Book Section:

Wentworth, Richard and Wilkin, Graeme Peter Desmond (2011) Morse theory and stable pairs. In: Bielawski, Roger, Houston, Kevin and Speight, Martin, (eds.) Variation problems in differential geometry. Cambridge University Press , pp. 142-180.

MORSE THEORY AND STABLE PAIRS

RICHARD A. WENTWORTH AND GRAEME WILKIN

Abstract. We study the Morse theory of the Yang-Mills-Higgs functional on the space of pairs (A, Φ), where A is a unitary connection on a rank 2 hermitian vector bundle over a compact Riemann surface, and Φ is a holomorphic section of (E, d''_A). We prove that a certain explicitly defined substratification of the Morse stratification is perfect in the sense of $\mathcal{G}$-equivariant cohomology, where $\mathcal{G}$ denotes the unitary gauge group. As a consequence, Kirwan surjectivity holds for pairs. It also follows that the twist embedding into higher degree induces a surjection on equivariant cohomology. This may be interpreted as a rank 2 version of the analogous statement for symmetric products of Riemann surfaces. Finally, we compute the $\mathcal{G}$-equivariant Poincaré polynomial of the space of τ-semistable pairs. In particular, we recover an earlier result of Thaddeus. The analysis provides an interpretation of the Thaddeus flips in terms of a variation of Morse functions.

Contents

Date: June 26, 2010.
2010 *Mathematics Subject Classification.* Primary: 58E15 ; Secondary: 14D20, 53D20.
R.W. supported in part by NSF grant DMS-0805797.

1. Introduction

In this paper we revisit the notion of a stable pair on a Riemann surface. We introduce new techniques for the computation of the equivariant cohomology of moduli spaces. The main ingredient is a version of Morse theory in the spirit of Atiyah and Bott [1] adapted to the *singular* infinite dimensional space of holomorphic pairs.

Recall first the basic idea. Let E be a hermitian vector bundle over a closed Riemann surface M of genus $g \geq 2$. The space $\mathcal{A}(E)$ of unitary connections on E is an infinite dimensional affine space with an action of the group $\mathcal{G}$ of unitary gauge transformations. Via the Chern connection there is an isomorphism $A \mapsto d''_A$ between $\mathcal{A}(E)$ and the space of (integrable) Dolbeault operators (i.e. holomorphic structures) on E. One of the key observations of Atiyah-Bott is that the Morse theory of a suitable $\mathcal{G}$-invariant functional on $\mathcal{A}(E)$, namely the Yang-Mills functional, gives rise to a smooth stratification (see also [6]). Moreover, this stratification is $\mathcal{G}$-equivariantly perfect in the sense that the long exact sequences for the equivariant cohomology of successive pairs split. Since $\mathcal{A}(E)$ is contractible, this gives an effective method, inductive on the rank of E, for computing the equivariant cohomology of the minimum, which consists of projectively flat connections.

Consider now a configuration space $\mathcal{B}(E)$ consisting of pairs (A, Φ), where $A \in \mathcal{A}(E)$ and Φ is a section of a vector bundle associated to E. We impose the condition that Φ be d''_A-holomorphic. Note that $\mathcal{B}(E)$ is still contractible, since an equivariant retraction of $\mathcal{B}(E)$ to $\mathcal{A}(E)$ is given by simply scaling Φ. It is therefore reasonable to attempt an inductive computation of equivariant cohomology as above. A problem arises, however, from the singularities caused by jumps in the dimension of the kernel as A varies. Nevertheless, the methods introduced in [8] for the case of Higgs bundles demonstrate that in certain cases this difficulty can be managed.

Below we apply this approach to the moduli space of rank 2, degree d, τ-semistable pairs $\mathfrak{M}_{\tau,d} = \mathcal{B}^{\tau}_{ss}(E) /\!\!/ \mathcal{G}^{\mathbb{C}}$ introduced by Bradlow [3] and Bradlow-Daskalopoulos [4]. In this case, Φ is holomorphic section of E, and the Yang-Mills functional $\mathrm{YM}(A)$ is replaced by the Yang-Mills-Higgs functional $\mathrm{YMH}(A, \Phi)$. We give a description of the algebraic and Morse theoretic stratifications of $\mathcal{B}(E)$. These stratifications, as well as the moduli space, depend on a real parameter τ, and since $\mathfrak{M}_{\tau,d}$ is nonempty only for $d/2 < \tau < d$, we shall always assume this bound for τ. For generic τ, $\mathcal{G}$ acts freely, and the quotient is geometric.

We will see that, as in [6, 7, 8], the algebraic and Morse stratifications agree (see Theorem 3.9). Because of singularities, however, the Morse stratification actually fails to be perfect in this case. We identify precisely how this comes about, and in fact we will show that this "failure of perfection" exactly cancels between different strata, so that there is a substratification that is indeed perfect (see Theorem 3.11). We formulate this result as

Theorem 1.1 (Equivariantly perfect stratification)**.** *For every τ, $d/2 < \tau < d$, there is a $\mathcal{G}$-invariant stratification of $\mathcal{B}(E)$ defined via the Yang-Mills-Higgs flow that is perfect in $\mathcal{G}$-equivariant cohomology.*

The fact that perfection fails for the Morse stratification but holds for a substratification seems to be a new phenomenon. In any case, as with vector bundles, Theorem 1.1 allows us to compute the $\mathcal{G}$-equivariant cohomology of the open stratum $\mathcal{B}^\tau_{ss}(E)$. Explicit formulas in terms of symmetric products of M are given in Theorems 4.1 and 4.2.

There is a natural map (called the *Kirwan map*) from the cohomology of the classifying space $B\mathcal{G}$ of $\mathcal{G}$ to the equivariant cohomology of the stratum of τ-semistable pairs $\mathcal{B}^\tau_{ss}(E) \subset \mathcal{B}(E)$, coming from inclusion (see [13]). One of the consequences of the work of Atiyah-Bott is that the analogous map is surjective for the case of semistable bundles. The same is true for pairs:

Theorem 1.2 (Kirwan surjectivity)**.** *The Kirwan map $H^*(B\mathcal{G}) \to H^*_{\mathcal{G}}(\mathcal{B}^\tau_{ss}(E))$ is surjective. In particular, for generic τ, $H^*(B\mathcal{G}) \to H^*(\mathfrak{M}_{\tau,d})$ is surjective.*

As noted above, for noninteger values of τ, $d/2 < \tau < d$, $\mathfrak{M}_{\tau,d}$ is a smooth projective algebraic manifold of dimension $d + 2g - 2$, and the equivariant cohomology of $\mathcal{B}^\tau_{ss}(E)$ is identical to the ordinary cohomology of $\mathfrak{M}_{\tau,d}$. The computation of equivariant cohomology presented here then recovers the result of Thaddeus in [20], who computed the cohomology using different methods. Namely, he gives an explicit description of the modifications, or "flips", in $\mathfrak{M}_{\tau,d}$ as the parameter τ varies. At integer values there is a change in stability conditions. Below, we show how the change in cohomology arising from a flip may be reinterpreted as a variation of the Morse function. This is perhaps not surprising in view of the construction in [5]. However, here we work directly on the infinite dimensional space. The basic idea is that there is a one parameter choice of Morse functions f_τ on $\mathcal{B}$. The minimum $f_\tau^{-1}(0)/\mathcal{G} \simeq \mathfrak{M}_{\tau,d}$, and the cohomology of $\mathfrak{M}_{\tau,d}$ may, in principle, be computed from the cohomology of the higher critical sets. As τ varies past certain critical values, new critical sets are created while others merge. Moreover, indices of critical sets can jump. All this taken together accounts for the change in topology of the minimum.

There are several important points in this interpretation. One is that the subvarieties responsible for the change in cohomology observed by Thaddeus as the parameter varies are somehow directly built into the Morse theory, even for a fixed τ, in the guise of higher critical sets. This example also exhibits computations at critical strata that can be carried out in the presence of singular normal *cusps*, as opposed to the singular normal vector bundles in [8]. These ideas may be useful for computations in higher rank or for other moduli spaces.

The critical set corresponding to minimal Yang-Mills connections, regarded as a subset of $\mathcal{B}(E)$ by setting $\Phi \equiv 0$, is special from the point of view of the Morse theory. In particular, essentially because of issues regarding Brill-Noether loci in the moduli space of vector bundles, we can only directly prove the perfection of the stratification at this step, and the crucial Morse-Bott lemma (Theorem 3.18), for $d > 4g - 4$. This we do in Section 3.5. By contrast, for the other critical strata there is no such requirement on the degree. Using this fact, we then give an inductive argument by twisting E by a positive line bundle and embedding $\mathcal{B}(E)$ into the space of pairs for higher degree, thus indirectly concluding the splitting of the associated long exact sequence even at minimal Yang-Mills connections in low degree (see Section 3.7).

This line of reasoning leads to another interesting consequence. For τ close to $d/2$, there is a surjective holomorphic map from $\mathfrak{M}_{\tau,d}$ to the moduli space of semistable rank 2 bundles of degree d. This is the rank 2 version of the Abel-Jacobi map [4]. In this sense, $\mathfrak{M}_{\tau,d}$ is a generalization of the d-th symmetric product S^dM of M. Choosing an effective divisor on M of degree k, there is a natural inclusion $S^dM \hookrightarrow S^{d+k}M$, and it was shown by MacDonald in (14.3) of [16] that this inclusion induces a surjection on rational cohomology. A similar construction works for rank 2 pairs, except now $d \mapsto d+2k$, while there is also a shift in the parameter $\tau \mapsto \tau + k$. We will prove the following

Theorem 1.3 (Embedding in higher degree)**.** *Let* $\deg E = d$ *and* $\deg \widetilde{E} = d + 2k$*. Then for all* $d/2 < \tau < d$*, the inclusion* $\mathcal{B}^{\tau}_{ss}(E) \hookrightarrow \mathcal{B}^{\tau+k}_{ss}(\widetilde{E})$ *described above induces a surjection on rational* $\mathcal{G}$*-equivariant cohomology. In particular, for generic* τ*, the inclusion* $\mathfrak{M}_{\tau,d} \hookrightarrow \mathfrak{M}_{\tau+k,d+2k}$ *induces a surjection on rational cohomology.*

Remark 1.4. It is also possible to construct a moduli space of pairs for which the isomorphism class of $\det E$ is fixed, indeed this is the space studied by Thaddeus in [20]. The explicit calculations in this paper are all done for the non-fixed determinant case, however it is worth pointing out here that the idea is essentially the same for the fixed determinant case, and that the only major difference between the two cases is in the topology of the critical sets. In particular, the indexing set $\Delta_{\tau,d}$ for the stratification is the same in both cases.

Acknowledgements. Thanks to George Daskalopoulos for many discussions. R.W. is also grateful for the hospitality at the MPIM-Bonn, where some of the work on this paper was completed.

2. Stable pairs

2.1. **The Harder-Narasimhan stratification.** Throughout this paper, E will denote a rank 2 hermitian vector bundle on M of positive degree $d = \deg E$. We will regard E as a smooth complex vector bundle, and when endowed with a holomorphic structure that is understood, we will use the same notation for the holomorphic bundle.

Recall that a holomorphic bundle E of degree d is *stable* (resp. *semistable*) if $\deg L < d/2$ (resp. $\deg L \leq d/2$) for all holomorphic line subbundles $L \subset E$.

Definition 2.1. For a stable holomorphic bundle E, set $\mu_+(E) = d/2$. For E unstable, let

$$\mu_+(E) = \sup\{\deg L : L \subset E \text{ a holomorphic line subbundle}\}$$

For a holomorphic section $\Phi \not\equiv 0$ of E, define $\deg \Phi$ to be the number of zeros of Φ, counted with multiplicity. Finally, for a holomorphic pair (E, Φ) let

$$\mu_-(E,\Phi) = \begin{cases} d - \deg \Phi & \Phi \not\equiv 0 \\ d - \mu_+(E) & \Phi \equiv 0 \end{cases}$$

Definition 2.2 ([3])**.** Given τ, a holomorphic pair (E,Φ) is called *τ-stable* (resp. *τ-semistable*) if

$$\mu_+(E)<\tau<\mu_-(E,\Phi) \qquad (\text{resp. } \mu_+(E)\leq\tau\leq\mu_-(E,\Phi))$$

As with holomorphic bundles, there is a notion of s-equivalence of strictly semistable objects. The set $\mathfrak{M}_{\tau,d}$ of isomorphism classes of semistable pairs, modulo s-equivalence, has the structure of a projective variety. Note that $\mathfrak{M}_{\tau,d}$ is empty if $\tau\notin[d/2,d]$. For non-integer values of $\tau\in(d/2,d)$, semistable is equivalent to stable, and $\mathfrak{M}_{\tau,d}$ is smooth.

Let $\mathcal{A}=\mathcal{A}(E)$ denote the infinite dimensional affine space of holomorphic structures on E, $\mathcal{G}$ the group of unitary gauge transformations, and $\mathcal{G}^{\mathbb{C}}$ its complexification. The space $\mathcal{A}$ may be identified with Dolbeault operators $A\mapsto d_A'':\Omega^0(E)\to\Omega^{0,1}(E)$, with the inverse of d_A'' given by the Chern connection with respect to the fixed hermitian structure. When we want to emphasize the holomorphic bundle, we write (E,d_A'').

$$\mathcal{B}=\mathcal{B}(E)=\left\{(A,\Phi)\in\mathcal{A}\times\Omega^0(E)\,:\,d_A''\Phi=0\right\} \tag{2.1}$$

Let

$$\mathcal{B}_{ss}^{\tau}=\left\{(A,\Phi)\in\mathcal{B}:((E,d_A''),\Phi)\text{ is }\tau\text{-semistable}\right\}$$

Then $\mathfrak{M}_{\tau,d}=\mathcal{B}_{ss}^{\tau}/\!\!/\mathcal{G}^{\mathbb{C}}$, where the double slash identifies s-equivalent orbits. For generic values of τ, semistability implies stability and $\mathcal{G}$ acts freely, and so this is a geometric quotient.

We now describe the stratification of $\mathcal{B}$ associated to the Harder-Narasimhan filtration, which has an important relationship to the Morse theory picture that will be discussed below in Section 3.2. In the case of rank 2 bundles, this stratification is particularly easy to describe. For convenience, throughout this section we fix a generic τ, $d/2<\tau<d$ (it suffices to assume $4\tau\notin\mathbb{Z}$). Genericity is used only to give a simple description of the strata in terms of δ. The extension to special values of τ is straightforward (see Remark 2.11).

