This is a repository copy of *Density symmetries for a class of 2-D diffusions with applications to finance*.

Article:

DENSITY SYMMETRIES FOR A CLASS OF 2-D DIFFUSIONS WITH APPLICATIONS TO FINANCE

KONSTANTINOS DAREIOTIS AND ERIK EKSTRÖM

ABSTRACT. We study densities of two-dimensional diffusion processes with one non-negative component. For such diffusions, the density may explode at the boundary, thus making a precise specification of the boundary condition in the corresponding forward Kolmogorov equation problematic. We overcome this by extending a classical symmetry result for densities of one-dimensional diffusions to our case, thereby reducing the study of forward equations with exploding boundary data to the study of a related backward equation with non-exploding boundary data. We also discuss applications of this symmetry for option pricing in stochastic volatility models and in stochastic short rate models.

1. INTRODUCTION

We study the distribution of a special class of diffusions of the form

$$\begin{cases} dY_t = \beta_1(Y_t)\,dt + \sigma_1(Y_t)\,dV_t \\ dZ_t = \beta_2(Y_t)\,dt + \sigma_2(Y_t)\,dW_t, \end{cases} \tag{1.1}$$

where β_i, σ_i, $i = 1, 2$ are given functions, and V and W are two one-dimensional Brownian motions. Furthermore, the coefficients are specified so that Y is a non-negative process. This class of decoupled systems includes some common stochastic volatility models (such as the Heston model) for derivative pricing, as well as stochastic short rate models (such as the CIR-model) for derivative pricing. Denoting by $X = (Y, Z)$, for a given initial condition $X_0 = \eta$ with density $\rho \in C_c^\infty((0, \infty) \times \mathbb{R})$, the density

$$p(t, x) = \frac{\mathbb{P}(X_t \in dx)}{dx}$$

is expected to satisfy the associated forward Kolmogorov equation

$$\partial_t p = L^* p,$$

initial data $p(0, x) = \rho(x)$, where L^* is the formal adjoint of the infinitesimal generator of X. However, for a characterization of the density in terms of the forward equation, boundary conditions at the spatial boundary $\{0\} \times \mathbb{R}$ are needed. Moreover, it is well-known that in many cases of practical importance, the density suffers from exploding boundary behaviour, thus introducing instabilities to any numerical scheme based on discretizing the forward equation. To overcome this, one approach would be to first determine the exact blow-up rate of the density, and then factor out this

from the equation to, hopefully, arrive at more well-behaved boundary conditions. This, however, requires knowledge about the exact blow-up rate of the density.

Our approach, instead, builds on the extension of a classical symmetry of the transition density for one-dimensional diffusion processes. In fact, the density

$$p(t, x_0, x) := \mathbb{P}(X_t \in dx)/dx$$

of a one-dimensional diffusion $X = X^{x_0}$ with $X_0 = x_0$ satisfies

$$\mu(x_0)p(t, x_0, x) = \mu(x)p(t, x, x_0), \tag{1.2}$$

where $\mu(x)$ is the density of the speed measure (see [12, Section 4.11]). Along with its theoretical interest, this symmetry also has important applications for numerical treatments of the density for non-negative processes. Indeed, if one seeks the density p of X_t, rather than solving the *forward* Kolmogorov equation in the x-variable, one may instead employ (1.2) to solve a *backward* equation. The advantage of this procedure is in the specification of boundary conditions, since the density may explode close to the boundary $x = 0$, whereas the appropriate boundary condition of the backward equation is much more well-behaved, compare [5] and [7].

To the best of our knowledge, extensions of the symmetry relation (1.2) to higher dimensions are still missing in the literature. In the present article we provide such a symmetry relation for systems of the form (1.1) under certain conditions on the coefficients, see Theorem 2.1. Moreover, the stochastic representation appearing in (2.6) can typically be characterized as the unique solution of a backward equation with well-behaved boundary conditions. For completeness, we also include a study of the associated backward equation. In fact, in Theorem 2.3 we demonstrate that the stochastic representation appearing in (2.6) can be characterized as the unique solution of an associated backward equation for a class of systems that finds applications in mathematical finance.

The symmetry relation in Theorem 2.1 is first proved for processes with the whole plane as state space by approximating the coefficients with smooth coefficients defined on the whole real line. For such problems, the symmetry relation (2.5) is derived using fairly standard methods involving integration by parts, compare Equation (3.9). To pass to the limit, we invoke an approximation result of [1] for diffusion processes with Hölder continuous coefficients, see Lemma 3.1. In our study of the corresponding backward equation, one of the main difficulties is in specifying the boundary conditions at the plane $y = 0$. First, to establish C^1-regularity of the stochastic solution of the equation up to the boundary we again approximate the problem with smooth coefficients on the whole plane and then take the limit using appropriate parabolic estimates, compare Proposition 4.5. Another key step is to show that the second order terms with at least one derivative in the y-direction explode slower than the reciprocal of the corresponding diffusion

coefficient, see Proposition 4.7. This is obtained by using a combination of parabolic estimates and suitable scaling arguments.

Finally let us introduce some notation that will be used throughout the article. Let $T \in (0, \infty)$ and let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with the filtration $\mathbb{F} := (\mathscr{F}_t)_{t \in [0,T]}$ satisfying the usual conditions. On Ω we consider two $\mathbb{F}$-Wiener processes $(V_t)_{t \in [0,T]}$ and $(W_t)_{t \in [0,T]}$ with correlation $\lambda \in (-1, 1)$. For random variables X, X_n, $n \in \mathbb{N}$, we will write

$$X_n \xrightarrow{\mathbb{P}} X$$

if $X_n \to X$ in probability as $n \to \infty$. Let d be a positive integer. For an open set $Q \subset \mathbb{R}^d$ and an integer $k \in \mathbb{N}$, $W_2^k(Q)$ will denote the set of all functions in $L_2(Q)$ having distributional derivatives up to order k in $L_2(Q)$. We will denote by $C_b^\infty(Q)$ the set off all smooth real-valued functions on Q that are bounded along with their derivatives of any order. If $Q \subset \mathbb{R}^d$ is open, we will denote by $C_c^\infty(Q)$ the set of all smooth functions with compact support in Q. We also set $\mathbb{W} := C([0,T]; L_2(\mathbb{R}^2)) \cap L_2([0,T]; W_2^1(\mathbb{R}^2))$ and $\mathcal{W} := C^\infty([0,T] \times \mathbb{R}^2) \cap \left(\cap_{m=1}^\infty C([0,T]; W_2^m(\mathbb{R}^2)) \right)$. The notation $(\cdot, \cdot)_{L_2}$ will stand for the inner product in $L_2(Q)$. If $x \in \mathbb{R}^2$, then x_1 and x_2 will denote the first and second coordinates of x with respect to the standard basis in $\mathbb{R}^2$. Finally, we set $D := (0, \infty) \times \mathbb{R}$.

2. FORMULATION OF THE MAIN RESULTS

We consider functions $\beta = (\beta_1, \beta_2) \colon [0, \infty) \to \mathbb{R}^2$ and $\sigma = (\sigma_1, \sigma_2) \colon [0, \infty) \to \mathbb{R}^2$. The system

$$\begin{cases} dY_t = \beta_1(Y_t)dt + \sigma_1(Y_t)\, dV_t \\ dZ_t = \beta_2(Y_t)dt + \sigma_2(Y_t)\, dW_t, \end{cases} \tag{2.1}$$

with initial condition $(Y_0, Z_0) = (\psi, \xi) = \eta$, where $\psi \geq 0$ and ξ are $\mathscr{F}_0$-measurable random variables, will be denoted by $\Pi(\eta; \beta, \sigma)$. We denote by $\lambda \in (-1, 1)$ the instantaneous correlation between V and W, and we set

$$h := \lambda \frac{\sigma_2}{\sigma_1}$$

and

$$a_{ij} := \lambda_{ij} \frac{\sigma_i \sigma_j}{2} \tag{2.2}$$

for $i, j = 1, 2$, where $\lambda_{ij} = \lambda$ for $i \neq j$ and $\lambda_{ij} = 1$ otherwise. Often, coefficients of SDEs of the type (2.1) (say f) will be regarded as functions on subsets of $\mathbb{R}^2$ by the formula $f(x) := f(x_1)$.

Assumption 2.1. The functions β and σ satisfy:

(i) $\beta_i, \sigma_i \in C([0, \infty))$. Moreover, for every $R > 0$ there exists $N_R \in \mathbb{R}$ such that
$$|\beta_1(r) - \beta_1(r')| + |\sigma_1(r) - \sigma_1(r')|^2 \leq N_R|r - r'| \text{ for all } r, r' \in [0, R].$$

(ii) $\beta_1(0) \geq 0$, $\sigma_1(0) = 0$, $\sigma_1(r) > 0$ for $r > 0$, and $\sigma_2(r) \geq 0$ for $r \geq 0$. Moreover, there exists a constant $N \in \mathbb{R}$ such that

$$|\beta_1(r)| + \sigma_1(r) \leq N(1+r)$$

for all $r \geq 0$.

(iii) $h \in C^1((0,\infty))$, $\beta_1 h$, $a_{11}h' \in C([0,\infty))$ and $a_{11}h'(0) = 0$.

In addition, there exist functions σ_1^n, $\sigma_2^n \in C_b^\infty(\mathbb{R})$ such that

(iv) $\sigma_i^n \to \sigma_i$ uniformly on compacts of $(0,\infty)$ as $n \to \infty$,

(v) $\sigma_i^n(r) \geq 1/n$ for all $r \in \mathbb{R}$, and there exists a constant N such that

$$\sup_n |\sigma_1^n(r)| \leq N(1 + |r|)$$

for all $r \in \mathbb{R}$,

(vi) $(\lambda \sigma_2^n / \sigma_1^n)' \to h'$ uniformly on compacts of $(0,\infty)$ as $n \to \infty$.

Remark 2.1. Notice that Assumption 2.1 is satisfied if for example (i), (ii) and one of the following hold:

1) $\lambda = 0$,
2) $\sigma_2 = c\sigma_1$ for some constant $c \in \mathbb{R}$,
3) $\sigma_1, \sigma_2 \in C^1((0,\infty))$, $a_{11}h', \beta_1 h \in C([0,\infty))$ and $a_{11}h'(0) = 0$.

