ON FINITE DIFFERENCE SCHEMES FOR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS OF LÉVY TYPE

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Abstract. In this article we introduce a finite difference approximation for integro-differential operators of Lévy type. We approximate solutions of possibly degenerate integro-differential equations by treating the nonlocal operator as a second-order operator on the whole unit ball, eliminating the need for truncation of the Lévy measure which is present in the existing literature. This yields an approximation scheme with significantly reduced computational cost, especially for Lévy measures corresponding to processes with jumps of infinite variation.

1. Introduction

In the present article we consider a finite difference approximation scheme for partial integro-differential equations (PIDEs) of the form

$$\begin{aligned} &du_t(x) = [(L_t + J)u_t(x) + f_t(x)]\, dt, \qquad (t,x) \in [0,T] \times \mathbb{R}, \\ &u_0(x) = \psi(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where the operators are given by

$$L_t\phi(x) = a_t(x)\partial_x^2\phi(x) + b_t(x)\partial_x\phi(x) + c_t(x)\phi(x),$$

$$J\phi(x) = \int_{\mathbb{R}} \left(\phi(x+z) - \phi(x) - I_{|z|\leq 1} z\partial_x\phi(x)\right)\, \nu(dz).$$

The coefficient of the second derivative in L_t is allowed to degenerate. Here ν denotes a Lévy measure on $\mathbb{R}$, that is, a Borel measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \ \int_{\mathbb{R}} 1 \wedge z^2\, \nu(dz) < \infty.$$

Equations of of the type (1.1) arise, for example, in mathematical finance for pricing derivatives in models with Lévy noise (for further reading on the subject we refer to [1]). Simple examples of Lévy measures are

$$\nu_\alpha(dz) := |z|^{-(1+\alpha)}\, dz,$$

for $\alpha \in (0,2)$. Notice that for $\alpha \in (1,2)$ these measures correspond to Lévy processes whose jumps have infinite variation and the analysis typically becomes more challenging.

Finite difference schemes for equations of this form have previously been studied in [2], [5], and [14]. In addition to the standard space (time, resp.)

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discretization parameter h (τ, resp.), they introduced a further approximation parameter δ for truncation of the Lévy measure. In [2], using this approach and an explicit in time scheme, the error estimates obtained are of the form

$$\|u - u^{\delta,h,\tau}\| \lesssim \frac{\int_{|z|\leq\delta} |z|^3 \nu(dz)}{\int_{|z|\leq\delta} z^2 \nu(dz)} + \nu(\{|z| \geq \delta\})(\sqrt{\tau} + h),$$

which in the case of $\nu = \nu_\alpha$ yields an error of order $\delta + \delta^{-\alpha}(\sqrt{\tau} + h)$. A simple calculation shows that in order to achieve an accuracy of order $\varepsilon > 0$, the computational cost is of order $O(\varepsilon^{-4(\alpha+1)})$, which for $\alpha \sim 2$ is arbitrarily close to $O(\varepsilon^{-12})$. In [14] only spatial discretization is considered, δ is a function of h, and the corresponding error estimates for the spatial approximation, in the case of infinite activity, are of the form

$$\|u - u^h\| \lesssim h \int_{(-1,1)\setminus(-h/2,h/2)} |z| \nu(dz).$$

Again, in the case $\nu = \nu_\alpha$, a simple computation shows that the rate of convergence is of order $h^{2-\alpha}$, which for $\alpha \sim 2$ is arbitrarily slow. The approach in [5] is also similar (truncation of the integro-differential operator near zero). Under some technical conditions posed on the Lévy measure (it is assumed to have a density of a particular form, that is twice continuously differentiable and has a prescribed behaviour near zero), similar estimates are obtained (with constants blowing up as the truncation parameter $\delta \to 0$).

In contrast to these works, in the present article we do not truncate the operator near the origin. We introduce a negative semi-definite approximation that treats the integro-differential operator as a second order operator on the whole unit ball. Using this approximation and an implicit in time scheme we obtain estimates of the form

$$\|u - u^{h,\tau}\| \lesssim (h + \sqrt{\tau}).$$

Despite the fact that our scheme is implicit in time, the gain from not truncating the operator near the origin allows us to reduce the computational cost to $O(\varepsilon^{-4})$ if the coefficients of L are independent of t and $O(\varepsilon^{-5})$ otherwise. Moreover, under some more spatial regularity of the data, we show that

$$\|u - u^{h,\tau}\| \lesssim (h + \tau).$$

This further reduces the computational cost to $O(\varepsilon^{-3})$ and $O(\varepsilon^{-4})$, respectively. Hence, while for Lévy measures with finite mass near zero the explicit algorithm from [2] is faster, in the case of measures with infinite mass near zero, in particular for measures with behaviour similar to ν_α for $\alpha \in (1, 2)$, our scheme significantly reduces the computational cost.

Our approximation is similar to the one that we introduced in [3], [4]. However, in these works the results and their proofs rely on the non-degeneracy of the second order differential operator. We show that the approximate operator J^h that we suggest here is negative semi-definite, and this combined with estimates obtained in [9] for the difference operators lead to apriori estimates of the solution of the scheme independent of the discretization parameters without posing a non-degeneracy condition. This combined with consistency estimates for the operators leads to the desired error estimates. The analysis of the spatial approximation is done in the spirit of [15]. The equations are first discretized in space and solved as equations in Sobolev spaces over $\mathbb{R}$ (u^h) and as equations on the grid (v^h). Error estimates are obtained in Sobolev norms for the difference $u - u^h$. By using embedding theorems, the restriction of u^h on the grid is shown to agree with v^h. Hence, the error estimates in Sobolev norm imply pointwise error estimates for the difference $u - v^h$ by virtue of Sobolev embedding theorems. The discretized equations are further discretized in time (see also [7]), they are solved in Sobolev spaces ($u^{h,\tau}$) and on the grid ($v^{h,\tau}$), and estimates are obtained for $u^h - u^{h,\tau}$, which in turn imply estimates for $v^h - v^{h,\tau}$.

For degenerate equations not involving non-local operators we refer to [12], [10], [9], and [6]. In the last three articles the results are obtained in a more general, stochastic setting, but they remain optimal for deterministic equations as well. Moreover, in these articles Richardson's extrapolation is used in order to accelerate the convergence in the spatial approximation. This is left as future work for the non-local case.