Note that stability of the pair fails if either of the inequalities in Definition 2.2 fails. The two inequalities are not quite independent, but there are some cases where only one fails and others where both fail. If the latter, it seems natural to filter by the *most destabilizing of the two.* With this in mind, we make the following

Definition 2.3. For a holomorphic pair (E,Φ), let

$$\delta(E,\Phi)=\max\left\{\tau-\mu_-(E,\Phi),\mu_+(E)-\tau,0\right\}$$

Note that δ takes on a discrete and infinite set of nonnegative real values, and is upper semicontinuous, since both μ_+ and $-\mu_-$ are (observe that $\deg\Phi\leq\mu_+(E)$). We denote the ordered set of such δ by $\Delta_{\tau,d}$. Clearly, δ is an integer modulo $\pm\tau$, or $\delta=\tau-d/2$. Because of the genericity of τ, the former two possibilities are mutually exclusive:

Lemma 2.4. *There is a disjoint union $\Delta_{\tau,d}\setminus\{0\}=\Delta_{\tau,d}^+\cup\Delta_{\tau,d}^-$, with*

$$\delta\in\Delta_{\tau,d}^+\iff\delta=\tau-\mu_-(E,\Phi)\,,\ \textit{for some pair }(E,\Phi)$$
$$\delta\in\Delta_{\tau,d}^-\iff\delta=\mu_+(E)-\tau\,,\ \textit{for some pair }(E,\Phi)$$

Lemma 2.5. *Suppose* $(E,\Phi)\notin \mathcal{B}^\tau_{ss}$, $\Phi\not\equiv 0$. *Then*

(1) *if* $\deg\Phi\geq d/2$, $\delta(E,\Phi)=\mu_+(E)-d+\tau$.
(2) *if* $d-\tau\leq \deg\Phi< d/2$, $\delta(E,\Phi)=\deg\Phi-d+\tau$.
(3) *if* $0\leq\deg\Phi<d-\tau$, $\delta(E,\Phi)=\mu_+(E)-\tau$.

If $\Phi\equiv 0$, *then* $\delta(E,\Phi)=\mu_+(E)-d+\tau$.

Proof. If $\deg\Phi\geq d/2$, then the line subbundle generated by Φ is the maximal destabilizing subbundle of E. Hence, $\mu_+(E)=\deg\Phi$, $\mu_-(E,\Phi)=d-\mu_+(E)$, and so (1) follows from the fact that $\tau>d/2$. For (2), consider the extension $0\to L_1\to E\to L_2\to 0$, where $\Phi\in H^0(L_1)$. Then $\deg L_2=d-\deg\Phi$, so $\mu_+(E,\Phi)\leq d-\deg\Phi$. It follows that $\mu_+(E)-\tau\leq 0$. For part (3), $0\leq\deg\Phi<d-\tau$ implies $\tau<\mu_-(E,\Phi)$. The last statement is clear, since $\tau>d/2$ implies $\tau-\mu_-(E,\Phi)=\mu_+(E)-d+\tau>\mu_+(E)-\tau$. □

Corollary 2.6. $\Delta^-_{\tau,d}\subset(0,d-\tau]$.

Proof. Indeed, if (E,Φ) is unstable and $\delta(E,\Phi)=\mu_+(E)-\tau$, then by (3) it must be that E is unstable and $\Phi\not\equiv 0$. From the Harder-Narasimhan filtration (cf. [14]) $0\to L_2\to E\to L_1\to 0$, the projection of Φ to L_1 must also be nonzero, since $\deg\Phi<\deg L_2$. Hence, $\deg L_1=d-\mu_+(E)\geq 0$, and so $d-\tau\geq\delta(E,\Phi)$. □

Remark 2.7. If $\delta\in\Delta^+_{\tau,d}$ and $\delta<\tau-d/2$, then $\delta\leq\tau-d/2-1/2$. Indeed, if $\delta+d-\tau=k\in\mathbb{Z}$, the condition forces $k<d/2$; hence, $k\leq d/2-1/2$.

Let $I_{\tau,d}=[\tau-d/2,2\tau-d)$. We are ready to describe the τ-Harder-Narasimhan stratification. First, for $j>d/2$, let $\mathcal{A}_j\subset\mathcal{A}$ be the set of holomorphic bundles E of Harder-Narasimhan type $\mu_+(E)=j$. We also set $\mathcal{A}_{d/2}=\mathcal{A}_{ss}$. There is an obvious inclusion $\mathcal{A}_j\subset\mathcal{B}:A\mapsto(A,0)$.

(**0**) $\delta=0$: The open stratum $\mathcal{B}^\tau_0=\mathcal{B}^\tau_{ss}$ consists of τ-semistable pairs.

(**I**$_a$) $\delta\in\Delta^+_{\tau,d}\cap I_{\tau,d}$: Then we include the strata $\mathcal{A}_{\delta+d-\tau}$. Note that this includes the semistable stratum $\mathcal{A}_{ss}$. The bundles in this strata that are not semistable have a unique description as extensions

$$0\longrightarrow L_1\longrightarrow E\longrightarrow L_2\longrightarrow 0 \tag{2.2}$$

where $\deg L_1=\mu_+(E)=\delta+d-\tau$.

(**I**$_b$) $\delta\in\Delta^+_{\tau,d}\cap[2\tau-d,+\infty)$: Then $\mathcal{B}^\tau_\delta=\{(E,\Phi):\mu_+(E)=\delta+d-\tau\}$. These are extensions (2.2), $\deg L_1=\mu_+(E)=\delta+d-\tau$, $\Phi\subset H^0(L_1)$.

(**II**$^+$) $\delta\in\Delta^+_{\tau,d}\cap(0,2\tau-d)$: Then $\mathcal{B}^\tau_\delta=\{(E,\Phi):\deg\Phi=\delta+d-\tau\}$. These are extensions (2.2), $\deg L_1=\delta+d-\tau$, $\Phi\subset H^0(L_1)$.

(**II**$^-$) $\delta\in\Delta^-_{\tau,d}$: Then $\mathcal{B}^\tau_\delta=\{(E,\Phi):\mu_+(E)=\delta+\tau\ ,\ \deg\Phi<d/2\}$. These are extensions

$$0\longrightarrow L_2\longrightarrow E\longrightarrow L_1\longrightarrow 0$$

where $\deg L_2=\mu_+(E)$, and the projection of Φ to $H^0(L_1)$ is nonzero.

For simplicity of notation, when τ is fixed we will mostly omit the superscript: $\mathcal{B}_\delta = \mathcal{B}^\tau_\delta$.

Remark 2.8. It is simple to verify that the stratification obtained above coincides with the possible Harder-Narasimhan filtrations of pairs (E, Φ) considered as *coherent systems* (see [15, 18, 12]).

It will be convenient to organize $\Delta_{\tau,d}$ by the slope of the subbundle in the maximal destabilizing subpair. Define $j : \Delta_{\tau,d} \setminus \{0\} \to \{d/2\} \cup \{k \in \mathbb{Z} : k \geq d - \tau\}$ by

$$j(\delta) = \begin{cases} \delta + d - \tau\,, & \delta \in \Delta^+_{\tau,d} \\ \delta + \tau\,, & \delta \in \Delta^-_{\tau,d} \end{cases} \tag{2.3}$$

Notice that $j(\delta) = \deg L_1$ for $\delta \in \Delta^+_{\tau,d}$, and $j(\delta) = \deg L_2$ for $\delta \in \Delta^-_{\tau,d}$, where L_1, L_2 refer to the line subbundles of E in the filtrations above. Note that j is surjective. It is precisely 2-1 on the image of $\Delta^-_{\tau,d}$ and 1-1 elsewhere (if d odd; otherwise $d/2$ labels both the stratum $\mathcal{A}_{ss}$ and the strictly semistable bundles of type $\mathbf{II}^+$). It is not order preserving but is, of course, order preserving on each of $\Delta^\pm_{\tau,d}$ separately.

Definition 2.9. For $\delta \in \Delta_{\tau,d}$, let

$$X_\delta = \bigcup_{\delta' \leq \delta\,,\, \delta' \in \Delta_{\tau,d}} \mathcal{B}_{\delta'} \ \cup \bigcup_{\delta' \leq \delta\,,\, \delta' \in \Delta^+_{\tau,d} \cap I_{\tau,d}} \mathcal{A}_{j(\delta')}$$

For $\delta \in \Delta^+_{\tau,d} \cap I_{\tau,d}$, let

$$X'_\delta = \bigcup_{\delta' \leq \delta\,,\, \delta' \in \Delta_{\tau,d}} \mathcal{B}_{\delta'} \ \cup \bigcup_{\delta' < \delta\,,\, \delta' \in \Delta^+_{\tau,d} \cap I_{\tau,d}} \mathcal{A}_{j(\delta')}$$

For $\delta \notin \Delta^+_{\tau,d} \cap I_{\tau,d}$, let

$$X'_\delta = \bigcup_{\delta' < \delta\,,\, \delta' \in \Delta_{\tau,d}} \mathcal{B}_{\delta'} \ \cup \bigcup_{\delta' < \delta\,,\, \delta' \in \Delta^+_{\tau,d} \cap I_{\tau,d}} \mathcal{A}_{j(\delta')}$$

We call the collection $\{X_\delta, X'_\delta\}_{\delta \in \Delta_{\tau,d}}$ the *τ-Harder-Narasimhan stratification* of $\mathcal{B}$.

Note that $X_{\delta_1} \subset X'_\delta \subsetneq X_\delta \subset X'_{\delta_2}$, where δ_1 is the predecessor and δ_2 is the successor of δ in $\Delta_{\tau,d}$. If $\delta \notin \Delta^+_{\tau,d} \cap I_{\tau,d}$, then $X'_\delta = X_{\delta_1}$ and $X_\delta = X'_{\delta_2}$. In the special case $\delta = \tau - d/2$, we have

$$X_{\tau - d/2} = X'_{\tau - d/2} \cup \mathcal{A}_{ss} \tag{2.4}$$

$$X'_{\tau - d/2} = \begin{cases} X_{\delta_1} & \text{if } d \text{ is odd} \\ X_{\delta_1} \cup \mathcal{B}_{\tau - d/2} & \text{if } d \text{ is even} \end{cases} \tag{2.5}$$

The following is clear.

Proposition 2.10. *The sets* $\{X_\delta, X'_\delta\}_{\delta\in\Delta_{\tau,d}}$, *are locally closed in* $\mathcal{B}$, $\mathcal{G}$-*invariant, and satisfy*

$$\mathcal{B} = \bigcup_{\delta\in\Delta_{\tau,d}} X_\delta = \bigcup_{\delta\in\Delta_{\tau,d}} X'_\delta$$

$$\overline{\mathcal{B}}_\delta \subset \bigcup_{\delta\leq\delta'\,,\,\delta'\in\Delta_{\tau,d}} \mathcal{B}_{\delta'} = \mathcal{B}_\delta \cup \bigcup_{\delta<\delta'\,,\,\delta'\in\Delta_{\tau,d}} \mathcal{B}'_{\delta'}$$

$$\overline{\mathcal{B}}'_\delta \subset \bigcup_{\delta\leq\delta'\,,\,\delta'\in\Delta_{\tau,d}} \mathcal{B}'_{\delta'} = \mathcal{B}'_\delta \cup \bigcup_{\delta<\delta'\,,\,\delta'\in\Delta_{\tau,d}} \mathcal{B}_{\delta'}$$

Remark 2.11. To extend this stratification in the case of nongeneric τ, we define the sets $\Delta^\pm_{\tau,d}$ and the corresponding strata as above. For $\delta\in\Delta^+_{\tau,d}\cap\Delta^-_{\tau,d}$ there are two or possibly three components with the same label.

Let us note the following behavior as τ varies. For $\tau_1\leq\tau_2$, there is a well-defined map $\Delta_{\tau_1,d}\to\Delta_{\tau_2,d}$ given by $\delta\mapsto\max\{\delta\pm(\tau_2-\tau_1),0\}$, where $\pm$ depends on $\delta\in\Delta^\pm_{\tau,d}$. Hence, elements of $\Delta^+_{\tau,d}$ (white circles in Figure 1 below) "move" to the right, and elements of $\Delta^-_{\tau,d}$ (dark circles) "move" to the left as τ increases. The map is an order preserving bijection *provided* τ_1, τ_2 are in a connected component of $(d/2,d)\setminus C_d$, where

$$C_d=\{\tau_c\in(d/2,d) : 2\tau_c\in\mathbb{Z} \text{ if } d \text{ even}, 4\tau_c\in\mathbb{Z} \text{ if } d \text{ odd}\} \tag{2.6}$$

However, as τ_2 crosses an element of C_d, there is a "flip" in the stratification. When this flip occurs at $\delta=0$, this is the phenomenon discovered by Thaddeus [20]; the discussion here is an extension of this effect to the entire stratification.

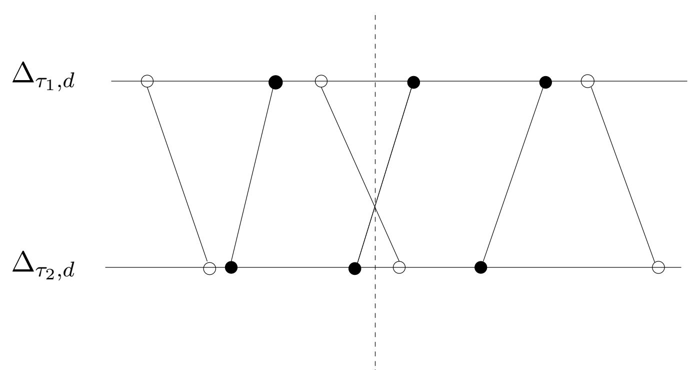

Figure 1. A "flip"

Finally, we will also have need to refer to the Harder-Narasimhan stratification of the space $\mathcal{A}$ of unitary connections on E. We denote this by

$$X^{\mathcal{A}}_j = \bigcup_{d/2\leq j'\leq j} \mathcal{A}_{j'} \tag{2.7}$$

The following statement will be used later on. It is an immediate consequence of the descriptions of the strata above.