This shows that the Heston model (in which $\sigma_2 = c\sigma_1$) is included in the analysis, compare Example 2.1 below. Similarly, Remark 2.3 discusses derivative pricing models with stochastic interest rate for which $\lambda = 0$.

Also notice that under (i) and the linear growth condition from (ii) of Assumption 2.1, there exists a unique solution $X := (Y, Z)$ of $\Pi(\eta; \beta, \sigma)$ (see, e.g., [11]). Moreover, due to the assumptions $\sigma_1(0) = 0$ and $\beta_1(0) \geq 0$, we have $Y_t \geq 0$ for all times $t \in [0, T]$.

For the statement of our main theorem, let $\mu \colon (0, \infty) \to \mathbb{R}$ be given by

$$\mu(r) = \frac{1}{a_{11}(r)} \exp\left(\int_1^r \frac{\beta_1(l)}{a_{11}(l)} dl\right). \tag{2.3}$$

We also introduce the function

$$\tilde{\beta}_2 := 2a_{11}h' + 2\beta_1 h + \beta_2, \tag{2.4}$$

and we set $\tilde{\beta} = (\beta_1, \tilde{\beta}_2)$. By Assumption 2.1 we have that $\tilde{\beta}_2 \in C([0, \infty))$.

Theorem 2.1. *Let Assumption 2.1 hold and let X be the unique solution of $\Pi(\eta; \beta, \sigma)$. Assume that η has a density $\rho \in C_c^\infty(D)$. Then for any $g \in C_c^\infty(D)$ we have*

$$\mathbb{E}[g(X_T)] = \int_D g(x)q(T, x)dx, \tag{2.5}$$

where for $x \in D$,

$$q(T, x) := \mu(x_1)\mathbb{E}\left[\frac{\rho(\tilde{X}_T^x)}{\mu(Y_T^{x_1})}\right] \tag{2.6}$$

and $\tilde{X}^x = (Y^{x_1}, \tilde{Z}^x)$ is the unique solution of $\Pi(x; \tilde{\beta}, \sigma)$. Consequently, the restriction of the law of X_T on D has a density given by $q(T, \cdot)$.

Corollary 2.2. (Symmetry of densities.) *For each $\xi \in D$, let $X^\xi = (Y^{\xi_1}, Z^\xi)$ and $\tilde{X}^\xi = (Y^{\xi_1}, \tilde{Z}^\xi)$ denote the unique solutions of $\Pi(\xi; \beta, \sigma)$ and $\Pi(\xi; \tilde{\beta}, \sigma)$, respectively. Suppose that the restriction of the laws of X_T^ξ and $\tilde{X}_T^\xi$ on D have densities $p(T, \xi, x)$ and $\tilde{p}(T, \xi, x)$, respectively, that are continuous in $(\xi, x) \in D \times D$. Then, for all $(\xi, x) \in D \times D$ we have*

$$\mu(\xi_1)p(T, \xi, x) = \mu(x_1)\tilde{p}(T, x, \xi).$$

Note that Theorem 2.1 transforms the problem of calculating a density with respect to the *forward* variables into a problem of solving a *backward* equation for a related process. Theorem 2.3 below provides the exact formulation of boundary conditions for backward equations corresponding to diffusions of the form (2.1); for related results, see [2] and [6].

Assumption 2.2. The functions $\sigma_i, \beta_i \colon [0, \infty) \to \mathbb{R}$ satisfy the following:
(1) There exists $N \in \mathbb{R}$ such that
$$|\sigma_1(r)| + |\beta_1(r)| \le N(1 + |r|),$$
for all $r \in [0, \infty)$.
(2) $\sigma_i(r) > 0$ for $r > 0$, $\sigma_1(0) = 0$, and $\beta_1(0) \ge 0$.
(3) $\sigma_i, \beta_i \in C^\infty((0, \infty)) \cap C([0, \infty))$.
(4) $\beta_i, a_{ii} \in C^1([0, \infty))$ with β_i', a_{22}' bounded, and a_{11}' is locally Lipschitz and has linear growth.
(5) Either $\lambda = 0$, or $a_{12} \in C^1([0, \infty))$ and there exists $N_0 \in (0, \infty)$ such that $\frac{1}{N_0}\sigma_2(r) \le \sigma_1(r) \le N_0\sigma_2(r)$ for all r sufficiently small.
(6) It holds that

$$\int_0^1 \left(\int_r^1 \exp\left(\int_1^s \frac{\beta_1(u)}{a_{11}(u)} \, du \right) ds \right) \exp\left(\int_r^1 \frac{\beta_1(s)}{a_{11}(s)} \, ds \right) \frac{1}{a_{11}(r)} \, dr = \infty.$$

As before, under Assumption 2.2, if $Y_0 = \psi \ge 0$ a.s., then (2.1) has a unique solution $X = (Y, Z)$, and $Y_t \ge 0$ a.s. for all $t \in [0, T]$. Let us introduce the differential operator L given by

$$L\phi(x) := \sum_{i,j} a_{ij}(x)\partial_{ij}\phi(x) + \sum_i \beta_i(x)\partial_i\phi(x), \tag{2.7}$$

and for a function $g \in C_c^\infty(D)$ let us consider the problem

$$\begin{cases} \partial_t u + Lu = 0 & \text{in } (0, T) \times D \\ u(T, x) = g(x) & \text{for } x \in D \\ \partial_t u + a_{22}\partial_{22}u + \sum_i \beta_i\partial_i u = 0 & \text{on } (0, T) \times \partial D. \end{cases} \tag{2.8}$$

Definition 2.1. A continuous function $u \colon [0, T] \times \overline{D} \to \mathbb{R}$, will be called a solution of equation (2.8) if $u \in C^{1,2}((0, T) \times D)$, $\partial_i u, \partial_{22} u, \partial_t u \in C((0, T) \times \overline{D})$, and the equalities in (2.8) are satisfied.

Theorem 2.3. *Let Assumption 2.2 hold and let X^x be the unique solution of* (2.1) *with initial condition $X_0 = x \in \overline{D}$. Then the function $u(t, x) = \mathbb{E}g(X^x_{T-t})$ is a solution of* (2.8). *Moreover, u is the unique solution of equation* (2.8) *in the class of functions of at most polynomial growth.*

Remark 2.2. Notice that in order to characterize the quantity $\mathbb{E}\left[\rho(\tilde{X}^x_T)/\mu(Y^{x_1}_T)\right]$ from Theorem 2.1 as a solution of a parabolic PDE, Theorem 2.3 should be applied with $\tilde{\beta}$ and ρ/μ in place of β and g, respectively.

Example 2.1. (The Heston stochastic volatility model) We illustrate Theorems 2.1-2.3 by considering the problem of calculating densities in stochastic volatility models. For that, assume that a stock price S is modelled by

$$dS_t = \sqrt{Y_t}S_t \, dW_t, \qquad S_0 = \zeta,$$

where the instantaneous variance Y is a CIR process given by

$$dY_t = (a - bY_t) \, dt + \sigma\sqrt{Y_t} \, dV_t, \qquad Y_0 = \eta_1.$$

Here V and W are two Brownian motions with correlation $\lambda \in (-1, 1)$, and $a \geq 0$, b and $\sigma > 0$ are constants. Notice that under the assumption that $a \geq 0$, Y stays non-negative but may hit zero (if $2a \leq \sigma^2$). In particular, we do not need to impose the usual, more strict, condition $2a > \sigma^2$. Introducing $Z_t := \ln S_t$ gives the system

$$\begin{cases} dY_t = (a - bY_t) \, dt + \sigma\sqrt{Y_t} \, dV_t, & Y_0 = \eta_1, \\ dZ_t = -(Y_t/2) \, dt + \sqrt{Y_t} \, dW_t, & Z_0 = \eta_2, \end{cases}$$

where we assume that $\eta = (\eta_1, \eta_2)$ has a smooth density ρ. The density

$$p(t, x) = \frac{\mathbb{P}(X_t \in dx)}{dx}$$

then satisfies the forward equation

$$\begin{cases} \partial_t p = L^* p & \text{on } (0, \infty) \times D \\ p(0, x) = \rho(x) & \text{for } x \in D, \end{cases}$$

where

$$L^* p = \frac{1}{2}\partial_{11}(\sigma^2 x_1 p) + \lambda\sigma\partial_{12}(x_1 p) + \frac{x_1}{2}\partial_{22}p - \partial_1((a - bx_1)p) + \frac{x_1}{2}\partial_2 p.$$

To calculate the density using the forward equation, however, is not straightforward since the boundary conditions at the boundary plane $\{x_1 = 0\}$ are not known (in fact, the density in the Heston model is known to explode for some parameter regimes, see the classical reference [9]). Instead, the symmetry relation in Theorem 2.1 may be used to translate the forward equation with boundary explosion into a backward equation with well-behaved boundary conditions.

More precisely, let

$$\mu(x_1) = \frac{2}{\sigma^2 x_1} \exp\left(\int_1^{x_1} \frac{2(a - bl)}{\sigma^2 l} dl\right),$$

and let u be the unique bounded solution (compare Theorem 2.3) of the backward equation

$$\begin{cases} \partial_t u + Lu = 0 & \text{on } (0, \infty) \times D \\ u(T, x) = \frac{\rho(x)}{\mu(x_1)} & \text{for } x \in D \\ \partial_t u + a\partial_1 u + \frac{\lambda a}{\sigma}\partial_2 u = 0 & \text{on } (0, T) \times \partial D, \end{cases}$$

where

$$Lu = \frac{\sigma^2 x_1}{2}\partial_{11}u + \lambda\sigma x_1\partial_{12}u + \frac{x_1}{2}\partial_{22}u + (a - bx_1)\partial_1 u + (\frac{2\lambda}{\sigma}(a - bx_1) - \frac{x_1}{2})\partial_2 u.$$

Then, by Theorem 2.1, the density is given by

$$p(t, x) = \mu(x_1)u(T - t, x).$$

Remark 2.3. Another situation in which the above methodology may be useful is in the case of derivative pricing models with stochastic interest rate. In fact, consider the system

$$\begin{cases} dY_t = \beta(Y_t)\,dt + \sigma(Y_t)\,dV_t \\ dZ_t = (Y_t - \frac{\nu^2}{2})\,dt + \nu dW_t, \end{cases}$$

with the interpretation that Y is a stochastic interest rate and Z is the log-price of a risky asset. To calculate option prices of the form

$$v = \mathbb{E}\left[\exp\left\{-\int_0^T Y_t\,dt\right\}g(Z_T)\right]$$

in this model, the density of the process Z, killed at the stochastic rate Y_t, is needed. This killed density satisfies a Kolmogorov forward equation; however, if Y is a non-negative diffusion (such as in the Cox-Ingersoll-Ross model, see [4]), density explosion is expected at the boundary $\{y = 0\}$. Theorems 2.1 and 2.3 can be modified (by adding zero-order terms in the equations) in order to cover also the case of derivative pricing models with stochastic interest rates. Note, however, that the specification of the volatility $\sigma_2 = \nu = constant$ suggests that the conditions of Assumptions 2.1 and 2.2 are only fulfilled in the case of uncorrelated Wiener processes. For ease of presentation, we refrain from including the extension of Theorems 2.1 and 2.3 to the case of killed processes.

