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Optimal stopping with f-expectations: the irregular case

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Abstract

We consider the optimal stopping problem with non-linear f-expectation (induced by a BSDE) without making any regularity assumptions on the payoff process ξ and in the case of a general filtration. We show that the value family can be aggregated by an optional process Y. We characterize the process Y as the $\mathcal{E}^f$-Snell envelope of ξ. We also establish an infinitesimal characterization of the value process Y in terms of a Reflected BSDE with ξ as the obstacle. To do this, we first establish some useful properties of irregular RBSDEs, in particular an existence and uniqueness result and a comparison theorem.

Keywords: backward stochastic differential equation, optimal stopping, f-expectation, non-linear expectation, aggregation, dynamic risk measure, American option, strong $\mathcal{E}^f$-supermartingale, Snell envelope, reflected backward stochastic differential equation, comparison theorem, Tanaka-type formula, general filtration

1. Introduction

The *classical* optimal stopping probem with *linear expectations* has been largely studied. General results on the topic can be found in El Karoui (1981) ([12]) where no regularity assumptions on the reward process ξ are made.

In this paper, we are interested in a generalization of the classical optimal stopping problem where the linear expectation is replaced by a possibly non-linear functional, the so-called f-expectation (f-evaluation), induced by a BSDE with Lipschitz driver f. For a stopping time S such that $0 \leq S \leq T$ a.s. (where $T > 0$ is a fixed terminal horizon), we define

$$V(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau), \tag{1.1}$$

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where $\mathcal{T}_{S,T}$ denotes the set of stopping times valued a.s. in $[S,T]$ and $\mathcal{E}^f_{S,\tau}(\cdot)$ denotes the conditional f-expectation/evaluation at time S when the terminal time is τ.

The above non-linear problem has been introduced in [16] in the case of a Brownian filtration and a continuous financial position/pay-off process ξ and applied to the (non-linear) pricing of American options. It has then attracted considerable interest, in particular, due to its links with dynamic risk measurement (cf., e.g., [2]). In the case of a financial position/payoff process ξ, only supposed to be right-continuous, this non-linear optimal stopping problem has been studied in [37] (the case of Brownian-Poisson filtration), and in [1] where the non-linear expectation is supposed to be convex. To the best of our knowledge, [19] is the first paper addressing the stopping problem (1.1) in the case of a non-right-continuous process ξ (with a Brownian-Poisson filtration); in [19] the assumption of right-continuity of ξ from the previous literature is replaced by the weaker assumption of right- uppersemicontinuity (r.u.s.c.).

In the present paper, we study problem (1.1) in the case of a general filtration and without making any regularity assumptions on ξ, which allows for more flexibility in the modelling (compared to the cases of more regular payoffs and/or of particular filtrations).

The usual approach to address the classical optimal stopping problem (i.e., the case $f \equiv 0$ in (1.1), or the case when f is linear) is a *a direct approach*, based on a direct study of the value family $(V(S))_{S\in\mathcal{T}_{0,T}}$. [1] An important step in this approach is the aggregation of the value family by an optional process.

The approach used in the literature to address the non-linear case (where f is not necessarily linear) is *an RBSDE-approach*, based on the study of a related non-linear Reflected BSDE and on linking directly the solution of the non-linear Reflected BSDE with the value family $(V(S), S \in \mathcal{T}_{0,T})$ (and thus avoiding, in particular, more technical aggregation questions). This approach was first introduced in [16] in the case when the reward process is continuous, and later used to study the right-continuous case (cf. [37]) and the right-uppersemicontinuity (cf. [19]). [2]

Neither of the two approaches is applicable in the general framework of the present paper and we adopt *a new approach which combines* some aspects of both the approaches. Our combined approach is the following: First, with the help of some results from the general theory of processes, we show that the value family $(V(S), S \in \mathcal{T}_{0,T})$ can be aggregated by a unique right-uppersemicontinuous (right-u.s.c.) optional process $(V_t)_{t\in[0,T]}$. We characterize the value process $(V_t)_{t\in[0,T]}$ as the $\mathcal{E}^f$*-Snell envelope* of ξ, that is, the smallest

[1] This direct approach was later used in [1] in the case of a convex non-linear expectation. However, it is not adapted to deal with the non-convex case, as noted in Remark 9.1.

[2] Actually, this *RBSDE approach* is not appropriate to the completely irregular case (cf. Remark 10.1).

strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ. Then, we turn to establishing an infinitesimal characterization of the value process $(V_t)_{t\in[0,T]}$ in terms of a RBSDE where the pay-off process ξ from (1.1) plays the role of a lower obstacle. We emphasize that this *RBSDE-part* of our approach is far from mimicking the one from the r.u.s.c. case; we have to rely on very different arguments here due to the complete irregularity of the process ξ.

Let us recall that Reflected BSDEs have been introduced by El Karoui et al. in the seminal paper [13] in the case of a Brownian filtration and a continuous obstacle, and then generalized to the case of a right-continuous obstacle and/or a larger stochastic basis than the Brownian one in [23], [4], [17], [24], [37]. In [19], we have formulated a notion of Reflected BSDE in the case where the obstacle is only right-u.s.c. (but possibly not right-continuous) and the filtration is the Brownian-Poisson filtration and we have shown existence and uniqueness of the solution. In the present paper, we show that the existence and uniqueness result from [19] still holds in the case of a completely irregular obstacle and a general filtration. In the recent paper [28], existence and uniqueness of the solution (in the Brownian framework) is shown by using a different approach, namely a penalization method.

We also establish a comparison result for RBSDEs with irregular obstacles and general filtration. Due to the complete irregularity of the obstacles and the presence of jumps, we are led to using an approach which differs from those existing in the literature on comparison of RBSDEs (cf. also Remark 9.2); in particular, we first prove a generalization of Gal'chouk-Lenglart's formula (cf. [18] and [32]) to the case of convex functions, which we then astutely apply in our framework in order to establish the comparison theorem. We also show an $\mathcal{E}^f$-Mertens decomposition for strong $\mathcal{E}^f$-supermartingales, which generalizes to our framework the ones provided in the literature (cf. [19] or [3]). This result, together with our comparison theorem, helps in the study of the non-linear operator $\mathcal{R}ef^f$ which maps a given (completely irregular) obstacle to the solution of the RBSDE with driver f. By using the properties of the operator $\mathcal{R}ef^f$, we show that $\mathcal{R}ef^f[\xi]$, that is, the (first component of the) solution to the reflected BSDE with irregular obstacle ξ and driver f, is equal to the $\mathcal{E}^f$-*Snell envelope* of ξ, from which we derive that it coincides with the value process $(V_t)_{t\in[0,T]}$ of problem (1.1).

Finally, we illustrate how this result can be applied to the problem of pricing of American options with irregular pay-off in an imperfect market model with jumps. Some examples of digital American options are given as particular cases.

The rest of the paper is organized as follows: In Section 2 we give some preliminary definitions and some notation. In Section 3 we revisit the classical optimal stopping problem with irregular pay-off process ξ and a general filtration. We first give some general results such as aggregation, Mertens decomposition of the value process, Skorokhod conditions satisfied by the associated non decreasing processes; then, we characterize the value process of the classical problem in terms of the solution of a RBSDE associated with a general filtration, with completely irregular obstacle and with a driver f *which does not depend on the solution*. In Section 4, we prove existence and uniqueness of the solution for *general Lipschitz driver* f, an irregular obstacle ξ and a general filtration. In Section 5, we present the formulation of our *non-linear* optimal stopping problem (1.1). In Section 6, we provide some results on the particular case where the payoff ξ is *right-u.s.c.*, from which we derive an $\mathcal{E}^f$-Mertens decomposition of $\mathcal{E}^f$-strong supermartingales in the (general) framework of a general filtration (cf. Section 7). We then turn to the study of the case where ξ is *completely irregular*. Section 8 is devoted to the direct part of our approach to this problem; in particular, we present the aggregation result and the Snell characterization. Section 9 is devoted to establishing some properties of RBSDEs with completely irregular obstacles, which will be used to establish an infinitesimal characterization of the value process of our problem (1.1) in the completely irregular case; more precisely, we first provide a comparison theorem (Subsection 9.2); then, using this result together with the $\mathcal{E}^f$-Mertens decomposition, we establish useful properties of the non-linear operator $\mathcal{R}ef^f$ (Subsection 9.3). In Section 10, using the results shown in the previous sections, we derive the infinitesimal characterization of the value of the non-linear optimal stopping problem (1.1) with a completely irregular payoff ξ in terms of the solution of our general RBSDE from Section 4. In Section 11 we give a financial application to the pricing of American options with irregular pay-off in an imperfect market model; we also give a useful corollary of the infinitesimal characterization, namely, *a priori estimates with universal constants* for RBSDEs with irregular obstacles and a general filtration.

2. Preliminaries

Let $T > 0$ be a fixed positive real number. Let $E = \mathbf{R}^n \setminus \{0\}, \mathscr{E} = \mathcal{B}(\mathbf{R}^n \setminus \{0\})$, which we equip with a σ-finite positive measure ν. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right-continuous complete filtration $I\!F = \{\mathcal{F}_t \colon t \in [0,T]\}$. Let W be a one-dimensional $I\!F$-Brownian motion W, and let $N(dt, de)$ an $I\!F$-Poisson random measure with compensator $dt \otimes \nu(de)$, supposed to be independent from W. We denote by $\tilde{N}(dt, de)$ the compensated measure, i.e. $\tilde{N}(dt, de) := N(dt, de) - dt \otimes \nu(de)$. We denote by $\mathcal{P}$ the predictable σ-algebra on $\Omega \times [0,T]$. The notation $L^2(\mathcal{F}_T)$ stands for the space of random variables which are $\mathcal{F}_T$-measurable and square-integrable. For $t \in [0,T]$, we denote by $\mathcal{T}_{t,T}$ the set of stopping times τ such that $P(t \le \tau \le T) = 1$. More generally, for a given stopping time $S \in \mathcal{T}_{0,T}$, we denote by $\mathcal{T}_{S,T}$ the set of stopping times τ such that $P(S \le \tau \le T) = 1$.

We use also the following notation:

- L^2_ν is the set of $(\mathscr{E}, \mathcal{B}(\mathbf{R}))$-measurable functions $\ell : E \to \mathbf{R}$ such that $\|\ell\|^2_\nu := \int_E |\ell(e)|^2 \nu(de) <$

∞. For $\ell \in \mathcal{L}^2_\nu$, $\ell\!\!\!k \in \mathcal{L}^2_\nu$, we define $\langle \ell, \ell\!\!\!k \rangle_\nu := \int_E \ell(e) \ell\!\!\!k(e) \nu(de)$.

• $I\!H^2$ is the set of $\mathbf{R}$-valued predictable processes ϕ with $\|\phi\|^2_{I\!H^2} := E\left[\int_0^T |\phi_t|^2 dt\right] < \infty$.

• $I\!H^2_\nu$ is the set of $\mathbf{R}$-valued processes $l : (\omega, t, e) \in (\Omega \times [0,T] \times E) \mapsto l_t(\omega, e)$ which are *predictable*, that is $(\mathcal{P} \otimes \mathscr{E}, \mathcal{B}(\mathbf{R}))$-measurable, and such that $\|l\|^2_{I\!H^2_\nu} := E\left[\int_0^T \|l_t\|^2_\nu \, dt\right] < \infty$.

• As in [19], we denote by $\mathcal{S}^2$ the vector space of $\mathbf{R}$-valued optional (not necessarily cadlag) processes ϕ such that $|||\phi|||^2_{\mathcal{S}^2} := E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,T}} |\phi_\tau|^2] < \infty$. By Proposition 2.1 in [19], the mapping $|||\cdot|||_{\mathcal{S}^2}$ is a norm on $\mathcal{S}^2$, and $\mathcal{S}^2$ endowed with this norm is a Banach space.

• Let $\mathcal{M}^2$ be the set of square integrable martingales $M = (M_t)_{t \in [0,T]}$ with $M_0 = 0$. This is a Hilbert space equipped with the scalar product $(M, M')_{\mathcal{M}^2} := E[M_T M'_T]$ $(= E[\langle M, M' \rangle_T] = E([M, M']_T))$, for $M, M' \in \mathcal{M}^2$ (cf., e.g., [35] IV.3). For each $M \in \mathcal{M}^2$, we set $\|M\|^2_{\mathcal{M}^2} := E(M_T^2)$.

• Let $\mathcal{M}^{2,\perp}$ be the subspace of martingales $h \in \mathcal{M}^2$ satisfying $\langle h, W \rangle_\cdot = 0$, and such that, for all predictable processes $l \in I\!H^2_\nu$,

$$\langle h, \int_0^\cdot \int_E l_s(e) \tilde{N}(ds, de) \rangle_t = 0, \quad 0 \le t \le T \quad \text{a.s.} \tag{2.1}$$

Remark 2.1. *Note that condition* (2.1) *is equivalent to the fact that the square bracket process* $[\, h \,, \int_0^\cdot \int_E l_s(e) \tilde{N}(ds, de) \,]_t$ *is a martingale (cf. the Appendix for additional comments on condition* (2.1)*).*

Recall also that the condition $\langle h, W \rangle_\cdot = 0$ *is equivalent to the orthogonality of* h *(in the sense of the scalar product* $(\cdot, \cdot)_{\mathcal{M}^2}$*) with respect to all stochastic integrals of the form* $\int_0^\cdot z_s dW_s$*, where* $z \in I\!H^2$ *(cf. e.g., [35] IV. 3 Lemma 2). Similarly, the condition* (2.1) *is equivalent to the orthogonality of* h *with respect to all stochastic integrals of the form* $\int_0^\cdot \int_E l_s(e) \tilde{N}(ds, de)$*, where* $l \in I\!H^2_\nu$ *(cf., e.g. , Lemma 12.1 in the Appendix).*

We recall the following orthogonal decomposition property of martingales in $\mathcal{M}^2$ (cf. Lemma III.4.24 in [26]).

Lemma 2.1. *For each* $M \in \mathcal{M}^2$*, there exists a unique triplet* $(Z, l, h) \in I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp}$ *such that*

$$M_t = \int_0^t Z_s dW_s + \int_0^t \int_E l_t(e) \tilde{N}(dt, de) + h_t \,, \quad \forall \, t \in [0,T] \quad a.s. \tag{2.2}$$

Definition 2.1 (Driver, Lipschitz driver). *A function* f *is said to be a* driver *if* $f : \Omega \times [0,T] \times \mathbf{R}^2 \times L^2_\nu \to \mathbf{R}$*;* $(\omega, t, y, z, \ell\!\!\!k) \mapsto f(\omega, t, y, z, \ell\!\!\!k)$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L^2_\nu)-$*measurable, with* $E[\int_0^T f(t, 0, 0, 0)^2 dt] < +\infty$.

A driver f *is called a* Lipschitz driver *if moreover there exists a constant* $K \ge 0$ *such that* $dP \otimes dt$*-a.e. , for each* $(y_1, z_1, \ell\!\!\!k_1) \in \mathbf{R}^2 \times L^2_\nu$*,* $(y_2, z_2, \ell\!\!\!k_2) \in \mathbf{R}^2 \times L^2_\nu$*,*

$$|f(\omega, t, y_1, z_1, \ell\!\!\!k_1) - f(\omega, t, y_2, z_2, \ell\!\!\!k_2)| \le K(|y_1 - y_2| + |z_1 - z_2| + \|\ell\!\!\!k_1 - \ell\!\!\!k_2\|_\nu).$$

Definition 2.2 (BSDE, conditional f-expectation). *We have (cf., e.g., Remark 12.1 in the Appendix) that if f is a Lipschitz driver and if ξ is in $L^2(\mathcal{F}_T)$, then there exists a unique solution $(X,\pi,l,h) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp}$ to the following BSDE:*
$-dX_t = f(t,X_t,\pi_t,l_t)dt - \pi_t dW_t - \int_E l_t(e)\tilde{N}(dt,de) - dh_t; \quad X_T = \xi.$
For $t \in [0,T]$, the (non-linear) operator $\mathcal{E}^f_{t,T}(\cdot) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)$ which maps a given terminal condition $\xi \in L^2(\mathcal{F}_T)$ to the position X_t (at time t) of the first component of the solution of the above BSDE is called conditional f-expectation *at time t. As usual, this notion can be extended to the case where the (deterministic) terminal time T is replaced by a (more general) stopping time $\tau \in \mathcal{T}_{0,T}$, the time t is replaced by a stopping time S such that $S \le \tau$ a.s. and the domain $L^2(\mathcal{F}_T)$ of the operator is replaced by $L^2(\mathcal{F}_\tau)$.*

We now pass to the notion of Reflected BSDE. Let $T > 0$ be a fixed terminal time. Let f be a driver. Let $\xi = (\xi_t)_{t\in[0,T]}$ be a process in $\mathcal{S}^2$.