In conclusion let us introduce some notation. By $u_t(x)$ we denote the value of a function $u\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ at $(t,x) \in [0,T] \times \mathbb{R}$, and when u is understood as a function of t with values in some function space (function of $x \in \mathbb{R}$) we will write $u_t := u_t(\cdot)$ for $t \in [0,T]$. By ∂_x we denote the derivative operator with respect to the spatial variable. The notation C_c^∞ stands for the set of all smooth, compactly supported, real functions on $\mathbb{R}$. We denote by $(\cdot,\cdot)_{L_2}$ and $\|\cdot\|_{L_2}$ the inner product and the norm respectively in $L_2(\mathbb{R})$. For an integer $l \geq 0$, H^l will be the Sobolev space of all function in $L_2(\mathbb{R})$ having distributional derivatives up to order l in $L_2(\mathbb{R})$, with the inner product $(f,g)_{H^l} = \sum_{j=0}^{l} (\partial_x^j f, \partial_x^j g)_{L_2}$, where $\partial_x^0 f := f$, and we denote the corresponding norm by $\|\cdot\|_{H^l}$. If X is a Hilbert space, $C([0,T];X)$ will denote the set of all functions that are norm-continuous in $t \in [0,T]$, and $C_w([0,T];X)$ will denote the set of all functions that are weakly-continuous in $t \in [0,T]$. For real numbers α, β, we use the notation $\alpha \wedge \beta := \min\{\alpha,\beta\}$ and $\alpha \vee \beta := \max\{\alpha,\beta\}$. We use the notation N for constants that may change from line to line. In the proofs of lemmas/theorems the dependence of N to certain parameters is given at the statement of the corresponding lemma/theorem.

2. Formulation of the main results

In this section we introduce our scheme and we state our main results. From now on we will use the following notations

$$\mu_0 := \nu(\mathbb{R} \setminus [-1,1]),\ \mu_2 := \int_{|z|\leq 1} z^2\, \nu(dz).$$

Assumption 2.1. Let $m \geq 1$ be an integer.

i) The functions $a, b, c \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable in (t,x). Moreover, the functions b and c are m-times continuously differentiable in x, the function a is $m \vee 2$-times continuously differentiable in x, and there exists a constant K such that for all $l \in \{1, \ldots, m\}$, $q \in \{1, \ldots, m \vee 2\}$, $(t,x) \in [0,T] \times \mathbb{R}$ it holds that
$$|\partial_x^q a_t(x)| + |\partial_x^l b_t(x)| + |\partial_x^l c_t(x)| \leq K.$$

ii) The initial condition ψ belongs to H^m and $f : [0,T] \to H^m$ is a measurable function such that
$$\mathcal{K}_m^2 = \|\psi\|_{H^m}^2 + \int_0^T \|f_t\|_{H^m}^2 \, dt < \infty.$$

Assumption 2.2. For all $(t,x) \in [0,T] \times \mathbb{R}$, we have $a_t(x) \geq 0$.

Notice that for $\phi, \varphi \in C_c^\infty$, by virtue of Taylor's formula and integration by parts we have
$$\begin{aligned}(J\phi, \varphi)_{L_2} = &- \int_{|z|\leq 1} \int_0^1 (1-\theta) z^2 (\partial_x \phi(\cdot + \theta z), \partial_x \varphi)_{L_2} \, d\theta \, \nu(dz) \\ &+ \int_{|z|>1} (\phi(\cdot + z) - \phi, \varphi)_{L_2} \, \nu(dz).\end{aligned}$$

The solution of (1.1) is understood in the following sense.

Definition 2.1. A solution of (1.1) is function $u \in C_w([0,T]; H^1)$ such that for all $\phi \in C_c^\infty$ and $t \in [0,T]$ we have
$$\begin{aligned}(u_t, \phi)_{L_2} = (\psi, \phi)_{L_2} &+ \int_0^t (\partial_x u_s, -\phi \partial_x a_s - a_s \partial_x \phi + b_s \phi)_{L_2} + (c_s u_s, \phi)_{L_2} \, ds \\ &- \int_0^t \int_{|z|\leq 1} \int_0^1 (1-\theta) z^2 (\partial_x u_s(\cdot + \theta z), \partial_x \phi)_{L_2} \, d\theta \nu(dz) ds \\ &+ \int_0^t \int_{|z|>1} (u_s(\cdot + z) + u_s, \phi)_{L_2} \, \nu(dz) ds.\end{aligned}$$

The following well-posedness result can be found in for example in [13].

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold. Then* (1.1) *has a unique solution $u \in C_w([0,T]; H^1)$. Moreover, $u \in C_w([0,T]; H^m) \cap C([0,T]; H^{m-1})$, and there exists a constant N, depending only on T, m, K, μ_0 and μ_2, such that*
$$\sup_{t \leq T} \|u_t\|_{H^m}^2 \leq N \mathcal{K}_m^2.$$

Remark 2.1. If Assumption 2.1 holds with $m \geq 2$ in the above theorem, then the solution is strongly continuous H^1 valued function, which by the continuous embedding $H^1 \hookrightarrow C^{0,1/2}$ (space of bounded 1/2-Hölder continuous functions with the usual norm) implies that the solution $u_t(x)$ is a continuous function of $(t,x) \in [0,T] \times \mathbb{R}$.

For $\lambda \in \mathbb{R} \setminus \{0\}$ we define the following operators

$$\delta_\lambda \phi(x) := \frac{\phi(x+\lambda) - \phi(x)}{\lambda}, \ \delta^\lambda \phi(x) := \frac{(\delta_\lambda + \delta_{-\lambda})\phi(x)}{2}.$$

We continue with the approximation of the integro-differential operator. For $h \in (0,1)$, we will denote our grid by $\mathbb{G}_h := h\mathbb{Z}$, and for integers $k \geq 1$ we define

$$B^h_k := ((k-1)h, kh],$$

while for integers $k \leq -1$ we define

$$B^h_k := [kh, (k+1)h).$$

Notice that B^h_0 is not defined. From now on we assume that $h \in \{1/n : n \in \mathbb{N}_+\} =: \mathfrak{N}$. We set $\mathbb{A}_h := \{m \in \mathbb{Z} : |m| \leq 1/h, m \neq 0\}$ and $\mathbb{B}_h := \mathbb{Z} \setminus (\mathbb{A}_h \cup \{0\})$. Let us define the operators

$$\begin{aligned} J^h_1 \phi(x) &:= \sum_{k \in \mathbb{A}_h} \zeta^h_k \sum_{l=0}^{|k|-1} \theta^l_k \delta_{-h} \delta_h \phi(x + s_k l h), \\ J^h_2 \phi(x) &:= \sum_{k \in \mathbb{B}_h} (\phi(x+hk) - \phi(x))\, \nu(B^h_k), \end{aligned}$$

where

$$s_k = \frac{k}{|k|}, \ \zeta^h_k := \int_{B^h_k} z^2 \ \nu(dz), \ \theta^l_k := \int_{l/|k|}^{(l+1)/|k|} (1-\theta) \ d\theta.$$

We denote $J^h := J^h_1 + J^h_2$.