Lemma 2.12. *Consider the projection* $\mathrm{pr} : \mathcal{B} \to \mathcal{A}$. *Then*

$$\begin{aligned} \mathrm{pr}(\mathcal{B}_\delta) &= \mathcal{A}_{j(\delta)}\ ,\ \ \delta \in \Delta^-_{\tau,d} \cup \Big(\Delta^+_{\tau,d} \cap [\tau - d/2, +\infty)\Big) \\ \mathrm{pr}(\mathcal{B}_\delta) &= X^{\mathcal{A}}_{d-j(\delta)}\ ,\ \ \delta \in \Delta^+_{\tau,d} \cap (0, \tau - d/2) \end{aligned}$$

2.2. **Deformation theory.** Fix a conformal metric on M, normalized[1] for convenience so that $\mathrm{vol}(M) = 2\pi$. Infinitesimal deformations of $(A, \Phi) \in \mathcal{B}$ modulo equivalence are described by the following elliptic complex, which we denote by $\mathcal{C}_{(A,\Phi)}$ (cf. [4]).

$$\begin{gathered} \mathcal{C}^0_{(A,\Phi)} \xrightarrow{D_1} \mathcal{C}^1_{(A,\Phi)} \xrightarrow{D_2} \mathcal{C}^2_{(A,\Phi)} \\ \Omega^0(\mathrm{End}\, E) \xrightarrow{D_1} \Omega^{0,1}(\mathrm{End}\, E) \oplus \Omega^0(E) \xrightarrow{D_2} \Omega^{0,1}(E) \\ D_1(u) = (-d''_A u, u\Phi)\ ,\ D_2(a, \varphi) = d''_A\varphi + a\Phi \end{gathered} \tag{2.8}$$

Here, D_1 is the linearization of the action of the complex gauge group $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{B}$, and D_2 is the linearization of the condition $d''_A\Phi = 0$. Note that $D_2 D_1 = 0$ if $(A, \Phi) \in \mathcal{B}$. The hermitian metric gives adjoint operators

$$D_1^*(a, \varphi) = -(d''_A)^* a + \varphi\Phi^*\ ,\ D_2^*(\beta) = (\beta\Phi^*, (d''_A)^*\beta) \tag{2.9}$$

The spaces of *harmonic forms* are by definition

$$\begin{aligned} \mathcal{H}^0(\mathcal{C}_{(A,\Phi)}) &= \ker D_1 \\ \mathcal{H}^1(\mathcal{C}_{(A,\Phi)}) &= \ker D_1^* \cap \ker D_2 \\ \mathcal{H}^2(\mathcal{C}_{(A,\Phi)}) &= \ker D_2^* \end{aligned}$$

Vectors in $\Omega^{0,1}(\mathrm{End}\, E) \oplus \Omega^0(E)$ that are orthogonal to the $\mathcal{G}^{\mathbb{C}}$-orbit through (A, Φ) are in $\ker D_1^*$, and a *slice* for the action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{B}$ is therefore given by

$$\mathcal{S}_{(A,\Phi)} = \ker D_1^* \cap \big\{(a, \varphi) \in \Omega^{0,1}(\mathrm{End}\, E) \oplus \Omega^0(E)\ :\ D_2(a, \varphi) + a\varphi = 0\big\} \tag{2.10}$$

Define the *slice map*

$$\begin{gathered} \Sigma : (\ker D_1)^\perp \times \mathcal{S}_{(A,\Phi)} \to \mathcal{B} \\ (u, a, \varphi) \mapsto e^u \cdot (A + a, \Phi + \varphi) \end{gathered} \tag{2.11}$$

The proof of the following may be modeled on [21, Proposition 4.12]. We omit the details.

Proposition 2.13. *The slice map* Σ *is a local homeomorphism from a neighborhood of* 0 *in* $(\ker D_1)^\perp \times \mathcal{S}_{(A,\Phi)}$ *to a neighborhood of* (A, Φ) *in* $\mathcal{B}$.

The *Kuranishi map* is defined by

$$\begin{gathered} \Omega^{0,1}(\mathrm{End}\, E) \oplus \Omega^0(E) \xrightarrow{k} \Omega^{0,1}(\mathrm{End}\, E) \oplus \Omega^0(E) \\ k(a, \varphi) = (a, \varphi) + D_2^* \circ G_2(a\varphi) \end{gathered}$$

[1] More generally, the scale invariant parameter is $\tau\, \mathrm{vol}(M)/2\pi$.

where G_2 denotes the Green's operator associated to the laplacian $D_2(D_2)^*$. We have the following standard result (cf. [14, Chapter VII] for the case of holomorphic bundles over a Kähler manifold and [4] for this case).

Proposition 2.14. *The Kuranishi map k maps $\mathcal{S}_{(A,\Phi)}$ to harmonics $\mathcal{H}^1(\mathcal{C}_{(A,\Phi)})$, and in a neighborhood of zero it is a local homeomorphism onto its image. Moreover, if $\mathcal{H}^2(\mathcal{C}_{(A,\Phi)}) = \{0\}$, then k is a local homeomorphism $\mathcal{S}_{(A,\Phi)} \to \mathcal{H}^1(\mathcal{C}_{(A,\Phi)})$.*

The following is immediate from (2.8) and (2.9).

Lemma 2.15. *Given $(A,\Phi) \in \mathcal{B}$, if $\Phi \neq 0$ then $\mathcal{H}^0(\mathcal{C}_{(A,\Phi)}) = \mathcal{H}^2(\mathcal{C}_{(A,\Phi)}) = \{0\}$. If $H^1(E) = \{0\}$ then $\mathcal{H}^2(\mathcal{C}_{(A,\Phi)}) = \{0\}$.*

We will be interested in the deformation complex along higher critical sets of the Yang-Mills-Higgs functional. As we will see in the next section, in addition to the Yang-Mills connections (where $\Phi \equiv 0$), the other critical sets correspond to split bundles $E = L_1 \oplus L_2$, $(A,\Phi) = (A_1 \oplus A_2, \Phi_1 \oplus \{0\})$, with $\deg L_1 = j \geq \deg L_2 = d - j$. Here, $j = j(\delta)$ for some $\delta \in \Delta^+_{\tau,d}$, or $j = d - j(\delta)$ for some $\delta \in \Delta^-_{\tau,d}$. The set of all such critical points will therefore be denoted by $\eta_\delta \subset \mathcal{B}$. We will denote the components of $\operatorname{End} E \simeq L_i \otimes L_j^*$ in the complex by u_{ij}, a_{ij}, φ_{ij}.

In this case, $\mathcal{H}^1(\mathcal{C}_{(A,\Phi)})$ consists of all (a,φ) satisfying

$$\begin{aligned} (d''_A)^* a_{12} &= 0 & (d'')^* a_{22} &= 0 \\ (d'')^* a_{11} - \varphi_1 \Phi_1^* &= 0 & (d''_A)^* a_{21} - \varphi_2 \Phi_1^* &= 0 \\ d''_{A_1}\varphi_1 + a_{11}\Phi_1 &= 0 & d''_{A_2}\varphi_2 + a_{21}\Phi_1 &= 0 \end{aligned} \tag{2.12}$$

We use this formalism to define deformation retractions in a neighborhood of $(A,\Phi) \in \mathcal{B}$ in two cases. First, we have

Lemma 2.16. *Suppose $(A,\Phi) = (A_1 \oplus A_2, \Phi_1 \oplus 0)$ is a split pair as above, $\Phi_1 \neq 0$. Let*

$$\begin{aligned} \mathcal{S}^{neg.}_{(A,\Phi)} &= \{(a,\varphi) \in \mathcal{S}_{(A,\Phi)} : a_{ij} = 0\,,\ (ij) \neq (21)\,, \text{ and } \varphi_1 = 0\} \\ \mathcal{S}'_{(A,\Phi)} &= \{(a,\varphi) \in \mathcal{S}_{(A,\Phi)} : (a_{21}, \varphi_2) \neq 0\} \end{aligned}$$

Then there is an equivariant deformation retraction $\mathcal{S}^{neg.}_{(A,\Phi)} \hookrightarrow \mathcal{S}_{(A,\Phi)}$ which restricts to a deformation retraction $\mathcal{S}^{neg.}_{(A,\Phi)} \setminus \{0\} \hookrightarrow \mathcal{S}'_{(A,\Phi)}$.

Proof. By Lemma 2.15 and Proposition 2.14, the Kuranishi map gives a homeomorphism of the slice with $\mathcal{H}^1(\mathcal{C}_{(A,\Phi)})$. Hence, it suffices to define the retraction there. For this we take

$$r_t(a_{11}, a_{12}, a_{21}, a_{22}; \varphi_1, \varphi_2) = (ta_{11}, ta_{12}, a_{21}, ta_{22}; t\varphi_1, \varphi_2)\,,\ t \in [0,1]$$

Notice that this preserves the equations in (2.12). □

Second, near minimal Yang-Mills connections, we find a similar retraction under the assumption that $\mathcal{H}^2(\mathcal{C}_{(A,\Phi)})$ vanishes.

Lemma 2.17. *Suppose $d > 4g-4$ and A is semistable. Let*

$$\mathcal{S}^{neg.}_{(A,0)} = \{(a,\varphi) \in \mathcal{S}_{(A,0)} : a = 0\}$$
$$\mathcal{S}'_{(A,0)} = \{(a,\varphi) \in \mathcal{S}_{(A,0)} : \varphi \neq 0\}$$

Then there is an equivariant deformation retraction $\mathcal{S}^{neg.}_{(A,0)} \hookrightarrow \mathcal{S}_{(A,0)}$ which restricts to a deformation retraction $\mathcal{S}^{neg.}_{(A,0)} \setminus \{0\} \hookrightarrow \mathcal{S}'_{(A,0)}$.

Proof. Let E be the holomorphic bundle given by A. Since E is semistable, so is $E^* \otimes K_M$, where K_M is the canonical bundle of M. On the other hand, by the assumption, $\deg(E^* \otimes K_M) = 4g-4-\deg E < 0$. Hence, by Serre duality, $H^1(E) \simeq H^0(E^* \otimes K_M)^* = \{0\}$. Given a, let $\mathcal{H}_a$ denote harmonic projection to $\ker d''_{A+a}$. It follows that for a in a small neighborhood of the origin in the slice, $\mathcal{H}_a$ is a continuous family. We can therefore define the deformation retraction explicitly by

$$r_t(a,\varphi) = (ta, \mathcal{H}_{ta}(\varphi)) \, , \, t \in [0,1]$$

For a sufficiently small neighborhood of the origin in the slice, this preserves the set $\mathcal{S}'_{(A,0)}$. It is also clearly equivariant. □

3. Morse theory

3.1. **The τ-vortex equations.** Let $\mu(A,\Phi) = *F_A - i\Phi\Phi^*$. Then $*\mu$ is a moment map for the action of $\mathcal{G}$ on $\mathcal{B} \subset \mathcal{A} \times \Omega^0(E)$. Let $\tau > 0$ be a positive parameter and define the Yang-Mills-Higgs functional

$$f_\tau(A,\Phi) = \|\mu + i\tau \cdot \mathrm{id}\,\|^2 \tag{3.1}$$

Solutions to the τ*-vortex equations* are the absolute minima of f_τ:

$$\mu(A,\Phi) + i\tau \cdot \mathrm{id} = 0 \tag{3.2}$$

Theorem 3.1 (Bradlow [3])**.** $\mathfrak{M}_{\tau,d} = \{(A,\Phi) \in \mathcal{B} \,:\, \mu(A,\Phi) + i\tau \cdot \mathrm{id} = 0\}/\mathcal{G}$.

If the space of solutions to the τ-vortex equations is nonempty, then τ must satisfy the following restriction.

$$\begin{aligned} &\mu + i\tau \cdot \mathrm{id} = *F_A - i\Phi\Phi^* + i\tau \cdot \mathrm{id} = 0 \\ \Longrightarrow \quad &\frac{i}{2\pi}\int_M \mathrm{Tr}(*F_A - i\Phi\Phi^*) = 2\tau \iff \quad \deg E + \|\Phi\|^2 = 2\tau \end{aligned} \tag{3.3}$$

Therefore $2\tau \geq d$ (with strict inequality if we want to ensure that $\Phi \neq 0$). Theorem 2.1.6 of [3] shows that a solution to the τ-vortex equations which is not τ-stable must split. Moreover, since $\mathrm{rk}\, E = 2$ the solutions can only split if τ is an integer. In particular, for a generic choice of τ solutions to (3.2) must be τ-stable. In general, critical sets of f_τ can be characterized in terms of a decomposition of the holomorphic structure of E. The critical point equations for the functional f_τ are

$$d''_A(\mu + i\tau \cdot \mathrm{id}) = 0 \tag{3.4}$$

$$(\mu + i\tau \cdot \mathrm{id})\,\Phi = 0 \tag{3.5}$$

There are three different types of critical points.

(**0**) Absolute minimum $f_\tau^{-1}(0)$.

(**I**) Yang-Mills connections with $\Phi = 0$. Then either A is an irreducible Yang-Mills minimum or E splits holomorphically as $E = L_1 \oplus L_2$. The latter exist for all values of $\deg L_1 \geq d/2$ and the existence of the critical points is independent of the choice of τ. However, as shown below the Morse index does depend on τ. If E is semistable (resp. $\deg L_1 < \tau$) we call this a critical point of type $\mathbf{I}_a$, and we label it $\delta = \tau - d/2$ (resp. $\delta = \deg L_1 - d + \tau$). If $\deg L_1 > \tau$ it is of type $\mathbf{I}_b$, and set $\delta = \deg L_1 - d + \tau$.

(**II**) E splits holomorphically as $E = L_1 \oplus L_2$, and $\Phi \in H^0(L_1) \setminus \{0\}$. On L_1 we have

$$*F_{A_1} - i\Phi\Phi^* = -i\tau\ ,\ \|\Phi\|^2 = 2\pi(\tau - \deg L_1)$$

Therefore $\deg L_1 < \tau$. Further subdivide these depending upon $\deg L_1$.

($\mathbf{II}^-$) $\deg L_1 \leq d - \tau$, $\delta = d - \deg L_1 - \tau$;

($\mathbf{II}^+$) $d - \tau < \deg L_1 < \tau$, $\delta = \deg L_1 - d + \tau$;

Let S^dM denote the d-th symmetric product of the Riemann surface M, and $J_d(M)$ the Jacobian variety of degree d line bundles on M. For future reference we record the following

Proposition 3.2. *For $\delta \in \Delta_{\tau,d} \setminus \{0\}$,*

$$H^*_{\mathcal{G}}(\eta_\delta) = \begin{cases} H^*_{\mathcal{G}}(\mathcal{A}_{ss}) & \textit{Type } \mathbf{I},\ \delta = \tau - d/2 \\ H^*(J_{j(\delta)}(M) \times J_{d-j(\delta)}(M)) \otimes H^*(BU(1) \times BU(1)) & \textit{Type } \mathbf{I},\ \delta \neq \tau - d/2 \\ H^*(S^{j(\delta)}M \times J_{d-j(\delta)}(M)) \otimes H^*(BU(1)) & \textit{Type } \mathbf{II}^+ \\ H^*(S^{d-j(\delta)}M \times J_{j(\delta)}(M)) \otimes H^*(BU(1)) & \textit{Type } \mathbf{II}^- \end{cases}$$

3.2. **The gradient flow.** Consider the negative gradient flow of the Yang-Mills-Higgs functional f_τ defined on the space $\mathcal{B} \subset \mathcal{A} \times \Omega^0(E)$. Since the functional is very similar to that studied in [10], we only sketch the details of the existence and convergence of the flow and focus on showing that the Morse stratification induced by the flow is equivalent to the Harder-Narasimhan stratification described in Section 2.1.

