq s^2.$$

Lemma 1 *Let Q be a bounded open set and let $u \in \Psi(Q)$. If there exists $M \in \mathbb{R}$, such that $u_0(x) \leq M$ for a.e. $x \in Q$ and $\limsup_{(t,x) \rightarrow (t_0,x_0)} u_t(x) \leq M$ for any $(t_0, x_0) \in (0, T] \times \partial Q$, then*

$$\sup_{t \in [0, T]} \sup_Q u_t(x) \leq M.$$

Proof Let us fix $t' \in (0, T)$, and let $\zeta \in C_c^\infty(B_1)$ be a positive function with unit integral. For $\varepsilon > 0$ and $\delta > 0$, set $\zeta^\varepsilon(x) = \varepsilon^{-1}\zeta(x/\varepsilon)$ and $M^\delta := M + \delta$. For $x \in Q^\varepsilon := \{x \in Q | \text{dist}(x, \partial Q) > \varepsilon\}$, we can plug $\zeta^\varepsilon(x - \cdot)$ in (6) in place of ϕ to obtain

$$u_t^\varepsilon(x) - M^\delta = u_{t'}^\varepsilon(x) - M^\delta + \int_{t'}^t (a_s^{ij} \partial_j u_s, \partial_i \zeta^\varepsilon(x - \cdot)) ds,$$

for all $t \in [t', T]$, where $u^\varepsilon = u * \zeta^\varepsilon$. Let also $g^n \in C_c^\infty(Q)$ with $0 \leq g^n \leq 1$, $g^n = 1$ on $Q^{1/n}$, $g^n = 0$ on $Q \setminus Q^{1/2n}$ and choose $\varepsilon < 1/2n$. We can then multiply the equation with g^n , and by the chain rule we have

$$\begin{aligned} \int_Q \alpha_r((u_t^\varepsilon - M^\delta)g_n) dx &= \int_Q \alpha_r((u_{t'}^\varepsilon - M^\delta)g_n) dx \\ &\quad - \int_{t'}^t \int_Q (a_s^{ij} \partial_j u_s)^\varepsilon \partial_i (g_n \beta_r((u_s^\varepsilon - M^\delta)g_n)) dx ds. \end{aligned}$$

By standard arguments (see e.g. [8]), letting $\varepsilon \rightarrow 0$, leads to

$$\begin{aligned} \int_Q \alpha_r((u_t - M^\delta)g_n) dx &= \int_Q \alpha_r((u_{t'} - M^\delta)g_n) dx \\ &\quad - \int_{t'}^t \int_Q g_n^2 a_s^{ij} \partial_j u_s \gamma_r(u_s - M^\delta) \partial_i (u_s - M^\delta) dx ds \end{aligned}$$

$$\begin{aligned}
 & - \int_{t'}^t \int_Q a_s^{ij} \partial_j u_s \partial_i g_n \beta_r ((u_s - M^\delta) g_n) dx ds \\
 & - \int_{t'}^t \int_Q a_s^{ij} \partial_j u_s \gamma_r ((u_s - M^\delta) g_n) (u_s - M^\delta) g_n \partial_i g_n dx ds
 \end{aligned} \tag{7}$$

Let us also introduce the notation

$$U_t^\delta = \{x \in Q \mid u_t(x) > M + \delta\}.$$

We claim that there exists $\rho > 0$ such that $\text{dist}(\overline{U_t^\delta}, \partial Q) > \rho$ for any $t \in [t', T]$. For each $t \in [t', T]$, we have $\overline{U_t^\delta} \subset Q \cup \partial Q$. If $\inf_{t \in [t', T]} \text{dist}(\overline{U_t^\delta}, \partial Q) = 0$, then we can find $(s, y) \in [t', T] \times \partial Q$, and a sequence $(t_n, x_n) \in [t', T] \times U_{t_n}^\delta$ such that $(t_n, x_n) \rightarrow (s, y)$ as $n \rightarrow \infty$. Then we have by the definition of $U_{t_n}^\delta$,

$$\limsup_{(x_n, t_n) \rightarrow (s, y)} u_{t_n}(x_n) \geq M + \delta,$$

while by assumption we have that

$$\limsup_{(x_n, t_n) \rightarrow (s, y)} u_{t_n}(x_n) \leq M,$$

which is a contradiction, and therefore

$$\inf_{t \in [t', T]} \text{dist}(\overline{U_t^\delta}, \partial Q) = \theta > 0. \tag{8}$$