We define the process $(\overline{\xi}_t)_{t\in(0,T]}$ by $\overline{\xi}_t := \limsup_{s\uparrow t, s<t} \xi_s$, for all $t \in (0,T]$. We recall that $\overline{\xi}$ is a predictable process (cf. [6, Thm. 90, page 225]). The process $\underline{\xi}$ is left-u.s.c. and is called the left upper-semicontinuous envelope of ξ.

Definition 2.3 (Reflected BSDE). *A process (Y,Z,k,h,A,C) is said to be a solution to the reflected BSDE with parameters (f,ξ), where f is a driver and ξ is a process in $\mathcal{S}^2$, if, $(Y,Z,k,h,A,C) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp} \times \mathcal{S}^2 \times \mathcal{S}^2$,*

$$-\,dY_t = f(t,Y_t,Z_t,k_t)dt + dA_t + dC_{t-} - Z_t dW_t - \int_E k_t(e)\tilde{N}(dt,de) - dh_t,\ 0 \le t \le T, \tag{2.3}$$

$Y_T = \xi_T$ a.s., and $Y_t \ge \xi_t$ for all $t \in [0,T]$, a.s.,
A is a nondecreasing right-continuous predictable process with $A_0 = 0$ and such that

$$\int_0^T \mathbf{1}_{\{Y_{t-} > \overline{\xi}_t\}} dA^c_t = 0 \text{ a.s. and } (Y_{\tau-} - \overline{\xi}_\tau)(A^d_\tau - A^d_{\tau-}) = 0 \text{ a.s. for all predictable } \tau \in \mathcal{T}_{0,T}, \tag{2.4}$$

C is a nondecreasing right-continuous adapted purely discontinuous process with $C_{0-} = 0$ and such that $(Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0$ a.s. for all $\tau \in \mathcal{T}_{0,T}$. (2.5)

Here A^c denotes the continuous part of the process A and A^d its discontinuous part.
Equations (2.4) and (2.5) are referred to as *minimality conditions* or *Skorokhod conditions.*

For real-valued random variables X and X_n, $n \in I\!N$, the notation "$X_n \uparrow X$" stands for "the sequence (X_n) is nondecreasing and converges to X a.s.".
For a ladlag process ϕ, we denote by ϕ_{t+} and ϕ_{t-} the right-hand and left-hand limit of ϕ at t. We denote by $\Delta_+\phi_t := \phi_{t_+} - \phi_t$ the size of the right jump of ϕ at t, and by $\Delta\phi_t := \phi_t - \phi_{t-}$ the size of the left jump of ϕ at t.

Remark 2.2. *In the particular case where ξ has left limits, we can replace the process $(\overline{\xi}_t)$ by the process of left limits (ξ_{t-}) in the Skorokhod condition* (2.4)*.*

Remark 2.3. *If (Y, Z, k, h, A, C) is a solution to the RBSDE defined above, by* (2.3)*, we have $\Delta C_t = Y_t - Y_{t+}$, which implies that $Y_t \geq Y_{t+}$, for all $t \in [0, T)$. Hence, Y is r.u.s.c. Moreover, from $C_\tau - C_{\tau-} = -(Y_{\tau+} - Y_\tau)$, combined with the Skorokhod condition* (2.5)*, we derive $(Y_\tau - \xi_\tau)(Y_{\tau+} - Y_\tau) = 0$, a.s. for all $\tau \in \mathcal{T}_{0,T}$. This, together with $Y_\tau \geq \xi_\tau$ and $Y_\tau \geq Y_{\tau+}$ a.s., leads to $Y_\tau = Y_{\tau+} \vee \xi_\tau$ a.s. for all $\tau \in \mathcal{T}_{0,T}$.*

Definition 2.4. *Let $\tau \in \mathcal{T}_{0,T}$. An optional process (ϕ_t) is said to be* right upper-semicontinuous (resp. left upper-semicontinuous*)* along stopping times *if for all stopping time $\tau \in \mathcal{T}_{0,T}$ and for all non increasing (resp. non decreasing) sequence of stopping times (τ_n) such that $\tau_n \downarrow \tau$ (resp. $\tau_n \uparrow \tau$) a.s. , $\phi_\tau \geq \limsup_{n\to\infty} \phi_{\tau_n}$* a.s..

Remark 2.4. *If ξ is left-u.s.c. along stopping times, then the process A is continuous.*[3] *Indeed, let $\tau \in \mathcal{T}_{0,T}$ be a predictable stopping time. For each martingale M, we have $E[\Delta M_\tau / \mathcal{F}_{\tau^-}] = 0$ a.s. Moreover, since A is predictable, we have $E[\Delta A_\tau / \mathcal{F}_{\tau^-}] = \Delta A_\tau$ a.s. By* (2.3)*, we get*

$$E[\Delta Y_\tau / \mathcal{F}_{\tau^-}] = -\Delta A_\tau = -\Delta A_\tau \mathbf{1}_{\{Y_{\tau^-} = \overline{\xi}_\tau\}} \quad \text{a.s.} \tag{2.6}$$

Hence, on $\{Y_{\tau^-} = \overline{\xi}_\tau\}$, we have $E[Y_\tau / \mathcal{F}_{\tau^-}] - Y_{\tau^-} = E[\Delta Y_\tau / \mathcal{F}_{\tau^-}] = -\Delta A_\tau \leq 0$ a.s. Since ξ is left-u.s.c. along stopping times, we thus derive that $\overline{\xi}_\tau \leq E[\xi_\tau / \mathcal{F}_{\tau^-}] \leq E[Y_\tau / \mathcal{F}_{\tau^-}] \leq Y_{\tau^-}$ a.s. on $\{Y_{\tau^-} = \overline{\xi}_\tau\}$, and the inequalities are even equalities. Hence, $E[Y_\tau / \mathcal{F}_{\tau^-}] = Y_{\tau^-}$ a.s. on $\{Y_{\tau^-} = \overline{\xi}_\tau\}$. By (2.6)*, we derive that $\Delta A_\tau = 0$ a.s. This equality being true for every predictable stopping time $\tau \in \mathcal{T}_{0,T}$, it follows that A is continuous.*

3. The classical optimal stopping problem

In this section, we revisit the classical (linear) optimal stopping problem with irregular pay-off process and a general filtration.

3.1. The classical linear optimal stopping problem revisited

Let $(\xi_t)_{t\in[0,T]}$ be a process belonging to $\mathcal{S}^2$, called the *reward process* or the *pay-off process*. For each $S \in \mathcal{T}_{0,T}$, we define the value $v(S)$ at time S by

$$v(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau \mid \mathcal{F}_S]. \tag{3.1}$$

[3] This property (together with our main result Theorem 10.1) generalizes a well-known result shown under an additional (right-u.s.c.) assumption on the process ξ in the literature on classical linear optimal stopping problems (cf. [12] or Proposition B.10 in [29]), and in the literature on reflected BSDEs (cf. Theorem 3.4 in [19]).

Lemma 3.1. **(i)** *There exists a ladlag optional process $(v_t)_{t\in[0,T]}$ which aggregates the family $(v(S))_{S\in\mathcal{T}_{0,T}}$ (i.e. $v_S = v(S)$ a.s. for all $S \in \mathcal{T}_{0,T}$).*
Moreover, the process $(v_t)_{t\in[0,T]}$ is the smallest strong supermartingale greater than or equal to $(\xi_t)_{t\in[0,T]}$.

(ii) *We have $v_S = \xi_S \vee v_{S+}$ a.s. for all $S \in \mathcal{T}_{0,T}$.*

(iii) [4] *For each $S \in \mathcal{T}_{0,T}$ and for each $\lambda \in (0,1)$, the process $(v_t)_{t\in[0,T]}$ is a martingale on $[S, \tau_S^\lambda]$, where $\tau_S^\lambda := \inf\{t \geq S\,, \lambda v_t \leq \xi_t\}$.*

Proof. These results are due to classical results of optimal stopping theory. For a sketch of the proof of the first two assertions, the reader is referred to the proof of Proposition A.5 in the Appendix of [19] (which still holds for a general process $\xi \in \mathcal{S}^2$). The last assertion corresponds to a result of optimal stopping theory (cf. [33], [12] or Lemma 2.7 in [29]). Its proof is based on a penalization method (used in convex analysis), introduced by Maingueneau (1978) (cf. the proof of Theorem 2 in [33]), which does not require any regularity assumption on the reward process ξ. □

Remark 3.1. *It follows from (ii) in the above lemma that $\Delta_+ v_S = \mathbf{1}_{\{v_S=\xi_S\}}\Delta_+ v_S$ a.s.*

Remark 3.2. *Let us note for further reference that Maingueneau's penalization approach for showing the martingale property on $[S, \tau_S^\lambda]$ (property (iii) in the above lemma) relies heavily on the convexity of the problem.*

Lemma 3.2. **(i)** *The value process V of Lemma 3.1 belongs to $\mathcal{S}^2$ and admits the following (Mertens) decomposition:*

$$v_t = v_0 + M_t - A_t - C_{t-},\ \text{for all } t \in [0,T] \text{ a.s.,} \tag{3.2}$$

where $M \in \mathcal{M}^2$, A is a nondecreasing right-continuous predictable process such that $A_0 = 0$, $E(A_T^2) < \infty$, and C is a nondecreasing right-continuous adapted purely discontinuous process such that $C_{0-} = 0$, $E(C_T^2) < \infty$.

(ii) *For each $\tau \in \mathcal{T}_{0,T}$, we have $\Delta C_\tau = \mathbf{1}_{\{v_\tau=\xi_\tau\}}\Delta C_\tau$ a.s.*

(iii) *For each predictable $\tau \in \mathcal{T}_{0,T}$, we have $\Delta A_\tau = \mathbf{1}_{\{v_{\tau-}=\bar{\xi}_\tau\}}\Delta A_\tau$ a.s.*

[4] Note that in the case of a not necessarily non-negative pay-off process ξ this result holds up to a translation by the martingale $X_S := E[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \xi_\tau^- | \mathcal{F}_S]$ (cf. e.g. Appendix A in [31]). More precisely, the property holds for $\tilde{v} := v + X$ and $\tilde{\xi} := \xi + X$.

Proof. By Lemma 3.1 (i), the process $(v_t)_{t\in[0,T]}$ is a strong supermartingale. Moreover, by using martingale inequalities, it can be shown that $E[\operatorname{ess\,sup}_{S\in\mathcal{T}_{0,T}}|v_S|^2]\leq c|||\xi|||^2_{\mathcal{S}^2}$. Hence, the process $(v_t)_{t\in[0,T]}$ is in $\mathcal{S}^2$ (a fortiori, of class (D)). Applying Mertens decomposition for strong supermartingales of class (D) (cf., e.g., [7, Appendix 1, Thm.20, equalities (20.2)]) gives the decomposition (3.2), where M is a cadlag uniformly integrable martingale, A is a nondecreasing right-continuous predictable process such that $A_0=0$, $E(A_T)<\infty$, and C is a nondecreasing right-continuous adapted purely discontinuous process such that $C_{0-}=0$, $E(C_T)<\infty$. Based on some results of Dellacherie-Meyer [7] (cf., e.g., Theorem A.2 and Corollary A.1 in [19]), we derive that $A\in\mathcal{S}^2$ and $C\in\mathcal{S}^2$, which gives the assertion (i).

Let $\tau\in\mathcal{T}_{0,T}$. By Remark 3.1 together with Mertens decomposition (3.2), we get $\Delta C_\tau=-\Delta_+v_\tau$ a.s. It follows that $\Delta C_\tau=\mathbf{1}_{\{v_\tau=\xi_\tau\}}\Delta C_\tau$ a.s. , which corresponds to (ii).

Assertion (iii) (concerning the jumps of A) is due to El Karoui [5] ([12, Proposition 2.34]) Its proof is based on the equality $A_S=A_{\tau_S^\lambda}$ a.s. , for each $S\in\mathcal{T}_{0,T}$ and for each $\lambda\in(0,1)$ (which follows from Lemma 3.1 (iii) together with Mertens decomposition (3.2)). □

The following minimality property for the continuous part A^c is well-known from the literature in the "more regular" cases (cf., e.g., [30] for the right-u.s.c. case). In the case of completely irregular ξ, this minimality property was not explicitly available. Only recently, it was proved by [28] (cf. Proposition 3.7) in the Brownian framework. Here, we generalize the result of [28] to the case of a general filtration by using different analytic arguments.

Lemma 3.3. *The continuous part A^c of A satisfies the equality $\int_0^T\mathbf{1}_{\{v_{t-}>\overline{\xi}_t\}}dA^c_t=0$ a.s.*

Proof. As for the discontinuous part of A, the proof is based on Lemma 3.1 (iii) , and also on some analytic arguments similar to those used in the proof of Theorem D13 in [27].

We have to show that $\int_0^T(v_{t-}-\overline{\xi}_t)dA^c_t=0$ a.s. Lemma 3.1 (iii) yields that for each $S\in\mathcal{T}_{0,T}$ and for each $\lambda\in(0,1)$, we have $A_S=A_{\tau_S^\lambda}$ a.s. Without loss of generality, we can assume that for each ω, the map $t\mapsto A^c_t(\omega)$ is continuous, that the map $t\mapsto v_t(\omega)$ is left-limited, and that, for all $\lambda\in(0,1)\cap\mathbb{Q}$ and $t\in[0,T)\cap\mathbb{Q}$, we have $A_t(\omega)=A_{\tau_t^\lambda}(\omega)$.

[5] Note that the proof in El Karoui [12] is given for nonnegative pay-off ξ. To pass from this to the more general case where ξ might take also negative values, we apply the result by El Karoui [12] with $\tilde{\xi}:=\xi+X$ (which is non-negative) and $\tilde{v}:=v+X$, where the process $X=(X_t)$ is defined by $X_t:=E[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}\xi_\tau^-|\mathcal{F}_t]$. We then notice that the Mertens process (A,C) from the Mertens decomposition of v is the same as the Mertens process $(\tilde{A},\tilde{C})$ from the Mertens decomposition of $\tilde{v}$ (indeed, only the martingale parts of the two decompositions differ by X). Moreover, we see that the set $\{v_{\tau-}=\overline{\xi}_\tau\}$ is the same as the set where v is replaced by $\tilde{v}$ and ξ is replaced by $\tilde{\xi}$ (this is due to the fact that X is a martingale and thus has left limits; so $\overline{X}_t=X_{t-}$).