3. Proofs of Theorem 2.1 and Corollary 2.2

For the proof of Theorem 2.1, we will need the following lemma which is a straightforward consequence of [1, Theorem 2.5].

Lemma 3.1. *For $n \in \mathbb{N}$, let $h^n = (h_1^n, h_2^n): \mathbb{R} \to \mathbb{R}^2$ and $f^n = (f_1^n, f_2^n): \mathbb{R} \to \mathbb{R}^2$ be continuous functions such that:*

1) *h_1^n is locally Lipschitz continuous and f_1^n is locally 1/2-Hölder continuous for each $n \in \mathbb{N}$,*
2) *there exists a constant K such that $|h_1^n(r)| + |f_1^n(r)| \le K(1 + |r|)$ for all $r \in \mathbb{R}$ and all $n \in \mathbb{N}$,*

3) f_i^n and h_i^n converge to f_i^0 and h_i^0, respectively, uniformly on compact subsets of $\mathbb{R}$, as $n \to \infty$.

Let $(x^n)_{n=0}^\infty \subset \mathbb{R}^2$, $(t^n)_{n=1}^\infty \subset [0, T]$ be sequences such that $\lim_{n\to\infty}(t^n, x^n) = (t^0, x^0)$, and for each $n \in \mathbb{N}$, let $X^n = (Y^n, Z^n)$ be the unique solution of $\Pi(x^n; h^n, f^n)$. Then we have the following:

(i) It holds that

$$\sup_{t \leq T} |X_t^n - X_t^0| \overset{\mathbb{P}}{\to} 0, \ \text{as } n \to \infty. \tag{3.1}$$

(ii) Let $g^n, \gamma^n : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions, bounded and bounded above respectively, uniformly in $n \in \mathbb{N}$, such that $g^n \to g^0$ and $\gamma^n \to \gamma^0$ uniformly on compact subsets of $\mathbb{R}^2$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \mathbb{E}\left[g^n(X_{t^n}^n) e^{\int_0^{t^n} \gamma^n(X_s^n)\,ds} \right] = \mathbb{E}\left[g^0(X_{t^0}^0) e^{\int_0^{t^0} \gamma^0(X_s^0)\,ds} \right]. \tag{3.2}$$

Proof. By [1, Theorem 2.5] we have $\lim_{n\to\infty} \mathbb{E}\sup_{t\leq T} |Y_t^n - Y_t^0|^2 = 0$, and consequently, $\sup_{t\leq T} |Y_t^n - Y_t^0| \overset{\mathbb{P}}{\to} 0$. Moreover, for a subsequence we have $\lim_{k\to\infty} \sup_{t\leq T} |Y_t^{n_k} - Y_t^0|^2 = 0$ almost surely. This, combined with 3) and the uniform continuity of h_2^0 and f_2^0 on compacts, imply that almost surely

$$\lim_{k\to\infty} \sup_{t\leq T} \left(|h_2^{n_k}(Y_t^{n_k}) - h_2^0(Y_t^0)| + |f_2^{n_k}(Y_t^{n_k}) - f_2^0(Y_t^0)| \right) = 0.$$

In particular, almost surely

$$\lim_{k\to\infty} \left(\int_0^T |h_2^{n_k}(Y_t^{n_k}) - h_2^0(Y_t^0)|\,dt + \int_0^T |f_2^{n_k}(Y_t^{n_k}) - f_2^0(Y_t^0)|^2\,dt \right) = 0,$$

which implies (see, e.g., [13, Theorem 5, p. 181])

$$\sup_{t\leq T} |Z_t^{n_k} - Z_t^0| \leq |z^{n_k} - z^0| + \int_0^T |h_2^{n_k}(Y_t^{n_k}) - h_2^0(Y_t^0)|\,dt$$

$$+ \sup_{t\leq T} \left| \int_0^t \left(f_2^{n_k}(Y_s^{n_k}) - f_2^0(Y_s^0) \right) dW_s \right| \overset{\mathbb{P}}{\to} 0$$

as $k \to \infty$. Moreover, for a further subsequence n_{k_l} the convergence takes place almost surely. Notice that any subsequences $(x^{n_k})_{k\in\mathbb{N}}$, $(h^{n_k})_{k\in\mathbb{N}}$ and $(f^{n_k})_{k\in\mathbb{N}}$ with $n_0 = 0$ satisfy the conditions of the lemma, so the convergence above is true along the whole sequence, that is

$$\sup_{t\leq T} |Z_t^n - Z_t^0| \overset{\mathbb{P}}{\to} 0,$$

as $n \to \infty$, which proves (3.1). The equality in (3.2) is a direct consequence of (3.1). $\qquad\square$

For the proof of Theorem 2.1, let $\vartheta \in C^\infty(\mathbb{R})$ such that $0 \leq \vartheta \leq 1$, $\vartheta(y) = 0$ for $y \leq 0$ and $\vartheta(y) = 1$ for $y \geq 1$. Also let

$$\varrho_m(y) = 1/m + \int_{1/m}^y \vartheta(mr)\,dr.$$

Remark 3.1. Notice that the function ϱ_m has the following properties:

(1) There exist $c_m > 0$ such that $\varrho_m(r) > c_m$ for all $r \in \mathbb{R}$,
(2) $\varrho_m(r) = r$ for $r \geq 1/m$,
(3) $0 \leq \varrho_m(r) \leq 1/m$ for $r \leq 1/m$.

Proof of Theorem 2.1. Let us extend the coefficients β and σ on $(-\infty, 0)$ by setting them identically equal to their value at 0. Let $\beta_i^n \in C_b^\infty(\mathbb{R})$ such that there exists a constant N with $\sup_n |\beta_1^n(r)| \leq N(1 + |r|)$ for all $r \in \mathbb{R}$, $\beta_1^n(r) = 0$ for $|r| \geq 2n$, and

$$\beta_i^n \to \beta_i \text{ uniformly on compacts of } \mathbb{R} \text{ as } n \to \infty. \tag{3.3}$$

Let σ_i^n be approximations of σ_i having the properties of Assumption 2.1, and let us set

$$\sigma_i^{n,m} := \sigma_i^n \circ \varrho_m, \quad \beta_1^{n,m} := \beta_1^n \circ \varrho_m, \quad \beta_2^{n,m} := \beta_2^n$$

and

$$a_{ij}^{n,m} = \lambda_{ij} \frac{\sigma_i^{n,m} \sigma_j^{n,m}}{2}, \qquad h^{n,m} = \lambda \frac{\sigma_2^{n,m}}{\sigma_1^{n,m}}.$$

We introduce the differential operators,

$$L_{n,m}\phi = \sum_{ij} a_{ij}^{n,m} \partial_{ij}\phi + \sum_i \beta_i^{n,m} \partial_i \phi$$

$$\tilde{L}_{n,m}\phi = \sum_{ij} a_{ij}^{n,m} \partial_{ij}\phi + \sum_i \tilde{\beta}_i^{n,m} \partial_i \phi$$

where $\tilde{\beta}_1^{n,m} = \beta_1^{n,m}$ and $\tilde{\beta}_2^{n,m}$ is defined similarly to (2.4) with β_i and σ_i replaced by $\beta_i^{n,m}$ and $\sigma_i^{n,m}$ respectively. We consider the equation

$$\begin{cases} \partial_t v(t,x) &= \tilde{L}_{n,m} v(t,x) \quad \text{on } (0,T) \times \mathbb{R}^2 \\ v(0,x) &= \frac{\rho(x)}{\mu_{n,m}(x_1)} \quad \text{on } \mathbb{R}^2, \end{cases} \tag{3.4}$$

where for $r \in \mathbb{R}$

$$\mu_{n,m}(r) := \frac{1}{a_{11}^{n,m}(r)} \exp\left(\int_1^r \frac{\beta_1^{n,m}(l)}{a_{11}^{n,m}(l)} \, dl \right).$$

Notice that $\rho/\mu_{n,m} \in C_c^\infty(\mathbb{R}^2)$ and that $\tilde{L}_{n,m}$ is strongly elliptic (due to (v) of Assumption 2.1), with coefficients of class $C_b^\infty(\mathbb{R}^2)$. Therefore, equation (3.4) has a unique solution $v_{n,m} \in \mathbb{W}$, which moreover belongs to $\mathcal{W}$. By the Feynman-Kac formula we have

$$v_{n,m}(t,x) = \mathbb{E}\left[\frac{\rho(\tilde{X}_t^{x;n,m})}{\mu_{n,m}(Y_t^{x_1;n,m})} \right], \tag{3.5}$$

where we have denoted by $\tilde{X}^{x;n,m} = (Y^{x_1;n,m}, \tilde{Z}^{x;n,m})$ the unique solution of $\Pi(x; \tilde{\beta}^{n,m}, \sigma^{n,m})$. Let us now set

$$q_{n,m}(t,x) = \mu_{n,m}(x_1) v_{n,m}(t,x).$$

Notice that since $\sigma_1^{n,m} \in C_b^\infty(\mathbb{R})$, $\sigma_1^{n,m} \geq 1/n$, $\beta_1^{n,m} \in C_b^\infty(\mathbb{R})$ and $\beta_1^n = 0$ for $|r| \geq 2n$, we have that $\mu_{n,m} \in C_b^\infty(\mathbb{R}^2)$ and therefore $q_{n,m} \in \mathcal{W}$. It is easily seen that $q_{n,m}$ is the unique (in $\mathbb{W}$) solution of

$$\begin{cases} \partial_t v(t,x) &= L_{n,m}^* v(t,x) \quad \text{on } (0,T) \times \mathbb{R}^2 \\ v(0,x) &= \rho(x) \quad \text{on } \mathbb{R}^2, \end{cases} \tag{3.6}$$

where

$$L_{n,m}^* \phi = \sum_{ij} \partial_{ij}(a_{ij}^{n,m} \phi) - \sum_i \partial_i(\beta_i^{n,m} \phi).$$