Going back to (7), for any $n > 1/\theta$, we have that for all $s \in [t', T]$

$$\int_Q \partial_i u_s \partial_i g_n \beta_r ((u_s - M^\delta) g_n) dx = \int_{U_s^\delta} \partial_i u_s \partial_i g_n \beta_r ((u_s - M^\delta) g_n) dx = 0,$$

since $\partial_i g_n = 0$ on $Q^{1/n}$ and $U_s^\delta \subset Q^{1/n}$ by (8). Similarly for the last term on the right hand side of (7). Therefore, letting $n \rightarrow \infty$ and $r \rightarrow 0$ in (7) gives

$$\begin{aligned}
 \|(u_t - M^\delta)_+\|_{L_2(Q)}^2 &= \|(u_{t'} - M^\delta)_+\|_{L_2(Q)}^2 - \int_{t'}^t \int_Q a_s^{ij} \partial_i u_s \partial_j u_s I_{u_s > M^\delta} dx ds \\
 &\leq \|(u_{t'} - M^\delta)_+\|_{L_2(Q)}^2.
 \end{aligned}$$

The above inequality holds for any $t' \in (0, T]$, and therefore by letting $t' \downarrow 0$ and using the continuity of u (in $L_2(Q)$) we have

$$\|(u_t - M^\delta)_+\|_{L_2(Q)}^2 \leq \|(u_0 - M^\delta)_+\|_{L_2(Q)}^2 \leq 0,$$

since $u_0 \leq M$. Consequently

$$\sup_Q u_t(x) \leq M + \delta,$$

for any $t \in [0, T]$. Since δ was arbitrary, the lemma is proved. □

We now continue with the continuity properties of the solution map. Let us fix $\xi \in L_2(\mathbb{R}^d)$ and $f \in L_2([0, T] \times \mathbb{R}^d)$. We will say that u solves the problem $\Pi_0(A, \xi, f)$ if

- (i) $u \in \mathcal{H}_0^1(\mathbb{R}^d \setminus A)$, and
- (ii) for each $\phi \in C_c^\infty(\mathbb{R}^d \setminus A)$,

$$(u_t, \phi) = (\xi, \phi) + \int_0^t \left((f_s, \phi) - (a_s^{ij} \partial_i u_s, \partial_j \phi) \right) ds,$$

for all $t \in [0, T]$.

For $n \in \mathbb{N}$, let $\xi^n \in L_2(\mathbb{R}^d)$, $f^n \in L_2([0, T] \times \mathbb{R}^d)$ and let $A_n \subset \mathbb{R}^d$ be compact sets.

Assumption 1

- (i) $\xi^n \rightarrow \xi$ weakly in $L_2(\mathbb{R}^d)$
- (ii) $f^n \rightarrow f$ weakly in $L_2([0, T]; L_2(\mathbb{R}^d))$
- (iii) $A_{n+1} \subset A_n$ For each $n \in \mathbb{N}$, and $\bigcap_n A_n = A$.

Lemma 2 *Suppose Assumption 1 holds, and let u^n and u be the solutions of the problems $\Pi_0(A_n, \xi^n, f^n)$ and $\Pi_0(A, \xi, f)$ respectively. Let us extend u^n and u to zero on A_n and A respectively. Then*

- (i) $u^n \rightarrow u$ weakly in $\mathbb{H}_0^1(\mathbb{R}^d)$ as $n \rightarrow \infty$,
- (ii) $u_t^n \rightarrow u_t$, weakly in $L_2(\mathbb{R}^d)$ as $n \rightarrow \infty$, for any $t \in [0, T]$.

Proof Let us set $C_n = \mathbb{R}^d \setminus A_n$ and $C = \mathbb{R}^d \setminus A$. Clearly, for (i) it suffices to show that there exists a subsequence with u^{n_k} such that $u^{n_k} \rightarrow u$ weakly in $\mathbb{H}_0^1(C)$. By standard estimates we have that there exists a constant N depending only on d, K, κ , and T , such that for all n

$$\sup_{t \leq T} \|u_t^n\|_{L_2(C_n)}^2 + \int_0^T \|u_t^n\|_{H_0^1(C_n)}^2 dt \leq N \left(\|\xi^n\|_{L_2(C_n)}^2 + \int_0^T \|f_t^n\|_{L_2(C_n)}^2 \right). \tag{9}$$

Since u^n are zero on A_n , we can replace C_n by C in the above inequality, to obtain that there exists a subsequence $(u^{n_k})_{k=1}^\infty \subset \mathbb{H}_0^1(C)$, and a function $v \in \mathbb{H}_0^1(C)$ such that $u^{n_k} \rightarrow v$ weakly in $\mathbb{H}_0^1(C)$.