The gradient flow equations are

$$\frac{\partial A}{\partial t} = 2 * d_A(\mu + i\tau)\ ,\ \frac{\partial \Phi}{\partial t} = -4i(\mu + i\tau)\Phi \tag{3.6}$$

Theorem 3.3. *The gradient flow of f_τ with initial conditions in $\mathcal{B}$ exists for all time and converges to a critical point of f_τ in the smooth topology.*

A standard calculation (cf. [3, Section 4]) shows that f_τ can be re-written as

$$f_\tau = \int_X \left(|F_A|^2 + \left|d_A'\Phi\right|^2 + |\Phi\Phi^*|^2 - 2\tau |\Phi|^2 + |\tau|^2 \right) dvol + 4\tau \deg E \tag{3.7}$$

This is very similar to the functional YMH studied in [10], and the proof for existence of the flow for all positive time follows the same structure (which is in turn modeled on Donaldson's proof for the Yang-Mills functional in [9]), therefore we omit the details. An important part of the proof worth mentioning here is that the flow is generated by the action of $\mathcal{G}^{\mathbb{C}}$, i.e. for all $t \in [0, \infty)$ there exists $g(t) \in \mathcal{G}^{\mathbb{C}}$ such that the solution $(A(t), \Phi(t))$ to the flow equations (3.6) with initial condition (A, Φ) is given by $(A(t), \Phi(t)) = g(t) \cdot (A, \Phi)$.

To show that the gradient flow converges, one can use the results of Theorem B of [11] (where again, the functional is not exactly the same as f_τ, but it has the same structure and so the proof of convergence is similar). The statement of [11, Theorem B] only describes smooth convergence along a subsequence (since they also study the higher dimensional case where bubbling occurs), and to extend this to show that the limit is unique we use the Lojasiewicz inequality technique of [19] and [17]. The key estimate is contained in the following proposition.

Proposition 3.4. *Let (A_∞, Φ_∞) be a critical point of f_τ. Then there exist $\varepsilon_1 > 0$, a positive constant C, and $\theta \in \left(0, \frac{1}{2}\right)$, such that $\|(A, \Phi) - (A_\infty, \Phi_\infty)\|$ implies that*

$$\|\nabla f_\tau(A, \Phi)\|_{L^2} \geq C \left| f_\tau(A, \Phi) - f_\tau(A_\infty, \Phi_\infty)\right|^{1-\theta} \tag{3.8}$$

The proof is similar to that in [21], and so is omitted.

The rest of the proof of convergence then follows the analysis in [21] for Higgs bundles. The key result is the following proposition, which is the analog of [21, Proposition 3.7] (see also [19] or [17, Proposition 7.4]).

Proposition 3.5. *Each critical point (A, Φ) of f_τ has a neighborhood U such that if $(A(t), \Phi(t))$ is a solution of the gradient flow equations for f_τ and $(A(T), \Phi(T)) \in U$ for some T, then either $f_\tau(A(t), \Phi(t)) < f_\tau(A, \Phi)$ for some t, or $(A(t), \Phi(t))$ converges to a critical point (A', Φ') such that $f_\tau(A', \Phi') = f_\tau(A, \Phi)$. Moreover, there exists ε (depending on U) such that $\|(A', \Phi') - (A, \Phi)\| < \varepsilon$.*

The next step is the main result of this section: The Morse stratification induced by the gradient flow of f_τ is the same as the τ-Harder-Narasimhan stratification described in Section 2.1. First recall the Hitchin-Kobayashi correspondence from Theorem 3.1, and the distance-decreasing result from [10], which can be re-stated as follows.

Lemma 3.6 (Hong [10])**.** *Let (A_1, Φ_1) and (A_2, Φ_2) be two pairs related by an element $g \in \mathcal{G}^{\mathbb{C}}$. Then the distance between the $\mathcal{G}$-orbits of $(A_1(t), \Phi_1(t))$ and $(A_2(t), \Phi_2(t))$ is non-increasing along the flow.*

Recall that the critical sets associated to each stratum are given in Section 3.1, and that the critical set associated to the stratum $\mathcal{B}_\delta$ is denoted η_δ. Define $S_\delta \subset \mathcal{B}$ to be the subset of pairs that converge to a point in C_δ under the gradient flow of f_τ. The next lemma gives some standard results about the critical sets of f_τ.

Lemma 3.7. (1) *The critical set η_δ is the minimum of the functional f_τ on the stratum $\mathcal{B}_\delta$.*
(2) *The closure of each $\mathcal{G}^{\mathbb{C}}$ orbit in $\mathcal{B}_\delta$ intersects the critical set η_δ.*
(3) *There exists $\varepsilon > 0$ (depending on τ) such that $(A, \Phi) \in \eta_\delta$ and $(A', \Phi') \in \eta_{\delta'}$ with $\delta \neq \delta'$ implies that $\|(A, \Phi) - (A', \Phi')\| \geq \varepsilon$.*

Proof. Since these results are analogous to standard results for the Yang-Mills functional (see for example [1], [6], or [7]), and the proof for holomorphic pairs is similar, we only sketch the idea of the proof here.

- The first statement follows by noting that the convexity of the norm-square function $\|\cdot\|^2$ shows that the minimum of f_τ on each extension class occurs at a critical point. This can be checked explicitly for each of the types $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{II}^+$, and $\mathbf{II}^-$.
- To see the second statement, simply scale the extension class and apply Theorem 3.1 (the Hitchin-Kobayashi correspondence) to the graded object of the filtration (cf. [7, Theorem 3.10] for the Yang-Mills case).
- The third statement can be checked by noting that (modulo the $\mathcal{G}$-action) the critical sets are compact, and then explicitly computing the distance between distinct critical sets.

□

As a consequence we have

Proposition 3.8. (1) *Each critical set η_δ has a neighborhood V_δ such that $V_\delta \cap \mathcal{B}_\delta \subset S_\delta$.*
(2) *$S_\delta \cap \mathcal{B}_\delta$ is $\mathcal{G}^{\mathbb{C}}$-invariant.*

Proof. Proposition 3.5 implies that there exists a neighborhood V_δ of each critical set η_δ such that if $(A, \Phi) \in V_\delta$ then the flow with initial conditions (A, Φ) either flows below η_δ, or converges to a critical point close to η_δ. Since f_τ is minimized on each Harder-Narasimhan stratum $\mathcal{B}_\delta$ by the critical set η_δ, the flow is generated by the action of $\mathcal{G}^{\mathbb{C}}$, and the strata $\mathcal{B}_\delta$ are $\mathcal{G}^{\mathbb{C}}$-invariant, then the first alternative cannot occur if $(A, \Phi) \in \mathcal{B}_\delta \cap V_\delta$. Since the critical sets are a finite distance apart, then (by shrinking V_δ if necessary) the limit must be contained in η_δ. Therefore $V_\delta \cap \mathcal{B}_\delta \subset S_\delta$, which completes the proof of (1).

To prove (2), for each pair $(A, \Phi) \in S_\delta \cap \mathcal{B}_\delta$, let $Y_{(A,\Phi)} = \left\{ g \in \mathcal{G}^{\mathbb{C}} : g \cdot (A, \Phi) \in S_\delta \cap \mathcal{B}_\delta \right\}$. The aim is to show that $Y_{(A,\Phi)} = \mathcal{G}^{\mathbb{C}}$. Firstly we note that since the group Γ of components of $\mathcal{G}^{\mathbb{C}}$ is the same as that for the unitary gauge group $\mathcal{G}$, the flow equations (3.6) are $\mathcal{G}$-equivariant, and the critical sets η_δ are $\mathcal{G}$-invariant, then it is sufficient to consider the connected component of $\mathcal{G}^{\mathbb{C}}$ containing the identity. Therefore the problem reduces to showing that $Y_{(A,\Phi)}$ is open and closed. Openness follows from the continuity of the group action, the distance-decreasing result of Lemma 3.6, and the result in part (1). Closedness follows by taking a sequence of points $\{g_k\} \subset Y_{(A,\Phi)}$ that converges to some $g \in \mathcal{G}^{\mathbb{C}}$, and observing that the distance-decreasing result of Lemma 3.6

implies that the flow with initial conditions $g \cdot (A, \Phi)$ must converge to a limit close to the $\mathcal{G}$-orbit of the limit of the flow with initial conditions $g_k \cdot (A, \Phi)$ for some large k. Since the critical sets are $\mathcal{G}$-invariant, and critical sets of different types are a finite distance apart, then by taking k large enough (so that $g_k \cdot (A, \Phi)$ is close enough to $g \cdot (A, \Phi)$) we see that the limit of the flow with initial conditions $g \cdot (A, \Phi)$ must be in η_δ. Therefore $Y_{(A,\Phi)}$ is both open and closed. $\square$

Theorem 3.9. *The Morse stratification by gradient flow is the same as the Harder-Narasimhan stratification in Definition 2.9.*

Proof. The goal is to show that $\mathcal{B}_\delta \subseteq S_\delta$ for each δ. Let $x \in \mathcal{B}_\delta$. By Lemma 3.7 (2) the closure of the orbit $\mathcal{G}^{\mathbb{C}} \cdot x$ intersects η_δ, therefore there exists $g \in \mathcal{G}^{\mathbb{C}}$ such that $g \cdot x \in V_\delta \cap \mathcal{B}_\delta \subseteq S_\delta$ by Proposition 3.8 (1). Since $S_\delta \cap \mathcal{B}_\delta$ is $\mathcal{G}^{\mathbb{C}}$-invariant by Proposition 3.8 (2), then $x \in \mathcal{B}_\delta \cap S_\delta$ also, and therefore $\mathcal{B}_\delta \subseteq S_\delta$. Since $\{\mathcal{B}_\delta\}$ and $\{S_\delta\}$ are both stratifications of $\mathcal{B}$, then we have $\mathcal{B}_\delta = S_\delta$ for all δ. $\square$

Remark 3.10. While we have identified the stable strata of the critical sets with the Harder-Narasimhan strata, the ordering on the set $\Delta_{\tau,d}$ coming from the values of YMH is more complicated. Since this will not affect the calculations, we continue to use the ordering already defined in Section 2.

We may now reformulate the main result, Theorem 1.1. The key idea is to define a substratification of $\{X_\delta, X'_\delta\}_{\delta\in\Delta_{\tau,d}}$ by combining $\mathcal{B}_\delta$ and $\mathcal{A}_{j(\delta)}$ for $\delta \in \Delta^+_{\tau,d} \cap I_{\tau,d}$. In other words, this is simply $\{X_\delta\}_{\delta\in\Delta_{\tau,d}}$. We call this the *modified* Morse stratification.

Theorem 3.11. *The modified Morse stratification $\{X_\delta\}_{\delta\in\Delta_{\tau,d}}$ is* $\mathcal{G}$-equivariantly perfect *in the following sense: For all $\delta \in \Delta_{\tau,d}$, the long exact sequence*

$$\cdots \longrightarrow H^*_{\mathcal{G}}(X_\delta, X_{\delta_1}) \longrightarrow H^*_{\mathcal{G}}(X_\delta) \longrightarrow H^*_{\mathcal{G}}(X_{\delta_1}) \longrightarrow \cdots \tag{3.9}$$

splits. Here, δ_1 denotes the predecessor of δ in $\Delta_{\tau,d}$.

3.3. **Negative normal spaces.** For critical points $(A, \Phi) \in \eta_\delta$, a tangent vector

$$(a, \varphi) \in \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^0(E)$$

is an eigenvector for the Hessian of f_τ if

$$i[\mu + i\tau \cdot \mathrm{id}, a] = \lambda a \tag{3.10}$$

$$i(\mu + i\tau \cdot \mathrm{id})\varphi = \lambda\varphi \tag{3.11}$$

Let $V^{neg.}_{(A,\Phi)} \subset \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^0(E)$ denote the span of all such (a, φ) with $\lambda < 0$. This is clearly $\mathcal{G}$-invariant, since f_τ is. Let $\mathcal{S}_{(A,\Phi)}$ be the slice at (A, Φ). Then we set $\nu_\delta \cap \mathcal{S}_{(A,\Phi)} = V^{neg.}_{(A,\Phi)} \cap \mathcal{S}_{(A,\Phi)}$. Using Proposition 2.13, this gives a well-defined $\mathcal{G}$-invariant subset $\nu_\delta \subset \mathcal{B}$, which we call the *negative normal space at η_δ*. By definition, η_δ is a closed subset of ν_δ.

We next describe ν_δ in detail for each of the critical sets:

($\mathbf{I}_a$) Recall that in this case $\Phi \equiv 0$. If E semistable, the negative eigenspace of the Hessian is $H^0(E)$. To see this, note that since $\Phi = 0$ then $i(\mu + i\tau \cdot \mathrm{id}) = (d/2 - \tau) \cdot \mathrm{id}$ is a negative constant multiple of the identity (by assumption $\tau > d/2$). Therefore $i[\mu + i\tau \cdot \mathrm{id}, a] = 0$, and $a = 0$. Then the slice equations imply $\varphi \in H^0(E)$. If $E = L_1 \oplus L_2$, then $\mathcal{H}^2(\mathcal{C}_{(A,0)})$ is nonzero in general. From the slice equations, we see that the negative eigendirections ν_δ of the Hessian are given by

$$d''_{A_2}\varphi_2 + a_{21}\varphi_1 = 0\,,\ (a_{21}, \varphi_1) \in H^{0,1}(L_1^* L_2) \oplus H^0(L_1) \tag{3.12}$$

($\mathbf{I}_b$) This is similar to the case above, except now for negative directions, $\varphi_1 \equiv 0$. We therefore conclude that ν_δ is given by

$$H^{0,1}(L_1^* L_2) \oplus H^0(L_2) \tag{3.13}$$

Note that if $\delta > \tau$, then $\deg L_2 = d - j(\delta) < 0$, and so ν_δ^- has constant dimension $\dim_{\mathbb{C}} H^{0,1}(L_1^* L_2) = 2j(\delta) - d + g - 1$.

($\mathbf{II}^+$) In this case, $\Phi \not\equiv 0$, so by Lemma 2.15, $\mathcal{H}^2(\mathcal{C}_{(A,0)}) = 0$, and the slice is homeomorphic to $\mathcal{H}^1(\mathcal{C}_{(A,0)})$ via the Kuranishi map. The negative eigenspace of the Hessian is then just

$$(d''_A)^* a_{21} - \varphi_2 \Phi_1^* = 0\,,\ d''_A \varphi_2 + a_{21}\Phi_1 = 0 \tag{3.14}$$

($\mathbf{II}^-$) This is similar to above, except now $\varphi_2 \equiv 0$. Hence, the fiber of ν_δ is given by

$$H^{0,1}(L_2^* L_1) \tag{3.15}$$

Note that $\dim_{\mathbb{C}} H^{0,1}(L_2^* L_1) = 2j(\delta) - d + g - 1$.