For $g \in C_c^\infty((0,\infty) \times \mathbb{R})$, the problem

$$\begin{cases} \partial_t u(t,x) + L_{n,m} u(t,x) = 0 \quad \text{on } (0,T) \times \mathbb{R}^2 \\ u(T,x) = g(x) \quad \text{on } \mathbb{R}^2 \end{cases} \tag{3.7}$$

has a unique solution $u_{n,m} \in \mathbb{W}$, for which also holds that $u_{n,m} \in \mathcal{W}$. By the Feynman-Kac formula we have

$$u_{n,m}(t,x) = \mathbb{E}\left[g(X_{T-t}^{x;n,m})\right], \tag{3.8}$$

where by $X^{x;n,m} = (Y^{x_1;n,m}, Z^{x;n,m})$ we have denoted the unique solution of $\Pi(x; \beta^{n,m}, \sigma^{n,m})$. By the Îto formula for $\|\cdot\|_{L_2}^2$ (see, e.g., [14]), and the polarization identity $4ab = (a+b)^2 - (a-b)^2$ we get

$$(u_{n,m}(T), q_{n,m}(T))_{L_2} = (u_{n,m}(0), q_{n,m}(0))_{L_2} + \int_0^T (L_{n,m}^* q_{n,m}(t), u_{n,m}(t))_{L_2} \, dt$$

$$- \int_0^T (L_{n,m} u_{n,m}(t), q_{n,m}(t))_{L_2} \, dt,$$

and since $(L_{n,m}^* q_{n,m}(t), u_{n,m}(t))_{L_2} = (L_{n,m} u_{n,m}(t), q_{n,m}(t))_{L_2}$ for all $t \in [0,T]$, we obtain by virtue of (3.8) that

$$\int_{\mathbb{R}^2} g(x) q_{n,m}(T,x) dx = \int_{\mathbb{R}^2} \mathbb{E}\left[g(X_T^{x;n,m})\right] \rho(x) dx. \tag{3.9}$$

We want to let $n \to \infty$ in the above relation. Let us set

$$\sigma_i^{\infty,m} := \sigma_i \circ \varrho_m, \quad \beta_1^{\infty,m} := \beta_1 \circ \varrho_m, \quad \beta_2^{\infty,m} := \beta_2,$$

and we denote by $X^{x;m} = (Y^{x_1;m}, Z^{x;m})$ and $\tilde{X}^{x;m} = (Y^{x_1;m}, \tilde{Z}^{x;m})$ the unique solutions of $\Pi(x; \beta^{\infty,m}, \sigma^{\infty,m})$ and $\Pi(x; \tilde{\beta}^{\infty,m}, \sigma^{\infty,m})$ respectively, where $\tilde{\beta}_1^{\infty,m} = \beta_1^{\infty,m}$, and

$$\tilde{\beta}_2^{\infty,m} := |\sigma_1^{\infty,m}|^2 \left(\lambda \frac{\sigma_2^{\infty,m}}{\sigma_1^{\infty,m}}\right)' + 2\lambda \frac{\beta_1^{\infty,m} \sigma_2^{\infty,m}}{\sigma_1^{\infty,m}} + \beta_2.$$

Notice that since $\sigma_i^n \to \sigma_i$, and $\beta_i^n \to \beta_i$, uniformly on compacts of $(0,\infty)$ and $\mathbb{R}$ respectively, we have that

$$\sigma_i^{n,m} \to \sigma_i^{\infty,m} \quad \text{and} \quad \beta_i^{n,m} \to \beta_i^{\infty,m}, \tag{3.10}$$

uniformly on compacts of $\mathbb{R}$ as $n \to \infty$. Moreover, by (v) of Assumption 2.1 and the properties of ϱ_m, there exists a constant N such that

$\sup_n \left(|\beta_1^n(r)| + |\sigma_1^{n,m}(r)| \right) \leq N(1 + |r|)$ for any $r \in \mathbb{R}$. Therefore, by Lemma 3.1 combined with the fact that ρ is compactly supported, we get that

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} \mathbb{E}\left[g(X_T^{x;n,m}) \right] \rho(x) dx = \int_{\mathbb{R}^2} \mathbb{E}\left[g(X_T^{x;m}) \right] \rho(x) dx.$$

For the left hand side of (3.9) we proceed as follows. Let us set

$$\mu_m(r) = \frac{2}{|\sigma_1 \circ \varrho_m(r)|^2} \exp\left(\int_1^r \frac{2\beta_1 \circ \varrho_m(l)}{|\sigma_1 \circ \varrho_m(l)|^2} dl \right).$$

Let K be a compact subset of $(0, \infty)$ and set $K_m := \varrho_m(K)$ which is also a compact set of $(0, \infty)$. We have

$$\sup_{r \in K} \left| \frac{1}{|\sigma_1 \circ \varrho_m(r)|^2} - \frac{1}{|\sigma_1^n \circ \varrho_m(r)|^2} \right| = \sup_{r \in K_m} \left| \frac{|\sigma_1(r)|^2 - |\sigma_1^n(r)|^2}{|\sigma_1(r)|^2 |\sigma_1^n(r)|^2} \right| \quad (3.11)$$

By the strict positivity of σ_1 on $(0, \infty)$ and the uniform convergence $\sigma_1^n \to \sigma_1$ on the compacts of $(0, \infty)$, there exist $c > 0$ such that for all n large enough it holds that $\inf_{r \in K_m} |\sigma_1(r)|^2 |\sigma_1^n(r)|^2 > c$. Consequently,

$$\lim_{n \to \infty} \sup_{r \in K} \left| \frac{1}{|\sigma_1 \circ \varrho_m(r)|^2} - \frac{1}{|\sigma_1^n \circ \varrho_m(r)|^2} \right| = 0. \quad (3.12)$$

This combined with the uniform convergence $\beta_1^n \to \beta_1$ on the compacts of $\mathbb{R}$ gives

$$\lim_{n \to \infty} \sup_{r \in K} \left| \frac{2\beta_1 \circ \varrho_m(l)}{|\sigma_1 \circ \varrho_m(l)|^2} - \frac{2\beta_1^n \circ \varrho_m(l)}{|\sigma_1^n \circ \varrho_m(l)|^2} \right| = 0. \quad (3.13)$$

By (3.12) and (3.13) it follows that $\mu_{n,m} \to \mu_m$ uniformly on compacts of $(0, \infty)$, as $n \to \infty$. Similarly, one can easily see that $1/\mu_{n,m} \to 1/\mu_m$, uniformly on compacts of $(0, \infty)$, as $n \to \infty$. Since ρ is compactly supported in $(0, \infty) \times \mathbb{R}$, we have that

$$\lim_{n \to \infty} \|\rho/\mu_{n,m} - \rho/\mu_m\|_{L_\infty(\mathbb{R}^2)} = 0. \quad (3.14)$$

Moreover, by the strict positivity of σ_1 on $(0, \infty)$ and (iv) of Assumption 2.1, we have that $\sigma_2^n/\sigma_1^n \to \sigma_2/\sigma_1$ uniformly on compacts of $(0, \infty)$, which implies that

$$\frac{\beta_1^{n,m} \sigma_2^{n,m}}{\sigma_1^{n,m}} \to \frac{\beta_1^{\infty,m} \sigma_2^{\infty,m}}{\sigma_1^{\infty,m}}, \quad (3.15)$$

uniformly on compacts of $\mathbb{R}$. In addition, by (iv) and (vi) of Assumption 2.1 and the properties of ϱ_m we have that

$$|\sigma_1^{n,m}|^2 \left(\lambda \frac{\sigma_2^{n,m}}{\sigma_1^{n,m}} \right)' = |\sigma_1^{n,m}|^2 \left(\left(\lambda \frac{\sigma_2^n}{\sigma_1^n} \right)' \circ \varrho_m \right) \varrho_m' \to |\sigma_1^{\infty,m}|^2 \left(\lambda \frac{\sigma_2^{\infty,m}}{\sigma_1^{\infty,m}} \right)' \quad (3.16)$$

uniformly on compacts of $\mathbb{R}$. Thus, from (3.15), (3.16), and the properties of β_2^n, we get that $\tilde{\beta}_2^{n,m} \to \tilde{\beta}_2^m$ uniformly on compacts of $\mathbb{R}$. This combined

with (3.10) and (3.14), imply by virtue of Lemma 3.1 that for any $x \in \mathbb{R}^2$

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\rho(\tilde{X}_T^{x;n,m})}{\mu_{n,m}(Y_T^{x_1;n,m})}\right] = \mathbb{E}\left[\frac{\rho(\tilde{X}_T^{x;m})}{\mu_m(Y_T^{x_1;m})}\right]$$

Notice that $\rho/\mu_{n,m}$ are bounded uniformly in $n \in \mathbb{N}$, $\mu_{n,m} \to \mu_m$ uniformly on compacts of $(0,\infty)$, and $g \in C_c^\infty((0,\infty) \times \mathbb{R})$. Consequently, we obtain by Lebesgue's theorem on dominated convergence that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} g(x)q_{n,m}(T,x)dx = \int_{\mathbb{R}^2} g(x)q_m(T,x)dx,$$

where

$$q_m(T,x) := \mu_m(x_1)\mathbb{E}\left[\frac{\rho(\tilde{X}_T^{x;m})}{\mu_m(Y_T^{x_1;m})}\right].$$