For $\phi \in C_c^\infty(\mathbb{R}^d \setminus A)$ we have that for all k large enough $\text{supp}(\phi) \subset C_{n_k}$. Also, u^{n_k} solves $\Pi_0(A_{n_k}, \xi^{n_k}, f^{n_k})$, and therefore

$$(u_t^{n_k}, \phi) = (\xi^{n_k}, \phi) + \int_0^t \left((f_s^{n_k}, \phi) - (a_s^{ij} \partial_i u_s^{n_k}, \partial_j \phi) \right) ds \text{ for all } t \in [0, T]. \tag{10}$$

which by letting $k \rightarrow \infty$ gives

$$(v_t, \phi) = (\xi, \phi) + \int_0^t \left((f_s, \phi) - (a_s^{ij} \partial_i v_s, \partial_j \phi) \right) ds \text{ for a.e. } t \in [0, T], \quad (11)$$

which also holds for any $\phi \in H_0^1(C)$, since $C_c^\infty(C)$ is dense in the latter. Hence v belongs to the space $\mathcal{H}_0^1(D)$ (by Theorem 2.16 in [12] for example), and is a solution of $\Pi_0(A, \xi, f)$. By the uniqueness of the solution we get $u = v$ (as elements of $\mathcal{H}_0^1(C)$), and this proves (i).

Let us fix $t \in [0, T]$. It suffices to show that there exists a subsequence $u_t^{n_k}$ such that $u_t^{n_k} \rightarrow u_t$ weakly in $L_2(C)$ as $k \rightarrow \infty$. Notice that by (9), there exists a subsequence $u_t^{n_k}$ which converges weakly to some $v' \in L_2(C)$. Again, for $\phi \in C_c^\infty(C)$ and k large enough, we have that (10) holds. As $k \rightarrow \infty$, the right hand side of (10) converges to the right hand side of (11) (for our fixed $t \in [0, T]$), which is equal to (u_t, ϕ) , while the left hand side of (11) converges to (v', ϕ) . Hence, $v' = u_t$ on C , and since $u_t^{n_k}$ converges weakly in $L_2(C)$ to v' , the lemma is proved. \square

Corollary 1 *Suppose that (i) and (iii) from Assumption 1 hold, and let u^n and u be the solutions of the problems $\Pi(A_n, \psi^n)$ and $\Pi(A, \psi)$. Set $u^n = 1$ and $u = 1$ on A_n and A respectively. Then for each t , $u_t^n \rightarrow u_t$ weakly in $L_2(\mathbb{R}^d)$ as $n \rightarrow \infty$.*

Proof Let $g \in C_c^\infty(\mathbb{R}^d)$ with $g = 1$ on a compact set B such that $A_0 \subset B$. Then $u^n - g$ and $u - g$ solve the problems $\Pi_0(A_n, \psi^n - g, -\frac{1}{2} \Delta g)$ and $\Pi_0(A, \psi - g, -\frac{1}{2} \Delta g)$ and the result follows by Lemma 2. \square

For two subsets of \mathbb{R}^d , A_1 and A_2 , we denote by $d(A_1, A_2)$ the Hausdorff distance, that is

$$d(A_1, A_2) = \inf \{ \rho \geq 0 \mid A_1 \subset (A_2 + \overline{B}_\rho), A_2 \subset (A_1 + \overline{B}_\rho) \}.$$

In Lemma 3 below we will need the following:

Remark 2 Let $A \subset \mathbb{R}^d$ be compact such that $\mathbb{R}^d \setminus A$ is a Carathéodory set (i.e. $\partial(\mathbb{R}^d \setminus A) = \partial(\mathbb{R}^d \setminus A)$). If $u \in H^1(\mathbb{R}^d)$ and $u = 0$ a.e. on A , then $u \in H_0^1(\mathbb{R}^d \setminus A)$. To see this, suppose first that $\text{supp}(u) \subset B_R$, where R is large enough, so that $A \subset B_R$. It follows that $B_R \setminus A$ is a Carathéodory set, and by Theorem 7.3(ii), p. 436 in [9], if $u \in H_0^1(B_R)$, and $u = 0$ a.e. on A , then $u \in H_0^1(B_R \setminus A)$, and therefore $u \in H_0^1(\mathbb{R}^d \setminus A)$. For general u we can take $\zeta \in C_c^\infty(\mathbb{R}^d)$, such that $0 \leq \zeta \leq 1$ and $\zeta(x) = 1$ for $|x| \leq 1$, and set $\zeta^n(x) = \zeta(x/n)$. Then by the previous discussion $\zeta^n u \in H_0^1(\mathbb{R}^d \setminus A)$ and since $\zeta^n u \rightarrow u$ in $H^1(\mathbb{R}^d \setminus A)$ we get that $u \in H_0^1(\mathbb{R}^d \setminus A)$.

Assumption 2

- (i) $\xi^n \rightarrow \xi$ weakly in $L_2(\mathbb{R}^d)$
- (ii) $f^n \rightarrow f$ weakly in $L_2([0, T]; L_2(\mathbb{R}^d))$
- (iii) $d(A, A_n) \rightarrow 0, |A \setminus A_n| \rightarrow 0$, as $n \rightarrow \infty$, and $\mathbb{R}^d \setminus A$ is a Carathéodory set.

Lemma 3 *Suppose Assumption 2 holds, and let u^n and u be the solutions of the problems $\Pi_0(A_n, \xi^n, f^n)$ and $\Pi_0(A, \xi, f)$. Let us extend u^n and u to 0 on A_n and A respectively. Then*

- (i) $u^n \rightarrow u$ weakly in $\mathbb{H}_0^1(\mathbb{R}^d)$,
- (ii) $u_t^n \rightarrow u_t$ weakly in $L_2(\mathbb{R}^d)$, as $n \rightarrow \infty$, for any $t \in [0, T]$.

Proof As in the proof of Lemma 2 it suffices to find a subsequences such that the corresponding convergences take place. By standard estimates, there exists a constant N depending only on d, κ, T and K , such that for all $n \in \mathbb{N}$

$$\sup_{t \leq T} \|u_t^n\|_{L_2(\mathbb{R}^d)}^2 + \int_0^T \|u_t^n\|_{H_0^1(\mathbb{R}^d)}^2 dt \leq N \left(\|\xi^n\|_{L_2(\mathbb{R}^d)}^2 + \int_0^T \|f_t^n\|_{L_2(\mathbb{R}^d)}^2 \right). \tag{12}$$

Therefore, there exists a subsequence $(u^{n_k})_{k=1}^\infty \subset \mathbb{H}_0^1(\mathbb{R}^d)$, and a function $v \in \mathbb{H}_0^1(\mathbb{R}^d)$ such that $u^{n_k} \rightarrow v$ weakly in $\mathbb{H}_0^1(\mathbb{R}^d)$.

For $\phi \in C_c^\infty(\mathbb{R}^d \setminus A)$, since $d(A, A_n) \rightarrow 0$ as $n \rightarrow \infty$, we have that for all k large enough $\text{supp}(\phi) \subset \mathbb{R}^d \setminus A_{n_k}$. Also, u^{n_k} solves $\Pi_0(A_{n_k}, \xi^{n_k}, f^{n_k})$, and therefore

$$(u_t^{n_k}, \phi) = (\xi^{n_k}, \phi) + \int_0^t \left((f_s^{n_k}, \phi) - (a_s^{ij} \partial_i u_s^{n_k}, \partial_j \phi) \right) ds, \text{ for all } t \in [0, T]. \tag{13}$$

which by letting $k \rightarrow \infty$ gives

$$(v_t, \phi) = (\xi, \phi) + \int_0^t \left((f_s, \phi) - (a_s^{ij} \partial_i v_s, \partial_j \phi) \right) ds \text{ for a.e. } t \in [0, T], \tag{14}$$