Example 2.1. In the case that the Lévy measure ν is given by $\nu(dz) = |z|^{-(1+\alpha)}\, dz$ the operator J^h takes the following form

$$\begin{aligned} &J^h \phi(x) \\ &= \sum_{k=1}^{\lfloor \frac{1}{h} \rfloor} \sum_{l=0}^{|k|-1} \mathcal{C}_\alpha(h,k) \left(\phi(x+lh+h) - 2\phi(x+lh) + \phi(x+lh-h)\right) \\ &+ \sum_{k=-\lfloor \frac{1}{h} \rfloor}^{-1} \sum_{l=0}^{|k|-1} \mathcal{C}_\alpha(h,k) \left(\phi(x-lh+h) - 2\phi(x-lh) + \phi(x-lh-h)\right) \\ &+ \sum_{\substack{k \\ |k| > \lfloor \frac{1}{h} \rfloor}} \frac{1}{\alpha} \left(\frac{1}{|(|k|-1)h|^\alpha} - \frac{1}{|kh|^\alpha} \right) \left(\phi(x+hk) - \phi(x)\right), \end{aligned} \tag{2.1}$$

where the coefficients $\mathcal{C}_\alpha(h,k)$ are given by

$$\mathcal{C}_\alpha(h,k) := \frac{\left(|k|^{2-\alpha} - (|k|-1)^{2-\alpha}\right)\left(2(|k|-l)-1\right)}{2(2-\alpha)h^\alpha k^2}.$$

The differential operator L_t is approximated by L^h_t, given by

$$L^h_t\phi(x) := a_t(x)\delta^h\delta^h\phi(x) + b_t(x)\delta^h\phi(x) + c_t\phi(x).$$

We will write $l_2(\mathbb{G}_h)$ for the set of all real valued function ϕ on $\mathbb{G}_h$ such that

$$\|\phi\|^2_{l_2(\mathbb{G}_h)} := h\sum_{x\in\mathbb{G}_h}|\phi(x)|^2 < \infty.$$

We will denote the corresponding inner product by $(\cdot,\cdot)_{l_2(\mathbb{G}_h)}$. Let us now consider in $l_2(\mathbb{G}_h)$ the scheme

$$\begin{aligned} dv^h_t &= \Big((L^h_t + J^h)v^h_t + f_t\Big)\,dt \\ v^h_0 &= \psi. \end{aligned} \tag{2.2}$$

Definition 2.2. A solution of (2.2) is a function $v^h \in C([0,T]; l_2(\mathbb{G}_h))$ such that for all $t \in [0,T]$

$$v^h_t = \phi + \int_0^t (L^h_t + J^h)v^h_s + f_s\ ds,$$

where the equality is understood in $l_2(\mathbb{G}_h)$.

Remark 2.2. For $l \geq 1$ we have the continuous embedding $H^l \hookrightarrow l_2(\mathbb{G}_h)$ (see [9]). Therefore under Assumption 2.1 we have

$$\|\phi\|^2_{l_2(\mathbb{G}_h)} + \int_0^T \|f_t\|^2_{l_2(\mathbb{G}_h)}dt < \infty.$$

Under the same assumption it is easy to see that $L^h_t + J$ is a bounded linear operator on $l_2(\mathbb{G}_h)$ into itself (with norm bounded by a constant uniformly in $t \in [0,T]$). Hence, under Assumption 2.1, (2.2) has a unique solution.

Next is our main result concerning the spatial approximation.

Theorem 2.2. *Let Assumptions 2.1 and 2.2 hold with $m \geq 4$. Let u and v^h be the unique solutions of* (1.1) *and* (2.2) *respectively. Then there exists a constant N, depending only on m, K, μ_0, μ_2, and T, such that for all $h \in \mathfrak{N}$ we have*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{G}_h}|u_t(x) - v^h_t(x)|^2 + \sup_{t\in[0,T]}\|u_t - v^h_t\|^2_{l_2(\mathbb{G}_h)} \leq Nh^2\mathcal{K}^2_m.$$

We now move to the temporal discretization. Let $n \geq 1$ be an integer and let $\tau = T/n$. In $l_2(\mathbb{G}_h)$ we consider the implicit scheme

$$\begin{aligned} v_i &= v_{i-1} + \tau[(L^h_{i\tau} + J^h)v_i + f_{i\tau}],\ i = 1, ..., n \\ v_0 &= \psi. \end{aligned} \tag{2.3}$$

Definition 2.3. A solution of (2.3) is a function $v^{h,\tau}\colon \{0,\ldots,n\} \to l_2(\mathbb{G}_h)$ such that the equalities in (2.3) are satisfied.

Theorem 2.3. *Let Assumptions 2.1 and 2.2 hold. Then there exists a constant N_0, depending only on K and T, such that for all $n > N_0$ and $h \in \mathfrak{N}$ there exists a unique solution $v^{h,\tau}$ of* (2.3).

Assumption 2.3. Let $l \geq 0$ be an integer. There exist constants C and $\gamma > 0$ such that for all $x \in \mathbb{R}$, $t, s \in [0, T]$, and $0 \leq j \leq l$ we have

$$|\partial_x^j a_t(x) - \partial_x^j a_s(x)|^2 + |\partial_x^j b_t(x) - \partial_x^j b_s(x)|^2 + |\partial_x^j c_t(x) - \partial_x^j c_s(x)|^2 \leq C|t-s|^\gamma$$

and

$$\|f_t - f_s\|_{H^l}^2 \leq C|t-s|^\gamma.$$

Assumption 2.4. There exists a constant K' such that for all $t \in [0, T]$ we have $\|f_t\|_{H^{m-2}}^2 \leq K'$.

Next is our result concerning the temporal approximation.