Hence,

$$\int_{\mathbb{R}^2} \mathbb{E}\left[g(X_T^{x;m})\right]\rho(x)dx = \int_{\mathbb{R}^2} g(x)q_m(T,x)dx. \tag{3.17}$$

Now we want to let $m \to \infty$. One can easily see that $\sigma_i^{\infty,m} \to \sigma_i$ and $\beta_i^{\infty,m} \to \beta_i$ uniformly on compacts of $\mathbb{R}$, which combined with the properties of β_1, σ_1, and ϱ_m, by virtue of Lemma 3.1 together with the boundedness of g and the fact that ρ has compact support imply that

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} \mathbb{E}\left[g(X_T^{x;m})\right]\rho(x)\,dx = \int_{\mathbb{R}^2} \mathbb{E}\left[g(X_T^{x})\right]\rho(x)\,dx \tag{3.18}$$

Notice that

$$|\sigma_1^{\infty,m}|^2\left(\lambda\frac{\sigma_2^{\infty,m}}{\sigma_1^{\infty,m}}\right)' = |\sigma_1|^2\left(\left(\lambda\frac{\sigma_2}{\sigma_1}\right)'\circ\varrho_m\right)\varrho_m'.$$

By (iii) of Assumption 2.1 and (3) of Remark 3.1 we have

$$\sup_{r\in\mathbb{R}}\left||\sigma_1|^2\left((\lambda\sigma_2/\sigma_1)'\circ\varrho_m\right)\varrho_m'(r) - |\sigma_1|^2\left(\lambda\sigma_2/\sigma_1\right)'(r)\right|$$

$$= \sup_{r\in[0,1/m]}\left||\sigma_1|^2\left((\lambda\sigma_2/\sigma_1)'\circ\varrho_m\right)\varrho_m'(r) - |\sigma_1|^2\left(\lambda\sigma_2/\sigma_1\right)'(r)\right|$$

$$\leq \sup_{r\in[0,1/m]}\left||\sigma_1|^2\left(\lambda\sigma_2/\sigma_1\right)'\circ\varrho_m(r)\right| + \sup_{r\in[0,1/m]}\left||\sigma_1|^2\left(\lambda\sigma_2/\sigma_1\right)'(r)\right| \to 0,$$

as $m \to \infty$. In addition, by (iii) of Assumption 2.1, we have that

$$\lambda\beta_1^{\infty,m}\sigma_2^{\infty,m}/\sigma_1^{\infty,m} \to \lambda\beta_1\sigma_2/\sigma_1$$

uniformly on compacts of $\mathbb{R}$. Consequently, $\tilde{\beta}_2^m \to \tilde{\beta}_2$ uniformly on compacts of $\mathbb{R}$. As before one can easily check that $\mu_m \to \mu$ and $1/\mu_m \to 1/\mu$ uniformly on compacts of $(0,\infty)$, and that

$$\lim_{n\to\infty} \|\rho/\mu_m - \rho/\mu\|_{L_\infty(\mathbb{R}^2)} = 0.$$

Putting these facts together implies by virtue of Lemma 3.1 that

$$\lim_{m \to \infty} \int_{\mathbb{R}^2} g(x) q_m(T, x) dx = \int_{\mathbb{R}^2} g(x) q(T, x) dx,$$

which combined with (3.18) brings the proof to an end. $\square$

Proof of Corollary 2.2. Let us fix $(\xi, x) \in D \times D$. Without loss of generality we can assume that on Ω there exist $\mathscr{F}_0$-measurable random variables $\eta^n = (\eta_1^n, \eta_2^n)$, $n \in \mathbb{N}$, having density $\rho^n(\xi - \cdot)$, where $\rho^n(\zeta) = n^2 \rho(n\zeta)$ for a smooth mollifier ρ supported in the unit ball of $\mathbb{R}^2$. Let $X^{(n)} = (Y^{(n)}, Z^{(n)})$ be the unique solution of $\Pi(\eta^n; \beta, \sigma)$. Notice that we have almost surely

$$\xi_1 - \frac{1}{n} \le \eta_1^n \le \xi_1 + \frac{1}{n}.$$

Consequently, by a comparison principle (see, e.g., [16, pp.292]) we have almost surely for all $t \in [0, T]$

$$Y_t^{\xi_1 - \frac{1}{n}} \le Y_t^{(n)} \le Y_t^{\xi_1 + \frac{1}{n}}.$$

By [1, Theorem 2.5] we have that

$$\lim_{n \to \infty} \mathbb{E} \sup_{t \le T} |Y_t^{\xi_1 \pm \frac{1}{n}} - Y_t^{\xi_1}| = 0,$$

which combined with the above inequality gives

$$\lim_{n \to \infty} \mathbb{E} \sup_{t \le T} |Y_t^{(n)} - Y_t^{\xi_1}| = 0.$$

Then one can easily see (as in the proof of Lemma 3.1) that this implies that $\sup_{t \le T} |Z_t^{(n)} - Z_t^{\xi}| \xrightarrow{\mathbb{P}} 0$, as $n \to \infty$. Consequently, for $g \in C_c^\infty(D)$ we have

$$\lim_{n \to \infty} \mathbb{E} g(X_T^{(n)}) = \mathbb{E} g(X_T^\xi). \tag{3.19}$$

On the other hand, we have

$$\int_D g(x) \mu(x) \mathbb{E} \left[\frac{\rho^n(\xi - \tilde{X}_T^x)}{\mu(Y_T^{x_1})} \right] dx = \int_D g(x) \mu(x) \left(\int_{\mathbb{R}^2} \rho^n(\xi - \zeta) \frac{\tilde{p}(T, x, \zeta)}{\mu(\zeta)} d\zeta \right) dx.$$

Next notice that for each $x \in D$

$$\lim_{n \to \infty} \int_{\mathbb{R}^2} \rho^n(\xi - \zeta) \frac{\tilde{p}(T, x, \zeta)}{\mu(\zeta)} d\zeta = \frac{\tilde{p}(T, x, \xi)}{\mu(\xi)},$$

and for all $n \ge n_0$ and $x \in \text{supp}(g)$

$$\left| g(x) \mu(x) \int_{\mathbb{R}^2} \rho^n(\xi - \zeta) \frac{\tilde{p}(T, x, \zeta)}{\mu(\zeta)} d\zeta \right|$$

$$\le \sup_{x \in \text{supp}(g)} \sup_{\zeta \in B_{1/n_0}(\xi)} \left| \frac{\tilde{p}(T, x, \zeta)}{\mu(\zeta)} \right| |g(x) \mu(x)| < \infty,$$

where n_0 is such that $B_{1/n_0}(\xi)$ (the ball of radius $1/n_0$ centered at ξ) is compactly supported in D. Lebesgue's theorem gives

$$\lim_{n \to \infty} \int_D g(x)\mu(x)\mathbb{E}\left[\frac{\rho^n(\xi - \tilde{X}_T^x)}{\mu(Y_T^{x_1})}\right] dx = \int_D g(x)\mu(x)\frac{\tilde{p}(T, x, \xi)}{\mu(\xi)} dx.$$

This, combined with (3.19) and Theorem 2.1 imply that

$$\mathbb{E}g(X_T^\xi) = \int_D g(x)\mu(x)\frac{\tilde{p}(T, x, \xi)}{\mu(\xi)} dx.$$

Since g was arbitrary, the claim follows. $\qquad\square$

4. Proof of Theorem 2.3

The next proposition is an obvious consequence of Lemma 3.1.

Proposition 4.1. *Under the assumption of Theorem 2.3, the function u is continuous on $[0, T] \times \overline{D}$.*

Proposition 4.2. *Under the assumption of Theorem 2.3, the function u belongs to $C^{1,2}((0, T) \times D)$ and satisfies $\partial_t u + Lu = 0$ for all $(t, x) \in (0, T) \times D$.*

Proof. Let $(t, x) \in (0, T) \times D$, let $Q \subset\subset D$ be an open rectangle containing x, and set $\mathcal{R} = (0, T) \times Q$. The problem

$$\begin{cases} \partial_t f + Lf = 0 & \text{in } \mathcal{R}; \\ f = u & \text{on } \partial_p \mathcal{R} \end{cases} \tag{4.1}$$

has a unique classical solution (since L has smooth coefficients and is strongly elliptic in $\mathcal{R}$). Let $X^{x,t}$ be the solution of (2.1) starting from x at time t. For $\varepsilon > 0$, set

$$\tau^\varepsilon = \inf\{s \geq t \mid (s, X_s^{x,t}) \notin \mathcal{R}_\varepsilon\},$$

$$\tau = \inf\{s \geq t \mid (s, X_s^{x,t}) \notin \mathcal{R}\},$$

where $\mathcal{R}_\varepsilon := \{(s, y) \in \mathcal{R} \mid \text{dist}((s, y), \partial \mathcal{R}) > \varepsilon\}$. By Ito's formula we have that the process $(f(s \wedge \tau^\varepsilon, X_{s \wedge \tau^\varepsilon}^{x,t}))_{s \geq t}$, is a local martingale and bounded (since f is bounded), hence a martingale. Thus, for any $\varepsilon > 0$, $s \geq t$,

$$f(t, x) = \mathbb{E}f(s \wedge \tau^\varepsilon, X_{s \wedge \tau^\varepsilon}^{x,t}),$$

which by letting $\varepsilon \downarrow 0$, by virtue of the continuity of f up to the parabolic boundary and due to the fact that $\tau_\varepsilon \uparrow \tau$, gives

$$f(t, x) = \mathbb{E}f(s \wedge \tau, X_{s \wedge \tau}^{x,t}).$$

Choosing $s = T$ in the above equality gives

$$f(t, x) = \mathbb{E}f(\tau, X_\tau^{x,t}) = \mathbb{E}u(\tau, X_\tau^{x,t}) = \mathbb{E}g(X_T^{x,t}) = u(t, x)$$

where the second equality follows from the fact that $f = u$ on $\partial_p \mathcal{R}$ and the third equality follows from the strong Markov property. As $(t, x) \in (0, T) \times D$ and $Q \subset\subset D$ were arbitrary, this brings the proof to an end. $\qquad\square$

Proposition 4.3. *Under Assumption 2.2, we have* $\partial_2 u(t,x) = \mathbb{E}(\partial_2 g)(X^x_{T-t})$ *and* $\partial_{22} u(t,x) = \mathbb{E}(\partial_{22} g)(X^x_{T-t})$. *In particular,* $\partial_2 u, \partial_{22} u \in C([0,T] \times \overline{D})$.