Notice that for $\phi \in L_\infty(A), \psi \in L_\infty((0, T))$

$$\begin{aligned} \left| \int_0^T \int_A v_t \phi \psi_t dx dt \right| &= \lim_{k \rightarrow \infty} \left| \int_0^T \int_{A \setminus A_{n_k}} u_t^{n_k} \phi \psi_t dx dt \right| \\ &\leq T \|\phi\|_{L_\infty(A)} \|\psi\|_{L_\infty((0, T))} \lim_{k \rightarrow \infty} \sup_{t \leq T} \|u_t^{n_k}\|_{L_2(\mathbb{R}^d)} |A \setminus A_{n_k}|^{1/2} = 0, \end{aligned}$$

by assumption and (12). Consequently for almost all $t \in (0, T), v_t = 0$ for a.e. $x \in A$. By virtue of Remark 2, we have that $v \in \mathbb{H}_0^1(\mathbb{R}^d \setminus A)$, which combined with (14) implies that $v \in \mathcal{H}_0^1(\mathbb{R}^d \setminus A)$ and is the unique solution of the problem $\Pi_0(A, \xi, f)$. This proves (i).

Let us fix $t \in [0, T]$. By (12) there exists a subsequence $u_t^{n_k}$ that converges weakly to some $v' \in L_2(\mathbb{R}^d)$. Again, for $\phi \in C_c^\infty(C)$ and k large enough, we have that (13) holds. As $k \rightarrow \infty$, the right hand side of (13) converges to the right hand side of (14), which is equal to (u_t, ϕ) , while for our fixed t , the left hand side of (11) converges to (v', ϕ) . Hence, $v' = u_t$ on $\mathbb{R}^d \setminus A$. Also if $\phi \in L_\infty(A)$

$$\int_A v' \phi dx = \lim_{k \rightarrow \infty} \int_A u_t^{n_k} \phi dx \leq \|\phi\|_{L_\infty(A)} \lim_{k \rightarrow \infty} \|u_t^{n_k}\|_{L_2(\mathbb{R}^d)} |A \setminus A_{n_k}|^{1/2} = 0.$$

Therefore $v' = 0 = u_t$ on A . This shows that $v' = u_t$ on \mathbb{R}^d and the lemma is proved. \square

As with Lemma 2, we have the following corollary, whose proof is similar to the one of Corollary 1.

Corollary 2 *Suppose that (i) and (iii) from Assumption 2 hold and let u^n and u be the solutions of the problems $\Pi(A_n, \psi^n)$ and $\Pi(A, \psi)$. Set $u^n = 1$ and $u = 1$ on A_n and A respectively. Then for each t , $u_t^n \rightarrow u_t$ weakly in $L_2(\mathbb{R}^d)$ as $n \rightarrow \infty$.*

4 Proofs of Theorems 3 and 4

Proof of Theorem 3 Let us assume for now that $\mathbb{R}^d \setminus A$ has smooth boundary, ψ is compactly supported and smooth. It follows under these extra conditions that $u \in C^\infty([0, T] \times \overline{\mathbb{R}^d \setminus A})$. Also, by the De Giorgi-Moser-Nash theorem v is continuous in $(0, T) \times (\mathbb{R}^d \setminus P_H A)$.

First notice that $0 \leq u, v \leq 1$ (see the remark below). Let us extend $u = 1$ and $v = 1$ on A and $P_H A$ respectively so that they are defined on the whole \mathbb{R}^d , and for a function f let us use the notation $\bar{f}(x) := f(\sigma_H(x))$. Clearly it suffices to show that for each $t \in (0, T]$

$$w_t := v_t + \bar{v}_t - u_t - \bar{u}_t \leq 0, \text{ for a.e. } x \in H^c.$$

Suppose that the opposite holds, that is,

$$\sup_{(0, T]} \sup_{H^c} w_t(x) = \sup_{(0, T]} \sup_{\mathbb{R}^d} w_t(x) =: \alpha > 0.$$

Then we have that

$$\sup_{(0, T]} \sup_{\Gamma_i} w_t(x) = \alpha, \tag{15}$$

for some $i \in \{1, 2, 3, 4\}$, where

$$\begin{aligned} \Gamma_1 &:= A \cap A_H \cap H^c, & \Gamma_2 &:= (\underline{A} \setminus A_H) \cap H^c \\ \Gamma_3 &:= (\underline{A}_H \setminus A) \cap H^c, & \Gamma_4 &:= H^c \setminus (A \cup A_H). \end{aligned}$$