Theorem 2.4. *Let Assumptions 2.1, 2.2 and 2.4 hold with $m \geq 4$, and let Assumption 2.3 hold with $l \geq 1$. Let v^h and $v^{h,\tau}$ be the unique solutions of equations* (2.2) *and* (2.3) *respectively (for $n > N_0$). Then there exists a constant N_0' such that for all $n > N_0'$ and $h \in \mathfrak{N}$*

(i) *the following estimate holds,*

$$\max_{i \leq n} \sup_{x \in \mathbb{G}_h} |v_{i\tau}^h(x) - v_i^{h,\tau}(x)|^2 + \max_{i \leq n} \|v_{i\tau}^h - v_i^{h,\tau}\|_{l_2(\mathbb{G}_h)}^2 \leq \tau^{1\wedge\gamma} N(K' + \mathcal{K}_m^2)$$

(ii) *if moreover $m \geq 5$, then*

$$\max_{i \leq n} \sup_{x \in \mathbb{G}_h} |v_{i\tau}^h(x) - v_i^{h,\tau}(x)|^2 + \max_{i \leq n} \|v_{i\tau}^h - v_i^{h,\tau}\|_{l_2(\mathbb{G}_h)}^2 \leq \tau^{2\wedge\gamma} N(K' + \mathcal{K}_m^2),$$

where N is a constant depending only on K, C, T, m, μ_0 and μ_2.

A direct consequence of the theorem above is the following:

Theorem 2.5. *Under the assumptions of Theorem 2.4, for all $n > N_0'$ and all $h \in \mathfrak{N}$ we have*

(i) *the following estimate holds,*

$$\max_{i \leq n} \sup_{x \in \mathbb{G}_h} |u_{i\tau}(x) - v_i^{h,\tau}(x)|^2 + \max_{i \leq n} \|u_{i\tau} - v_i^{h,\tau}\|_{l_2(\mathbb{G}_h)}^2 \leq N(h^2 + \tau^{1\wedge\gamma})\mathcal{N}_m^2$$

(ii) *if moreover $m \geq 5$, then*

$$\max_{i \leq n} \sup_{x \in \mathbb{G}_h} |u_{i\tau}(x) - v_i^{h,\tau}(x)|^2 + \max_{i \leq n} \|u_{i\tau} - v_i^{h,\tau}\|_{l_2(\mathbb{G}_h)}^2 \leq N(h^2 + \tau^{2\wedge\gamma})\mathcal{N}_m^2,$$

where $\mathcal{N}_m^2 = K' + \mathcal{K}_m^2$, and N is a constant depending only on K, C, T, m, μ_0 and μ_2.

Remark 2.3. Consider the case $\nu(dz) = |z|^{-(1+\alpha)}\, dz$, where the operator J^h is given by (2.1). It is clear from the above theorem that while the constant N depends on α, the rate of converge does not.

3. Auxiliary Facts

In this section we prove some results that will be used in order to prove the main theorems.

Lemma 3.1. *For any integer $l \geq 0$, $\phi \in H^l$, and any $j \in \{0, ..., l\}$, we have*

$$(\partial_x^j J^h \phi, \partial_x^j \phi)_{L_2} \leq 0.$$

Proof. Since $\partial_x J^h \phi = J^h \partial_x \phi$, it clearly suffices to show the conclusion with $l = j = 0$. We have

$$\begin{aligned}(J_2^h \phi(x), \phi(x))_{L_2} &= \sum_{k \in \mathbb{B}_h} \left((\phi(\cdot + hk), \phi)_{L_2} - \|\phi\|_{L_2}^2 \right) \nu(B_k^h) \\ &\leq \sum_{k \in \mathbb{B}_h} \left(\|\phi\|_{L_2}^2 - \|\phi\|_{L_2}^2 \right) \nu(B_k^h) = 0,\end{aligned}$$

where the inequality is due to Hölder's inequality and the translation invariance of the Lebesgue measure. In order to show that $(J_1^h \phi, \phi) \leq 0$, clearly it suffices to show that for each $k \in \mathbb{A}_h$

$$\left(\sum_{l=0}^{|k|-1} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k l h), \phi(x) \right)_{L_2} \leq 0. \tag{3.1}$$

If $s_k = 1$, then a simple calculation shows that

$$\sum_{l=0}^{|k|-1} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k l h)$$

$$= \sum_{l=0}^{|k|-1} \frac{2|k| - (2l+1)}{2k^2 h^2} \left[\phi(x + (l-1)h) - 2\phi(x + lh) + \phi(x + (l+1)h) \right]$$

$$= \frac{1}{2k^2 h^2} \left(\phi(x + kh) + \phi(x + (k-1)h) + (2k-1)\phi(x-h) - (2k+1)\phi(x) \right),$$

which combined with Hölder's inequality imply (3.1). If $s_k = -1$, then

$$\sum_{l=0}^{|k|-1} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k l h)$$

$$= \sum_{l=0}^{|k|-1} \frac{2|k| - (2l+1)}{2k^2 h^2} \left[\phi(x - (l+1)h) - 2\phi(x - lh) + \phi(x - (l-1)h) \right]$$

$$= \frac{1}{2k^2 h^2} \left(\phi(x + kh) + \phi(x + (k+1)h) + (2|k|-1)\phi(x+h) - (2|k|+1)\phi(x) \right),$$

which again by virtue of Hölder's inequality implies (3.1). □

Lemma 3.1 combined with Lemma 3.4 from [9] implies the following:

Lemma 3.2. *Suppose Assumption 2.1 (i) holds. Then there exists a constant N, depending only on K and m, such that for all $l \in \{0, .., m\}$ and all $\phi \in H^m$ we have*

$$(\partial_x^l (L_t^h + J^h)\phi, \partial_x^l \phi)_{L_2} \leq N \|\phi\|_{H^m}^2.$$

The following is very well known (see e.g. [9], [11]).

Lemma 3.3. *For each integer $l \geq 0$, there is a constant N depending only on l, such that for all $u \in H^{l+2}$, $v \in H^{l+3}$, and $\lambda \in \mathbb{R} \setminus \{0\}$ we have*

$$\|\delta^\lambda u - \partial_x u\|_{H^l} + \|\delta_\lambda u - \partial_x u\|_{H^l} \leq N|\lambda| \|u\|_{H^{l+2}},$$

$$\|\delta^\lambda \delta^\lambda v - \partial_x^2 v\|_{H^l} + \|\delta_\lambda \delta_{-\lambda} v - \partial_x^2 v\|_{H^l} \leq N|\lambda| \|v\|_{H^{l+3}}.$$

We have the following consistency estimates for our approximation.