To see ($\mathbf{II}^+$) and ($\mathbf{II}^-$), we need to compute the solutions to (3.10) and (3.11), which involves knowing the value of $i(\mu + i\tau \cdot \mathrm{id})$ on the critical set. Equation (3.4) shows that

$$i(\mu + i\tau \cdot \mathrm{id}) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1 \in \Omega^0(L_1^* L_1)$ and $\lambda_2 \in \Omega^0(L_2^* L_2)$ are constant. Since $\Phi \in H^0(L_1) \setminus \{0\}$, then (3.5) shows that $\lambda_1 = 0$. Since λ_2 is constant, the integral over M becomes

$$\lambda_2 = \frac{1}{2\pi}\int_M \lambda_2 \, dvol = \frac{i}{2\pi}\int_M F_{A_2} - \frac{1}{2\pi}\int_M \tau \, dvol = \deg L_2 - \tau$$

Therefore, if $d - \tau < \deg L_1 = d - \deg L_2$, then $\deg L_2 < \tau$ and so λ_2 is negative. Similarly, if $\deg L_1 < d - \tau$ then λ_2 is positive. Equation (3.10) then shows that $a \in \Omega^{0,1}(L_1^* L_2)$ if $d - \tau < \deg L_1$, and $a \in \Omega^{0,1}(L_2^* L_1)$ if $\deg L_1 < d - \tau$. Similarly, if $d - \tau < \deg L_1$ then $\varphi \in \Omega^0(L_2)$, and if $\deg L_1 < d - \tau$ then $\varphi = 0$. Equations (3.14) and (3.15) then follow from the slice equations.

The following lemma describes the space of solutions to (3.12) when φ_1 is fixed.

Lemma 3.12. *Fix φ_1. When $\varphi_1 = 0$ then the space of solutions $\{(a_{21}, \varphi_2)\}$ to* (3.12) *is isomorphic to $H^{0,1}(L_1^* L_2) \oplus H^0(L_2)$. When $\varphi_1 \neq 0$ then the space of solutions $\{(a_{21}, \varphi_2)\}$ to* (3.12) *has dimension* $\deg L_1$*.*

Proof. The first case (when $\varphi_1 = 0$) is easy, since the equations for $a \in \Omega^{0,1}(L_1^* L_2)$ and $\varphi_2 \in \Omega^0(L_2)$ become

$$d_A''^* a = 0, \quad d_A'' \varphi_2 = 0. \tag{3.16}$$

In the second case (when $\varphi_1 \neq 0$ is fixed), note (3.12) implies that $\mathcal{H}(a\varphi_1) = 0$, where $\mathcal{H}$ denotes the harmonic projection $\Omega^{0,1}(L_2) \to H^{0,1}(L_2)$. Hence, it suffices to show that the map

$$H^{0,1}(L_1^* L_2) \to H^{0,1}(L_2) \tag{3.17}$$

given by multiplication with φ_1 (followed by harmonic projection) is surjective. For then, since $\deg L_1^* L_2 < 0$, we have by Riemann-Roch that the dimension of (3.12) is $h^0(L_2) + h^1(L_1^* L_2) - h^1(L_2) = \deg L_1$. By Serre duality, (3.17) is surjective if and only if $H^0(KL_2^*) \to H^0(KL_2^* L_1)$ is injective. But since $\varphi_1 \neq 0$, multiplication gives an injection of sheaves $\mathcal{O} \hookrightarrow L_1$, and the result follows by tensoring and taking cohomology. □

Lemma 3.13. *The space of solutions to* (3.14) *has constant dimension* $= \deg L_1 = j(\delta)$.

Proof. Consider the subcomplex $\mathcal{C}^{LT}_{(A,\Phi)}$

$$\Omega^0(L_1^* L_2)) \xrightarrow{D_1} \Omega^{0,1}(L_1^* L_2) \oplus \Omega^0(L_2) \xrightarrow{D_2} \Omega^1(L_2) \tag{3.18}$$

Since $\Phi \neq 0$, by Lemma 2.15 the cohomology at the ends of the complex (3.18) vanishes, and we have (by Riemann-Roch)

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{H}^1(\mathcal{C}^{LT}_{(A,\Phi)}) &= \dim_{\mathbb{C}}(\ker D_1^* \cap \ker D_2) \\
&= h^1(L_1^* L_2) + h^0(L_2) - h^1(L_2) - h^0(L_1^* L_2) \\
&= -\deg L_1^* L_2 + g - 1 + \deg L_2 + (1-g) \\
&= \deg L_1
\end{aligned}$$

□

We summarize the the above considerations with

Corollary 3.14. *The fiber of* ν_δ *is linear of constant dimension for critical sets of type* $\mathbf{II}^\pm$*, and for those of type* $\mathbf{I}_b$ **provided** $\delta \notin \Delta^+_{\tau,d} \cap [\tau - d/2, \tau]$*. The complex dimension of the fiber in these cases is* $\sigma(\delta)$*, where*

$$\sigma(\delta) = \begin{cases} 2j(\delta) - d + g - 1 & \text{if type } \mathbf{I}_b \text{ or } \mathbf{II}^- \\ j(\delta) & \text{if type } \mathbf{II}^+ \end{cases}$$

Remark 3.15. The strata for $\delta \in I_{\tau,d}$ have two components corresponding to the strata $\mathcal{A}_{j(\delta)}$ and $\mathcal{B}_\delta$. When there is a possible ambiguity, we will distinguish these by the notation $\nu_{I,\delta}$ for the negative normal spaces to strata of type $\mathbf{I}_a$ or $\mathbf{I}_b$, and $\nu_{II,\delta}$ for the negative normal spaces to strata of type $\mathbf{II}^+$ or $\mathbf{II}^-$.

3.4. **Cohomology of the negative normal spaces.** As in [8], at certain critical sets – namely, those of type $\mathbf{I}_a$, $\mathbf{I}_b$ where $\delta \in \Delta^+_{\tau,d} \cap [\tau - d/2, \tau]$ – the negative normal directions are not necessarily constant in dimension. In the present case, they are not even linear. In order to carry out the computations, we appeal to a relative sequence by considering special subspaces with better behavior.

Definition 3.16. For $\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, \tau]$, let $\nu_{I,\delta}$ be the negative normal space to a critical set with $\Phi \equiv 0$, as in Section 3.3. Define

$$\begin{aligned} \nu'_{I,\delta} &= \{(a, \varphi_1, \varphi_2) \in \nu_{I,\delta} : (a, \varphi_1, \varphi_2) \neq 0\} \\ \nu''_{I,\delta} &= \{(a, \varphi_1, \varphi_2) \in \nu_{I,\delta} : a \neq 0\} \end{aligned}$$

The goal of this section is the proof of the following

Proposition 3.17.

$$\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, \tau] \ : H^*_{\mathcal{G}}(\nu_{I,\delta}, \nu''_{I,\delta}) \simeq H^{*-2(2j(\delta)-d+g-1)}_{S^1 \times S^1}(\eta^{\mathcal{A}}_{j(\delta)}) \tag{3.19}$$

$$\delta \in \Delta^+_{\tau,d} \cap (2\tau - d, \tau] \ : H^*_{\mathcal{G}}(\nu'_{I,\delta}, \nu''_{I,\delta}) \simeq H^{*-2(2j(\delta)-d+g-1)}_{S^1}(S^{d-j(\delta)}M \times J_{j(\delta)}(M)) \tag{3.20}$$

$$\begin{aligned} \delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, 2\tau - d) \ : H^*_{\mathcal{G}}(\nu'_{I,\delta}, \nu''_{I,\delta}) &\simeq H^{*-2j(\delta)}_{S^1}(S^{j(\delta)}M \times J_{d-j(\delta)}(M)) \\ &\oplus H^{*-2(2j(\delta)-d+g-1)}_{S^1}(S^{d-j(\delta)}M \times J_{j(\delta)}(M)) \end{aligned} \tag{3.21}$$

Proof. Fix $E = L_1 \oplus L_2$. Consider first the case $\tau > \deg L_1 = j(\delta) > d/2$, and $\deg L_2 = d - j(\delta) < d/2$. Define the following spaces

$$\begin{aligned} \omega_\delta &= \{(A_1, A_2, a, \varphi_1, \varphi_2) \in \nu_{I,\delta} \ : \ (a, \varphi_2) \neq 0\} \\ Z^-_\delta &= \{(A_1, A_2, a, \varphi_1, \varphi_2) \in \nu_{I,\delta} \ : \ \varphi_1 = 0\} \\ Z'_\delta &= \{(A_1, A_2, a, \varphi_1, \varphi_2) \in \nu_{I,\delta} \ : \ \varphi_1 = 0, (a, \varphi_2) \neq 0\} \\ Y'_\delta &= \{(A_1, A_2, a, \varphi_1, \varphi_2) \in \nu_{I,\delta} \ : \ \varphi_1 \neq 0\} \\ Y''_\delta &= \{(A_1, A_2, a, \varphi_1, \varphi_2) \in \nu_{I,\delta} \ : \ \varphi_1 \neq 0, (a, \varphi_2) \neq 0\} \\ T_\delta &= \{(A_1, A_2, a, \varphi_1, \varphi_2) \in \nu_{I,\delta} \ : \ \varphi_1 \neq 0, (a, \varphi_2) = 0\} \end{aligned}$$

Note that $Y'_\delta = \nu_{I,\delta} \backslash Z^-_\delta = \nu'_{I,\delta} \backslash Z'_\delta$ and $Y''_\delta = \omega_\delta \backslash Z'_\delta$. Consider the following commutative diagram.

(3.22)
$$
\begin{tikzcd}
 & & \vdots \arrow[d] & & & \\
 & \cdots \arrow[r] & H^p_{\mathcal{G}}(\nu_{I,\delta},\nu'_{I,\delta}) \arrow[r] \arrow[d] & H^p_{\mathcal{G}}(\nu_{I,\delta}) \arrow[r] & H^p_{\mathcal{G}}(\nu'_{I,\delta}) \arrow[r] & \cdots \\
 & & H^p_{\mathcal{G}}(\nu_{I,\delta},\nu''_{I,\delta}) \arrow[ur, "\xi"] \arrow[dr, "\xi''"] \arrow[d] & & & \\
\cdots \arrow[r] & H^p_{\mathcal{G}}(\nu'_{I,\delta},\omega_\delta) \arrow[r] & H^p_{\mathcal{G}}(\nu'_{I,\delta},\nu''_{I,\delta}) \arrow[r, "\beta"] \arrow[d] & H^p_{\mathcal{G}}(\omega_\delta,\nu''_{I,\delta}) \arrow[r] & \cdots & \\
 & & \vdots & & &
\end{tikzcd}
$$

- First, it follows as in the proof of [8, Thm. 2.3] that the pair $(\nu_{I,\delta},\nu''_{I,\delta})$ is homotopic to the Atiyah-Bott pair $(X^{\mathcal{A}}_{j(\delta)}, X^{\mathcal{A}}_{j(\delta)-1})$. Hence, (3.19) follows from [1].
- Consider the pair $(\nu'_{I,\delta},\omega_\delta)$. Excision of Z'_δ gives the isomorphism

$$H^*_{\mathcal{G}}(\nu'_{I,\delta},\omega_\delta) \cong H^*_{\mathcal{G}}(\nu'_{I,\delta}\setminus Z'_\delta,\omega_\delta\setminus Z'_\delta) \cong H^*_{\mathcal{G}}(Y'_\delta,Y''_\delta) \tag{3.23}$$

The space $Y''_\delta = Y'_\delta \setminus T_\delta$, and Lemma 3.12 shows that Y'_δ is a vector bundle over T_δ with fibre dimension $= \deg L_1$. Therefore the Thom isomorphism implies

$$H^*_{\mathcal{G}}(Y'_\delta,Y''_\delta) = H^*_{\mathcal{G}}(Y'_\delta,Y'_\delta\setminus X'_\delta) \cong H^{*-2j(\delta)}_{\mathcal{G}}(T_\delta)$$

and therefore

$$H^*_{\mathcal{G}}(\nu'_{I,\delta},\omega_\delta) = H^*_{\mathcal{G}}(Y'_\delta,Y''_\delta) = H^{*-2j(\delta)}_{S^1}(S^{j(\delta)}M \times J_{d-j(\delta)}(M)) \tag{3.24}$$

- Consider $(\omega_\delta,\nu''_{I,\delta})$. By retraction, the pair is homotopic to the intersection with $\varphi_1 = 0$. It then follows exactly as in [8] (or the argument above) that

$$H^*_{\mathcal{G}}(\omega_\delta,\nu''_{I,\delta}) \cong H^{*-2(2j(\delta)-d+g-1)}_{S^1}(S^{d-j(\delta)}M \times J_{j(\delta)}(M)) \tag{3.25}$$

(Recall that $\dim H^{0,1}(L_1^*L_2) = 2j(\delta) - d + g - 1$ by Riemann-Roch, and that $\deg L_2 = d - j(\delta)$).

It then follows as in [8] that ξ'', and hence also β, is surjective. This implies that the lower horizontal exact sequence splits, and (3.21) follows from (3.24) and (3.25). This completes the proof in this case. The case where $\deg L_1 > \tau$ is simpler, since $\varphi_1 \equiv 0$ from (3.11). Hence, $\omega_\delta = \nu'_{I,\delta}$, and the proof proceeds as above. $\square$

3.5. **The Morse-Bott lemma.** In this section we prove the fundamental relationship between the relative cohomology of successive strata and the relative cohomology of the negative normal spaces. From this we derive the proof of the main result. In the following we use δ_1 to denote the predecessor of δ in $\Delta_{\tau,d}$.

Theorem 3.18. *For all* $\delta \in \Delta_{\tau,d} \setminus \left(\Delta^+_{\tau,d} \cap I_{\tau,d}\right)$,

$$H^*_{\mathcal{G}}(X_\delta, X_{\delta_1}) \simeq H^*_{\mathcal{G}}(\nu_\delta, \nu'_\delta) \tag{3.26}$$

For all $\delta \in \Delta^+_{\tau,d} \cap I_{\tau,d}$,

$$H^*_{\mathcal{G}}(X'_\delta, X_{\delta_1}) \simeq H^*_{\mathcal{G}}(\nu_{II,\delta}, \nu'_{II,\delta}) \tag{3.27}$$

For all $\delta \in \Delta^+_{\tau,d} \cap I_{\tau,d}$, $\delta \neq \tau - d/2$,

$$H^*_{\mathcal{G}}(X_\delta, X'_\delta) \simeq H^*_{\mathcal{G}}(\nu_{I,\delta}, \nu'_{I,\delta}) \tag{3.28}$$

Eq. (3.28) *also holds for* $\delta = \tau - d/2$, *provided* $d > 4g - 4$. *In the statements above,* δ_1 *denotes the predecessor of* δ *in* $\Delta_{\tau,d}$.