Proof. The result follows immediately from straightforward differentiation, from the fact that $X^x = (Y^{x_1}, Z^x)$, where $Z^x = x_2 + f(Y^{x_1})$ for some functional f, combined with the fact that $g \in C_c^\infty((0,\infty) \times \mathbb{R})$. $\qquad\square$

We proceed with the continuity of $\partial_1 u$ up to the boundary ∂D. If we formally differentiate the equation $\partial_t u + Lu = 0$ with respect to x_1, we obtain
$$\partial_t(\partial_1 u) + \hat{L}(\partial_1 u) + f = 0,$$
where the operator $\hat{L}$ is given by
$$\hat{L}\phi := \sum_{i,j} a_{ij}\partial_{ij}\phi + \sum_i \hat{\beta}_i \partial_i \phi + c\phi, \tag{4.2}$$
with $\hat{\beta}_i = \beta_i + i\partial_1 a_{1i}$, $c = \partial_1 \beta_1$, and the free term f is given by
$$f = (\partial_1 \beta_2)\partial_2 u + (\partial_1 a_{22})\partial_{22}u. \tag{4.3}$$
For any $x \in \overline{D}$, let $\hat{X}^x = (\hat{Y}^{x_1}, \hat{Z}^x)$ be the unique solution of $\Pi(x; \hat{\beta}, \sigma)$, where $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)$, and notice that $\partial_1 a_{11}(0) \geq 0$ (since $\sigma_1(0) = 0$). Consequently, for all $t \in [0,T]$ we have that $\hat{X}^x_t \in \overline{D}$. Let us set
$$\begin{aligned}
v(t,x) :=& \mathbb{E}\left[(\partial_1 g)(\hat{X}^x_{T-t}) \exp\left(\int_0^{T-t} c(\hat{X}^x_s)\,ds\right)\right] \\
&+ \mathbb{E}\left[\int_0^{T-t} f(t+s, \hat{X}^x_s) \exp\left(\int_0^s c(\hat{X}^x_r)\,dr\right)\,ds\right].
\end{aligned} \tag{4.4}$$

To prove that $\partial_1 u$ is continuous on $[0,T] \times \overline{D}$ we show that v is continuous and that $v = \partial_1 u$.

Proposition 4.4. *Under Assumption 2.2, the function v defined in (4.4) is continuous on $[0,T] \times \overline{D}$.*

Proof. Let $(t^n, x^n) \in [0,T] \times \overline{D}$, $n \in \mathbb{N}$, converging to $(t,x) \in [0,T] \times \overline{D}$. By (3.1) of Lemma 3.1 combined with the continuity and the boundedness of $\partial_1 g$ and c, we have
$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}&\left[(\partial_1 g)(\hat{X}^{x^n}_{T-t^n}) \exp\left(\int_0^{T-t^n} c(\hat{X}^{x^n}_s)\,ds\right)\right] \\
&= \mathbb{E}\left[(\partial_1 g)(\hat{X}^x_{T-t}) \exp\left(\int_0^{T-t} c(\hat{X}^x_s)\,ds\right)\right].
\end{aligned}$$
Also, by (3.1) and the continuity of f and c we have that
$$\begin{aligned}
I_{s<T-t^n} f(t^n+s, \hat{X}^{x^n}_s) &\exp\left(\int_0^s c(\hat{X}^{x^n}_r)\,dr\right) \\
&\to I_{s<T-t} f(t+s, \hat{X}^x_s) \exp\left(\int_0^s c(\hat{X}^x_r)\,dr\right),
\end{aligned}$$

in measure (on $(0, T) \times \Omega$) as $n \to \infty$ (where we have set $f(r, \cdot) = f(T, \cdot)$ for $r > T$). The result now follows by the boundedness of c and f (recall (4) from Assumption 2.2). □

For the proof of the next proposition we will need to define some approximation functions. Let $\zeta_i^n \in C^\infty(\mathbb{R})$ with $0 \le \zeta_i^n \le 1$ such that

(1) $\zeta_1^n = 1$ on $(-\infty, 1/(2n)]$, $\zeta_1^n = 0$ on $(1/n, \infty]$, $\zeta_1^n > 0$ on $(1/(2n), 1/n)$, and $|\partial \zeta_1^n| \le Nn$
(2) $\zeta_2^n = 1 - \zeta_1^n$ on $(-\infty, n]$, $\zeta_2^n = 0$ on $[n^4, \infty)$, $\zeta_2^n > 0$ on (n, n^4), and $|\partial \zeta_2^n| \le n^{-4}$ on (n, n^4)
(3) $\zeta_3^n = 0$ on $(-\infty, n]$, $\zeta_3^n = 1$ on $(n^4, \infty]$, $\zeta_3^n > 0$ on (n, n^4), and $|\partial \zeta_3^n| \le Nn^{-4}$

For $i \in \{1, 2\}$ let us extend σ_i and β_i on $\mathbb{R}$ by setting $\sigma_i(r) = \sigma_i(0)$ and $\beta_i(r) = \beta_i(0)$ for $r < 0$, and let us set

$$\sigma_i^n(r) := \left[2a_{ii}\left(\frac{1}{2n}\right) \zeta_1^n(r) + 2a_{ii}(r)\zeta_2^n(r) + \zeta_3^n(r) \right]^{1/2},$$

$$\beta_i^n(r) = \beta_i(\frac{1}{2n})\zeta_1^n(r) + \beta_i(r)\zeta_2^n(r),$$

and as usual

$$a_{ij}^n := \lambda_{ij} \frac{\sigma_i^n \sigma_j^n}{2}.$$

Proposition 4.5. *Under Assumption 2.2, we have $\partial_1 u \in C([0, T] \times \overline{D})$.*

Proof. For $n \in \mathbb{N}_+$ let σ_i^n and β_i^n be the functions defined above. Under Assumption 2.2 it is not difficult to see that $\sigma_i^n, \beta_i^n \in C_b^\infty(\mathbb{R})$, $\inf_\mathbb{R} \sigma_i^n > 0$, and the following hold:

(i) $\sigma_i^n = \sigma_i$ and $\beta_i^n = \beta_i$ on $[1/n, n]$,
(ii) $\beta_i^n \to \beta_i$ and $\sigma_i^n \to \sigma_i$ uniformly on compacts subsets of $\mathbb{R}$ as $n \to \infty$, and there exists a constant N such that $|\beta_1^n(r)| + |\sigma_1^n(r)| \le N(1 + |r|)$, for all $n \in \mathbb{N}_+$, $r \in \mathbb{R}$
(iii) $(a_{22}^n)'$, $(\beta_1^n)'$, and $(\beta_2^n)'$ are bounded, uniformly in $n \in \mathbb{N}_+$.

Let us set $\beta^n := (\beta_1^n, \beta_2^n)$, $\sigma^n := (\sigma_1^n, \sigma_2^n)$, and for every $n \in \mathbb{N}_+$ let L_n denote the generator of $\Pi(\cdot; \beta^n, \sigma^n)$. For every $n \in \mathbb{N}_+$, the equation

$$\begin{cases} \partial_t u^n + L_n u^n = 0 & \text{in } (0, T) \times \mathbb{R}^2; \\ u^n(T, x) = g(x) & \text{for } x \in \mathbb{R}^2 \end{cases} \qquad (4.5)$$

has a unique solution $u^n \in \mathbb{W}$ which moreover belongs to $\mathscr{W}$, and by the Feynman-Kac formula we have for all $(t, x) \in [0, T] \times \mathbb{R}^2$

$$u^n(t, x) = \mathbb{E}g(X_{T-t}^{x; n}), \qquad (4.6)$$

where $X^{x; n}$ is the unique solution of $\Pi(x; \beta^n, \sigma^n)$. Let K be a compact subset of $(0, T) \times D$ and notice that on K, for all $n \in \mathbb{N}_+$ large enough, it holds that $L_n = L$. Moreover, L is strongly elliptic on K and its coefficients

and all their derivatives are bounded. By virtue of Proposition 4.2, for any $Q \subset\subset \text{int}(K)$, we obtain by standard parabolic estimates

$$\|\nabla u^n - \nabla u\|_{L_2(Q)}^2 \leq N\|u^n - u\|_{L_2(K)}^2 + N\|(L_n - L)u\|_{L_2(K)}^2$$
$$= N\|u^n - u\|_{L_2(K)}^2, \tag{4.7}$$

for n large enough, with a constant N independent of $n \in \mathbb{N}$. By the properties of β^n and σ^n, Lemma 3.1 and (4.6) we have that $u^n(t,x) \to u(t,x)$ for all $(t,x) \in K$, and since $|u^n(t,x)| \leq \|g\|_{L_\infty} < \infty$ for all $(t,x) \in K$, $n \in \mathbb{N}_+$, we get that $\lim_{n\to\infty} \|u^n - u\|_{L_2(K)}^2 = 0$, which due to (4.7) implies that

$$\lim_{n\to\infty} \|\nabla u^n - \nabla u\|_{L_2(Q)}^2 = 0. \tag{4.8}$$

On the other hand, by differentiating u^n with respect to x_1 we easily see that $v^n := \partial_1 u^n$ belongs to $\mathcal{W}$ and satisfies

$$\begin{cases} \partial_t v^n + \hat{L}_n v^n + f^n = 0 & \text{on } (0,T) \times \mathbb{R}^2 \\ v^n(T) = \partial_1 g & \text{on } \mathbb{R}^2, \end{cases}$$

where

$$\hat{L}_n \phi := \sum_{ij} a_{ij}^n \partial_{ij} \phi + \sum_i \hat{\beta}_i^n \partial_i \phi + c^n \phi$$

with

$$\hat{\beta}_1^n = \beta_1^n + \partial_1 a_{11}^n, \ \hat{\beta}_2^n = \beta_2^n + 2\partial_1 a_{12}^n, \ c^n = \partial_1 \beta_1^n$$

and

$$f^n = (\partial_1 \beta_2^n)\partial_2 u^n + (\partial_1 a_{22}^n)\partial_{22} u^n.$$