Lemma 3.4. *Let $l \geq 0$ be an integer. Then there exists a constant N, depending only on l, μ_0, and μ_2, such that for all $\phi \in H^{l+3}$ we have*

$$\|J^h \phi - J\phi\|_{H^l} \leq Nh \|\phi\|_{l+3}. \tag{3.2}$$

Proof. Again, we can and we will assume that $l = 0$. We have

$$\begin{aligned} J_2^h \phi(x) - J_2 \phi(x) &= \sum_{k \in \mathbb{B}_h} \int_{B_k^h} (\phi(x + hk) - \phi(x + z))\, \nu(dz) \\ &= \sum_{k \in \mathbb{B}_h} \int_{B_k^h} \int_0^1 (hk - z) \partial_x \phi(x + z + \theta(hk - z)\, d\theta \nu(dz), \end{aligned}$$

which combined with the fact that $|hk - z| \leq h$ for $z \in B_k^h$ gives

$$\|J_2^h \phi - J_2 \phi\|_{L_2} \leq h\mu_0 \|\phi\|_{H^1} \tag{3.3}$$

by virtue of Minkowski's integral inequality. For $J_1^h - J_1$, we have

$$\begin{aligned} &J_1^h \phi - J_1 \phi \\ &= \sum_{k \in \mathbb{A}_h} \zeta_k^h \sum_{l=0}^{|k|-1} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k lh) - \int_{|z| \leq 1} \int_0^1 (1 - \theta) z^2 \partial_x^2 \phi(x + \theta z)\, d\theta \nu(dz) \\ &= \sum_{k \in \mathbb{A}_h} \int_{B_k^h} \sum_{l=1}^{|k|-1} \int_{l/|k|}^{(l+1)/|k|} z^2 (1 - \theta) \left(\delta_{-h} \delta_h \phi(x + s_k lh) - \partial_x^2 \phi(x + \theta z) \right) d\theta \nu(dz). \end{aligned} \tag{3.4}$$

Then we have for the integrand in the above quantity

$$\begin{aligned} &\delta_{-h} \delta_h \phi(x + s_k lh) - \partial_x^2 \phi(x + \theta z) \\ =& \delta_{-h} \delta_h \phi(x + s_k lh) - \delta_{-h} \delta_h \phi(x + \theta z) + \delta_{-h} \delta_h \phi(x + \theta z) - \partial_x^2 \phi(x + \theta z) \\ =& \int_0^1 (s_k lh - \theta z) \delta_{-h} \delta_h \partial_x \phi(x + \theta z + \eta(s_k lh - \theta z))\, d\eta \\ &+ \delta_{-h} \delta_h \phi(x + \theta z) - \partial_x^2 \phi(x + \theta z). \end{aligned} \tag{3.5}$$

Notice that for $\theta \in [l/|k|,(l+1)/|k|)$ and $z \in B^h_k$ we have

$$|s_k lh - \theta z| \le |s_k lh - \theta kh| + |\theta kh - \theta z| \le 2h.$$

Hence, for the first term at the right hand side of (3.5) we have

$$\Big\| \int_0^1 (s_k lh - \theta z)\delta_{-h}\delta_h \partial_x \phi(\cdot + \theta z \eta(s_k lh - \theta z))d\eta \Big\|_{L_2} \le 2h\|\phi\|_{H^3},$$

while for the second one we have by Lemma 3.3

$$\|\delta_{-h}\delta_h \phi(\cdot + \theta z) - \partial_x^2 \phi(\cdot + \theta z)\|_{L_2} \le h\|\phi\|_{H^3}.$$

Therefore,

$$\|\delta_{-h}\delta_h \phi(\cdot + s_k lh) - \partial_x^2 \phi(\cdot + \theta z)\|_{L_2} \le Nh\|\phi\|_{H^3},$$

which combined with 3.4 and Minkowski's inequality gives

$$\|J^h_1 \phi - J_1\phi\|_{L_2} \le Nh\|\phi\|_{H^3}.$$

By this inequality and (3.3) we obtain (3.2). $\square$

Lemma 3.5. *Let (i) from Assumption 2.1 hold. Then there exists a constant N, depending only on K, m, μ_0, and μ_2, such that for all $l \in \{0,\dots,\}$, $\phi \in H^{l+2}$, and $t \in [0,T]$ we have*

$$\|L^h_t \phi\|^2_{H^l} + \|J^h \phi\|^2_{H^l} \le N\|\phi\|^2_{H_{l+2}}.$$

Proof. Clearly it suffices to show the inequality for $\phi \in C^\infty_c$. We have for $\lambda \neq 0$

$$\delta_\lambda \phi(x) = \int_0^1 \partial_x \phi(x + \theta\lambda)\, d\theta.$$

Hence, by Minkowski's inequality we get $\|\delta_\lambda \phi\|_{L_2} \le \|\partial_x \phi\|_{L_2}$, which implies

$$\|\delta^\lambda \phi\|_{L_2} \le N\|\phi\|_{H^1},\ \|\delta^\lambda\delta^\lambda \phi\|_{L_2} \le N\|\phi\|_{H^2}, \|\delta_\lambda\delta_{-\lambda}\phi\|_{L_2} \le N\|\phi\|_{H^2}.$$

Hence, by Assumption 2.1 (i) we have

$$\|L^h_t \phi\|^2_{H^l} \le N\|\phi\|^2_{H^{l+2}}.$$

By Minkowski's inequality, we have

$$\|J^h_1 \phi\|_{L_2} \le \sum_{k\in\mathbb{A}_h} \zeta^h_k \sum_{l=0}^{|k|-1} \theta^l_k \|\delta_{-h}\delta_h \phi(\cdot + s_k lh)\|_{L_2} \le \frac{1}{2}\mu_2\|\phi\|_{H^2}$$

and

$$\|J^h_2 \phi\|_{L_2} \le \sum_{k\in\mathbb{B}^h} \|\phi(x+hk) - \phi\|_{L_2} \le 2\mu_0\|\phi\|_{L_2}.$$

These estimates combined with the fact that $\partial_x J^h = J^h \partial_x$ give

$$\|J^h \phi\|_{H^l} \le N\|\phi\|_{H^{l+2}}.$$

This finishes the proof. $\square$

Next we consider in $L_2(\mathbb{R})$ the following scheme

$$
\begin{aligned}
du_t^h &= \Big((L_t^h + J^h)u_t^h + f_t\Big)\,dt \\
u_0^h &= \psi.
\end{aligned}
\tag{3.6}
$$