First, we give a proof of (3.26) in the case $\delta \notin \Delta^+_{\tau,d} \cap [\tau - d/2, \tau]$. By excision and convergence of the gradient flow, there is a neighborhood U of η_δ such that

- U is $\mathcal{G}$-invariant;
- U is the union of images of slices $\mathcal{S}_{(A,\Phi)}$, where $(A, \Phi) \in \eta_\delta$;
- $H^*_{\mathcal{G}}(X_\delta, X_{\delta_1}) \simeq H^*_{\mathcal{G}}(U, U \setminus (U \cap \mathcal{B}_\delta))$

Notice that for each slice $\mathcal{S}_{(A,\Phi)} \cap U \setminus (U \cap \mathcal{B}_\delta) = \mathcal{S}'_{(A,\Phi)} \cap U$, where the latter is defined as in Lemma 2.16. By the lemma, it follows that the pair $(U, U \setminus (U \cap \mathcal{B}_\delta))$ locally retracts to $(\nu_\delta, \nu'_\delta)$. On the other hand, by Corollary 3.14, ν_δ is a bundle over η_δ. It follows by continuity as in [2], that there is a $\mathcal{G}$-equivariant retraction of the pair $(\nu_\delta, \nu'_\delta) \hookrightarrow (U, U \setminus (U \cap \mathcal{B}_\delta))$. The result therefore follows in this case. We also note that by Corollary 3.14 and the Thom isomorphism,

$$H^*_{\mathcal{G}}(\nu_\delta, \nu'_\delta) \simeq H^{*-2\sigma(\delta)}_{\mathcal{G}}(\eta_\delta) \tag{3.29}$$

Remark 3.19. Notice that by Corollary 3.14 the same argument also proves (3.27). For $d > 4g-4$, we can use Lemma 2.17 in the same way to derive (3.28) for $\delta = \tau - d/2$. In this case, by the Thom isomorphism, we have

$$H^*_{\mathcal{G}}(X_{\tau-d/2}, X'_{\tau-d/2}) \simeq H^{*-2(d+2-2g)}_{\mathcal{G}}(\mathcal{A}_{ss}) \tag{3.30}$$

Lemma 3.20. *For* $\delta \notin \Delta^+_{\tau,d} \cap [\tau - d/2, \tau]$, *or if* $\delta = \tau - d/2$ *and* $d > 4g - 4$, *then the long exact sequence* (3.9) *splits. Similarly, the long exact sequence*

$$\cdots \longrightarrow H^p_{\mathcal{G}}(X'_\delta, X_{\delta_1}) \longrightarrow H^p_{\mathcal{G}}(X'_\delta) \longrightarrow H^p_{\mathcal{G}}(X_{\delta_1}) \longrightarrow \cdots \tag{3.31}$$

splits for all $\delta \in I_{\tau,d}$.

Proof. Indeed, since (3.26) holds in this case, we have

$$
\begin{tikzcd}
\cdots \arrow[r] & H^p_{\mathcal{G}}(X_\delta, X_{\delta_1}) \arrow[r, "\alpha"] \arrow[d, "\cong"] & H^p_{\mathcal{G}}(X_\delta) \arrow[r] \arrow[d] & H^p_{\mathcal{G}}(X_{\delta_1}) \arrow[r] & \cdots \\
& H^p_{\mathcal{G}}(\nu_\delta, \nu'_\delta) \arrow[r, "\beta"] & H^p_{\mathcal{G}}(\eta_\delta) & &
\end{tikzcd}
\tag{3.32}
$$

Now $\nu_\delta \to \eta_\delta$ is a complex vector bundle with a $\mathcal{G}$-action and a circle subgroup that fixes η_δ and acts freely on $\nu_\delta \setminus \eta_\delta$, so by [1, Prop. 13.4], β is injective. It follows that is α is injective as well, and hence the sequence splits. The second statement follows by Remark 3.19 and the same argument as above. □

It remains to prove (3.28) and the remaining cases of (3.26). As noted above, in these cases the negative normal spaces are no longer constant in dimension, and indeed they are not even linear in the fibers. From the point of view of deformation theory, the Kuranishi map near these critical sets is not surjective, and defining an appropriate retraction is more difficult than in the situation just considered. Instead, we resort to the analog of the decomposition used in Section 3.3. Let $X''_\delta = X_\delta \setminus \mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)})$. Note that by Lemma 2.12, $X''_\delta \subset X'_\delta$. We will prove the following

Proposition 3.21. *Suppose $\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, \tau]$. Then*

$$H^*_{\mathcal{G}}(X_\delta, X''_\delta) \cong H^*_{\mathcal{G}}(\nu_{I,\delta}, \nu''_{I,\delta})) \tag{3.33}$$

$$H^*_{\mathcal{G}}(X'_\delta, X''_\delta) \cong H^*_{\mathcal{G}}(\nu'_{I,\delta}, \nu''_{I,\delta}) \tag{3.34}$$

Proof of (3.33). By [1] and (3.19), it suffices to prove

$$H^*_{\mathcal{G}}(X_\delta, X''_\delta) \cong H^*_{\mathcal{G}}(X^{\mathcal{A}}_{j(\delta)}, X^{\mathcal{A}}_{j(\delta)-1}) \tag{3.35}$$

We first note that the pair (X_δ, X''_δ) is not necessarily invariant under scaling $t\Phi$, $t \to 0$, in particular because of the strata in $\Delta^-_{\tau,d}$ (cf. Lemma 2.12). However, if we set

$$\widehat{X}_\delta = X_\delta \cup \bigcup_{\delta' \leq \delta\,,\, \delta' \in \Delta^-_{\tau,d}} X^{\mathcal{A}}_{\delta'+\tau}\ ,\ \ \widehat{X}''_\delta = \widehat{X}_\delta \setminus \mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)})$$

then by excision on the closed subset

$$\bigcup_{\substack{j(\delta)-\tau < \delta' \leq \delta \\ \delta' \in \Delta^-_{\tau,d}}} \mathcal{A}_{\delta'+\tau}$$

it follows that $H^*_{\mathcal{G}}(X_\delta, X''_\delta) = H^*_{\mathcal{G}}(\widehat{X}_\delta, \widehat{X}''_\delta)$. Then for the pair $(\widehat{X}_\delta, \widehat{X}''_\delta)$, projection to $\mathcal{A}$ is a deformation retraction (by scaling the section Φ), and we have

$$H^*_{\mathcal{G}}(X_\delta, X''_\delta) = H^*_{\mathcal{G}}(\mathrm{pr}(\widehat{X}_\delta), \mathrm{pr}(\widehat{X}''_\delta)) \tag{3.36}$$

Next, let

$$\mathcal{K}_\delta = \mathrm{pr}\big(\widehat{X}_{(\tau-d/2)} \cup \bigcup_{\delta' \leq \delta\,,\, \delta' \in \Delta^-_{\tau,d}} \mathcal{B}_{\delta'} \cup \bigcup_{\delta' < \tau - d/2\,,\, \delta' \in \Delta^+_{\tau,d}} \mathcal{B}_{\delta'}\big) \cap \bigcup_{k > j(\delta)} \mathcal{A}_k$$

Note that $\mathcal{K}_\delta \subset \mathrm{pr}(\widehat{X}''_\delta)$. We claim that it is actually a closed subset of $\mathrm{pr}(\widehat{X}_\delta)$. Indeed, suppose $(A_i,\Phi_i)\in X_{(\tau-d/2)}$, $(A_i,\Phi_i)\to(A,\Phi)\in\widehat{X}_\delta$, and suppose that $\mu_+(A_i)>j(\delta)$ for each i. By semicontinuity, it follows that $\mu_+(A)>j(\delta)$. On the other had, either $A\in\mathcal{K}_\delta$ or $(A,\Phi)\in\mathcal{B}_{\delta'}$, $\tau-d/2<\delta'\le\delta$ and $\delta'\in\Delta^+_{\tau,d}$. But by Lemma 2.12, this would imply $A\in\mathcal{A}_{j(\delta')}$; which is a contradiction, since $j(\delta')\le j(\delta)$. It follows that the latter cannot occur, and hence, $\mathcal{K}_\delta$ is closed. Similarly,

$$\begin{aligned}\mathrm{pr}(\widehat{X}_\delta) &= \mathrm{pr}\big(\widehat{X}_{(\tau-d/2)}\cup\bigcup_{\delta'\le\delta\,,\,\delta'\in\Delta^-_{\tau,d}}\mathcal{B}_{\delta'}\cup\bigcup_{\delta'<\tau-d/2\,,\,\delta'\in\Delta^+_{\tau,d}}\mathcal{B}_{\delta'}\big)\\ &\qquad\cup\bigcup_{d/2<k\le j(\delta)}\mathcal{A}_k\cup\bigcup_{\tau-d/2<\delta'\le\delta\,,\,\delta'\in\Delta^+_{\tau,d}}\mathrm{pr}(\mathcal{B}_\delta)\\ &=\mathcal{K}_\delta\cup\bigcup_{d/2\le k\le j(\delta)}\mathcal{A}_k\end{aligned}$$

and the union is disjoint. It follows also that

$$\mathrm{pr}(\widehat{X}''_\delta)=\mathcal{K}_\delta\cup\bigcup_{d/2\le k<j(\delta)}\mathcal{A}_k$$

Hence, $\mathrm{pr}(\widehat{X}_\delta)\setminus\mathcal{K}_\delta=X^{\mathcal{A}}_{j(\delta)}$, $\mathrm{pr}(\widehat{X}''_{j(\delta)})\setminus\mathcal{K}_\delta=X^{\mathcal{A}}_{j(\delta)-1}$, and (3.35) follows from (3.36) by excision. $\square$

Proof of (3.34). First consider the case $\delta\in\Delta^+_{\tau,d}\cap(\tau-d/2,2\tau-d)$. We have

$$\begin{aligned}X''_\delta=\big(X_{(\tau-d/2)}\cup\bigcup_{\delta'\le\delta\,,\,\delta'\in\Delta^-_{\tau,d}}\mathcal{B}_{\delta'}\cup\bigcup_{\delta'<\tau-d/2\,,\,\delta'\in\Delta^+_{\tau,d}}\mathcal{B}_{\delta'}\big)\setminus\mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)})\\ \cup\bigcup_{d/2<k<j(\delta)}\mathcal{A}_k\cup\bigcup_{\tau-d/2<\delta'<\delta\,,\,\delta'\in\Delta^+_{\tau,d}}\mathcal{B}_\delta\end{aligned}$$

whereas $X'_\delta=X_{\delta_1}\cup\mathcal{B}_\delta$, where δ_1 is the predecessor of δ in $\Delta_{\tau,d}$. Also, $X''_\delta=X_{\delta_1}\setminus\mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)})$. We then have the following diagram

$$\begin{tikzcd}\cdots\arrow[r] & H^p_{\mathcal{G}}(X'_\delta,X''_\delta)\arrow[r]\arrow[d,"f"] & H^p_{\mathcal{G}}(X'_\delta)\arrow[r]\arrow[d,"g"] & H^p_{\mathcal{G}}(X''_\delta)\arrow[r]\arrow[d,"\cong"] & \cdots\\ \cdots\arrow[r] & H^p_{\mathcal{G}}(X_{\delta_1},X''_\delta)\arrow[r] & H^p_{\mathcal{G}}(X_{\delta_1})\arrow[r] & H^p_{\mathcal{G}}(X''_\delta)\arrow[r] & \cdots\end{tikzcd} \tag{3.37}$$

where f and g are induced by the inclusion $X_{\delta_1}\hookrightarrow X'_\delta$. By Lemma 3.20 and (3.27) (see Remark 3.19), it follows that g is surjective and

$$\ker g=H^*_{\mathcal{G}}(\nu_{II,\delta},\nu'_{II,\delta})\simeq H^{*-2j(\delta)}_{\mathcal{G}}(\mathcal{B}_\delta)\simeq H^{*-2j(\delta)}_{S^1}(S^{j(\delta)}M\times J_{d-j(\delta)}M)$$

by Thom isomorphism. Chasing through the diagram, it follows that f is also surjective with the same kernel. We conclude that

$$H^*_{\mathcal{G}}(X'_\delta,X''_\delta)\simeq H^*_{\mathcal{G}}(X_{\delta_1},X''_\delta)\oplus H^{*-2j(\delta)}_{S^1}(S^{j(\delta)}M\times J_{d-j(\delta)}M) \tag{3.38}$$

It remains to compute the first factor on the right hand side. To begin, notice that

$$\bigcup_{d/2<k<j(\delta)} \mathcal{A}_k \cup \bigcup_{\tau-d/2<\delta'<\delta\,,\,\delta'\in\Delta^-_{\tau,d}} \mathcal{B}_{\delta'} \cup \bigcup_{\tau-d/2<\delta'<\delta\,,\,\delta'\in\Delta^+_{\tau,d}} \mathcal{B}_\delta$$

is contained in X''_δ and closed in X_{δ_1}. It follows by excision that

$$H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \simeq H^*_{\mathcal{G}}(X_{\tau-d/2}, X_{\tau-d/2} \setminus \mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)}))$$

Next, we observe that

$$\mathcal{A}_{ss} \cup \bigcup_{\delta'<\tau-d/2\,,\,\delta'\in\Delta^-_{\tau,d}} \mathcal{B}_{\delta'}$$

is contained in $X_{\tau-d/2} \setminus \mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)})$ and closed in $X_{\tau-d/2}$. This is clear for $\mathcal{A}_{ss}$. More generally, if (E,Φ) in this set and $\Phi \not\equiv 0$, then $\mu_+(E) > \tau > j(\delta)$, and elements in the strata of type $\mathbf{II}^-$ cannot specialize to points in $\mathbf{II}^+$. Again applying excision, we have

$$H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \simeq H^*_{\mathcal{G}}(Y_\delta, Y_\delta \setminus \mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)}))$$

where

$$Y_\delta = \mathcal{B}^\tau_{ss} \cup \bigcup_{0<\delta'\leq\tau-d/2\,,\,\delta'\in\Delta^+_{\tau,d}} \mathcal{B}_{\delta'}$$

We make a third excision of the closed set

$$\bigcup_{\tau-j(\delta)<\delta'\leq\tau-d/2\,,\,\delta'\in\Delta^+_{\tau,d}} \mathcal{B}_{\delta'}$$

and a final excision of the subset

$$\mathcal{D}_\delta = \big\{\mathcal{B}^\tau_{ss} \cup \bigcup_{0<\delta'\leq\tau-j(\delta)\,,\,\delta'\in\Delta^+_{\tau,d}} \mathcal{B}_\delta\big\} \cap \big(\bigcup_{k>j(\delta)} \mathrm{pr}^{-1}(\mathcal{A}_k)\big)$$