Let us denote by $\mathcal{J}(\omega)$ the set on which the nondecreasing function $t \mapsto A_t^c(\omega)$ is "flat": $\mathcal{J}(\omega) := \{t \in (0,T)\,,\ \exists \delta > 0 \text{ with } A_{t-\delta}^c(\omega) = A_{t+\delta}^c(\omega)\}$. Since the set $\mathcal{J}(\omega)$ is open, it can be written as a countable union of disjoint intervals: $\mathcal{J}(\omega) = \cup_i (\alpha_i(\omega), \beta_i(\omega))$. We consider

$$\hat{\mathcal{J}}(\omega) := \cup_i (\alpha_i(\omega), \beta_i(\omega)] = \{t \in (0,T]\,,\ \exists \delta > 0 \text{ with } A_{t-\delta}^c(\omega) = A_t^c(\omega)\}. \tag{3.3}$$

We have $\int_0^T \mathbf{1}_{\hat{\mathcal{J}}(\omega)} dA_t^c(\omega) = \sum_i (A^c_{\beta_i(\omega)}(\omega) - A^c_{\alpha_i(\omega)}(\omega)) = 0$. Hence, the nondecreasing function $t \mapsto A_t^c(\omega)$ is "flat" on $\hat{\mathcal{J}}(\omega)$. We introduce

$$\mathcal{K}(\omega) := \{t \in (0,T] \text{ s.t. } v_{t-}(\omega) > \overline{\xi}_t(\omega)\}$$

We next show that for almost every ω, $\mathcal{K}(\omega) \subset \hat{\mathcal{J}}(\omega)$, which clearly provides the desired result. Let $t \in \mathcal{K}(\omega)$. Let us prove that $t \in \hat{\mathcal{J}}(\omega)$. By (3.3), we thus have to show that there exists $\delta > 0$ such that $A_{t-\delta}^c(\omega) = A_t^c(\omega)$. Since $t \in \mathcal{K}(\omega)$, we have $v_{t-}(\omega) > \overline{\xi}_t(\omega)$. Hence, there exists $\delta > 0$ and $\lambda \in (0,1) \cap \mathbb{Q}$ such that $t - \delta \in [0,T) \cap \mathbb{Q}$ and for each $r \in [t-\delta, t)$, $\lambda v_r(\omega) > \xi_r(\omega)$. By definition of $\tau^\lambda_{t-\delta}(\omega)$, it follows that $\tau^\lambda_{t-\delta}(\omega) \geq t$. Now, we have $A^c_{\tau^\lambda_{t-\delta}}(\omega) = A^c_{t-\delta}(\omega)$. Since the map $s \mapsto A_s^c(\omega)$ is nondecreasing, we get $A_t^c(\omega) = A^c_{t-\delta}(\omega)$, which implies that $t \in \hat{\mathcal{J}}(\omega)$. We thus have $\mathcal{K}(\omega) \subset \hat{\mathcal{J}}(\omega)$, which completes the proof. $\square$

Remark 3.3. *We note that the martingale property from assertion (iii) of Lemma 3.1 is crucial for the proof of the minimality conditions for the process A (namely, for the proofs of Lemma 3.2 assertion (iii), and for Lemma 3.3).*

3.2. The classical linear optimal stopping problem with an additional instantaneous reward

In this subsection, we extend the previous results to the case where, besides the reward process ξ, there is an additional running (or instantaneous) reward process $f \in I\!H^2$. More precisely, let $(\xi_t)_{t\in[0,T]}$ be a process belonging to $\mathcal{S}^2$, called the *reward process* or the *pay-off process*. Let $f = (f_t)_{t\in[0,T]}$ be a predictable process with $E[\int_0^T f_t^2 dt] < +\infty$, called the *instantaneous reward process*. For each $S \in \mathcal{T}_{0,T}$, we define the value $V(S)$ at time S by

$$V(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau + \int_S^\tau f_u du \mid \mathcal{F}_S]. \tag{3.4}$$

This is equivalent to $V(S) + \int_0^S f_u du := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau + \int_0^\tau f_u du \mid \mathcal{F}_S]$. Hence, the results of the previous subsection can be applied with $\xi_\cdot$ replaced by $\xi_\cdot + \int_0^\cdot f_u du$ and $v(S)$ replaced by $V(S) + \int_0^S f_u du$. Here is a brief summary.

Lemma 3.4. (i) *There exists a ladlag optional process $(V_t)_{t\in[0,T]}$ which aggregates the family $(V(S))_{S\in\mathcal{T}_{0,T}}$ (i.e. $V_S = V(S)$ a.s. for all $S \in \mathcal{T}_{0,T}$).*
Moreover, the process $(V_t + \int_0^t f_u du)_{t\in[0,T]}$ is the smallest strong supermartingale greater than or equal to $(\xi_t + \int_0^t f_u du)_{t\in[0,T]}$.

(ii) *We have $V_S = \xi_S \vee V_{S+}$ a.s. for all $S \in \mathcal{T}_{0,T}$.*

Remark 3.4. *It follows from (ii) in the above lemma that $\Delta_+ V_S = \mathbf{1}_{\{V_S = \xi_S\}} \Delta_+ V_S$ a.s.*

Lemma 3.5. **(i)** *The value process V of Lemma 3.4 belongs to $\mathcal{S}^2$ and admits the following (Mertens) decomposition:*

$$V_t = V_0 - \int_0^t f_u du + M_t - A_t - C_{t-}, \text{ for all } t \in [0,T] \text{ a.s.}, \tag{3.5}$$

where $M \in \mathcal{M}^2$, A is a nondecreasing right-continuous predictable process such that $A_0 = 0$, $E(A_T^2) < \infty$, and C is a nondecreasing right-continuous adapted purely discontinuous process such that $C_{0-} = 0$, $E(C_T^2) < \infty$.

(ii) *For each $\tau \in \mathcal{T}_{0,T}$, we have $\Delta C_\tau = \mathbf{1}_{\{V_\tau = \xi_\tau\}} \Delta C_\tau$ a.s.*

(iii) *For each predictable $\tau \in \mathcal{T}_{0,T}$, we have $\Delta A_\tau = \mathbf{1}_{\{V_{\tau-} = \overline{\xi}_\tau\}} \Delta A_\tau$ a.s.*

Lemma 3.6. *The continuous part A^c of A satisfies the equality $\int_0^T \mathbf{1}_{\{V_{t-} > \overline{\xi}_t\}} dA_t^c = 0$ a.s.*

3.3. Characterization of the value function as the solution of an RBSDE

In this subsection, we show, using the above lemmas, that the value process V of the classical optimal stopping problem (3.4) solves the RBSDE from Definition 2.3 with parameters the driver process (f_t) and the obstacle (ξ_t). We also prove the uniqueness of the solution of this RBSDE. To this aim, we first provide *a priori* estimates for RBSDEs in our general framework.

Lemma 3.7 (A priori estimates). *Let $(Y^1, Z^1, k^1, h^1, A^1, C^1)$ (resp. $(Y^2, Z^2, k^2, h^2, A^2, C^2)$) $\in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp} \times \mathcal{S}^2 \times \mathcal{S}^2$ be a solution to the RBSDE associated with driver $f^1(\omega, t)$ (resp. $f^2(\omega, t)$) and with obstacle ξ. We set $\tilde{Y} := Y^1 - Y^2$, $\tilde{Z} := Z^1 - Z^2$, $\tilde{A} := A^1 - A^2$, $\tilde{C} := C^1 - C^2$, $\tilde{k} := k^1 - k^2$, $\tilde{h} := h^1 - h^2$, and $\tilde{f}(\omega, t) := f^1(\omega, t) - f^2(\omega, t)$. There exists $c > 0$ such that for all $\varepsilon > 0$, for all $\beta \geq \frac{1}{\varepsilon^2}$ we have*

$$\|\tilde{Z}\|_\beta^2 \leq \varepsilon^2 \|\tilde{f}\|_\beta^2, \ \|\tilde{k}\|_{\nu,\beta}^2 \leq \varepsilon^2 \|\tilde{f}\|_\beta^2 \text{ and } \|\tilde{h}\|_{\beta,\mathcal{M}^2}^2 \leq \varepsilon^2 \|\tilde{f}\|_\beta^2. \tag{3.6}$$

$$|||\tilde{Y}|||_\beta^2 \leq 4\varepsilon^2 (1 + 12c^2) \|\tilde{f}\|_\beta^2. \tag{3.7}$$

The proof is given in the Appendix.

Using these *a priori* estimates, the lemmas from the previous subsection, and the orthogonal martingale decomposition (Lemma 2.1), we derive the following "infinitesimal characterization" of the value process V.

Theorem 3.1. *Let V be the value process of the optimal stopping problem* (3.4)*. Let A and C be the non decreasing processes associated with the Mertens decomposition* (3.5) *of V. There exists a unique triplet $(Z, k, h) \in I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp}$ such that the process (V, Z, k, h, A, C) is a solution of the RBSDE from Definition 2.3 associated with the driver process $f(\omega, t, y, z, \kappa) = f_t(\omega)$ and the obstacle (ξ_t). Moreover, the solution of this RBSDE is unique.*

Proof. By Lemma 3.4 (ii), the value process V corresponding to the optimal stopping problem (3.4) satisfies $V_T = V(T) = \xi_T$ a.s. and $V_t \geq \xi_t$, $0 \leq t \leq T$, a.s. By Lemma 3.5 (ii), the process C of the Mertens decomposition of V (3.5) satisfies the minimality condition (2.5). Moreover, by Lemma 3.5 (iii) and Lemma 3.6, the process A satisfies the minimality condition (2.4). By Lemma 2.1, there exists a unique triplet $(Z, k, h) \in I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp}$ such that $dM_t = Z_t dW_t + \int_E k_t(e)\tilde{N}(dt, de) + dh_t$. The process (V, Z, k, h, A, C) is thus a solution of the RBSDE (2.3) associated with the driver process (f_t) and the obstacle ξ.

The uniqueness of the solution follows from the *a priori estimates* (cf. Lemma 3.7), together with classical arguments (cf. step 5 of the proof of Lemma 3.3 in [19]). □

We are interested in generalizing this result to the case of the optimal stopping problem (1.1) with *non-linear* f-expectation (associated with a non-linear driver $f(\omega, t, y, z, \kappa)$). To this purpose, we first establish an existence and uniqueness result for the RBSDE from Definition 2.3 in the case of a general (non-linear) Lipschitz driver $f(\omega, t, y, z, \kappa)$.

4. Existence and uniqueness of the solution of the RBSDE with an irregular obstacle and a general filtration in the case of a general driver

In Theorem 3.1, we have shown that, in the case where the driver does not depend on y, z, and κ, the RBSDE from Definition 2.3 admits a unique solution. Using this result together with the above *a priori* estimates from Lemma 3.7, we derive the following existence and uniqueness result in the case of a general Lipschitz driver $f(t, y, z, \kappa)$.

By the a priori estimates from Lemma 3.7 and using similar arguments to those used in the right-u.s.c. case (cf. proof of Theorem 3.4 in [19]), we derive the following result:

Theorem 4.1 (Existence and uniqueness). *Let ξ be a process in $\mathcal{S}^2$ and let f be a Lipschitz driver. The RBSDE with parameters (f, ξ) from Definition 2.3 admits a unique solution $(Y, Z, k, h, A, C) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp} \times \mathcal{S}^2 \times \mathcal{S}^2$.*

Remark 4.1. *In [28], the above result is shown in a Brownian framework by using a penalization method. Our approach provides an alternative proof of this result.*

We now provide a useful property of the solution of an RBSDE.

Lemma 4.1 (**$\mathcal{E}^f$-martingale property of** Y)**.** *Let ξ be a process in $\mathcal{S}^2$ and let f be a Lipschitz driver. Let (Y, Z, k, h, A, C) be the solution to the reflected BSDE with parameters (f, ξ) as in Definition 2.3. For each $S \in \mathcal{T}_{0,T}$ and for each $\varepsilon > 0$, we set*

$$\tau_S^\varepsilon := \inf\{t \geq S\,, Y_t \leq \xi_t + \varepsilon\}. \tag{4.1}$$

The process (Y_t) is an $\mathcal{E}^f$-martingale on $[S, \tau_S^\varepsilon]$.

Proof. The proof in our case is identical to that of Lemma 4.1 (ii) in [19] (which does not require any regularity assumption of ξ). It is therefore omitted. □

Remark 4.2. *In the case where ξ is nonnegative, the above result holds true also on the stochastic interval $[S, \tau_S^\lambda]$, where $\lambda \in (0,1)$ and $\tau_S^\lambda := \inf\{t \geq S : \lambda Y_t \leq \xi_t\}$. Note that in the case where $\xi \geq 0$, we have $Y \geq 0$ (as $Y \geq \xi \geq 0$); hence, $\lambda Y_T \leq Y_T = \xi_T$ a.s. and τ_S^λ is finite a.s.*

5. Optimal stopping with non-linear f-expectation: formulation of the problem

Let $(\xi_t)_{t\in[0,T]}$ be a process in $\mathcal{S}^2$. Let f be a Lipschitz driver. For each $S \in \mathcal{T}_{0,T}$, we define *the value at time* S by

$$V(S) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau). \tag{5.1}$$

We make the following assumption on the driver (cf., e.g., Theorem 4.2 in [36]).

Assumption 5.1. *Assume that $dP \otimes dt$-a.e. for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$,*

$$f(t, y, z, k_1) - f(t, y, z, k_2) \geq \langle \theta_t^{y,z,k_1,k_2}\,,\, k_1 - k_2\rangle_\nu,$$

where $\theta : [0,T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \to L^2_\nu\,;\, (\omega, t, y, z, k_1, k_2) \mapsto \theta_t^{y,z,k_1,k_2}(\omega, \cdot)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L^2_\nu)^2)$-measurable mapping, satisfying $\|\theta_t^{y,z,k_1,k_2}(\cdot)\|_\nu \leq C$ for all $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$, $dP \otimes dt$-a.e. , where C is a positive constant, and such that $\theta_t^{y,z,k_1,k_2}(e) \geq -1$, for all $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L^2_\nu)^2$, $dP \otimes dt \otimes d\nu(e) -$ a.e.

We recall that under Assumption 5.1 on the driver f, the functional $\mathcal{E}^f_{S,\tau}(\cdot)$ is nondecreasing (cf. [36, Thm. 4.2] and Remark 12.1 in the Appendix).
As mentioned in the introduction, the above optimal stopping problem has been largely studied: in [16], and in [2], in the case of a continuous pay-off process ξ; in [37] in the case of a right-continuous pay-off; and recently in [19] in the case of a right-u.s.c. pay-off process ξ. In this section, we do not make any regularity assumptions on ξ (cf. also Remark 2.2).

We begin by addressing the simpler case where the payoff is assumed to be *right u.s.c.* This preliminary study of the right u.s.c. case will allow us to establish an $\mathcal{E}^f$-Mertens decomposition for strong $\mathcal{E}^f$-supermartingales with respect to a general filtration (extending the existing results from the literature; cf. [3] and [19]). This will be an important result for the treatment of the non-linear optimal stopping problem in the case of a *completely irregular* pay-off.

6. Optimal stopping with non-linear f-expectation: the right u.s.c. case

Let f be a Lipschitz driver satisfying Assumption 5.1. The following result relies crucially on an assumption of right-uppersemicontinuity of ξ.

Lemma 6.1. *Let $\xi \in \mathcal{S}^2$, supposed to be* right u.s.c. *Let (Y, Z, k, h, A, C) be the solution to the reflected BSDE with parameters (f, ξ) as in Definition 2.3. Let $S \in \mathcal{T}_{0,T}$ and let $\varepsilon > 0$. Let τ_S^ε be the stopping time defined by (4.1), that is, $\tau_S^\varepsilon := \inf\{t \geq S \,, Y_t \leq \xi_t + \varepsilon\}$. We have*

$$Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \quad \text{a.s.} \tag{6.1}$$

Proof. The proof of this result in our case of a general filtration is identical to that of [19, Lemma 4.1(i)] in the case of a Brownian-Poisson filtration. □

By the previous lemma together with Lemma 4.1, we derive the following result:

Theorem 6.1 (Characterization theorem in the r.u.s.c. case). *Let $(\xi_t)_{t\in[0,T]}$ be a process in $\mathcal{S}^2$, supposed to be* right u.s.c. *Let (Y, Z, k, h, A, C) be the solution to the reflected BSDE with parameters (f, ξ) as in Definition 2.3.*
(i) For each stopping time $S \in \mathcal{T}_{0,T}$, we have [6]

$$Y_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) \quad \text{a.s.} \tag{6.2}$$

(ii) Moreover, the stopping time τ_S^ε, defined by $\tau_S^\varepsilon := \inf\{t \geq S, \; Y_t \leq \xi_t + \varepsilon\}$, satisfies

$$Y_S \;\leq\; \mathcal{E}^f_{S,\tau_S^\varepsilon}(\xi_{\tau_S^\varepsilon}) + L\varepsilon \quad \text{a.s.}\,, \tag{6.3}$$

where L is a constant which only depends on T and the Lipschitz constant K of f. In other words, τ_S^ε is an $L\varepsilon$-optimal stopping time for problem (6.2).

[6] In other words, the process (Y_t) aggregates the value family $(V(S), S \in \mathcal{T}_{0,T})$ defined by (5.1), that is $Y_S = V(S)$ a.s. for all $S \in \mathcal{T}_{0,T}$.

Proof. Let us show the inequality (6.3). Since by Lemma 4.1, the process (Y_t) is an $\mathcal{E}^f$-martingale on $[S, \tau_S^\varepsilon]$, we get $Y_S = \mathcal{E}^f_{S,\tau_S^\varepsilon}(Y_{\tau_S^\varepsilon})$ a.s. Since ξ is right u.s.c. , we can apply Lemma 6.1. Using this, the monotonicity property of the conditional f-expectation and the *a priori estimates* for BSDEs (cf. [36] which still hold in our case of a general filtration), we derive that $Y_S = \mathcal{E}^f_{S,\tau_S^\varepsilon}(Y_{\tau_S^\varepsilon}) \leq \mathcal{E}^f_{S,\tau_S^\varepsilon}(\xi_{\tau_S^\varepsilon} + \varepsilon) \leq \mathcal{E}^f_{S,\tau_S^\varepsilon}(\xi_{\tau_S^\varepsilon}) + L\varepsilon$ a.s., where L is a positive constant depending only on T and the Lipschitz constant K of the driver f; this gives the desired inequality (6.3). Moreover, as ε is an arbitrary nonnegative number, we get $Y_S \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s.
It remains to show the converse inequality. Let $\tau \in \mathcal{T}_{S,T}$. By Lemma 12.2 in the Appendix, the process (Y_t) is a strong $\mathcal{E}^f$-supermartingale. Hence, for each $\tau \in \mathcal{T}_{S,T}$, we have $Y_S \geq \mathcal{E}^f_{S,\tau}(Y_\tau) \geq \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s., where the second inequality follows from the inequality $Y \geq \xi$ and the monotonicity property of $\mathcal{E}^f(\cdot)$. By taking the supremum over $\tau \in \mathcal{T}_{S,T}$, we get $Y_S \geq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s. We thus get the equality (6.2), which ends the proof. □

We now investigate the question of the existence of optimal stopping times for the optimal stopping problem (6.2). We first provide an optimality criterion.