An L_2-solution of (3.6) is a function $u^h \in C([0,T]; L_2(\mathbb{R}))$ such that for all $t \in [0,T]$

$$
u_t^h = \psi + \int_0^t \Big((L_s^h + J^h)u_s^h + f_s\Big)\,ds.
$$

Lemma 3.6. *Let Assumption 2.1 hold with some integer $l \geq 1$ instead of m. Then* (3.6) *has a unique L_2-solution u^h which also belongs to $C([0,T]; H^l)$. If moreover Assumption 2.2 holds, then there exists a constant N, depending only on l,T, and K, such that for all $h \in \mathfrak{N}$*

$$
\sup_{t \leq T} \|u_t^h\|_{H^l}^2 \leq N\mathcal{K}_l^2. \tag{3.7}
$$

Proof. Equation (3.6) is a differential equation on L_2 with Lipschitz continuous coefficients, and it therefore has a unique L_2-valued continuous solution u^h. Similarly, it is a differential equation on H^l with Lipschitz continuous coefficients, and it therefore has a unique H^l-valued continuous solution w^h. Since $H^l \subset L_2$ we have that $w^h = u^h$.

For (3.7), we have for any $t \in [0,T]$

$$
\begin{aligned}
\|u_t^h\|_{H_l}^2 &= \|\psi\|_{H^l}^2 + \int_0^t \Big[\Big((L_s^h + J^h)u_s^h, u_s^h\Big)_{H^l} + (f_s, u_s^h)_{H^l}\Big]\,ds \\
&\leq \|\psi\|_{H^l}^2 + N\int_0^t \|u_s^h\|_{H^l}^2\,ds + \int_0^T \|f_s\|_{H^l}^2\,ds < \infty,
\end{aligned}
$$

where the last inequality is by virtue of Lemma 3.2 and Young's inequality. Gronwall's lemma finishes the proof. □

Theorem 3.7. *Let Assumptions 2.1 and 2.2 with $m \geq 4$, and let u^h and u be the unique solutions of* (3.6) *and* (1.1) *respectively. Then there exists a constant N, depending only on m, T, μ_0, μ_2, and K, such that for all $h \in \mathfrak{N}$ we have*

$$
\sup_{t \leq T} \|u_t - u_t^h\|_{H^{m-3}}^2 \leq N\mathcal{K}_m^2 h^2. \tag{3.8}
$$

Proof. We have that $u^h - u$ satisfies the conditions of Lemma 3.6 with $l = m-3$, $\psi = 0$, and $f_t = (L_t^h - L_t)u_t + (I^h - I)u_t$. Therefore, we have

$$
\sup_{t \leq T} \|u_t^h - u_t\|_{H^{m-3}}^2 \leq N\int_0^T \|(L_t^h - L_t)u_t + (J^h - J)u_t\|_{H^{m-3}}^2\,dt \tag{3.9}
$$

$$
\leq N h^2 \int_0^T \|u\|_{H^m}^2\,dt \leq N h^2 \mathcal{K}_m^2. \tag{3.10}
$$

where the second inequality follows from Lemmata 3.3 and 3.4. This finishes the proof. □

Next we continue with the time discretization. Let us consider on $L_2(\mathbb{R})$ the following implicit scheme.

$$\begin{aligned} u_i &= u_{i-1} + \tau[(L^h_{i\tau} + J^h)u_i + f_{i\tau}],\ i = 1, ..., n \\ u_0 &= \psi. \end{aligned} \tag{3.11}$$

An L_2-solution of the above scheme is a function $u^{h,\tau}\colon \{0, \dots, n\} \to L_2(\mathbb{R})$.

The following is well known (see Proposition 3.4 in [8] for a statement that is in fact more general).

Lemma 3.8. *Let $\mathbb{D}$ be a bounded linear operator on a Hilbert space X into itself. If there exists $\delta > 0$ such that $(\mathbb{D}\phi, \phi)_X \geq \delta\|\phi\|^2_X$ for all $\phi \in X$, then for each $f \in X$ there exists a unique $g \in X$ such that $\mathbb{D}g = f$.*

Theorem 3.9. *Let Assumptions 2.1 and 2.2 hold. Then there exists a constant N', depending only on K, T, and m, such that for all $n > N'$ and $h \in \mathfrak{N}$ there exists a unique L_2-solution $u^{h,\tau}$ of* (3.11) *which in addition satisfies $u^{h,\tau}_i \in H^m$ for each $i = 0, ..., n$.*

Proof. Let us write (3.11) in the form

$$\mathbb{D}_i u_i = F_i,\ i = 1, ..., n,$$

where

$$\mathbb{D}_i = I - \tau(L^h_{i\tau} + J^h),\ F_i = v_{i-1} + \tau f_{i\tau}.$$

For each $i = 1, ..., n$, $\mathbb{D}_i$ is a bounded linear operator from H^k to H^k for all $k = 0, ..., m$. By Lemma 3.2 we have

$$(\mathbb{D}_i\phi, \phi)_{H^k} = \|\phi\|^2_{H^k} - \tau\left((L^h_{i\tau} + J^h)\phi, \phi\right)_{H^k} \geq \|\phi\|^2_{H^k} - \tau N\|\phi\|^2_{H^k},$$

for all $k = 0, .., m$, with N depending only on K and m. Hence, if $n > TN$, then we have with $\lambda := 1 - (\tau/N) > 0$

$$(\mathbb{D}_i\phi, \phi)_{H^k} \geq \lambda\|\phi\|^2_{H^k}.$$

The conclusion follows from the lemma above.

□

Theorem 3.10. *Let Assumptions 2.1, 2.2 and 2.4 hold with $m \geq 4$ and let u^h and $u^{h,\tau}$ be the unique L_2-solutions of equations* (3.6) *and* (3.11) *respectively (for $n > N'$). Then there exists a constant N_1 such that for all $n > N_1$ and $h \in \mathfrak{N}$ the following hold*

(i) *if Assumption 2.3 holds with $l = m - 3$, then*

$$\max_{i \leq n} \|u^h_{i\tau} - u^{h,\tau}_i\|^2_{H^{m-3}} \leq \tau^{1\wedge\gamma} N(K' + \mathcal{K}^2_m) \tag{3.12}$$

(ii) *if Assumption 2.3 holds with $l = m - 4$, then*

$$\max_{i \leq n} \|u^h_{i\tau} - u^{h,\tau}_i\|^2_{H^{m-4}} \leq \tau^{2\wedge\gamma} N(K' + \mathcal{K}^2_m), \tag{3.13}$$

where N is a constant depending only on K, C, T, m, μ_0 and μ_2.