Notice that

$$\big\{\mathcal{B}^\tau_{ss} \cup \bigcup_{0<\delta'\leq\tau-j(\delta)\,,\,\delta'\in\Delta^+_{\tau,d}} \mathcal{B}_\delta\big\} \setminus \mathcal{D}_\delta = \mathcal{B}^{j(\delta)}_{ss}$$

We conclude that

$$H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \simeq H^*_{\mathcal{G}}(\mathcal{B}^{j(\delta)}_{ss}, \mathcal{B}^{j(\delta)}_{ss} \setminus \mathrm{pr}^{-1}(\mathcal{A}_{j(\delta)}))$$

Choose $\varepsilon > 0$ small, and let $\tau' = j(\delta) - \varepsilon$. Then with respect to the τ'-*stratification*, the right hand side above is $\simeq H^*_{\mathcal{G}}(\mathcal{B}^{\tau'}_{ss} \cup \mathcal{B}^{\tau'}_{\varepsilon}\,,\, \mathcal{B}^{\tau'}_{ss})$ where $\varepsilon \in \Delta^-_{\tau'}$ is the lowest τ'-critical set. Since $\varepsilon < \tau' - d/2$, it follows from Lemma 3.20 that the long exact sequence (3.9) splits for this stratum. Hence, we have

$$H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \simeq H^*_{\mathcal{G}}(\mathcal{B}^{\tau'}_{ss} \cup \mathcal{B}^{\tau'}_{\varepsilon}\,,\, \mathcal{B}^{\tau'}_{ss}) \simeq H^{*-2(2j(\delta)-d+g-1)}_{S^1}(S^{d-j(\delta)}M \times J_{j(\delta)}(M)) \tag{3.39}$$

(notice that $j_{\tau'}(\varepsilon) = j_\tau(\delta)$). Eqs. (3.38) and (3.39), combined with Proposition 3.17, complete the proof. In case $\delta \notin I_{\tau,d}$, note that by definition $H^*_{\mathcal{G}}(X'_\delta, X''_\delta) \simeq H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta)$. The part of the proof following (3.38) now applies verbatim to this case. □

Proof of Theorem 3.18. For $\delta \notin \Delta^+_{\tau,d} \cap [\tau - d/2, \tau]$, or $\delta = \tau - d/2$ and $d > 4g-4$, we have proven the result directly (see the discussion following Theorem 3.18 and also Remark 3.19). For $\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, \tau]$, the result follows from Proposition 3.21 and the five lemma. □

3.6. **Perfection of the stratification for large degree.** Note that Lemma 3.20 shows that the long exact sequence (3.9) splits for all $\delta \notin \Delta^+_{\tau,d} \cap [\tau - d/2, \tau]$, and also for $\delta = \tau - d/2$ if $d > 4g-4$. Therefore it remains to show that (3.9) splits for $\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, \tau]$.

Firstly we consider the case where $\delta \in \Delta^+_{\tau,d} \cap [2\tau - d, \tau]$, which corresponds to a stratum of type $\mathbf{I}_b$. Proposition 3.21 shows that the vertical long exact sequence splits and the map ξ is injective in the following commutative diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & \\
\cdots \arrow[r] & H^p_{\mathcal{G}}(X_\delta, X_{\delta_1}) \arrow[r, "\alpha^p"] \arrow[d, "\zeta^p"] & H^p_{\mathcal{G}}(X_\delta) \arrow[r] & H^p_{\mathcal{G}}(X_{\delta_1}) \arrow[r] & \cdots \\
H^p_{\mathcal{G}}(\nu_{I,\delta}, \nu''_{I,\delta}) \arrow[r, "\cong"] \arrow[d] & H^p_{\mathcal{G}}(X_\delta, X''_\delta) \arrow[ur, "\xi"] \arrow[d] & & & \\
H^p_{\mathcal{G}}(\omega_\delta, \nu''_{I,\delta}) \arrow[r, "\cong"] \arrow[d] & H^p_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \arrow[d] & & & \\
0 & \vdots & & &
\end{tikzcd}$$

Therefore the map α^p is injective, and so the horizontal long exact sequence splits also.

Next, suppose $\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, 2\tau - d)$. For this we need the following lemma.

Lemma 3.22. *When $\delta \in \Delta^+_{\tau,d} \cap (\tau - d/2, 2\tau - d)$, then the isomorphisms $H^*_{\mathcal{G}}(X_\delta, X''_\delta) \cong H^*_{\mathcal{G}}(\nu_{I,\delta}, \nu''_{I,\delta})$ and $H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \cong H^*_{\mathcal{G}}(\omega_\delta, \nu''_{I,\delta})$ in equivariant cohomology are induced by an inclusion of triples $(\nu_{I,\delta}, \omega_\delta, \nu''_{I,\delta}) \hookrightarrow (X_\delta, X_{\delta_1}, X''_\delta)$.*

Proof. The first isomorphism is contained in (3.33). To see the second isomorphism, note that the results of the last section show that $H^*_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \cong H^*_{\mathcal{G}}(\mathcal{B}^{\tau'}_{ss} \cup \mathcal{B}^{\tau'}_{\varepsilon}, \mathcal{B}^{\tau'}_{ss})$, where $\varepsilon \in \Delta^-_{\tau'}$ is the lowest τ' critical set. Excise all but a neighborhood of $\mathcal{B}^{\tau'}_{\varepsilon}$, and deformation retract Φ so that $\|\Phi\|$ is small. Call these new sets W and W_0, respectively. Then

$$H^*_{\mathcal{G}}(\mathcal{B}^{\tau'}_{ss} \cup \mathcal{B}^{\tau'}_{\varepsilon}, \mathcal{B}^{\tau'}_{ss}) \cong H^*_{\mathcal{G}}(W, W_0)$$

Since $\Phi \neq 0$, then we can apply Lemma 2.16 to the slices within the spaces W and W_0, and the resulting spaces are homeomorphic to ω_δ and $\nu''_{I,\delta}$ respectively. □

The previous lemma together with the surjection $\xi'' : H^*_{\mathcal{G}}(\nu^-_{I,\delta}, \nu''_{I,\delta}) \to H^*_{\mathcal{G}}(\omega_\delta, \nu''_{I,\delta})$ from (3.22) implies that the map ξ''_g is surjective in the following commutative diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & \\
\cdots \arrow[r] & H^p_{\mathcal{G}}(X_\delta, X_{\delta_1}) \arrow[r] \arrow[d] & H^p_{\mathcal{G}}(X_\delta) \arrow[r] & H^p_{\mathcal{G}}(X_{\delta_1}) \arrow[r] & \cdots \\
 & H^p_{\mathcal{G}}(X_\delta, X''_\delta) \arrow[ur, "\xi_g"] \arrow[d, "\xi''_g"] & & & \\
 & H^p_{\mathcal{G}}(X_{\delta_1}, X''_\delta) \arrow[d] & & & \\
 & \vdots & & &
\end{tikzcd}$$

The isomorphism (3.35) together with the results of [1] show that the map ξ_g is injective, and so the same argument as before shows that the horizontal long exact sequence splits.

3.7. **The case of low degree.** By the results of the previous section, there is only one critical stratum unaccounted for on the way to completing the proof of Theorem 3.11 for $1 \leq d \leq 4g-4$. Namely, we need to analyze what happens when we attach the minimal Yang-Mills stratum $\mathcal{A}_{ss}$, which is the lowest critical set of Type **I**. More precisely, from (2.4), we need to show that the inclusion $X'_{\tau-d/2} \hookrightarrow X_{\tau-d/2}$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology for all $\tau \in (d/2, d)$. Notice that by (2.5), $X'_{\tau-d/2} = X_{\delta_1}$ for d odd, so this is precisely what we need to prove; and if d is even, then the above statement together with Lemma 3.20 will prove that $X_{\delta_1} \hookrightarrow X_{\tau-d/2}$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology in this case as well.

In low degree, the negative normal directions exist only over a Brill-Noether subset of $\mathcal{A}_{ss}$, whose cohomology is unknown, and the dimension of the fiber jumps in a complicated way; it is not even clear that there is a good Morse-Bott lemma of the type (3.28) in this case.

Hence, in order to prove surjectivity in this case we will use an indirect argument via embeddings of the space of pairs of degree d into corresponding pairs of larger degree. More precisely, this is defined as follows. Choose a point $p \in M$, and let $\mathcal{O}(p)$ denote the holomorphic line bundle with divisor p. We also choose a hermitian metric on $\mathcal{O}(p)$. Choose a holomorphic section σ_p of $\mathcal{O}(p)$ with a simple zero at p. Note that σ_p is unique up to a nonzero multiple. A holomorphic (and hermitian) structure on the complex vector bundle E induces one on the bundle $\widetilde{E} = E \otimes \mathcal{O}(p)$. Moreover, if $\Phi \in H^0(E)$, then $\widetilde{\Phi} = \Phi \otimes \sigma_p \in H^0(\widetilde{E})$. The unitary gauge group $\mathcal{G}$ of E is canonically isomorphic to that of $\widetilde{E}$. Hence, we have a $\mathcal{G}$-equivariant embedding $\mathcal{B}(E) \hookrightarrow \mathcal{B}(\widetilde{E})$. For simplicity, we will use the notation $\mathcal{B} = \mathcal{B}(E)$ and $\widetilde{\mathcal{B}} = \mathcal{B}(\widetilde{E})$.

Let $\tilde{d} = d + 2$ and $\tilde{\tau} = \tau + 1$. Then we note the following properties:

$$\begin{aligned} \deg \widetilde{E} &= \tilde{d} & \Delta_{\tilde{\tau},\tilde{d}} &= \Delta_{\tau,d} \\ \deg \widetilde{\Phi} &= \deg \Phi + 1 & I_{\tilde{\tau},\tilde{d}} &= I_{\tau,d} \\ \mu_+(\widetilde{E}) &= \mu_+(E) + 1 & j_{\tilde{\tau},\tilde{d}}(\delta) &= j_{\tau,d} + 1 \end{aligned}$$

It follows easily that the inclusion respects the Harder-Narasimhan stratification, i.e. for all $\delta \in \Delta_{\tau,d}$, $\mathcal{B}_\delta \hookrightarrow \widetilde{\mathcal{B}}_\delta$, $X_\delta \hookrightarrow \widetilde{X}_\delta$, and $X'_\delta \hookrightarrow \widetilde{X}'_\delta$, where the tilde's have the obvious meaning. In particular, if we fix $\tau_{max} = d - \varepsilon$, for ε small, then $\mathcal{B}_{ss}^{\tau_{max}} \hookrightarrow \widetilde{\mathcal{B}}_{ss}^{\tilde{\tau}_{max}}$. Notice that while $\mathcal{B}_{ss}^{\tau_{max}}$ gives the "last" moduli space in the sense that there are no critical values between τ_{max} and d (provided ε is sufficiently small), $\widetilde{\mathcal{B}}_{ss}^{\tilde{\tau}_{max}}$ gives the "second to last" moduli space in the sense that there is precisely one critical value between $\tilde{\tau}_{max}$ and $\tilde{d}$.

Lemma 3.23. *The inclusion $\mathcal{B}_{ss}^{\tau_{max}} \hookrightarrow \widetilde{\mathcal{B}}_{ss}^{\tilde{\tau}_{max}}$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology.*

Proof. Since τ is generic, it suffices to prove the result on the level of moduli spaces, i.e. that the inclusion $\imath : \mathfrak{M}_{\tau_{max},d} \hookrightarrow \widetilde{\mathfrak{M}}_{\tilde{\tau}_{max},\tilde{d}}$ induces a surjection in cohomology. Consider the determinant map $(E, \Phi) \mapsto \det E$. We have the following diagram

$$\begin{tikzcd} \mathfrak{M}_{\tau_{max},d} \arrow[r, "\imath"] \arrow[d, "\det"] & \widetilde{\mathfrak{M}}_{\tilde{\tau}_{max},\tilde{d}} \arrow[d, "\det"] \\ J_d(M) \arrow[r, "\jmath"] & J_{\tilde{d}}(M) \end{tikzcd} \tag{3.40}$$

Now $\mathfrak{M}_{\tau_{max},d}$ is the projectivization of a vector bundle (cf. [20]). Hence, by the Leray-Hirsch theorem its cohomology ring is generated by the embedding $(\det)^*(H^*(J_d(M)))$, and a 2-dimensional class generating the cohomology of the fiber. Since $\imath^*(\det)^* = (\det)^*\jmath^*$, and $\jmath^*$ is an isomorphism, it follows that $\imath^*$ is surjective onto $(\det)^*(H^*(J_d(M)))$. It remains to show that the 2-dimensional class is in the image of $\imath^*$. But since $\imath$ is holomorphic and $\mathfrak{M}_{\tilde{\tau}_{max},\tilde{d}}$ is projective, the Kähler class of $\mathfrak{M}_{\tilde{\tau}_{max},\tilde{d}}$ restricted to the image generates the cohomology of the fiber. $\square$

Lemma 3.24. *Suppose $\delta \in \Delta_{\tau_{max},d}$, $\delta < \tau_{max} - d/2$. Then the inclusion $X_\delta \hookrightarrow \widetilde{X}_\delta$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology. The same holds for $X'_{\tau-d/2} \hookrightarrow \widetilde{X}'_{\tilde{\tau}-\tilde{d}/2}$.*

Proof. By Lemma 3.23, the result holds for the semistable stratum. Fix $\delta < \tau - d/2$, and let δ_1 be its predecessor in $\Delta_{\tau_{max},d}$. By induction, we may assume the result holds for δ_1. By Lemma 3.20 we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^p_{\mathcal{G}}(\widetilde{X}_\delta, \widetilde{X}_{\delta_1}) \arrow[r] \arrow[d, "f"] & H^p_{\mathcal{G}}(\widetilde{X}_\delta) \arrow[r] \arrow[d, "g"] & H^p_{\mathcal{G}}(\widetilde{X}_{\delta_1}) \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & H^p_{\mathcal{G}}(X_\delta, X_{\delta_1}) \arrow[r] & H^p_{\mathcal{G}}(X_\delta) \arrow[r] & H^p_{\mathcal{G}}(X_{\delta_1}) \arrow[r] & 0 \end{tikzcd} \tag{3.41}$$

By the inductive hypothesis, h is surjective. On the other hand, by (3.26) and (3.29), surjectivity of f is equivalent to surjectivity of the map $H^*_{\mathcal{G}}(\tilde{\eta}_\delta) \to H^*_{\mathcal{G}}(\eta_\delta)$. From the description of critical sets (cf. Proposition 3.2), this map is induced by the inclusion $S^{j(\delta)}M \hookrightarrow S^{j(\delta)+1}M$. Surjectivity then follows by the argument in [8, Sect. 4]. Since both f and h are surjective, so is g. The result for any $\delta < \tau - d/2$ now follows by induction. If d is even, the exact same argument, with $\delta_1 =$ the predecessor of $\tau - d/2$, proves the statement for $X'_{\tau-d/2}$ as well. □