Lemma 6.2 (Optimality criterion). *Let ξ be a process belonging to $\mathcal{S}^2$, and let f be a Lipschitz driver satisfying Assumption 5.1. Let $S \in \mathcal{T}_{0,T}$ and $\tau^* \in \mathcal{T}_{S,T}$. If Y is a strong $\mathcal{E}^f$-martingale on $[S, \tau^*]$ with $Y_{\tau^*} = \xi_{\tau^*}$ a.s., then the stopping time τ^* is optimal at time S (i.e. $Y_S = \mathcal{E}^f_{S,\tau^*}(\xi_{\tau^*})$ a.s.). The converse statement also holds true, if, in addition, the inequality from Assumption 5.1 is strict (that is, $\theta_t^{y,z,k_1,k_2} > -1$).*

Proof. The proof of this result is identical to the one in the case of a Brownian-Poisson filtration, given in [19, Proposition 4.1] (which does not require any regularity assumption on ξ). It is therefore omitted. □

We now show that if ξ is assumed to be r.u.s.c. and also l.u.s.c. along stopping times, then there exists an optimal stopping time. Let $S \in \mathcal{T}_{0,T}$. Let us recall the definition of τ_S^ε from before: $\tau_S^\varepsilon := \inf\{t \geq S \,, Y_t \leq \xi_t + \varepsilon\}$. We notice that τ_S^ε is non-increasing in ε. Let (ε_n) be a non-increasing positive sequence converging to 0. We set

$$\hat{\tau}_S := \lim_{n\to\infty} \uparrow \tau_S^{\varepsilon_n}.$$

The random time $\hat{\tau}_S$ is a stopping time in $\mathcal{T}_{S,T}$. We also set

$$\tau_S^0 := \inf\{t \geq S \,, Y_t = \xi_t\}.$$

We notice that $\tau_S^{\varepsilon_n} \leq \tau_S^0$ a.s. for all n. Hence, by passing to the limit, we get $\hat{\tau}_S \leq \tau_S^0$ a.s.

In the following theorem we show that, under the additional assumption that ξ is *l.u.s.c. along stopping times*, the stopping time $\hat{\tau}_S$ is an *optimal* stopping time at time S. We also show that the stopping times $\hat{\tau}_S$ and τ_S^0 coincide.

Theorem 6.2 (Existence of optimal stopping time). *Let $(\xi_t, 0 \leq t \leq T)$ be an r.u.s.c. process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 5.1. We assume, in addition, that (ξ_t) is* l.u.s.c. along stopping times. *Then, the stopping time $\hat{\tau}_S$ is S*-optimal, *in the sense that it attains the supremum in* (6.2). *Moreover, $\hat{\tau}_S = \tau_S^0$ a.s.*

Proof. By applying Fatou's lemma for BSDEs (cf. Lemma A.5 in [11] [7]), we obtain

$$\limsup_{n\to\infty} \mathcal{E}^f_{S,\tau_S^{\varepsilon_n}}\left(\xi_{\tau_S^{\varepsilon_n}}\right) \leq \mathcal{E}^f_{S,\hat{\tau}_S}\left(\limsup_{n\to\infty} \xi_{\tau_S^{\varepsilon_n}}\right) \leq \mathcal{E}^f_{S,\hat{\tau}_S}\left(\xi_{\hat{\tau}_S}\right) \text{ a.s.,} \tag{6.4}$$

where the last inequality follows from the l.u.s.c. (along stopping times) property of ξ and from the monotonicity of $\mathcal{E}^f_{S,\hat{\tau}_S}(\cdot)$. On the other hand, from Eq. (6.3) in Theorem 6.1, we have $Y_S \leq \limsup_{n\to\infty} \mathcal{E}^f_{S,\tau_S^{\varepsilon_n}}\left(\xi_{\tau_S^{\varepsilon_n}}\right)$ a.s. From this, together with (6.4), we get $Y_S \leq \mathcal{E}^f_{S,\hat{\tau}_S}\left(\xi_{\hat{\tau}_S}\right)$ a.s., which shows that $\hat{\tau}_S$ is an optimal stopping time.
Let us now prove the equality $\hat{\tau}_S = \tau_S^0$ a.s. We have already noticed that $\hat{\tau}_S \leq \tau_S^0$ a.s. It remains to show the converse inequality. Note that for each $S \in \mathcal{T}_{0,T}$, Y_S is equal a.s. to the value at time S of the linear optimal stopping problem associated with the pay-off process (ξ_t) and the instantaneous reward process $(\bar{f}_t)$ defined by $\bar{f}_t(\omega) := f(\omega, t, Y_{t-}(\omega), Z_t(\omega), k_t(\omega))$, that is

$$Y_S = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} E[\xi_\tau + \int_S^\tau \bar{f}_u du \mid \mathcal{F}_S] \text{ a.s..} \tag{6.5}$$

It is not difficult to see that $\hat{\tau}_S$ is also optimal for this linear optimal stopping problem. Now, from classical results on linear optimal stopping, τ_S^0 is the minimal optimal stopping time for problem (6.5); hence, we have $\hat{\tau}_S \geq \tau_S^0$ a.s., which completes the proof. $\square$

7. $\mathcal{E}^f$-Mertens decomposition of strong $\mathcal{E}^f$-supermartingales with respect to a general filtration

By using the above characterization of the solution of the RBSDE with an r.u.s.c. obstacle as the value function of the non-linear optimal stopping problem (5.1) (cf. Theorem 6.1),

[7] Note that Fatou's lemma for (non-reflected) BSDEs, shown in [11] in the case of a Brownian-Poisson filtration, still holds true in our framework of a general filtration.

we derive an $\mathcal{E}^f$-Mertens decomposition of strong $\mathcal{E}^f$-supermartingales, which generalizes the one provided in [19] (cf. Theorem 5.2 in [19]) to the case of a general filtration.[8]

As mentioned before, this is an important property in the present work which will allow us to address the non-linear optimal stopping problem in the completely irregular case (cf. Section 9.3, more precisely the proof of Proposition 9.1, and also Theorem 10.1).

Theorem 7.1 ($\mathcal{E}^f$-Mertens decomposition). *Let (Y_t) be a process in $\mathcal{S}^2$. Let f be a Lipschitz driver satisfying Assumption 5.1. The process (Y_t) is a strong $\mathcal{E}^f$-supermartingale if and only if there exists a nondecreasing right-continuous predictable process A in $\mathcal{S}^2$ with $A_0 = 0$ and a nondecreasing right-continuous adapted purely discontinuous process C in $\mathcal{S}^2$ with $C_{0-} = 0$, as well as three processes $Z \in I\!H^2$, $k \in \mathbb{H}^2_\nu$ and $h \in \mathcal{M}^{2,\perp}$, such that*

$$-dY_t = f(t, Y_t, Z_t, k_t)dt + dA_t + dC_{t-} - Z_t dW_t - \int_E k_t(e)\tilde{N}(dt, de) - dh_t, \quad 0 \le t \le T. \tag{7.1}$$

This decomposition is unique. Moreover, a strong $\mathcal{E}^f$-supermartingale is necessarily r.u.s.c.

Proof. Assume that (Y_t) is a strong $\mathcal{E}^f$-supermartingale. By the same arguments as in [19] (cf. Lemma 5.1 in [19]), it can be shown that the process (Y_t) is *r.u.s.c.* Let $S \in \mathcal{T}_{0,T}$. Since (Y_t) is a strong $\mathcal{E}^f$-supermartingale, we have $Y_S \ge \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. We derive that $Y_S \ge \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s. Now, by definition of the essential supremum, $Y_S \le \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s., since $S \in \mathcal{T}_{S,T}$. Hence, $Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^f_{S,\tau}(Y_\tau)$ a.s. By Theorem 6.1, the process (Y_t) coincides with the solution of the reflected BSDE associated with the (r.u.s.c.) obstacle (Y_t), and thus admits the decomposition (7.1).
The converse follows from Lemma 12.2 in the Appendix. $\square$

8. Optimal stopping with non-linear f-expectation in the completely irregular case: the direct part of the approach

We now turn to the study of the non-linear optimal stopping problem (5.1) in the more difficult case where (ξ_t) is *completely irregular*. Since the process (ξ_t) is not r.u.s.c., the inequality $Y_{\tau_S^\varepsilon} \le \xi_{\tau_S^\varepsilon} + \varepsilon$ (i.e. inequality (6.1)) does not necessarily hold (not even in the simplest case of linear expectations; cf., e.g., [12]). This prevents us from adopting here the approach used in the r.u.s.c. case to prove an infinitesimal characterization of the value of the non-linear optimal stopping problem in terms of the solution of an RBSDE. Thus,

[8] An $\mathcal{E}^f$-Mertens decomposition was also shown in [3] (at the same time as in [19]) in the case of a driver $f(t, y, z)$ which does not depend on k by using a different approach.

when ξ is completely irregular, we have to proceed differently. We use a combined approach which consists in a direct part and an RBSDE-part. This section is devoted to the *direct part* of our approach to the non-linear optimal stopping problem (5.1).

8.1. Preliminary results on the value family

Let us first introduce the definition of an admissible family of random variables indexed by stopping times in $\mathcal{T}_{0,T}$ (or $\mathcal{T}_{0,T}$-system in the vocabulary of Dellacherie and Lenglart [5]).

Definition 8.1. *We say that a family $U = (U(S),\, S \in \mathcal{T}_{0,T})$ is* admissible *if it satisfies the following conditions*

1. *for all $S \in \mathcal{T}_{0,T}$, $U(S)$ is a real-valued $\mathcal{F}_S$-measurable random variable.*
2. *for all $S, S' \in \mathcal{T}_{0,T}$, $U(S) = U(S')$ a.s. on $\{S = S'\}$.*

Moreover, we say that an admissible family U is square-integrable *if for all $S \in \mathcal{T}_{0,T}$, $U(S)$ is square-integrable.*

Lemma 8.1 (*Admissibility of the family V*). *The family $V = (V(S), S \in \mathcal{T}_{0,T})$ defined in* (5.1) *is a square-integrable admissible family.*

Proof. The proof uses arguments similar to those used in the "classical" case of linear expectations, combined with some properties of f-expectations.
For each $S \in \mathcal{T}_{0,T}$, V(S) is an $\mathcal{F}_S$-measurable square-integrable random variable, due to the definitions of the conditional f-expectation and of the essential supremum. Let us prove Property 2 of the definition of admissibility. Let S and S' be two stopping times in $\mathcal{T}_{0,T}$. We set $A := \{S = S'\}$ and we show that $V(S) = V(S')$, P-a.s. on A. For each $\tau \in \mathcal{T}_{S,T}$, we set $\tau_A := \tau \mathbf{1}_A + T\mathbf{1}_{A^c}$. We have $\tau_A \geq S'$ a.s. By using the fact that $S = S'$ a.s. on A, the fact that $\tau_A = \tau$ a.s. on A, and a standard property of conditional f-expectations (cf., e.g., Proposition A.3 in [21] which can be extended without difficulty to the framework of general filtration), we obtain

$$\mathbf{1}_A \mathcal{E}^f_{S,\tau}[\xi_\tau] = \mathbf{1}_A \mathcal{E}^f_{S',\tau}[\xi_\tau] = \mathcal{E}^{f^\tau \mathbf{1}_A}_{S',T}[\xi_\tau \mathbf{1}_A] = \mathcal{E}^{f^{\tau_A} \mathbf{1}_A}_{S',T}[\xi_{\tau_A} \mathbf{1}_A] = \mathbf{1}_A \mathcal{E}^f_{S',\tau_A}[\xi_{\tau_A}] \leq \mathbf{1}_A V(S'),$$

where $f^\tau(t, y, z, k) := f(t, y, z, k)\mathbf{1}_{\{t \leq \tau\}}$. By taking the ess sup over $\mathcal{T}_{S,T}$, we get $\mathbf{1}_A V(S) \leq \mathbf{1}_A V(S')$. We obtain the converse inequality by interchanging the roles of S and S'. $\square$

Lemma 8.2 (Optimizing sequence). *For each $S \in \mathcal{T}_{0,T}$, there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times in $\mathcal{T}_{S,T}$ such that the sequence $(\mathcal{E}^f_{S,\tau_n}(\xi_{\tau_n}))_{n \in \mathbb{N}}$ is nondecreasing and $V(S) = \lim_{n \to \infty} \uparrow \mathcal{E}^f_{S,\tau_n}(\xi_{\tau_n})$ a.s.*

Proof. Due to a classical result on essential suprema, it is sufficient to show that, for each $S \in \mathcal{T}_{0,T}$, the family $(\mathcal{E}_{S,\tau}(\xi_\tau),\ \tau \in \mathcal{T}_{S,T})$ is stable under pairwise maximization. Let us fix $S \in \mathcal{T}_{0,T}$. Let $\tau \in \mathcal{T}_{S,T}$ and $\tau' \in \mathcal{T}_{S,T}$. We define $A := \{\, \mathcal{E}^f_{S,\tau'}(\xi_{\tau'}) \leq \mathcal{E}^f_{S,\tau}(\xi_\tau) \,\}$ and $\nu := \tau \mathbf{1}_A + \tau' \mathbf{1}_{A^c}$. We have $A \in \mathcal{F}_S$ and $\nu \in \mathcal{T}_{S,T}$. We compute $\mathbf{1}_A \mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathcal{E}^{f^\nu \mathbf{1}_A}_{S,T}(\xi_\nu \mathbf{1}_A) = \mathcal{E}^{f^\tau \mathbf{1}_A}_{S,T}(\xi_\tau \mathbf{1}_A) = \mathbf{1}_A \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s. Similarly, we show $\mathbf{1}_{A^c} \mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathbf{1}_{A^c} \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})$. It follows that $\mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathcal{E}^f_{S,\tau}(\xi_\tau)\mathbf{1}_A + \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})\mathbf{1}_{A^c} = \mathcal{E}^f_{S,\tau}(\xi_\tau) \vee \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})$, which shows the stability under pairwise maximization and concludes the proof. $\square$

Definition 8.2 (***$\mathcal{E}^f$-supermartingale family***)**.** *An admissible square-integrable family $U := (U(S),\ S \in \mathcal{T}_{0,T})$ is said to be an $\mathcal{E}^f$*-supermartingale family *if for all $S, S' \in \mathcal{T}_{0,T}$ such that $S \leq S'$ a.s., we have $\mathcal{E}^f_{S,S'}(U(S')) \leq U(S)$ a.s.*

Definition 8.3 (***Right-uppersemicontinuous family***)**.** *An admissible family $U := (U(S),\ S \in \mathcal{T}_{0,T})$ is said to be a* right-uppersemicontinuous *(along stopping times)* family *if, for any (τ_n) nonincreasing sequence in $\mathcal{T}_{0,T}$ and any τ in $\mathcal{T}_{0,T}$ such that $\tau = \lim \downarrow \tau_n$, we have $U(\tau) \geq \limsup_{n\to\infty} U(\tau_n)$ a.s.*

Lemma 8.3. *Let $U := (U(S),\ S \in \mathcal{T}_{0,T})$ be an $\mathcal{E}^f$-supermartingale family. Then, $(U(S),\ S \in \mathcal{T}_{0,T})$ is a right-uppersemicontinuous (along stopping times) family.*

Proof. Let $\tau \in \mathcal{T}_{0,T}$ and let (τ_n) be a nonincreasing sequence of stopping times such that $\lim_{n\to+\infty} \tau_n = \tau$ a.s. and for all $n \in \mathbb{N}$, $\tau_n > \tau$ a.s. on $\{\tau < T\}$, and such that $\lim_{n\to+\infty} U(\tau_n)$ exists a.s. As U is an $\mathcal{E}^f$-supermartingale family and as the sequence (τ_n) is nonincreasing, we have $\mathcal{E}^f_{\tau,\tau_n}(U(\tau_n)) \leq \mathcal{E}^f_{\tau,\tau_{n+1}}(U(\tau_{n+1})) \leq U(\tau)$ a.s. Hence, the sequence $(\mathcal{E}^f_{\tau,\tau_n}(U(\tau_n)))_n$ is nondecreasing and $U(\tau) \geq \lim \uparrow \mathcal{E}^f_{\tau,\tau_n}(U(\tau_n))$. Hence, by the property of continuity of BSDEs with respect to terminal time and terminal condition (cf. [36, Prop. A.6] which still holds in the case of a general filtration), we get $U(\tau) \geq \lim_{n\to+\infty} \mathcal{E}^f_{\tau,\tau_n}(U(\tau_n)) = \mathcal{E}^f_{\tau,\tau}(\lim_{n\to+\infty} U(\tau_n)) = \lim_{n\to+\infty} U(\tau_n)$ a.s. By Lemma 5 of Dellacherie and Lenglart [5] [9], the family U is thus right-u.s.c. (along stopping times). $\square$

Theorem 8.1. *The value family $V = (V(S),\ S \in \mathcal{T}_{0,T})$ defined in* (5.1) *is an $\mathcal{E}^f$-supermartingale family. In particular, $V = (V(S),\ S \in \mathcal{T}_{0,T})$ is a right-u.s.c. (along stopping times) family in the sense of Definition 8.3.*

[9] The chronology Θ (in the vocabulary and notation of [5]) which we work with here is the chronology of all stopping times, that is, $\Theta = \mathcal{T}_{0,T}$; hence $[\Theta] = \Theta = \mathcal{T}_{0,T}$.