(Notice that the boundaries of A and A_H are of measure zero, since they are smooth). On Γ_1 , by definition $w_t = 0$ for any $t \in [0, T]$, and therefore (15) holds for some $i \in \{2, 3, 4\}$. Suppose it holds for $i = 2$. Since the initial conditions are compactly supported, we can find an open rectangle R with $A \cup A_H \subset R$, such that

$$\sup_{(0, T) \times \overline{R^c}} \max\{u_t(x), v_t(x)\} \leq \alpha/10. \tag{16}$$

Since $w_t = v_t - \bar{u}_t =: \hat{w}_t$ on Γ_2 we have

$$\sup_{(0,T]} \sup_{\Theta} \hat{w}_t \geq \alpha, \tag{17}$$

where $\Theta = (H^c \setminus A_H) \cap R$. Since

- (i) $\limsup_{(0,T] \times \Theta \ni (t,x) \rightarrow (t_0,x_0)} \hat{w}_t \leq 0$, for any $(t_0, x_0) \in (0, T) \times \partial A_H$,
- (ii) $P_H \psi - \bar{\psi} \leq 0$ on H^c ,
- (iii) inequality (16) holds,

we obtain by virtue of Lemma 1 that for any $\varepsilon > 0$, there exists $(t_0, x_0) \in (0, T) \times \partial H$ (in fact $x_0 \in \partial H \setminus A_H$ due to (i) above) such that

$$\limsup_{(0,T] \times \Theta \ni (t,x) \rightarrow (t_0,x_0)} \hat{w}_t \geq \alpha - \varepsilon.$$

Notice that \hat{w} is continuous at (t_0, x_0) and therefore $\hat{w}_{t_0}(x_0) \geq \alpha - \varepsilon$. This implies that

$$w_{t_0}(x_0) = 2\hat{w}_{t_0}(x_0) \geq 2(\alpha - \varepsilon) = 2 \sup_{(0,T]} \sup_{\mathbb{R}^d} w_t - 2\varepsilon,$$

which is a contradiction for ε small enough. If (15) holds for $i = 3$ then in the same way we have that

$$\sup_{(0,T]} \sup_{\Theta'} \tilde{w}_t \geq \alpha, \tag{18}$$

where $\tilde{w}_t = v_t - u_t$ and $\Theta' = (H^c \setminus A) \cap R$. This inequality leads to a similar contradiction.

Finally let us assume that (15) holds for $i = 4$. In particular then we have

$$\sup_{(0,T]} \sup_G w_t \geq \alpha, \text{ where } G := R \setminus (A \cup A_H).$$

By virtue of (16), and since $P_H \psi \triangleleft_H \psi$, Lemma 1 implies that for any $\varepsilon > 0$, there exists $(t_0, x_0) \in (0, T] \times \partial(A \cup A_H)$ such that

$$\limsup_{(0,T] \times G \ni (t,x) \rightarrow (t_0,x_0)} w_t(x) \geq \alpha - \varepsilon.$$

Notice that $x_0 \in \partial A \cap A_H^c$ or $x_0 \in \partial A_H \cap A^c$, because if it belongs to $\partial A \cap \partial A_H$ then the lim sup above is less than or equal to zero. Let us consider the first case. We can assume further that $(t_0, x_0) \in \partial A \cap A_H^c \cap H^c$ because of symmetry. Let $(t_n, x_n) \in G$ be a sequence converging to (t_0, x_0) such that $w_{t_n}(x_n) \rightarrow \alpha - \varepsilon$. For all $n \in \mathbb{N}$ sufficiently large, we have $w_{t_n}(x_n) \geq \alpha - 2\varepsilon$ and $u_{t_n}(x_n) \geq 1 - \varepsilon$, the last by the continuity of u up to the parabolic boundary. Then we have for all n large

$$\alpha - 2\varepsilon \leq w_{t_n}(x_n) \leq v_{t_n}(x_n) + 1 - (1 - \varepsilon) - \bar{u}_{t_n}(x_n).$$

This now implies (17) which we showed leads to a contradiction. For the second case, we can assume again that $(t_0, x_0) \in \partial A_H \cap A^c \cap H^c$. This in the same manner leads to (18), which also leads to contradiction.