By the Feynman-Kac formula we have

$$v^n(t,x) = \mathbb{E}\left[(\partial_1 g)(\hat{X}_{T-t}^{x;n})e^{\int_0^{T-t} c^n(\hat{X}_s^{x;n})ds}\right]$$
$$+ \mathbb{E}\left[\int_0^{T-t} f^n(t+s, \hat{X}_s^{x;n})e^{\int_0^s c^n(\hat{X}_r^{x;n})dr} ds\right], \tag{4.9}$$

where $\hat{X}^{x;n} = (\hat{Y}^{x_1;n}, \hat{Z}^{x;n})$ is the unique solution of $\Pi(x; \hat{\beta}^n, \sigma^n)$. Let us set

$$\tau_n := \inf\{t \geq 0 \colon Y_t^{x_1} \notin (1/n, n)\} \wedge T,$$

and notice that on $[\![0, \tau_n]\!] := \{(\omega, t) \in \Omega \times [0,T] \colon t \leq \tau_n(\omega)\}$ both $\hat{X}^x$ and $\hat{X}^{x;n}$ satisfy $\Pi(x, \hat{\beta}^n, \sigma^n)$ and since $\hat{\beta}^n, \sigma^n$ are Lipschitz continuous we have that for all $n \in \mathbb{N}_+$, $\hat{X}^{x;n} = \hat{X}^x$ on $[\![0, \tau_n]\!]$. In addition, by virtue of (6) of Assumption 2.2 we have that zero is not an exit boundary for the diffusion $\hat{Y}^{x_1}$ (see, e.g., [3, pp. 14]). That is, if $x_1 > 0$, then $\inf_{t\in[0,T]} \hat{Y}_t^{x_1} > 0$, which in turn implies that almost surely $\tau_n = T$ for n sufficiently large. In particular, for almost all $\omega \in \Omega$, $\sup_{t\in[0,T]} |\hat{X}_t^x - \hat{X}_t^{x;n}| = 0$ for n large enough depending on $\omega \in \Omega$. Then notice that for each $(t,x) \in [0,T] \times D$, by the properties of σ^n and β^n we have

$$\lim_{n\to\infty} \mathbb{E}\left[(\partial_1 g)(\hat{X}_{T-t}^{x;n})e^{\int_0^{T-t} c^n(\hat{X}_s^{x;n})ds}\right] = \mathbb{E}\left[(\partial_1 g)(\hat{X}_{T-t}^x)e^{\int_0^{T-t} c(\hat{X}_s^x)ds}\right].$$

Moreover, notice that similarly to Proposition 4.3 we have $\partial_2 u^n(t,x) = \mathbb{E}\partial_2 g(X_{T-t}^{x;n})$ and $\partial_{22} u^n(t,x) = \mathbb{E}\partial_{22} g(X_{T-t}^{x;n})$, which by virtue of (ii) above and Lemma 3.1 implies that for any sequence $(x^n)_{n=1}^\infty \subset D$ with $\lim_{n\to\infty} x^n = x \in D$, we have $\lim_{n\to\infty} \partial_2 u^n(t,x^n) = \partial_2 u(t,x)$, and $\lim_{n\to\infty} \partial_{22} u^n(t,x^n) = \partial_{22} u(t,x)$. This combined with the properties of σ^n, β^n imply in turn that $\lim_{n\to\infty} f^n(t,x^n) = f(t,x)$ whenever $(x^n)_{n=1}^\infty \subset D$ with $\lim_{n\to\infty} x^n = x \in D$. In addition, f^n are bounded in $(t,x) \in [0,T] \times \overline{D}$, uniformly in $n \in \mathbb{N}$, and then one can easily see that for each $(t,x) \in [0,T] \times D$,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{T-t} f^n(t+s, \hat{X}_s^{x;n}) e^{\int_0^s c^n(\hat{X}_r^{x;n})dr}\, ds\right]$$
$$= \mathbb{E}\left[\int_0^{T-t} f(t+s, \hat{X}_s^x) e^{\int_0^s c(\hat{X}_r^x)dr}\, ds\right].$$

Consequently, for every $(t,x) \in [0,T] \times D$, we have $\lim_{n\to\infty} v^n(t,x) = v(t,x)$, which combined with (4.8) gives that $\partial_1 u = v$ on $[0,T] \times D$. Since $v \in C([0,T] \times \overline{D})$, it follows that u is differentiable with respect to x_1 on $[0,T] \times \overline{D}$ and $\partial_1 u = v \in C([0,T] \times \overline{D})$. This finishes the proof. $\square$

Let D be an open bounded domain in $\mathbb{R}^d$ where $d \in \mathbb{N}_+$, and for $i,j \in \{1,\ldots,d\}$, let $a^{ij}, b^i, c \colon [0,T] \times D \to \mathbb{R}$ be measurable functions. Let us set $\mathcal{L}\phi := \sum_{i,j} \partial_i(a^{ij}\partial_j\phi) + \sum_i b^i \partial_i + c\phi$ and $Q := (0,T) \times D$.

Assumption 4.1. The functions $a^{ij}, \partial_l a^{ij}, b^i, c$ are bounded in magnitude by a constant N_1. Moreover there exists a constant $\varkappa > 0$ such that $a^{ij}\xi_i\xi_j \geq \varkappa|\xi|^2$, for all $\xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$.

The following is well-known in the theory of parabolic PDEs and for a proof we refer the reader to [15, pp. 211, Theorem 11.1].

Lemma 4.6. *Suppose that Assumption 4.1 holds and let $f \in L_2((Q)$ and $Q' \subset Q$ such that $dist(Q', \partial_p Q) > 0$. Then there exists a constant N depending only on N_1, $\varkappa$, Q, and $dist(Q', \partial_p Q)$, such that for any $u \in C^{1,2}(Q)$ satisfying $\partial_t u = \mathcal{L}u + f$ on Q, the estimate*

$$\|\nabla u\|_{L_\infty(Q')} \leq N(\|u\|_{L_\infty(Q)} + \|f\|_{L_\infty(Q)})$$

holds.

Proposition 4.7. *Under Assumption 2.2, for each $i \in \{1,2\}$ we have*

$$\lim_{n\to\infty} a_{i1}(x^n)\partial_{i1} u(t^n, x^n) = 0,$$

for any sequence $(t^n, x^n)_{n=1}^\infty \subset (0,T) \times D$ with $\lim_{n\to\infty}(t^n, x^n) = (t^0, x^0)$, where $x^0 \in \partial D$, $t^0 \in (0,T)$.

Proof. Let $\{(t^n, x^n)\}_{n=1}^\infty \subset (0,T) \times D$ be a sequence converging to $(t^0, x^0) \in (0,T) \times \partial D$ and set $s^n = T - t^n$. Recall that $2a_{11}$ is Lipschitz near zero, with a Lipschitz constant $K > 0$. Around the point (t^n, x^n) consider the rectangle

$$R^n := (s^n - \theta^n, s^n + \theta^n) \times (x_1^n - \theta^n, x_1^n + \theta^n) \times (x_2^n - \theta^n, x_2^n + \theta^n),$$

where $\theta^n = a_{11}(x_1^n)/K$, and let $p_n \colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$p_n(t, x_1, x_2) = \left(\frac{Tt}{4\theta^n} + T\left(\frac{3}{4} - \frac{s^n}{4\theta^n} \right), \frac{x_1 + 2\theta^n - x_1^n}{2\theta^n}, \frac{x_2 - x_2^n}{2\theta^n} \right).$$

It follows that $p_n(R^n) = (T/2, T) \times (1/2, 3/2) \times (-1/2, 1/2) =: Q$, and $p_n(s^n, x_1^n, x_2^n) = (3T/4, 1, 0) =: q$, for all $n \in \mathbb{N}$. Since $u \in C^{1,2}((0,T) \times D)$ and satisfies $\partial_t u + Lu = 0$, we have for $v(t,x) := u(T-t, x)$ that $v \in C^{1,2}((0,T) \times D)$ and satisfies $\partial_t v = Lv$ on $(0,T) \times D$. Moreover, since the coefficients are smooth in D it follows that $v \in C^{1,k}(Q)$ for any $k \in \mathbb{N}$ (see, e.g. Theorem 10, page 72 in [10]); in particular, $\hat{v} := \partial_1 v \in C^{1,2}(R^n)$. It is easy to see that $\partial_t \hat{v} = \hat{L}\hat{v} + f$ on R^n, where $\hat{L}$ and f are given in (4.2) and (4.3) respectively. It follows then that for all $n \in \mathbb{N}$, $w^n := \hat{v} \circ p_n^{-1} \in C^{1,2}(Q)$ and it satisfies on Q

$$\partial_t w^n = \hat{L}_n w^n + f^n,$$

where

$$\hat{L}_n \phi := \sum_{ij} \frac{a_{ij} \circ p_n^{-1}}{T\theta^n} \partial_{ij}\phi + \sum_i \frac{2\hat{\beta}_i \circ p_n^{-1}}{T} \partial_i \phi + \frac{4\theta^n}{T} c \circ p_n^{-1} \phi \qquad (4.10)$$

and $f^n := (4\theta^n/T) f \circ p_n^{-1}$. Recall that a_{11} is Lipschitz continuous near zero with Lipschitz constant $K/2$. Consequently, for any $x_1 \in (1/2, 3/2)$ we have

$$a_{11}(2\theta^n(x_1 - 1) + x_1^n) \geq -K\theta^n |1 - x_1| + a_{11}(x_1^n),$$

which implies that on Q we have

$$\frac{a_{11} \circ p_n^{-1}}{\theta^n} \geq -K|1 - x_1| + K \geq \frac{K}{2}.$$