Proof. In order to ease the notation, let us introduce $e_i = u^h_{i\tau} - u^{h,\tau}_i$. We have that $(e_i)_{i=0}^n$ satisfies

$$\begin{aligned} &e_i = e_{i-1} + \tau \mathbb{R}_i e_i + \mathbf{F}_i,\ i = 1, , , .n, \\ &e_0 = 0, \end{aligned}$$

where

$$\mathbb{R}_i = L^h_{i\tau} + J^h,\ \mathbf{F}_i := \int_{(i-1)\tau}^{\tau} F_t\ dt$$

$$F_t := (L^h_t + J^h)u^h_t - (L^h_{k(t)} + J^h)u^h_{k(t)} + f_t - f_{k(t)},$$

and

$$k(t) = l\tau \text{ for } t \in ((l-1)\tau, l\tau],\ l = 1, ..., n,\ k(0) = 0. \tag{3.14}$$

By the identity $\|b\|^2 - \|a\|^2 = 2(b, b-a) - \|b-a\|^2$, we have for $j \le m-3$ and $i \ge 1$,

$$\|\partial^j_x e_i\|^2_{L_2} - \|\partial^j_x e_{i-1}\|^2_{L_2} \le 2\tau(\partial^j_x e_i, \partial^j_x \mathbb{R}_i e_i)_{L_2} + 2(\partial^j_x e_i, \partial^j_x \mathbf{F}_i)_{L_2} \tag{3.15}$$

By Lemma 3.2 we have

$$2\tau(\partial^j_x e_i, \partial^j_x \mathbb{R}_i e_i)_{L_2} \le \tau N \|\partial^j_x e_i\|^2_{L_2},$$

while by Young's inequality we have

$$\begin{aligned} 2(\partial^j_x e_i, \partial^j_x \mathbf{F}_i)_{L_2} &\le \tau \|\partial^j_x e_i, \|^2_{L_2} + \tau^{-1} \| \int_{(i-1)\tau}^{i\tau} \partial^j_x F_t\, dt\|^2_{L_2} \\ &\le \tau \|\partial^j_x e_i, \|^2_{L_2} + \int_{(i-1)\tau}^{i\tau} \|\partial^j_x F_t\|^2_{L_2}\, dt. \end{aligned}$$

By using these inequalities and summing up (3.15) over $0 \le j \le q$, where $q \in \{m-4, m-3\}$, and over $i \le \rho \le n$, we get

$$\|e_\rho\|^2_{H^q} \le \tau N \sum_{i=1}^{\rho} \|e_i\|^2_{H^q} + N \int_0^T \|F_t\|^2_{H^q}\ dt < \infty,$$

where N is a constant depending only on m and K. Let us set $N_1 := TN$. By the discrete Gronwall inequality we have for $n > N_1$ (i.e. for $\tau < 1/N$)

$$\max_{\rho \le n} \|e_\rho\|^2_{H^q} \le N \int_0^T \|F_t\|^2_{H^q}\ dt,$$

where N depends only on m, K and T. We only have to estimate the term at the right hand side of the above inequality. We have

$$\begin{aligned} \int_0^T \|F_t\|^2_{H^q} dt \le & N \int_0^T \|(L^h_t - L^h_{k(t)})u^h_t\|^2_{H^q}\, dt \\ &+ N \int_0^T \|(J^h + L^h_{k(t)})(u^h_t - u^h_{k(t)})\|^2_{H^q}\, dt \\ &+ N \int_0^T \|f_t - f_{k(t)}\|^2_{H^q}\, dt, \end{aligned} \tag{3.16}$$

where $k_n(t)$ is given by (3.14). Let us show first (3.12) under Assumption 2.3 with $l=m-3$. By Assumption 2.3 and (3.7) we have with $q=m-3$

$$\int_0^T \|(L^h_t - L^h_{k(t)})u^h_t\|^2_{H^q}\,dt \le \tau^\gamma N \int_0^T \|u^h_t\|^2_{H^{q+2}}\,dt \le \tau^\gamma N \mathcal{K}^2_{q+2} \tag{3.17}$$

$$\int_0^T \|f_t - f_{k(t)}\|^2_{H^q}\,dt \le \tau^\gamma T. \tag{3.18}$$

By Lemma 3.5 we have

$$\int_0^T \|(J^h + L^h_{k(t)})(u^h_t - u^h_{k(t)})\|^2_{H^q}\,dt \le N \int_0^T \|u^h_t - u^h_{k(t)}\|^2_{H^{q+2}}\,dt.$$

Therefore, in order to show (i) we only need to show that

$$\int_0^T \|u^h_t - u^h_{k(t)}\|^2_{H^{m-1}}\,dt \le N\tau(\mathcal{K}^2_m + K'). \tag{3.19}$$

For $\phi \in H^{m-1}$ and $\phi' \in H^m$, one has $|(\phi',\phi)_m| \le \|\phi'\|_{H^m}\|\phi\|_{H^{m-2}}$. By using this and Young's inequality, we obtain for $s,t\in[0,T]$ with $s\le t$

$$\begin{aligned}
\|u^h_t - u^h_s\|^2_{m-1} &= 2\int_s^t \Big(u^h_r - u^h_s, (L^h_r + J^h)u_r + f_r\Big)_{m-1}\,dr \\
&\le N\int_s^t \|u^h_r - u^h_s\|^2_{H^m} + \|(L^h_r + J^h)u^h_r\|^2_{H^{m-2}} + \|f_r\|^2_{H^{m-2}}dr \\
&\le N\int_s^t \sup_{t'\le T}\|u^h_{t'}\|^2_{H^m} + \|f_r\|^2_{H^{m-2}}dr \\
&\le N(\mathcal{K}^2_m + K')(t-s),
\end{aligned}$$

where the last inequality follows by Lemma 3.6 and Assumption 2.4. This shows (3.19) which combined with (3.17) and (3.18) (with $q=m-3$) imply (3.12) by virtue of (3.16). In order to show (3.13) under Assumption 2.3 with $l=m-4$, by virtue of (3.16), (3.17) and (3.18), with $q=m-4$, it suffices to show

$$\int_0^T \|u^h_t - u^h_{k(t)}\|^2_{H^{m-2}}\,dt \le N\tau^2(\mathcal{K}^2_m + K').$$

For $t,s\in[0,T]$ we have

$$\begin{aligned}
\|u^h_t - u^h_s\|^2_{H^{m-2}} &\le \Big\|\int_s^t (L^h_r + J^h)u^h_r + f_r\ dr\Big\|^2_{H^{m-2}} \\
&\le \Big(\int_s^t N\sup_{t'\le T}\|u^h_{t'}\|_{H^m} + \|f_r\|_{H^{m-2}}\ dr\Big)^2 \\
&\le N(t-s)^2(\mathcal{K}^2_m + K').
\end{aligned}$$

This brings the proof to an end.