Proof. We know from Lemma 8.1 that V is a square-integrable admissible family. Let $S \in \mathcal{T}_{0,T}$ and $S' \in \mathcal{T}_{S,T}$. We will show that $\mathcal{E}^f_{S,S'}(V(S')) \leq V(S)$ a.s., which will prove that V is an $\mathcal{E}^f$-supermartingale family. By Lemma 8.2, there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times such that $\tau_n \geq S'$ a.s. and $V(S') = \lim_{n\to\infty} \uparrow \mathcal{E}^f_{S',\tau_n}(\xi_{\tau_n})$ a.s. By the continuity and the consistency properties of f-expectation, we get

$$\mathcal{E}^f_{S,S'}(V(S')) = \mathcal{E}^f_{S,S'}(\lim_{n\to\infty} \uparrow \mathcal{E}^f_{S',\tau_n}(\xi_{\tau_n})) = \lim_{n\to\infty} \mathcal{E}^f_{S,S'}(\mathcal{E}^f_{S',\tau_n}(\xi_{\tau_n})) = \lim_{n\to\infty} \mathcal{E}^f_{S,\tau_n}(\xi_{\tau_n}) \leq V(S).$$

Hence, V is an $\mathcal{E}^f$-supermartingale family. This property, together with Lemma 8.3, yields that V is a right-u.s.c. (along stopping times) family. □

8.2. Aggregation and Snell characterization

Using the above results on the value family $V = (V(S),\ S \in \mathcal{T}_{0,T})$, we show the following theorem, which generalizes some results of classical optimal stopping theory (more precisely, Lemma 3.4 (i)) to the case of an optimal stopping problem with f-expectation.

Theorem 8.2 (Aggregation and Snell characterization). *There exists a unique right-uppersemicontinuous optional process, denoted by $(V_t)_{t\in[0,T]}$, which aggregates the value family $V = (V(S),\ S \in \mathcal{T}_{0,T})$. Moreover, $(V_t)_{t\in[0,T]}$ is the $\mathcal{E}^f$-*Snell envelope *of the pay-off process ξ, that is, the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.*

Proof. By Theorem 8.1, the value family $V = (V(S),\ S \in \mathcal{T}_{0,T})$ is a right-u.s.c. family (or a right-u.s.c. $\mathcal{T}_{0,T}$-system in the vocabulary of Dellacherie-Lenglart [5]). Applying Theorem 4 of Dellacherie-Lenglart ([5]), gives the existence of a unique (up to indistinguishability) right-u.s.c. optional process $(V_t)_{t\in[0,T]}$ which *aggregates* the value family $(V(S),\ S \in \mathcal{T}_{0,T})$. From this aggregation property, namely the property $V_S = V(S)$ a.s. for each $S \in \mathcal{T}_{0,T}$, and from Theorem 8.1, we deduce that the process $(V_t)_{t\in[0,T]}$ is a strong $\mathcal{E}^f$-supermartingale. Moreover, we have $V_S = V(S) \geq \xi_S$ a.s. for each $S \in \mathcal{T}_{0,T}$, which implies that $V_t \geq \xi_t$, for all $t \in [0,T]$, a.s.
Let us now prove that the process $(V_t)_{t\in[0,T]}$ is *the smallest* strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ. Let $(V'_t)_{t\in[0,T]}$ be a strong $\mathcal{E}^f$-supermartingale such that $V'_t \geq \xi_t$, for all $t \in [0,T]$, a.s. Let $S \in \mathcal{T}_{0,T}$. We have $V'_\tau \geq \xi_\tau$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. Hence, $\mathcal{E}^f_{S,\tau}(V'_\tau) \geq \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s., where we have used the monotonicity of the conditional f-expectation. On the other hand, by using the strong $\mathcal{E}^f$-supermartingale property of the process $(V'_t)_{t\in[0,T]}$, we have $V'_S \geq \mathcal{E}^f_{S,\tau}(V'_\tau)$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. Hence, $V'_S \geq \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. By taking the essential supremum over $\tau \in \mathcal{T}_{S,T}$ in the inequality, we get $V'_S \geq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) = V(S) = V_S$ a.s. Hence, for all $S \in \mathcal{T}_{0,T}$, we have $V'_S \geq V_S$ a.s., which yields that $V'_t \geq V_t$, for all $t \in [0,T]$, a.s. The proof is thus complete. □

9. Non-linear Reflected BSDE with completely irregular obstacle and general filtration: useful properties

Our aim now is to establish an infinitesimal characterization of the value process of the non-linear problem (5.1) in terms of the solution of a non-linear RBSDE (thus generalizing Theorem 3.1 from the classical linear case to the non-linear case). In order to do so, we first need to establish some results on non-linear RBSDEs with *completely irregular* obstacles, in particular, a comparison result for such RBSDEs. This section is devoted to these results (this is *the RBSDE-part* of our approach to problem (5.1)). The results from this section extend and complete our work from [19], where an assumption of right-uppersemicontinuity on the obstacle is made. Let us note that the proof of the comparison theorem from [19] cannot be adapted to the completely irregular framework considered here; instead, we rely on a Tanaka-type formula for strong (irregular) semimartingales which we also establish.

Remark 9.1. (A "bottle-neck" of the direct approach) *One might wonder whether the infinitesimal characterization for the non-linear optimal stopping problem* (5.1) *can be obtained by pursuing the direct study of the value process* (V_t) *of problem* (5.1)*, similarly to what was done in the classical linear case in Sub-section 3.1. In the classical case, we applied Mertens decomposition to* (V_t)*; then, we showed directly the minimality properties for the processes* A^d *and* A^c *(cf. Lemmas 3.2 and 3.3) by using the martingale property on the interval* $[S, \tau_S^\lambda]$ *from Lemma 3.1(iii), which itself relies on Maingueneau's penalization approach (cf. also Remarks 3.3 and 3.2). In the non-linear case, Mertens decomposition is generalized by the* $\mathcal{E}^f$*-Mertens decomposition (cf. Theorem 7.1). However, the analogue in the non-linear case of the martingale property of Lemma 3.4[(iii)] (namely, the* $\mathcal{E}^f$*-martingale property) cannot be obtained via Maingueneau's approach (not even in the case of nonnegative* ξ *and under the additional assumption* $f(t,0,0,0)=0$ *which ensures the non-negativity of* $\mathcal{E}^f$*) due to the* lack of convexity *of the functional* $\mathcal{E}^f$*.*

9.1. Tanaka-type formula

The following lemma will be used in the proof of the comparison theorem for RBSDEs with irregular obstacles. The lemma can be seen as an extension of Theorem 66 of [35, Chapter IV] from the case of right-continuous semimartingales to the more general case of strong optional semimartingales.

Lemma 9.1 (Tanaka-type formula). *Let* X *be a (real-valued) strong optional semimartingale with decomposition* $X = X_0 + M + A + B$*, where* M *is a local (cadlag) martingale,* A *is a right-continuous adapted process of finite variation such that* $A_0 = 0$*,* B *is a left-continuous adapted purely discontinuous process of finite variation such that* $B_0 = 0$*. Let* $f : \mathbf{R} \longrightarrow \mathbf{R}$ *be a convex function. Then,* $f(X)$ *is a strong optional semimartingale. Moreover, denoting* *by f' the left-hand derivative of the convex function f, we have*

$$f(X_t) = f(X_0) + \int_{(0,t]} f'(X_{s-})d(A_s + M_s) + \int_{[0,t)} f'(X_s)dB_{s+} + K_t,$$

where K is a nondecreasing adapted process (which is in general neither left-continuous nor right-continuous) such that

$$\Delta K_t = f(X_t) - f(X_{t-}) - f'(X_{t-})\Delta X_t \text{ and } \Delta_+ K_t = f(X_{t+}) - f(X_t) - f'(X_t)\Delta_+ X_t.$$

Proof. Our proof follows the proof of Theorem 66 of [35, Chapter IV] with suitable changes.

Step 1. We assume that X is bounded; more precisely, we assume that there exists $N \in \mathbb{N}$ such that $|X| \leq N$. We know (cf. [35]) that there exists a sequence (f_n) of twice continuously differentiable convex functions such that (f_n) converges to f, and (f_n') converges to f' from below. By applying Gal'chouk-Lenglart's formula (cf., e.g., Theorem A.3 in [19]) to $f_n(X_t)$, we obtain for all $\tau \in \mathcal{T}_{0,T}$

$$f_n(X_\tau) = f_n(X_0) + \int_{(0,\tau]} f_n'(X_{s-})d(A_s + M_s) + \int_{[0,\tau)} f_n'(X_s)dB_{s+} + K_\tau^n, \text{ a.s., where} \quad (9.1)$$

$$K_\tau^n := \sum_{0<s\leq\tau} \left[f_n(X_s) - f_n(X_{s-}) - f_n'(X_{s-})\Delta X_s\right] + \sum_{0\leq s<\tau} \left[f_n(X_{s+}) - f_n(X_s) - f_n'(X_s)\Delta_+ X_s\right]$$
$$+ \frac{1}{2}\int_{(0,\tau]} f_n''(X_{s-})d\langle M^c, M^c\rangle_s \text{ a.s.} \quad (9.2)$$

We show that (K_τ^n) is a convergent sequence by showing that the other terms in Equation (9.1) converge. The convergence $\int_{(0,\tau]} f_n'(X_{s-})d(A_s + M_s) \underset{n\to\infty}{\longrightarrow} \int_{(0,\tau]} f'(X_{s-})d(A_s + M_s)$ is shown by using the same arguments as in the proof of [35, Thorem 66, Ch. IV]. The convergence of the term $\int_{[0,\tau)} f_n'(X_s)dB_{s+}$, which is specific to the non-right-continuous case, is shown by using dominated convergence. We conclude that (K_τ^n) converges and we set $K_\tau := \lim_{n\to\infty} K_\tau^n$. The process (K_t) is adapted as the limit of adapted processes. Moreover, we have from Eq. (9.2) and from the convexity of f_n that, for each n, K_t^n is nondecreasing in t. Hence, the limit K_t is nondecreasing.

Step 2. We treat the general case where X is not necessarily bounded by using a localization argument similar to that used in [35, Th. 66, Ch. IV]. □

9.2. Comparison theorem

Theorem 9.1 (Comparison). *Let $\xi \in \mathcal{S}^2$, $\xi' \in \mathcal{S}^2$ be two processes. Let f and f' be Lipschitz drivers satisfying Assumption 5.1. Let (Y, Z, k, h, A, C) (resp. (Y', Z', k', h', A', C'))*

be the solution of the RBSDE associated with obstacle ξ (resp. ξ') and with driver f (resp. f'). If $\xi_t \leq \xi'_t$, $0 \leq t \leq T$ a.s. and $f(t, Y'_t, Z'_t, k'_t) \leq f'(t, Y'_t, Z'_t, k'_t)$, $0 \leq t \leq T$ $dP \otimes dt$-a.s., then, $Y_t \leq Y'_t$, $0 \leq t \leq T$ a.s.

Proof. We set $\bar{Y}_t = Y_t - Y'_t$, $\bar{Z}_t = Z_t - Z'_t$, $\bar{k}_t = k_t - k'_t$, $\bar{A}_t = A_t - A'_t$, $\bar{C}_t = C_t - C'_t$, $\bar{h}_t = h_t - h'_t$, and $\bar{f}_t = f(t, Y_{t-}, Z_t, k_t) - f'(t, Y'_{t-}, Z'_t, k'_t)$. Then, $-d\bar{Y}_t = \bar{f}_t dt + d\bar{A}_t + d\bar{C}_{t-} - \bar{Z}_t dW_t - \int_E \bar{k}_t(e)\tilde{N}(dt, de) - d\bar{h}_t$, with $\bar{Y}_T = \xi_T - \xi'_T$. Applying Lemma 9.1 to $\bar{Y}_t^+$ and noticing that $\bar{Y}_T^+ = 0$ (as $\xi_T \leq \xi'_T$), we obtain

$$\begin{aligned}\bar{Y}_t^+ = &- \int_{(t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{Z}_s dW_s - \int_{(t,T]} \int_E \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{k}_s(e)\tilde{N}(ds, de) - \int_{(t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{h}_s \\ &+ \int_{(t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{f}_s ds + \int_{(t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}_s + \int_{[t,T)} \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s + (K_t - K_T).\end{aligned} \tag{9.3}$$

We set $\delta_t := \frac{f(t,Y_{t-},Z_t,k_t)-f(t,Y'_{t-},Z_t,k_t)}{Y_{t-}-Y'_{t-}} \mathbf{1}_{\{\bar{Y}_{t-}\neq 0\}}$ and $\beta_t := \frac{f(t,Y'_{t-},Z_t,k_t)-f(t,Y'_{t-},Z'_t,k_t)}{Z_t - Z'_t} \mathbf{1}_{\{\bar{Z}_t\neq 0\}}$. Due to the Lipschitz-continuity of f, the processes δ and β are bounded. We note that $\bar{f}_t = \delta_t \bar{Y}_t + \beta_t \bar{Z}_t + f(Y'_{t-}, Z'_t, k_t) - f(Y'_{t-}, Z'_t, k'_t) + \varphi_t$, where $\varphi_t := f(Y'_{t-}, Z'_t, k'_t) - f'(Y'_{t-}, Z'_t, k'_t)$. Using this, together with Assumption 5.1, we obtain

$$\bar{f}_t \leq \delta_t \bar{Y}_t + \beta_t \bar{Z}_t + \langle \gamma_t \,, \, \bar{k}_t \rangle_\nu + \varphi_t \;\; 0 \leq t \leq T, \quad dP \otimes dt - \text{a.e.}, \tag{9.4}$$

where we have set $\gamma_t := \theta_t^{Y'_{t-}, Z'_t, k'_t, k_t}$. For $\tau \in \mathcal{T}_{0,T}$, let $\Gamma_{\tau,\cdot}$ be the unique solution of the following forward SDE $d\Gamma_{\tau,s} = \Gamma_{\tau,s-}[\delta_s ds + \beta_s dW_s + \int_E \gamma_s(e)\tilde{N}(ds, de)]$ with initial condition (at the initial time τ) $\Gamma_{\tau,\tau} = 1$. To simplify the notation, we denote $\Gamma_{\tau,s}$ by Γ_s for $s \geq \tau$.

By applying Gal'chouk-Lenglart's formula to the product $(\Gamma_t \bar{Y}_t^+)$, and by using that $\langle h^c, W \rangle = 0$, we get

$$\begin{aligned}\Gamma_\tau \bar{Y}_\tau^+ = &-(M_T - M_\tau) - \int_\tau^T \Gamma_s (\bar{Y}_{s-}^+ \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \beta_s - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}}) ds \\ &+ \int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}_s^c + \sum_{\tau<s\leq T} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{A}_s - \int_\tau^T \Gamma_{s-} dK_s^c - \int_\tau^T \Gamma_{s-} dK_s^{d,-} \\ &+ \int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s - \int_\tau^T \Gamma_s dK_s^{d,+} - \sum_{\tau<s\leq T} \Delta\Gamma_s \Delta\bar{Y}_s^+.\end{aligned} \tag{9.5}$$

where the process M is defined by $M := M^W + M^N + M^h$, with $M_t^W := \int_0^t \Gamma_{s-}(\mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{Z}_s + \bar{Y}_{s-}^+ \beta_s) dW_s$, and $M_t^N := \int_0^t \int_E \Gamma_{s-}(\bar{k}_s(e)\mathbf{1}_{\{\bar{Y}_{s-}>0\}} + \bar{Y}_{s-}^+ \gamma_s(e))\tilde{N}(ds, de)$, and $M_t^h := \int_0^t \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{h}_s$. Note that by classical arguments (which use Burkholder-Davis-Gundy inequalities), the stochastic integrals M^W, M^N and M^h are martingales. Hence, M is a martingale (equal to zero in expectation).