By (5) of Assumption 2.2 we have on Q (for all n sufficiently large)

$$\frac{a_{22} \circ p_n^{-1}}{\theta^n} \geq \frac{N_0 K}{2}.$$

Moreover, one can easily check that the coefficients of $\hat{L}_n$ are bounded on Q uniformly in $n \in \mathbb{N}$. Consequently the operators $\hat{L}_n$ satisfy the assumption of Lemma 4.6 with constants N_1 and $\varkappa$ independent of $n \in \mathbb{N}$. Let $Q' \subset Q$ be a cylinder with $\mathrm{dist}(Q', \partial_p Q) > 0$ and such that $q \in Q'$, and set $\varrho := \hat{v}(s^0, x^0)$. Notice that on Q we have

$$\partial_t(w^n - \varrho) = \hat{L}_n(w^n - \varrho) + \frac{4\theta^n}{T} \varrho c \circ p_n^{-1} + f^n,$$

and by virtue of Lemma 4.6 there exists a constant N such that for all $n \in \mathbb{N}$ large enough

$$\|\nabla w^n\|_{L_\infty(Q')} \leq N(\|w^n - \varrho\|_{L_\infty(Q)} + \theta^n \|c \circ p_n^{-1}\|_{L_\infty(Q)} + \|f^n\|_{L_\infty(Q)}).$$

Since f and c are bounded we have

$$\lim_{n \to \infty} \|f^n\|_{L_\infty(Q)} = \lim_{n \to \infty} \theta^n \|c \circ p_n^{-1}\|_{L_\infty(Q)} = 0.$$

Also, since $\hat{v} \in C([0,T] \times \overline{D})$ we have

$$\lim_{n \to \infty} \|w^n - \varrho\|_{L_\infty(Q)} = \lim_{n \to \infty} \|\hat{v} - \varrho\|_{L_\infty(R^n)} = 0.$$

Consequently, for each $i \in \{1,2\}$ we have

$$\begin{aligned}
|a_{i1}(x_1^n)\partial_{i1}u(t^n, x^n)| &= |a_{i1}(x_1^n)\partial_i \hat{v}(s^n, x^n)| \\
&\leq N\theta^n|\partial_i \hat{v}(s^n, x^n)| \\
&\leq N|\partial_i w^n(q)| \leq N\|\nabla w^n\|_{L_\infty(Q')} \to 0,
\end{aligned}$$

where the first inequality above follows from (5) of Assumption 2.2. This brings the proof to an end. □

Proposition 4.8. *Under Assumption 2.2 we have that $\partial_t u \in C((0,T) \times \overline{D})$.*

Proof. We show first that $u(t,x)$ is differentiable with respect to $t \in (0,T)$ for any $x \in \partial D$. For $t \in (0,T)$ and $x \in \partial D$ we have

$$\frac{u(t+h,x) - u(t,x)}{h} = \frac{u(t+h, x_{(h)}) - u(t, x_{(h)})}{h} + O(h)$$

where $x_{(h)} = (h^2, x_2)$. By the mean value theorem we have that the right hand side of the above inequality is equal to

$$(\partial_t u)(t + \xi(h), x_{(h)}) + O(h) = -(Lu)(t + \xi(h), x_{(h)}) + O(h)$$

for some $\xi(h) \in [0,h]$. Consequently, by virtue of Propositions 4.3, 4.5, and 4.7 we obtain

$$\lim_{h \to 0} \frac{u(t+h,x) - u(t,x)}{h} = -a_{22}(0)\partial_{22}u(t,x) - \sum_i \beta_i(0)\partial_i u(t,x).$$

Hence, the time derivative exists. Moreover, by the above equality combined again with Propositions 4.3, 4.5, and 4.7 and the fact that $\partial_t u = -Lu$ on $(0,T) \times D$, it follows that $\partial_t u$ is continuous on $(0,T) \times \overline{D}$.

□

Proof of Theorem 2.3. The fact that u is indeed a solution follows from Propositions 4.1 to 4.8. Hence we proceed with the uniqueness part. It suffices to show that if $g = 0$ and u is a solution of (2.8) having polynomial growth, then $u \geq 0$. To this end, let v be a solution of (4.1) such that for some constant N we have for all $(t,x) \in (0,T) \times D$ that $|v(t,x)| \leq N(1 + |x|^{m-1})$ with an integer $m \geq 2$. Let us also set $\tilde{v}(t,x) = v(T-t,x)$. Let $w(t,x) = 1 + |x_1|^m + |x_2|^m$ and notice that due to (4) of Assumption 2.2 we obtain that $Lw < cw - 1$ for a sufficiently large constant c. Let us set $\tilde{v}^\varepsilon = u + \varepsilon e^{ct}w$ and notice that on $(0,T) \times D$ we have

$$\partial_t \tilde{v}^\varepsilon - L\tilde{v}^\varepsilon = \varepsilon e^{ct}(cw - Lw) > \varepsilon e^{ct}. \qquad (4.11)$$

Assume that $\{\tilde{v}^\varepsilon < 0\} =: \Gamma$ is non-empty for some $\varepsilon > 0$ (otherwise there is nothing to prove) and notice that by the growth condition on v and the definition of w we have that Γ is bounded. Let $s = \inf \Gamma_{[0,T]}$, where $\Gamma_{[0,T]}$ is the projection of Γ on $[0,T]$. Since $\overline{\Gamma}$ is compact, there exists $z = (z_1, z_2) \in \overline{D}$

such that $(s, z) \in \overline{\Gamma}$ and by the continuity of $\tilde{v}^{\varepsilon}$ we get $\tilde{v}^{\varepsilon}(s, z) = 0$ which in particular implies that $s > 0$. First assume that $z \in \partial D$. By definition of s we have that $\tilde{v}^{\varepsilon}(t, z) \geq 0$ for $t < s$, $\tilde{v}^{\varepsilon}(s, 0, x_2) \geq 0$ for all $x_2 \in \mathbb{R}$, and $\tilde{v}^{\varepsilon}(s, x_1, z_2) \geq 0$ for all $x_1 \geq 0$. Consequently, $\partial_t \tilde{v}^{\varepsilon}(s, z) \leq 0$, $\partial_2 \tilde{v}^{\varepsilon}(s, z) = 0$, $\partial_{22} \tilde{v}^{\varepsilon}(s, z) \geq 0$, and $\partial_1 \tilde{v}^{\varepsilon}(s, z) \geq 0$. Since $\beta_1(0), a_{11}(0) \geq 0$ we obtain

$$\delta := \partial_t \tilde{v}^{\varepsilon}(s, z) - \beta_1(0)\partial_1 \tilde{v}^{\varepsilon}(s, z) - \beta_2(0)\partial_2 \tilde{v}^{\varepsilon}(s, z) - a_{22}(0)\partial_{22}\tilde{v}^{\varepsilon}(s, z) \leq 0$$
$$(4.12)$$

It follows from Definition 2.1 that

$$\lim_{(0,T)\times D \ni (t,x) \to (s,z)} a_{11}\partial_{11}\tilde{v} + a_{12}\partial_{12}\tilde{v} = 0$$

which in turn implies that

$$\lim_{(0,T)\times D \ni (t,x) \to (s,z)} a_{11}\partial_{11}\tilde{v}^{\varepsilon} + a_{12}\partial_{12}\tilde{v}^{\varepsilon} = 0,$$

where we have used that $a_{12}(0) = a_{11}(0) = 0$. Combining this with (4.11) and (4.12) gives

$$\varepsilon e^{cs} \leq \lim_{(0,T)\times D \ni (t,x) \to (s,z)} \partial_t \tilde{v}^{\varepsilon} - L\tilde{v}^{\varepsilon} = \delta \leq 0,$$

which is a contradiction. Hence, $z_1 > 0$ and $z \in D$ is a local minimum of the function $\tilde{v}^{\varepsilon}(s, \cdot)$. It follows then that $L\tilde{v}^{\varepsilon}(s, z) \geq 0$ or else, if $L\tilde{v}^{\varepsilon}(s, z) < 0$, then on a ball around z we have $L\tilde{v}^{\varepsilon}(s, z) < 0$ and since $\tilde{v}^{\varepsilon}(s, \cdot)$ has minimum on z we have by the Hopf maximum principle (see, e.g., [8, pp. 349, Theorem 3]) that $\tilde{v}^{\varepsilon}(s, \cdot) = 0$ near z and in particular $L\tilde{v}^{\varepsilon}(s, z) = 0$. Hence

$$\partial_t \tilde{v}^{\varepsilon}(s, z) - L\tilde{v}^{\varepsilon}(s, z) \leq 0,$$

which again contradicts (4.11). This shows that $\tilde{v}^{\varepsilon} \geq 0$, and since this is true for all $\varepsilon > 0$ we have $v \geq 0$. This brings the proof to an end. $\square$

References

[1] Khaled Bahlali, Brahim Mezerdi, and Youssef Ouknine. Pathwise uniqueness and approximation of solutions of stochastic differential equations. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 166–187. Springer, Berlin, 1998.

[2] Erhan Bayraktar, Constantinos Kardaras, and Hao Xing. Valuation equations for stochastic volatility models. *SIAM J. Financial Math.*, 3(1):351–373, 2012.

[3] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[4] John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.

[5] Erik Ekström, Per Lötstedt, and Johan Tysk. Boundary values and finite difference methods for the single factor term structure equation. *Appl. Math. Finance*, 16(3-4):253–259, 2009.

[6] Erik Ekström and Johan Tysk. The Black-Scholes equation in stochastic volatility models. *J. Math. Anal. Appl.*, 368(2):498–507, 2010.

[7] Erik Ekström and Johan Tysk. Boundary conditions for the single-factor term structure equation. *Ann. Appl. Probab.*, 21(1):332–350, 2011.

[8] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[9] William Feller. Two singular diffusion problems. *Ann. of Math. (2)*, 54:173–182, 1951.

[10] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[11] István Gyöngy and Nicolai Krylov. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields*, 105(2):143–158, 1996.

[12] Kiyosi Itô and Henry P. McKean, Jr. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin-New York, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

[13] N. V. Krylov. *Introduction to the theory of random processes*, volume 43 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[14] N. V. Krylov and B. L. Rozovskiĭ. Stochastic evolution equations. In *Current problems in mathematics, Vol. 14 (Russian)*, pages 71–147, 256. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1979.

[15] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*, volume Vol. 23 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1968.

[16] Rong Situ. *Theory of stochastic differential equations with jumps and applications*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005. Mathematical and analytical techniques with applications to engineering.

(K. Dareiotis) Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

E-mail address: konstantinos.dareiotis@mis.mpg.de

(E. Ekström) Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden

E-mail address: Erik.Ekstrom@math.uu.se