□

4. Proofs of the main results

We are now ready to prove the main theorems.

Proof of Theorem 2.2. Let $\mathfrak{I}$, $\mathfrak{K}$ denote the continuous embeddings $H^{m-3} \hookrightarrow l_2(\mathbb{G}_h)$ and $H^{m-3} \hookrightarrow C^{0,1/2}$. Let u^h and v^h denote the solutions of (3.6) and (2.2) (the same equation, considered on $l_2(\mathbb{G}_h)$ and $L_2(\mathbb{G}_h)$). By applying $\mathfrak{I}$ to both sides of (3.6) we see that $\mathfrak{I}u^h$ satisfies (2.2). Therefore $\mathfrak{I}u^h = v^h$ by uniqueness. Notice also that $\mathfrak{K}u^h_t(x) = \mathfrak{I}u^h_t(x)$ and $u_t(x) = \mathfrak{I}u_t(x) = \mathfrak{K}u_t(x)$, for all $t \in [0,T]$ and $x \in \mathbb{G}_h$. Hence

$$\begin{aligned}\sup_{x\in\mathbb{G}_h} |v^h_t(x) - u_t(x)| &= \sup_{x\in\mathbb{G}_h} |\mathfrak{I}u^h_t(x) - u_t(x)| \\ &= \sup_{x\in\mathbb{G}_h} |\mathfrak{K}u^h_t(x) - \mathfrak{K}u_t(x)| \\ &\le N\|u^h_t - u_t\|_{H^{m-3}},\end{aligned}$$

and

$$\begin{aligned}\|v^h_t - u_t\|_{l_2(\mathbb{G}_h)} &= \|\mathfrak{I}u^h_t - \mathfrak{I}u_t\|_{l_2(\mathbb{G}_h)} \\ &\le N\|u^h_t - u_t\|_{H^{m-3}},\end{aligned}$$

where N depends only on m. The conclusion now follows from Theorem 3.7. □

We move to the proof of Theorem 2.3. Notice that the existence part follows easily from Theorem 3.9. Namely, if $u^{h,\tau}$ solves (3.11), then $\mathfrak{I}u^{h,\tau}$ solves (2.3). Also, the uniqueness part is immediate if for example one poses a Courant–Friedrichs–Lewy condition on τ and h (that is, τ/h^2 being sufficiently small). However, such a condition is obviously not necessary, therefore, in order to prove Theorem 2.3, we will proceed as in the proof of Theorem 3.9. Hence, we need the following, whose proof is essentially given in [10] but we give a sketch for the convenience of the reader.

Lemma 4.1. *Let Assumptions 2.1 and 2.2 hold with $m = 1$. Then there exists a constant N, depending only on K, such that for all $\phi \in l_2(\mathbb{G}_h)$ we have*

$$\Big((L^h_t + J^h)\phi, \phi\Big)_{l_2(\mathbb{G}_h)} \le N\|\phi\|^2_{l_2(\mathbb{G}_h)}.$$

Proof. One can replace $(\cdot,\cdot)$ with $(\cdot,\cdot)_{l_2(\mathbb{G}_h)}$ in the proof of Lemma 3.1 to obtain

$$(J^h\phi, \phi)_{l_2(\mathbb{G}_h)} \le 0.$$

Consequently we only need that $(L^h_t\phi, \phi)_{l_2(\mathbb{G}_h)} \le N\|\phi\|^2_{l_2(\mathbb{G}_h)}$. This is proved in [10]. In the proof of Lemma 3.3 in that article, one can replace $(\cdot,\cdot)$ with $(\cdot,\cdot)_{l_2(\mathbb{G}_h)}$ to obtain

$$|(\delta^h\phi, (\delta^h a_t)T^h\phi)_{l_2(\mathbb{G}_h)}| + |(b_t\delta^h\phi, \phi)_{l_2(\mathbb{G}_h)}| + |(c_t\phi, \phi)_{l_2(\mathbb{G}_h)}| \le N\|\phi\|^2_{l_2(\mathbb{G}_h)}, \tag{4.1}$$

where $T^h\phi(x) = (\phi(x+h) + \phi(x-h))/2$, and N depends only on K. It is shown also in [10] (see (3.3)) that for functions u, v

$$\delta^h(uv) = (\delta^h u)T^h v + (\delta^h v)T^h u.$$

Therefore,

$$\begin{aligned}(a_t\delta^h\delta^h u_t, u_t)_{l_2(\mathbb{G}_h)} &= -(\delta^h u_t, \delta^h(a_t u_t))_{l_2(\mathbb{G}_h)} \\ &= -(\delta^h u_t, (\delta^h a_t)T^h u_t)_{l_2(\mathbb{G}_h)} - (\delta^h u_t, (T^h a_t)\delta^h u_t)_{l_2(\mathbb{G}_h)}.\end{aligned} \tag{4.2}$$

Notice that by virtue of Assumption 2.2, we have

$$-(\delta^h u_t, (T^h a_t)\delta^h u_t)_{l_2(\mathbb{G}_h)} \leq 0.$$

Hence, (4.2) and (4.1) imply

$$(L_t^h\phi, \phi)_{l_2(\mathbb{G}_h)} \leq N\|\phi\|^2_{l_2(\mathbb{G}_h)}.$$

□

Proof of Theorem 2.3. The proof is the same as the one of Theorem 3.9, this time using Lemma 4.1 instead of Lemma 3.2. □

Proof of Theorem 2.4. The conclusion follows by Sobolev embeddings and Theorem 3.9 similarly to the proof of Theorem 2.2. □

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