By definition of Γ, we have $\Gamma_\tau = 1$, which gives that $\Gamma_\tau \bar{Y}_\tau^+ = \bar{Y}_\tau^+$. Moreover, we have $\int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s = \int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC_s - \int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC'_s$. For the first term, it holds $\int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC_s = 0$. Indeed, $\{\bar{Y}_s > 0\} = \{Y_s > Y'_s\} \subset \{Y_s > \xi_s\}$ (as $Y'_s \geq \xi'_s \geq \xi_s$). This, together with the Skorokhod condition for C gives the equality. For the second term, it holds $-\int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC'_s \leq 0$, as $\Gamma \geq 0$ and dC' is a nonnegative measure. Hence, $\int_\tau^T \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s \leq 0$. Similarly, we obtain $\int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}^c_s \leq 0$. Indeed, $\int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}^c_s = \int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} dA^c_s - \int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} dA'^c_s$. For the first term, we have $\int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} dA^c_s = 0$. This is due to the fact that $\{\bar{Y}_{s-} > 0\} = \{Y_{s-} > Y'_{s-}\} \subset \{Y_{s-} > \bar{\xi}_s\}$ (as $Y'_s \geq \xi'_s \geq \xi_s$, and hence $Y'_{s-} \geq \bar{\xi}_s$), together with the Skorokhod condition for A^c. For the second term, we have $-\int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} dA'^c_s \leq 0$. We also have $-\int_\tau^T \Gamma_{s-} dK^c_s \leq 0$ and $-\int_\tau^T \Gamma_s dK^{d,+}_s \leq 0$. Hence,

$$
\begin{aligned}
\bar{Y}_\tau^+ \leq & -(M_T - M_\tau) - \int_\tau^T \Gamma_s (\bar{Y}^+_{s-}\delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \beta_s - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}}) ds \\
& + \sum_{\tau<s\leq T} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta \bar{A}_s - \int_\tau^T \Gamma_{s-} dK^{d,-}_s - \sum_{\tau<s\leq T} \Delta\Gamma_s \Delta \bar{Y}^+_s.
\end{aligned}
\tag{9.6}
$$

We compute the last term $\sum_{\tau<s\leq T} \Delta\Gamma_s \Delta\bar{Y}^+_s$.
Let (p_s) be the point process associated with the Poisson random measure N (cf. [7, VIII Section 2. 67], or [25, Section III §d]). We have $\Delta\Gamma_s = \Gamma_{s-}\gamma_s(p_s)$ and $\Delta\bar{Y}^+_s = \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{k}_s(p_s) - \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{A}_s + \Delta K^{d,-}_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{h}_s$. Hence,

$$
\begin{aligned}
& \sum_{\tau<s\leq T} \Delta\Gamma_s \Delta\bar{Y}^+_s = \\
& = \sum_{\tau<s\leq T} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(p_s) \bar{k}_s(p_s) - \sum_{\tau<s\leq T} \Gamma_{s-}\gamma_s(p_s)(\mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{A}_s - \Delta K^{d,-}_s - \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{h}_s) \\
& = \int_\tau^T \int_E \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(e) \bar{k}_s(e) N(ds, de) - \sum_{\tau<s\leq T} \Gamma_{s-}\gamma_s(p_s)(\mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{A}_s - \Delta K^{d,-}_s - \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta\bar{h}_s) \\
& = \int_\tau^T \int_E \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(e) \bar{k}_s(e) \tilde{N}(ds, de) + \int_\tau^T \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \langle \gamma_s, \bar{k}_s \rangle_\nu ds \\
& - \sum_{\tau<s\leq T} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(p_s) \Delta\bar{A}_s + \sum_{\tau<s\leq T} \Gamma_{s-}\gamma_s(p_s) \Delta K^{d,-}_s + \sum_{\tau<s\leq T} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(p_s) \Delta\bar{h}_s.
\end{aligned}
\tag{9.7}
$$

By plugging this expression in equation (9.6) and by putting together the terms in "ds", the terms in "$dK_s^{d,-}$", and the terms in "$\Delta\bar{A}_s$", we get

$$\begin{aligned}\bar{Y}_\tau^+ \leq &-(M_T - M_\tau) - \int_\tau^T \Gamma_{s-}(\bar{Y}_{s-}^+\delta_s + \bar{Z}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\beta_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\langle\gamma_s, \bar{k}_s\rangle_\nu - \bar{f}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}})ds \\ &+ \sum_{\tau<s\leq T}\Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}(1+\gamma_s(p_s))\Delta\bar{A}_s - \sum_{\tau<s\leq T}\Gamma_{s-}(1+\gamma_s(p_s))\Delta K_s^{d,-} \\ &-(\tilde{M}_T - \tilde{M}_\tau) - \int_\tau^T d[\,\bar{h}\,, \int_0^\cdot\int_E \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(e)\tilde{N}(ds,de)\,]_s,\end{aligned} \tag{9.8}$$

where $\tilde{M}_t := \int_0^t\int_E \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(e)\bar{k}_s(e)\tilde{N}(ds,de)$. Note that by classical arguments (as for M above), the stochastic integral $\tilde{M}$ is a martingale, equal to zero in expectation.

We have $-\int_\tau^T \Gamma_{s-}(\bar{Y}_{s-}^+\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\delta_s + \bar{Z}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\beta_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\langle\gamma_s, \bar{k}_s\rangle_\nu - \bar{f}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}})ds \leq \int_\tau^T \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\varphi_s ds$, due to the inequality (9.4). The term $-\sum_{\tau<s\leq T}\Gamma_{s-}(1+\gamma_s(p_s))\Delta K_s^{d,-}$ is nonpositive, as $1+\gamma_s \geq 0$ by Assumption 5.1. The term $\sum_{\tau<s\leq T}\Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}(1+\gamma_s(p_s))\Delta\bar{A}_s$ is nonpositive, due to $1+\gamma_s \geq 0$, to the Skorokhod condition for ΔA_s and to $\Delta A'_s \geq 0$ (the details are similar to those for $d\bar{C}$ in the reasoning above). Since $\bar{h} \in \mathcal{M}^{2,\perp}$, by Remark 2.1, we derive that the expectation of the last term of the above inequality (9.8) is equal to 0. Moreover, the term $\int_\tau^T \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\varphi_s ds$ is nonpositive, as $\varphi_s = f(Y'_s, Z'_s, k'_s) - f'(Y'_s, Z'_s, k'_s) \leq 0$ $dP\otimes ds$-a.s. by the assumptions of the theorem. We conclude that $E[\bar{Y}_\tau^+] \leq 0$, which implies $\bar{Y}_\tau^+ = 0$ a.s. The proof is thus complete. □

Remark 9.2. *Note that due to the irregularity of the obstacles, together with the presence of jumps, we cannot adopt the approaches used up to now in the literature (see e.g. [13], [4], [37] and [19]) to show the comparison theorem for our RBSDE.*

9.3. Non-linear operator induced by an RBSDE. $\mathcal{E}^f$-Snell characterization

We introduce the non-linear operator $\mathcal{R}ef^f$ (associated with a given non-linear driver f) and provide some useful properties.

Definition 9.1 (Non-linear operator $\mathcal{R}ef^f$). *Let f be a Lipschitz driver. For a process $(\xi_t) \in \mathcal{S}^2$, we denote by $\mathcal{R}ef^f[\xi]$ the first component of the solution to the Reflected BSDE with (lower) barrier ξ and with Lipschitz driver f.*

The operator $\mathcal{R}ef^f[\cdot]$ is well-defined due to Theorem 4.1. Moreover, $\mathcal{R}ef^f[\cdot]$ is valued in $\mathcal{S}^{2,rusc}$, where $\mathcal{S}^{2,rusc} := \{\phi \in \mathcal{S}^2 : \phi \text{ is r.u.s.c.}\}$ (cf. Remark 2.3). In the following proposition we give some properties of the operator $\mathcal{R}ef^f$. Note that equalities (resp. inequalities) between processes are to be understood in the "up to indistinguishability"-sense. We recall the notion of a strong $\mathcal{E}^f$-supermartingale.

Definition 9.2. *Let ϕ be a process in $\mathcal{S}^2$. Let f be a Lipschitz driver. The process ϕ is said to be a strong $\mathcal{E}^f$-supermartingale (resp. a strong $\mathcal{E}^f$-martingale), if $\mathcal{E}^f_{\sigma,\tau}(\phi_\tau) \leq \phi_\sigma$ a.s. (resp. $\mathcal{E}^f_{\sigma,\tau}(\phi_\tau) = \phi_\sigma$ a.s.) on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_{0,T}$.*

Using the above comparison theorem and the $\mathcal{E}^f$-Mertens decomposition for strong (r.u.s.c.) $\mathcal{E}^f$-supermartingales in the case of a general filtration (cf. Theorem 7.1), we show that the operator $\mathcal{R}ef^f$ satisfies the following properties.

Proposition 9.1 (Properties of the operator $\mathcal{R}ef^f$). *Let f be a Lipschitz driver satisfying Assumption 5.1. The operator $\mathcal{R}ef^f : S^2 \to S^{2,rusc}$, defined in Definition 9.1, has the following properties:*
(i) *The operator $\mathcal{R}ef^f$ is nondecreasing, that is, for $\xi, \xi' \in S^2$ such that $\xi \leq \xi'$ we have $\mathcal{R}ef^f[\xi] \leq \mathcal{R}ef^f[\xi']$.*
(ii) *If $\xi \in S^2$ is a (r.u.s.c.) strong $\mathcal{E}^f$-supermartingale, then $\mathcal{R}ef^f[\xi] = \xi$.*
(iii) *For each $\xi \in S^2$, $\mathcal{R}ef^f[\xi]$ is a strong $\mathcal{E}^f$-supermartingale and satisfies $\mathcal{R}ef^f[\xi] \geq \xi$.*

Proof. The assertion **(i)** follows from our comparison theorem for reflected BSDEs with irregular obstacles (Theorem 9.1). Let us prove **(ii)**. Let ξ be a (r.u.s.c.) strong $\mathcal{E}^f$-supermartingale in $\mathcal{S}^2$. By definition of $\mathcal{R}ef^f$, we have to show that ξ is the solution of the reflected BSDE associated with driver f and obstacle ξ. By the $\mathcal{E}^f$-Mertens decomposition for strong (r.u.s.c.) $\mathcal{E}^f$-supermartingales in the case of a general filtration (Theorem 7.1), together with Lemma 2.1, there exists $(Z, k, h, A, C) \in I\!H^2 \times I\!H^2_\nu \times \mathcal{M}^{2,\perp} \times \mathcal{S}^2 \times \mathcal{S}^2$ such that

$$-d\xi_t = f(t, \xi_t, Z_t, k_t)dt - Z_t dW_t - \int_{\mathbf{E}} k_t(e)\tilde{N}(dt, de) - dh_t + dA_t + dC_{t-}, \quad 0 \leq t \leq T,$$

where A is predictable right-continuous nondecreasing with $A_0 = 0$, and C is adapted right-continuous nondecreasing and purely discontinuous, with $C_{0-} = 0$. Moreover, the Skorokhod conditions are here trivially satisfied. Hence, $\xi = \mathcal{R}ef^f[\xi]$, which is the desired conclusion. It remains to show **(iii)**. By definition, the process $\mathcal{R}ef^f[\xi]$ is equal to Y, where (Y, Z, k, h, A, C) is the solution our reflected BSDE. Hence, $\mathcal{R}ef^f[\xi] = Y$ admits the decomposition (7.1), which, by Theorem 7.1, implies that $\mathcal{R}ef^f[\xi] = Y$ is a strong $\mathcal{E}^f$-supermartingale. Moreover, by definition, $\mathcal{R}ef^f[\xi] = Y$ is greater than or equal to ξ. $\square$

With the help of the above proposition, we show that the process $\mathcal{R}ef^f[\xi]$ is characterized in terms of the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.

Theorem 9.2 (The operator $\mathcal{R}ef^f$ and the $\mathcal{E}^f$- Snell envelope operator). *Let $\xi \in \mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 5.1. The first component $Y = \mathcal{R}ef^f[\xi]$ of the solution to the RBSDE with parameters (ξ, f) coincides with the $\mathcal{E}^f$-*Snell envelope *of ξ, that is,* the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.

Proof. By Proposition 9.1 (iii), the process $Y = \mathcal{R}ef^f[\xi]$ is a strong $\mathcal{E}^f$-supermartingale satisfying $Y \geq \xi$. It remains to show the minimality property. Let Y' be a strong $\mathcal{E}^f$-supermartingale such that $Y' \geq \xi$. We have $\mathcal{R}ef^f[Y'] \geq \mathcal{R}ef^f[\xi]$, due to the nondecreasingness of the operator $\mathcal{R}ef^f$ (cf. Proposition 9.1 (i)) On the other hand, $\mathcal{R}ef^f[Y'] = Y'$ (cf. Proposition 9.1 (ii)) and $\mathcal{R}ef^f[\xi] = Y$. Hence, $Y' \geq Y$, which completes the proof. □

In the case of a right-continuous left-limited obstacle ξ the above characterization has been established in [37]; it has been generalized to the case of a right-upper-semicontinuous obstacle in [19, Prop. 4.4]. Let us note however that the arguments of the proofs given in [37] and in [19] cannot be adapted to our general framework.

10. Infinitesimal characterization of the value process in terms of an RBSDE

The following theorem is a direct consequence of the Theorems 9.2 and 8.2. It gives "an infinitesimal characterization" of the value process $(V_t)_{t\in[0,T]}$ of the non-linear problem (5.1) in the completely irregular case.

Theorem 10.1 (Characterization in terms of an RBSDE). *Let $(\xi_t)_{t\in[0,T]}$ be a process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 5.1. The value process $(V_t)_{t\in[0,T]}$ aggregating the family $V = (V(S),\ S \in \mathcal{T}_{0,T})$ defined by (5.1) coincides (up to indistinguishability) with the first component $(Y_t)_{t\in[0,T]}$ of the solution of our RBSDE with driver f and obstacle ξ. In other words, we have, for all $S \in \mathcal{T}_{0,T}$,*

$$Y_S = V_S = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) \ a.s. \tag{10.1}$$

Remark 10.1. *Let us summarize our* two-part *approach to the non-linear optimal stopping problem* (5.1) *in the case where ξ is* completely irregular*: First, we have applied* a direct approach *to the problem* (5.1)*, which consists in showing that the value family $(V(S))_{S\in\mathcal{T}_{0,T}}$ can be aggregated by an optional process $(V_t)_{t\in[0,T]}$ and, then, in characterizing (V_t) as the $\mathcal{E}^f$-Snell envelope of the (completely irregular) pay-off process (ξ_t). On the other hand, we have applied* an RBSDE-approach *which consists in establishing some results on RBSDEs with completely irregular obstacles (in particular, existence, uniqueness, and a comparison result) and some useful properties of the operator $\mathcal{R}ef^f$,* [10] *and then in using these properties to show that the unique solution (Y_t) of the RBSDE is equal to the $\mathcal{E}^f$-Snell envelope of the completely irregular obstacle. We have then deduced from those two parts (the direct part and the RBSDE-part) that (Y_t) and (V_t) coincide, which gives an infinitesimal characterization for the value process (V_t).*

[10] We emphasize that the proof of these properties (cf. Proposition 9.1) relies heavily on the $\mathcal{E}^f$-Mertens decomposition for strong $\mathcal{E}^f$-supermartingales (cf. Theorem 7.1), which is obtained as a direct consequence of the preliminary result (Theorem 6.1) established in the r.u.s.c. case.

Finally, let us put together some of the results for the non-linear optimal stopping problem (5.1):

i) • For any reward process $\xi \in \mathcal{S}^2$, we have the infinitesimal characterization $V_t = Y_t = \mathcal{R}ef_t^f[\xi]$, for all t, a.s. (Theorem 10.1).
   • Also, $(V_t)_{t\in[0,T]}$ is the $\mathcal{E}^f$-*Snell envelope* of the pay-off process ξ (Theorem 8.2).

ii) If, moreover, ξ is right-u.s.c., then, for any $S \in \mathcal{T}_{0,T}$, for any $\varepsilon > 0$, there exists an $L\varepsilon$-optimal stopping time for the problem at time S. (Theorem 6.1).

iii) If, moreover, ξ is also left-u.s.c. along stopping times, then, for any $S \in \mathcal{T}_{0,T}$, there exists an optimal stopping time for the problem at time S (Theorem 6.2).