For general A and ψ , let A_n be a sequence of compact sets such that for $n \in \mathbb{N}$, $\mathbb{R}^d \setminus A_n$ has smooth boundary, $A \subset A_{n+1} \subset A_n$, and $A = \bigcap_n A_n$ (see e.g. page 60 in [7]). Let $0 \leq \psi^n \leq 1$ be smooth with compact support such that $\psi^n = 1$ on A_n , and $\|\psi^n - \psi\|_{L_2(\mathbb{R}^d)} \rightarrow 0$ as $n \rightarrow \infty$. Then we also have that $\|P_H \psi^n - P_H \psi\|_{L_2(\mathbb{R}^d)} \rightarrow 0$ as $n \rightarrow \infty$, $P_H A \subset P_H A_{n+1} \subset P_H A_n$ for any $n \in \mathbb{N}$, and $P_H A = \bigcap_n P_H A_n$ (see [4]). Let u^n and v^n be the solutions of the problems $\Pi(A_n, \psi^n)$ and $\Pi(P_H A_n, P_H \psi^n)$ respectively. By Lemma 2 we have that u_t^n and v_t^n converge to u_t and v_t weakly in $L_2(\mathbb{R}^d)$. In particular $z^n := (u_t^n, v_t^n)$ converges weakly to $z := (v_t, u_t)$ in $L_2(\mathbb{R}^d; \mathbb{R}^2)$. By Mazur’s lemma there exists a sequence $(g_k = (g_k^1, g_k^2))_{k \in \mathbb{N}}$ of convex combinations of z^n such that the convergence takes place strongly. Then we can find a subsequence $g_{k(l)}, l \in \mathbb{N}$, where the convergence takes place for a.e. $x \in \mathbb{R}^d$. For each l we have

$$g_{k(l)}^1 + \overline{g_{k(l)}^1} = \sum_{i \in C} c_i (v_t^i + \overline{v_t^i}) \leq \sum_{i \in C} c_i (u_t^i + \overline{u_t^i}) = g_{k(l)}^2 + \overline{g_{k(l)}^2}$$

where $C \subset \mathbb{N}$ is a finite set and $c_i \geq 0, \sum_{i \in C} c_i = 1$. Letting $l \rightarrow \infty$ finishes the proof. □

Remark 3 With ψ and A as in Theorem 3, it is easily seen that if u solves $\Pi(A, \psi)$ then $0 \leq u \leq 1$, if for example $\mathbb{R}^d \setminus A$ has Lipschitz boundary. For general A , we can take A_n compact such that for all $n, \mathbb{R}^d \setminus A_n$ has smooth boundary, and $A_n \downarrow A$. Then the corresponding solutions u^n satisfy $0 \leq u^n \leq 1$, which by virtue of Corollary 1 implies that $0 \leq u \leq 1$.

Proof of of Theorem 4 First, let us assume that

$$\int_{\mathbb{R}^d} u_t(x) dx < \infty,$$

or else the conclusion of the theorem is obviously true. Since $|A| > 0$, it follows from [5] that there exist $H_i \in \mathcal{H}, i \in \mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} (\|\psi^* - \psi^n\|_{L_2(\mathbb{R}^d)} + |A^* \Delta A_n| + d(A^*, A_n)) = 0,$$

where

$$\psi^n := P_{H_n} \dots P_{H_1} \psi, A_n := P_{H_n} \dots P_{H_1} A.$$

Let u^n be the solution of the problem $\Pi(A_n, \psi^n)$. For $t \in [0, T]$, by virtue of Theorem 3, we have by induction

$$\int_{\mathbb{R}^d} u_t^n dx \leq \int_{\mathbb{R}^d} u_t dx, \tag{19}$$

for all $n \geq 0$. By Lemma 3 ($|A| > 0$ and therefore $\mathbb{R}^d \setminus A^*$ is obviously a Carathéodory set) we have that $u_t^n \rightarrow v_t$ weakly in $L_2(\mathbb{R}^d)$ as $n \rightarrow \infty$. Hence we can find a sequence of convex combination that converges strongly, and a subsequence of it, let us call it $(v^n)_{n=1}^\infty$, such that $v^n \rightarrow v_t$ for a.e. $x \in \mathbb{R}^d$. Since for each n , v^n is convex combination of elements from $(u_t^n)_{n=1}^\infty$, we have by (19)

$$\int_{\mathbb{R}^d} v^n dx \leq \int_{\mathbb{R}^d} u_t dx,$$

which combined with Fatou's lemma brings the proof to an end. \square

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