11. Applications of Theorem 10.1

11.1. Application to American options with a completely irregular payoff

In the following example, we set $E := \mathbf{R}$, $\nu(de) := \lambda\delta_1(de)$, where λ is a positive constant, and where δ_1 denotes the Dirac measure at 1. The process $N_t := N([0,t] \times \{1\})$ is then a Poisson process with parameter λ, and we have $\tilde{N}_t := \tilde{N}([0,t] \times \{1\}) = N_t - \lambda t$.

We assume that the filtration is the natural filtration associated with W and N.

We consider a financial market which consists of one risk-free asset, whose price process S^0 satisfies $dS_t^0 = S_t^0 r_t dt$, and two risky assets with price processes S^1, S^2 satisfying:

$$dS_t^1 = S_{t^-}^1[\mu_t^1 dt + \sigma_t^1 dW_t + \beta_t^1 d\tilde{N}_t]; \quad dS_t^2 = S_{t^-}^2[\mu_t^2 dt + \sigma_t^2 dW_t + \beta_t^2 d\tilde{N}_t].$$

We suppose that the processes σ^1, σ^2, β^1, β^2, r, μ^1, μ^2 are predictable and bounded, with $\beta_t^i > -1$ for $i = 1, 2$. Let $\mu_t := (\mu^1, \mu^2)'$ and let $\Sigma_t := (\sigma_t, \beta_t)$ be the 2×2-matrix with first column $\sigma_t := (\sigma_t^1, \sigma_t^2)'$ and second column $\beta_t := (\beta_t^1, \beta_t^2)'$. We suppose that Σ_t is invertible and that the coefficients of Σ_t^{-1} are bounded.

We consider an agent who can invest his/her initial wealth $x \in \mathbf{R}$ in the three assets.

For $i = 1, 2$, we denote by φ_t^i the amount invested in the i^{th} risky asset. A process $\varphi = (\varphi^1, \varphi^2)'$ belonging to $\mathbb{H}^2 \times \mathbb{H}_\nu^2$ will be called a *portfolio strategy.*

The value of the associated portfolio (or *wealth*) at time t is denoted by $X_t^{x,\varphi}$ (or simply by X_t). In the case of a perfect market, we have $dX_t = (r_t X_t + \varphi_t'(\mu_t - r_t\mathbf{1}))dt + \varphi_t'\sigma_t dW_t + \varphi_t'\beta_t d\tilde{N}_t$, where $\mathbf{1} = (1,1)'$. More generally, we will suppose that there may be some imperfections in the market, taken into account via the *nonlinearity* of the dynamics of the wealth and encoded in a Lipschitz driver f satisfying Assumption 5.1 (cf. e.g. Remark 6.1 in [20] in our market model, [16] and [9] in other frameworks). More precisely, we suppose that the *wealth* process $X_t^{x,\varphi}$ (also X_t) satisfies the forward differential equation: $-dX_t = f(t, X_t, {\varphi_t}'\sigma_t, {\varphi_t}'\beta_t)dt - {\varphi_t}'\sigma_t dW_t - {\varphi_t}'\beta_t d\tilde{N}_t$, with $X_0 = x$, or, equivalently, setting

$Z_t = {\varphi_t}'\sigma_t$ and $k_t = {\varphi_t}'\beta_t$,

$$-dX_t = f(t, X_t, Z_t, k_t)dt - Z_t dW_t - k_t d\tilde{N}_t;\ X_0 = x. \tag{11.1}$$

Note that $(Z_t, k_t) = {\varphi_t}'\Sigma_t$, which is equivalent to ${\varphi_t}' = (Z_t, k_t)\,\Sigma_t^{-1}$.
This model includes the case of a perfect market, for which f is a linear driver given by $f(t, y, z, k) = -r_t y - (z, k)\,\Sigma_t^{-1}(\mu_t - r_t\mathbf{1})$.

Remark 11.1. *Note that the wealth process $X^{x,\varphi}$ is an $\mathcal{E}^f$-martingale, since $X^{x,\varphi}$ is the solution of the BSDE with driver f, terminal time T and terminal condition $X_T^{x,\varphi}$.*

Let us consider an American option associated with terminal time T and payoff given by a process $(\xi_t) \in \mathcal{S}^2$. For each initial wealth x, we denote by $\mathcal{A}(x)$ the set of all portfolio strategies $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\nu$ such that $X_t^{x,\varphi} \geq \xi_t$, for all $t \in [0, T]$ a.s. The *superhedging price* of the American option at time 0 is defined by

$$u_0 := \inf\{x \in \mathbf{R},\ \exists\varphi \in \mathcal{A}(x)\}. \tag{11.2}$$

Using the infinitesimal characterization of the value function (5.1) (cf. Theorem 10.1), we derive the following characterizations of the superhedging price u_0, as well as the existence of a superhedging strategy.

Proposition 11.1. *Let (ξ_t) be an optional process such that $E[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\xi_\tau|^2] < \infty$.*
(i) The superhedging price *u_0 of the American option with payoff (ξ_t) is equal to the value function $V(0)$ of our optimal stopping problem* (1.1) *at time* 0*, that is*

$$u_0 = \sup_{\tau\in\mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau). \tag{11.3}$$

(ii) $u_0 = Y_0$, where (Y, Z, k, h, A, C) is the solution of the RBSDE (2.3) *(with $h = 0$).*
(iii) The portfolio strategy $\hat{\varphi}$, defined by ${\hat{\varphi}_t}' = (Z_t, k_t)\,\Sigma_t^{-1}$, is a superhedging strategy, that is, belongs to $\mathcal{A}(u_0)$.

Remark 11.2. *This result generalizes Theorem 3.4 in [10] which concerns the case when the payoff process ξ is right-continuous. Note also that, even in the case of a linear market, our result for a completely irregular pay-off is new. Some additional comments relative to this result are given in the Appendix (cf. Remark 12.2).*

Proof. The proof relies on Theorem 10.1 and similar arguments to those used in the proof of Theorem 3.4 in [10] in the case with RCLL payoff.

We first show that $u_0 \geq \sup_{\tau\in\mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau)$. Let $x \in \mathbf{R}$ be such that there exists $\varphi \in \mathcal{A}(x)$. We show that $x \geq Y_0$. Since $\varphi \in \mathcal{A}(x)$, we have, for each $\tau \in \mathcal{T}_{0,T}$, $X_\tau^{x,\varphi} \geq \xi_\tau$ a.s.

From this and Remark 11.1, we get $x = \mathcal{E}^f_{0,\tau}(X^{x,\varphi}_\tau) \geq \mathcal{E}^f_{0,\tau}(\xi_\tau)$. Hence, $x \geq \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau)$ By definition of u_0 (cf. (11.2)), we get the desired inequality, that is, $u_0 \geq \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau)$.

Since by Theorem 10.1, $Y_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau)$, we thus have $u_0 \geq Y_0$. In order to prove the three first assertions of the above theorem, it is thus sufficient to show that $\hat{\varphi} \in \mathcal{A}(Y_0)$, which, by definition of u_0, implies that $Y_0 \geq u_0$ (and hence, since $u_0 \geq Y_0$, the equality $Y_0 = u_0$). Now, by (11.1), the value $X^{Y_0,\hat{\varphi}}$ of the portfolio associated with initial wealth Y_0 and strategy $\hat{\varphi}$ satisfies: $dX^{Y_0,\hat{\varphi}}_t = -f(t, X^{Y_0,\hat{\varphi}}_t, Z_t, k_t)dt + dM_t$, with $X^{Y_0,\hat{\varphi}}_0 = Y_0$, where $M_t := \int_0^t Z_s dW_s + \int_0^t k_s d\tilde{N}_s$. Moreover, since Y is the solution of the reflected BSDE (2.3) (with $h = 0$), we have $dY_t = -f(t, Y_t, Z_t, k_t)dt + dM_t - dA_t - dC_{t-}$. Applying the comparison result for forward differential equations, we derive that $X^{Y_0,\hat{\varphi}}_\cdot \geq Y_\cdot$. Since $Y_\cdot \geq \xi_\cdot$, we thus get $X^{Y_0,\hat{\varphi}}_\cdot \geq \xi_\cdot$. It follows that $\hat{\varphi} \in \mathcal{A}(Y_0)$, which ends the proof. $\square$

Example 11.1. *(some examples of American options with completely irregular pay-off) We consider a pay-off process (ξ_t) of the form $\xi_t := h(S^1_t)$, for $t \in [0, T]$, where $h : \mathbf{R} \to \mathbf{R}$ is a (possibly irregular) Borel function such that the process $(h(S_t))$ is optional and $(h(S^1_t)) \in \mathcal{S}^2$. In general, the pay-off (ξ_t) is a completely irregular process. By the first two statements of Proposition 11.1, the superhedging price of the American option is equal to the value function of the optimal stopping problem* (11.3)*, and is also characterized as the solution of the reflected BSDE* (2.3) *with obstacle $\xi_t = h(S^1_t)$.*

If h is an uppersemicontinuous function on $\mathbf{R}$, then the process $(h(S^1_t))$ is optional, since an u.s.c. function can be written as the limit of a (non increasing) sequence of continuous functions. Moreover, the process $(h(S^1_t))$ is right-u.s.c. and also left-u.s.c. along stopping times. The right-uppersemicontinuity of (ξ_t) follows from the fact that the process S^1 is right-continuous; the left-uppersemicontinuity along stopping times of (ξ_t) follows from the fact that S^1 jumps only at totally inaccessible stopping times. In virtue of Proposition 11.1, last statement, there exists in this case an optimal exercise time *for the American option with payoff $\xi_t = h(S^1_t)$. A particular example is given by the American digital call option (with strike $K > 0$), where $h(x) := \mathbf{1}_{[K,+\infty)}(x)$. The function h is u.s.c. on $\mathbf{R}$. The corresponding payoff process $\xi_t := \mathbf{1}_{S^1_t \geq K}$ is thus r.u.s.c and left-u.s.c. along stopping times in this case, which implies the existence of an optimal exercise time.*

In the case of the American digital put option (with strike $K > 0$), the corresponding payoff $\xi_t := \mathbf{1}_{S^1_t < K}$ is not r.u.s.c. We note that the pay-off of the American digital call and put options is in general neither left-limited nor right-limited.

11.2. An application to RBSDEs

The characterization (Theorem 10.1) is also useful in the theory of RBSDEs in itself: it allows us to obtain a priori estimates with universal constants for RBSDEs with completely irregular obstacles.

Proposition 11.2 (A priori estimates with universal constants). *Let ξ and ξ' be two processes in $\mathcal{S}^2$. Let f and f' be two Lipschitz drivers satisfying Assumption 5.1 with common Lipschitz constant $K > 0$. Let (Y, Z, k) (resp. (Y', Z', k')) be the three first components of the solution of the reflected BSDE associated with driver f (resp. f') and obstacle ξ (resp. ξ'). Let $\overline{Y} := Y - Y'$, $\overline{\xi} := \xi - \xi'$, and $\delta f_s := f'(s, Y'_s, Z'_s, k'_s) - f(s, Y'_s, Z'_s, k'_s)$.*
Let $\eta, \beta > 0$ with $\beta \geq 2K + 3/\eta$ and $\eta \leq 1/K^2$. For each $S \in \mathcal{T}_{0,T}$, we have

$$\overline{Y_S}^2 \leq e^{\beta(T-S)} E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \overline{\xi_\tau}^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} (\delta f_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.} \tag{11.4}$$

Proof. The proof is divided into two steps.
Step 1: For each $\tau \in \mathcal{T}_{0,T}$, let $(X^\tau, \pi^\tau, l^\tau)$ (resp. $(X'^\tau, \pi'^\tau, l'^\tau)$) be the solution of the BSDE associated with driver f (resp. f'), terminal time τ and terminal condition ξ_τ (resp. ξ'_τ). Set $\overline{X}^\tau := X^\tau - X'^\tau$. By an estimate on BSDEs (cf. Proposition $A.4$ in [36]), we have

$$(\overline{X}^\tau_S)^2 \leq e^{\beta(T-S)} E[\overline{\xi}^2 \mid \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} [(f - f')(s, X'^\tau_s, \pi'^\tau_s, l'^\tau_s)]^2 ds \mid \mathcal{F}_S] \quad \text{a.s.}$$

from which we derive

$$(\overline{X}^\tau_S)^2 \leq e^{\beta(T-S)} E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \overline{\xi_\tau}^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} (\overline{f}_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.}, \tag{11.5}$$

where $\overline{f}_s := \sup_{y,z,k} |f(s, y, z, k) - f'(s, y, z, k)|$. Now, by Theorem 10.1, we have $Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} X^\tau_S$ and $Y'_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} X'^\tau_S$. We thus get $|\overline{Y}_S| \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} |\overline{X}^\tau_S|$. By (11.5), we derive the inequality (11.4) with δf_s replaced by $\overline{f}_s$.
Step 2: Note that (Y', Z', k') is the solution the RBSDE associated with obstacle ξ' and driver $f(t, y, z, k) + \delta f_t$. By applying the result of Step 1 to the driver $f(t, y, z, k)$ and the driver $f(t, y, z, k) + \delta f_t$ (instead of f'), we get the desired result. □

Remark 11.3. *We note that a similar result holds for a doubly reflected BSDE (cf. Proposition 6.6 in [8] in the case of RCLL barriers, and Corollary 5.2 in [20] in the case of completely irregular barriers).*

12. Appendix

Let $M, M' \in \mathcal{M}^2$. Recall that $MM' - [M, M']$ is a martingale, and that $\langle M, M' \rangle$ is defined as the *compensator* of the integrable finite variation process $[M, M']$. Using these properties we derive the following equivalent statements (cf., e.g.,[35] IV.3 for details):
$\langle M, M' \rangle_t = 0$, $0 \leq t \leq T$ a.s. $\Leftrightarrow$ $[M, M']$. is a martingale $\Leftrightarrow$ MM' is a martingale. [11]

[11] In this case, using he terminology of [35] IV.3, M and M' are said to be *strongly orthogonal.*

For the convenience of the reader, we state the following equivalences, which, to our knowledge, are not explicitly specified in the literature.

Lemma 12.1. *For each $h \in \mathcal{M}^2$, the following properties are equivalent:*
(i) *For all predictable process $l \in \mathbb{H}^2_\nu$, we have $\langle h, \int_0^\cdot l_s(e)\tilde{N}(dsde) \rangle_t = 0$, $t \in [0,T]$ a.s.*
(ii) *For all predictable process $l \in \mathbb{H}^2_\nu$, we have $(h, \int_0^\cdot \int_E l_s(e)\tilde{N}(dsde))_{\mathcal{M}^2} = 0$.*
(iii) *$M^P_N(\Delta h_. | \tilde{\mathcal{P}}) = 0$, where $M^P_N(\,.\,|\tilde{\mathcal{P}})$ is the conditional expectation given $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathscr{E}$ under the Doleans' measure M^P_N associated to probability P and random measure N.*[12]

Proof. Let us show that **(i)** $\Leftrightarrow$ **(ii)**. By definition of the scalar product $(\cdot,\cdot)_{\mathcal{M}^2}$, we have $(h, \int_0^\cdot \int_E l_s(e)\tilde{N}(dsde))_{\mathcal{M}^2} = E[\langle h, \int_0^\cdot \int_E l_s(e)\tilde{N}(dsde) \rangle_T]$. Hence, **(i)** $\Rightarrow$ **(ii)**. Let us show that **(ii)** $\Rightarrow$ **(i)**. If for all $l \in \mathbb{H}^2_\nu$, $E[\langle h, \int_0^\cdot l_s(e)\tilde{N}(dsde) \rangle_T] = 0$, then, for each bounded predictable process $\varphi \in \mathbb{H}^2$, we have

$$E[\int_0^T \varphi_t \, d\langle h, \int_0^\cdot \int_E l_s(e)\tilde{N}(dsde) \rangle_t] = E[\langle h, \int_0^\cdot \int_E \varphi_s l_s(e)\tilde{N}(dsde) \rangle_T] = 0.$$

since, for each $M \in \mathcal{M}^2$, $\varphi \cdot \langle h, M \rangle = \langle h, \varphi . M \rangle$ (using the notation of [7] or [25]). By [7] (Chap 6 II Th. 64 p141), this implies that the integrable-variation predictable process $A_. := \langle h, \int_0^\cdot l_s(e)\tilde{N}(dsde) \rangle_.$ is equal to 0, which gives that **(ii)** $\Rightarrow$ **(i)**. Hence **(i)** $\Leftrightarrow$ **(ii)**.

It remains to show that **(ii)** $\Leftrightarrow$ **(iii)**. Note first that $(h, \int_0^\cdot \int_E l_s(e)\tilde{N}(dsde))_{\mathcal{M}^2} = E([h, \int_0^\cdot \int_E l_s(e)\tilde{N}(dsde)]_T) = E(\int_{[0,T]\times E} \Delta h_s l_s(e) N(dsde)) = M^P_N(\Delta h_. \, l_.)$. Property **(ii)** can thus be written as $M^P_N(\Delta h_. \, l_.) = 0$ for all $l_. \in \mathbb{H}^2_\nu$, which means that $M^P_N(\Delta h_. | \tilde{\mathcal{P}}) = 0$. Hence, **(ii)** $\Leftrightarrow$ **(iii)**. $\square$

Proof of Lemma 3.7: Let $\beta > 0$ and $\varepsilon > 0$ be such that $\beta \geq \frac{1}{\varepsilon^2}$. We note that $\tilde{Y}_T = \xi_T - \xi_T = 0$; moreover, we have $-d\tilde{Y}_t = \tilde{f}(t)dt + d\tilde{A}_t + d\tilde{C}_{t-} - \tilde{Z}_t dW_t - \int_E \tilde{k}_t(e)\tilde{N}(dt, de) - d\tilde{h}_t$.

Thus we see that $\tilde{Y}$ is an *optional strong semimartingale* in the vocabulary of [18] with decomposition $\tilde{Y} = \tilde{Y}_0 + M + A + B$, where $M_t := \int_0^t \tilde{Z}_s dW_s + \int_0^t \int_E \tilde{k}_s(e)\tilde{N}(ds, de) + \tilde{\mathbf{h}}_{\mathbf{t}}$, $A_t := -\int_0^t \tilde{f}(s)ds - \tilde{A}_t$ and $B_t := -\tilde{C}_{t-}$. We set

$$\tilde{M}_t := 2\int_{(0,t]} e^{\beta s}\, \tilde{Y}_{s-}\tilde{Z}_s dW_s + 2\int_{(0,t]} e^{\beta s} \int_E \tilde{Y}_{s-}\tilde{k}_s(e)\tilde{N}(ds, de) + 2\int_{(0,t]} e^{\beta s}\, \tilde{Y}_{s-} d\tilde{h}_s. \quad (12.1)$$

Applying Gal'chouk-Lenglart's formula (more precisely Corollary A.2 in [19]) to $e^{\beta t}\, \tilde{Y}_t^2$, and

Note also that, if $M, M' \in \mathcal{M}^2$, using the terminology of [35] IV.3, the martingales M and M' are said to be *weakly orthogonal* if $(M, M')_{\mathcal{M}^2} = 0$, that is $E[M_T M'_T] = 0$.

[12] For the definitions of M^P_N and $M^P_N(\,.\,|\tilde{\mathcal{P}})$ see, e.g., chapter III.1 (3.10) and (3.25) in [25].

using that $\tilde{Y}_T = 0$, and the property $\langle h^c, W\rangle = 0$, we get, almost surely, for all $t \in [0,T]$,

$$\mathrm{e}^{\beta t}\,\tilde{Y}_t^2 + \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds + \int_{(t,T]} \mathrm{e}^{\beta s}\,d\langle \tilde{h}^c\rangle_s = -\int_{(t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + 2\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_s \tilde{f}(s) ds$$
$$+\, 2\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} d\tilde{A}_s - (\tilde{M}_T - \tilde{M}_t) - \sum_{t<s\leq T} \mathrm{e}^{\beta s}(\Delta \tilde{Y}_s)^2 + 2\int_{[t,T)} \mathrm{e}^{\beta s}\,\tilde{Y}_s d\tilde{C}_s - \sum_{t\leq s<T} \mathrm{e}^{\beta s}(\tilde{Y}_{s+} - \tilde{Y}_s)^2. \tag{12.2}$$

By the same arguments as in [19] (cf. the proof of Lemma 3.2 in [19] for details), since $\beta \geq \frac{1}{\varepsilon^2}$, we obtain the following estimate for the sum of the first and the second term on the r.h.s. of equality (12.2): $-\int_{(t,T]} \beta\,\mathrm{e}^{\beta s}(\tilde{Y}_s)^2 ds + 2\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_s\tilde{f}(s)ds \leq \varepsilon^2 \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds$. We also have that $\int_{[t,T)} \mathrm{e}^{\beta s}\,\tilde{Y}_s d\tilde{C}_s \leq 0$ and $\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} d\tilde{A}_s \leq 0$. We give the detailed arguments for the second inequality (the arguments for the first are similar). We have $\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} d\tilde{A}_s = \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} dA_s^1 - \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} dA_s^2$. For the first term, we write $\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} dA_s^1 = \int_{(t,T]} \mathrm{e}^{\beta s}(Y_{s-}^1 - Y_{s-}^2)dA_s^1 = \int_{(t,T]} \mathrm{e}^{\beta s}(Y_{s-}^1 - \overline{\xi}_s)dA_s^1 + \int_{(t,T]} \mathrm{e}^{\beta s}(\overline{\xi}_s - Y_{s-}^2)dA_s^1$. The second summand is nonpositive as $Y_{s-}^2 \geq \overline{\xi}_s$ (which is due to $Y_s^2 \geq \xi_s$, for all s). The first summand is equal to 0 due to the Skorokhod condition for A^1. Hence, $\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} dA_s^1 \leq 0$. By similar arguments, we see that $-\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} dA_s^2 \leq 0$. Hence, $\int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Y}_{s-} d\tilde{A}_s \leq 0$. The above observations, together with equation (12.2), yield that

$$\mathrm{e}^{\beta t}\,\tilde{Y}_t^2 + \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds + \int_{(t,T]} \mathrm{e}^{\beta s}\,d\langle \tilde{h}^c\rangle_s \leq \varepsilon^2 \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds - (\tilde{M}_T - \tilde{M}_t) - \sum_{t<s\leq T} \mathrm{e}^{\beta s}(\Delta \tilde{Y}_s)^2, \tag{12.3}$$

from which we derive estimates for $\|\tilde{Z}\|_\beta^2$, $\|\tilde{k}\|_{\nu,\beta}^2$, $\|\tilde{h}\|_{\beta,\mathcal{M}^2}^2$, and then an estimate for $|\!|\!|\tilde{Y}|\!|\!|_\beta^2$.

Estimate for $\|\tilde{Z}\|_\beta^2$, $\|\tilde{k}\|_{\nu,\beta}^2$ and $\|\tilde{h}\|_{\beta,\mathcal{M}^2}^2$. Note first that we have:

$$\sum_{t<s\leq T} \mathrm{e}^{\beta s}(\Delta \tilde{h}_s)^2 + \int_{(t,T]} \mathrm{e}^{\beta s}\,||\tilde{k}_s||_\nu^2 ds - \sum_{t<s\leq T} \mathrm{e}^{\beta s}(\Delta \tilde{Y}_s)^2 = -\sum_{t<s\leq T} \mathrm{e}^{\beta s}(\Delta \tilde{A}_s)^2$$
$$-\int_{(t,T]} \mathrm{e}^{\beta s} \int_E \tilde{k}_s^2(e)\tilde{N}(ds,de) - 2\sum_{t<s\leq T} \mathrm{e}^{\beta s}\,\Delta\tilde{A}_s \Delta\tilde{h}_s - 2\sum_{t<s\leq T} \mathrm{e}^{\beta s}\,\tilde{k}_s(p_s)\Delta\tilde{h}_s,$$

where, we have used the fact that the processes $A.$ and $N(\cdot, de)$ "do not have jumps in common", since A (resp. $N(\cdot, de)$) jumps only at predictable (resp. totally inaccessible) stopping times. By adding the term $\int_{(t,T]} \mathrm{e}^{\beta s}\,||\tilde{k}_s||_\nu^2 ds + \sum_{t<s\leq T} \mathrm{e}^{\beta s}(\Delta \tilde{h}_s)^2$ on both sides of inequality (12.3), by using the above computation and the well-known equality $[\tilde{h}]_t =$

$\langle \tilde{h}^c\rangle_t + \sum(\Delta\tilde{h})^2_s$, we get

$$\mathrm{e}^{\beta t}\,\tilde{Y}_t^2 + \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{Z}_s^2 ds + \int_{(t,T]} \mathrm{e}^{\beta s}\,||\tilde{k}_s||_\nu^2 ds + \int_{(t,T]} \mathrm{e}^{\beta s}\, d[\tilde{h}]_s \le \varepsilon^2 \int_{(t,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds - (M'_T - M'_t)$$
$$- 2\sum_{t<s\le T} \mathrm{e}^{\beta s}\,\Delta\tilde{A}_s\Delta\tilde{h}_s - 2\int_t^T d[\tilde{h}\,, \int_0^{\cdot}\int_E \mathrm{e}^{\beta s}\,\tilde{k}_s(e)\tilde{N}(ds,de)\,]_s, \tag{12.4}$$

with $M'_t = \tilde{M}_t + \int_{(t,T]} \mathrm{e}^{\beta s} \int_E \tilde{k}_s^2(e)\tilde{N}(ds,de)$ (where $\tilde{M}$ is given by (12.1)).
By classical arguments, which use Burkholder-Davis-Gundy inequalities, we can show that the local martingale M' is a martingale. Moreover, since $\tilde{h} \in \mathcal{M}^{2,\perp}$, by Remark 2.1, we derive that the expectation of the last term of the above inequality (12.4) is equal to 0. Furthermore, since $\tilde{h}$ is a martingale, for each predictable stopping time τ, we have $E[\Delta\tilde{h}_\tau/\mathcal{F}_{\tau-}] = 0$ (cf., e.g., Chapter I, Lemma (1.21) in [25]). Moreover, since $\tilde{A}$ is predictable, $\Delta\tilde{A}_\tau$ is $\mathcal{F}_{\tau-}$-measurable (cf., e.g., Chap I (1.40)-(1.42) in [25]), which implies that $E[\Delta\tilde{A}_\tau\Delta\tilde{h}_\tau/\mathcal{F}_{\tau-}] = \Delta\tilde{A}_\tau E[\Delta\tilde{h}_\tau/\mathcal{F}_{\tau-}] = 0$. We thus get $E[\sum_{0<s\le T} \mathrm{e}^{\beta s}\,\Delta\tilde{A}_s\Delta\tilde{h}_s] = 0$.

By applying (12.4) with $t = 0$, and by taking expectations on both sides of the resulting inequality, we obtain $\tilde{Y}_0^2 + \|\tilde{Z}\|^2_\beta + \|\tilde{k}\|^2_{\nu,\beta} + \|\tilde{h}\|^2_{\beta,\mathcal{M}^2} \le \varepsilon^2\|\tilde{f}\|^2_\beta$. We deduce that $\|\tilde{Z}\|^2_\beta \le \varepsilon^2\|\tilde{f}\|^2_\beta$, $\|\tilde{k}\|^2_{\nu,\beta} \le \varepsilon^2\|\tilde{f}\|^2_\beta$ and $\|\tilde{h}\|^2_{\beta,\mathcal{M}^2} \le \varepsilon^2\|\tilde{f}\|^2_\beta$, which are the desired estimates (3.6).

Estimate for $|||\tilde{Y}|||^2_\beta$. From inequality (12.3) we derive that, for all $\tau \in \mathcal{T}_{0,T}$, a.s.,
$\mathrm{e}^{\beta\tau}\,\tilde{Y}_\tau^2 \le \varepsilon^2 \int_{(\tau,T]} \mathrm{e}^{\beta s}\,\tilde{f}^2(s)ds - (\tilde{M}_T - \tilde{M}_\tau)$, where $\tilde{M}$ is given by (12.1).

Using first Chasles' relation for stochastic integrals, then taking the essential supremum over $\tau \in \mathcal{T}_{0,T}$ and the expectation on both sides of the above inequality, we obtain

$$E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau}\,\tilde{Y}_\tau^2] \le \varepsilon^2\|\tilde{f}\|^2_\beta + 2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s|] + 2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}d\tilde{h}_s|]$$
$$+ 2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_{(0,\tau]} \mathrm{e}^{\beta s} \int_E \tilde{Y}_{s-}\tilde{k}_s(e)\tilde{N}(ds,de)|]. \tag{12.5}$$

Let us consider the third term of the r.h.s. of the inequality (12.5). By Burkholder-Davis-Gundy inequalities, we have $E[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}d\tilde{h}_s|] \le cE[(\int_0^T \mathrm{e}^{2\beta s}\,\tilde{Y}^2_{s-}d[\tilde{h}]_s)^{1/2}]$. This inequality and the trivial inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ lead to

$$2E[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} |\int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}d\tilde{h}_s|] \le E[(\frac{1}{2}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \mathrm{e}^{\beta\tau}\,\tilde{Y}_\tau^2)^{1/2}(8c^2\int_0^T \mathrm{e}^{\beta s}\,d[\tilde{h}]_s)^{1/2}] \le \frac{1}{4}|||\tilde{Y}|||^2_\beta + 4c^2\|\tilde{h}\|^2_{\beta,\mathcal{M}^2}.$$

By using similar arguments, we get $2E[\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}} \int_0^\tau \mathrm{e}^{\beta s}\,\tilde{Y}_{s-}\tilde{Z}_s dW_s] \le \frac{1}{4}|||\tilde{Y}|||^2_\beta + 4c^2\|\tilde{Z}\|^2_\beta$, and a similar estimate for the last term in (12.5). By (12.5), we thus have $\frac{1}{4}|||\tilde{Y}|||^2_\beta \le$

$\varepsilon^2\|\tilde{f}\|_\beta^2+4c^2(\|\tilde{Z}\|_\beta^2+\|\tilde{k}\|_{\nu,\beta}^2+\|\tilde{h}\|_{\beta,\mathcal{M}^2}^2)$. Using the estimates for $\|\tilde{Z}\|_\beta^2$, $\|\tilde{k}\|_{\nu,\beta}^2$ and $\|\tilde{h}\|_{\beta,\mathcal{M}^2}^2$(cf. (3.6)), we thus get $|||\tilde{Y}|||_\beta^2 \leq 4\varepsilon^2(1+12c^2)\|\tilde{f}\|_\beta^2$, which is the desired result. □

Remark 12.1. *We note that this proof shows that the estimates* (3.6) *and* (3.7) *also hold in the simpler case of a non reflected BSDE. From this result, together with Lemma 2.1, and using the same arguments as in the proof of Theorem 4.1, we easily derive the existence and the uniqueness of the solution of the non reflected BSDE with general filtration from Definition 2.2. Similarly, we can show the comparison result for non reflected BSDEs with general filtration under the Assumption 5.1.*

Lemma 12.2. *Let f be a Lipschitz driver satisfying Assumption 5.1. Let A be a nondecreasing right-continuous predictable process in $\mathcal{S}^2$ with $A_0 = 0$ and let C be a nondecreasing right-continuous adapted purely discontinuous process in $\mathcal{S}^2$ with $C_{0-} = 0$.*
Let $(Y, Z, k, h) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{M}^{2,\perp}$ satisfy

$$-dY_t = f(t, Y_t, Z_t, k_t)dt + dA_t + dC_{t-} - Z_t dW_t - \int_E k_t(e)\tilde{N}(dt, de) - dh_t, \quad 0 \leq t \leq T.$$

Then the process (Y_t) is a strong $\mathcal{E}^f$-supermartingale.

The proof is omitted since it relies on the same arguments as those used in the proof of the same result shown in [19] in the particular case when the filtration is associated with W and N (cf. Proposition A.5 in [19]), as well as on some specific arguments, due to the general filtration, which are similar to those used in the proof of the previous lemma.

Remark 12.2. *(the non-linear case in the literature) Recall that the first studies on the pricing of Americans options in the non-linear case have been done in [15] in the case when f is convex, and in [16] in the case when f non-linear non-convex (both in the case of a continuous payoff). In these paper, the authors define the initial price $\mathbf{v_0}$ of the American option with payoff (ξ_t) by $\mathbf{v_0} := \sup_{\tau\in\mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau)$, and show that it is equal to the solution at time 0 of the associated non-linear reflected BSDE by using the* RBSDE approach.*This result has been extended in [37] to the case of an RCLL payoff. More recently, in [10], the authors have shown that the seller's superhedging price u_0 of the American option is equal to $\mathbf{v_0}(= \sup_{\tau\in\mathcal{T}_{0,T}} \mathcal{E}^f_{0,\tau}(\xi_\tau))$ in the case of an RCLL payoff.*

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