Lemma 3.25. *Suppose the inclusion $\widetilde{X}'_{\tilde{\tau}_{max}-\tilde{d}/2} \hookrightarrow \widetilde{X}_{\tilde{\tau}_{max}-\tilde{d}/2}$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology. Then the same is true for the inclusion $X'_{\tau_{max}-d/2} \hookrightarrow X_{\tau_{max}-d/2}$.*

Proof. Consider the diagram

$$\begin{tikzcd} H^p_{\mathcal{G}}(\widetilde{X}_{\tilde{\tau}_{max}-\tilde{d}/2}) \arrow[r] \arrow[d] & H^p_{\mathcal{G}}(\widetilde{X}'_{\tilde{\tau}_{max}-\tilde{d}/2}) \arrow[r] \arrow[d, "h"] & 0 \\ H^p_{\mathcal{G}}(X_{\tau_{max}-d/2}) \arrow[r] & H^p_{\mathcal{G}}(X'_{\tau_{max}-d/2}) \arrow[r] & \cdots \end{tikzcd} \tag{3.42}$$

By Lemma 3.24, h is surjective. The result then follows immediately. □

Lemma 3.26. *Suppose the inclusion $X'_{\tau-d/2} \hookrightarrow X_{\tau-d/2}$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology for $\tau = \tau_{max}$. Then the same is true for all $\tau \in (d/2, d)$. Moreover, $\dim H^p_{\mathcal{G}}(X_{\tau-d/2}, X'_{\tau-d/2})$ is independent of τ for all p.*

Proof. The sets $X'_{\tau-d/2}$, $X_{\tau-d/2}$ remain unchanged for τ in a connected component of $(d/2, d) \setminus C_d$, where C_d is given in (2.6). Fix $\tau_c \in C_d$, $2\tau_c - d/2 = k \in \mathbb{Z}$, and let $\tau_l < \tau_c < \tau_r$ be in components $(d/2, d) \setminus C_d$ containing τ_c in their closures. Let $\delta^{l,r} = 2\tau_c - d/2 - \tau_{l,r}$. Note that $\delta^{l,r} \in \Delta^-_{\tau_{l,r},d}$, $\delta^l > \tau_l - d/2$, and $\delta^r < \tau_r - d/2$. Also, we claim

$$X_{\tau_r-d/2} = X_{\tau_l-d/2} \cup \mathcal{B}^{\tau_l}_{\delta^l} \ , \ X'_{\tau_r-d/2} = X'_{\tau_l-d/2} \cup \mathcal{B}^{\tau_l}_{\delta^l} \tag{3.43}$$

To see this, we refer to Figure 1 and the discussion preceding it. Under the map $\Delta_{\tau_l,d} \to \Delta_{\tau_r,d}$, $\delta^l \mapsto \delta^r$ and $\tau_l - d/2 \mapsto \tau_r - d/2$. The claim then follows if we show that δ^r is the predecessor of $\tau_r - d/2$ in $\Delta_{\tau_r,d}$, and δ^l is the successor of $\tau_l - d/2$ in $\Delta_{\tau_l,d}$ (see Figure 1). So suppose $\delta \in \Delta_{\tau_r,d}$, $\delta < \tau_r - d/2$. By Remark 2.7, we may assume $\delta \in \Delta^-_{\tau_r,d}$. Write $\delta + \tau_r = \ell \in \mathbb{Z}$. Then $\ell \leq 2\tau_r - d/2$, which implies $\ell \leq k$, and $\delta \leq \delta_r$. The reasoning is similar for δ_l.

Now since the result holds by assumption for τ_{max}, we may assume by induction that the result holds for $\tau \geq \tau_r$. Then we have

$$\begin{tikzcd} 0 \arrow[r] & H^p_{\mathcal{G}}(X_{\tau_r-d/2}, X'_{\tau_r-d/2}) \arrow[r] \arrow[d, "f"] & H^p_{\mathcal{G}}(X_{\tau_r-d/2}) \arrow[r] \arrow[d, "g"] & H^p_{\mathcal{G}}(X'_{\tau_r-d/2}) \arrow[r] \arrow[d, "h"] & 0 \\ \cdots \arrow[r] & H^p_{\mathcal{G}}(X_{\tau_l-d/2}, X'_{\tau_l-d/2}) \arrow[r] & H^p_{\mathcal{G}}(X_{\tau_l-d/2}) \arrow[r] & H^p_{\mathcal{G}}(X'_{\tau_l-d/2}) \arrow[r] & \cdots \end{tikzcd} \tag{3.44}$$

By (3.43) and the proof of Lemma 3.24, h is surjective. Hence, the lower long exact sequence must split. Moreover, g is surjective as well, and $\ker g = \ker h$. As a consequence, f must be an isomorphism. The result now follows by induction. □

Proof of Theorems 3.11 and 1.2. We proceed by induction as follows. First, if $d > 4g-4$, then by Lemma 3.20, the hypothesis of Lemma 3.25 is satisfied. It then follows from Lemma 3.26 that the inclusion $X'_{\tau-d/2} \hookrightarrow X_{\tau-d/2}$ induces a surjection in $\mathcal{G}$-equivariant rational cohomology for any τ. In particular, this is true for the value $\tilde{\tau}_{max}$ corresponding to degree $d-2$. Hence, the inductive hypothesis holds, and the result is proven for all d. Kirwan surjectivity follows immediately. □

Proof of Theorem 1.3. This follows from Kirwan surjectivity, but more generally we prove this on each stratum. Clearly it suffices to prove the result for $k=1$. Since the gauge groups for E and $\widetilde{E}$ are canonically isomorphic, it suffices by induction to show that if the result holds for the inclusion $X_\delta \hookrightarrow \widetilde{X}_\delta$, then it also holds for $X_{\delta_1} \hookrightarrow \widetilde{X}_{\delta_1}$, where δ_1 is the predecessor of δ in $\Delta_{\tau,d}$. By Theorem 3.11, the diagram (3.41) holds for *all* δ. It follows that if g is surjective, then so is h. This completes the proof. □

4. Cohomology of moduli spaces

4.1. **Equivariant cohomology of τ-semistable pairs.** The purpose of this section is to complete the calculation of the $\mathcal{G}$-equivariant Poincaré polynomial of $\mathcal{B}^\tau_{ss}$. First we consider the case where τ is generic. Choose an integer N, $d/2 < N \leq d$, and let $\tau \in (\max\{d/2, N-1\}, N)$. Then the different allowable values of δ for each type of stratum and the cohomology are as follows (see Proposition 3.2).

($\mathbf{I}_a$) There is one stratum $\mathbf{I}_a^{d/2}$ corresponding to $\mathcal{A}_{ss}$ (indexed by $j = d/2$), and by Lemma 3.26 the contribution $\mathbf{I}_a^{d/2}(t)$ to the Poincaré polynomial is independent of τ. For $d > 4g-4$ it follows from (3.30) that

$$\mathbf{I}_a^{d/2}(t) = \frac{t^{2d+4-4g}}{(1-t^2)} P_t^{\overline{\mathcal{G}}}(\mathcal{A}_{ss}) \tag{4.1}$$

where $\overline{\mathcal{G}}$ is defined in [1, p. 577]. We compute $\mathbf{I}_a^{d/2}(t)$ in general in Lemma 4.3 below. The remaining strata are indexed by integers $j = j(\delta) = \mu_+$ such that $d/2 < j \leq N-1$ and $\delta = j - d + \tau$. The contribution to the $\mathcal{G}$-equivariant Poincaré polynomial is

$$\begin{aligned} \mathbf{I}_a^j(t) = {} & \frac{t^{2(2j(\delta)-d+g-1)}}{(1-t^2)^2} P_t\left(J_{j(\delta)}(M) \times J_{d-j(\delta)}(M)\right) - \frac{t^{2j(\delta)}}{1-t^2} P_t\left(S^{j(\delta)}M \times J_{d-j(\delta)}(M)\right) \\ & - \frac{t^{2(2j(\delta)-d+g-1)}}{1-t^2} P_t\left(S^{d-j(\delta)}M \times J_{j(\delta)}(M)\right) \end{aligned} \tag{4.2}$$

($\mathbf{I}_b$) There are an infinite number of strata indexed by integers $j = j(\delta) = \mu_+$ such that $N \leq j$ and $\delta = j - d + \tau$. The contribution is

$$
\begin{aligned}
\mathbf{I}_b^j(t) = \frac{t^{2(2j(\delta)-d+g-1)}}{(1-t^2)^2} P_t\left(J_{j(\delta)}(M) \times J_{d-j(\delta)}(M)\right) \\
- \frac{t^{2(2j(\delta)-d+g-1)}}{1-t^2} P_t\left(S^{d-j(\delta)} M \times J_{j(\delta)}(M)\right)
\end{aligned}
\tag{4.3}
$$

($\mathbf{II}^+$) These strata are indexed by integers $j = j(\delta) = \deg \Phi = \deg L_1$ such that $d - N + 1 \leq j \leq N - 1$, and $\delta = j - d + \tau$. The contribution is

$$
\mathbf{II}_j^+(t) = \frac{t^{2j(\delta)}}{1-t^2} P_t\left(S^{j(\delta)} M \times J_{d-j(\delta)}(M)\right) \tag{4.4}
$$

($\mathbf{II}^-$) These strata are indexed by integers $j = d - j(\delta)$ such that $0 \leq j \leq d - N$, where $\delta = j(\delta) - \tau = d - j - \tau$, and the contribution is

$$
\mathbf{II}_j^-(t) = \frac{t^{2(2j(\delta)-d+g-1)}}{1-t^2} P_t\left(S^{d-j(\delta)} M \times J_{j(\delta)}(M)\right) \tag{4.5}
$$

Then we have

Theorem 4.1. *For $\tau \in (\max\{d/2, N-1\}, N)$,*

$$
P_t(\mathfrak{M}_{\tau,d}) = P_t^{\mathcal{G}}(\mathcal{B}_{ss}^\tau) = P_t(B\mathcal{G}) - \mathbf{I}_a^{d/2}(t) - \sum_{j=\lfloor d/2+1 \rfloor}^{N-1} \mathbf{I}_a^j(t) - \sum_{j=N}^{\infty} \mathbf{I}_b^j(t) - \sum_{j=0}^{d-N} \mathbf{II}_j^-(t) - \sum_{j=d-N+1}^{N-1} \mathbf{II}_j^+(t)
$$

Proof. By Theorem 3.11 we have

$$
P_t^{\mathcal{G}}(\mathcal{B}_{ss}^\tau) = P_t(B\mathcal{G}) - \sum_{\delta \in \Delta_{\tau,d} \setminus \{0\}} P_t^{\mathcal{G}}(X_\delta, X_{\delta_1})
$$

If $\delta \notin \Delta_{\tau,d}^+ \cap (\tau - d/2, \tau]$, then by the Morse-Bott lemma (3.26) and (3.29),

$$
P_t^{\mathcal{G}}(X_\delta, X_{\delta_1}) = \frac{t^{2\sigma(\delta)}}{1-t^2} P_t^{\mathcal{G}}(\eta_\delta)
$$

where $\sigma(\delta)$ is given in Corollary 3.14. If $\delta \in \Delta_{\tau,d}^+ \cap (\tau - d/2, \tau]$ then by Section 3.6,

$$
P_t^{\mathcal{G}}(X_\delta, X_{\delta_1}) = P_t^{\mathcal{G}}(X_\delta, X_\delta'') - P_t^{\mathcal{G}}(X_{\delta_1}, X_\delta'')
$$

The first term on the right hand side is given by (3.35). For the second term, we have

$$
H_{\mathcal{G}}^*(X_{\delta_1}, X_\delta'') = \begin{cases} H_{\mathcal{G}}^*(\nu_{I,\delta}', \nu_{I,\delta}'') & \delta \in \Delta_{\tau,d}^+ \cap [2\tau - d, \tau] \\ H_{\mathcal{G}}^*(\omega_\delta, \nu_{I,\delta}'') & \delta \in \Delta_{\tau,d}^+ \cap (\tau - d/2, 2\tau - d) \end{cases}
$$

and the latter cohomology groups have been computed in (3.20) and (3.25). This completes the computation. $\square$

When the parameter τ is non-generic (i.e. $\tau = N$ for some integer $N \in [d/2, d]$) then the same analysis as above applies, however now there are split solutions to the vortex equations. These correspond to one of the critical sets of type **II**, where $E = L_1 \oplus L_2$ with $\phi \in H^0(L_1) \setminus \{0\}$, and $\deg L_2 = \tau$. Therefore, the only difference the generic and non-generic case is that we do not count any contribution from the critical set of type $\mathbf{II}^-$ with $j = d - N$. Therefore the Poincaré polynomial is

Theorem 4.2. *For $\tau = N$,*

$$P_t^{\mathcal{G}}(\mathcal{B}_{ss}^N) = P_t(B\mathcal{G}) - \mathbf{I}_a^{d/2}(t) - \sum_{j=\lfloor d/2+1 \rfloor}^{N-1} \mathbf{I}_a^j(t) - \sum_{j=N}^{\infty} \mathbf{I}_b^j(t) - \sum_{j=0}^{d-N-1} \mathbf{II}_j^-(t) - \sum_{j=d-N+1}^{N-1} \mathbf{II}_j^+(t)$$

Finally, using Theorem 4.1, we can give a computation of the remaining term which is as yet undetermined in low degree.

Lemma 4.3. *For all $d \geq 2$,*

$$\mathbf{I}_a^{d/2}(t) = \frac{1}{1-t^2} P_t^{\overline{\mathcal{G}}}(\mathcal{A}_{ss}) - \sum_{j=0}^{\lfloor d/2 \rfloor} \frac{t^{2j} - t^{2(d+g-1-2j)}}{1-t^2} P_t(S^j M \times J_{d-j}(M))$$
$$- \begin{cases} 0 & \textit{if } d \textit{ odd} \\ \dfrac{t^{2g-2}}{(1-t^2)} P_t(S^{d/2} M \times J_{d/2}(M)) & \textit{if } d \textit{ even} \end{cases}$$

Remark 4.4. It can be verified directly that for $d > 4g-4$, the expression above agrees with (4.1). See the argument of Zagier in [20, pp. 336-7].

Proof of Lemma 4.3. Take the special case $N = d$. Then $\mathfrak{M}_{\tau,d}$ is a projective bundle over $J_d(M)$, and so

$$P_t(\mathfrak{M}_{\tau,d}) = \frac{1 - t^{2(d+g-1)}}{1-t^2} P_t(J_d(M))$$

On the other hand, from Theorem 4.1 we have

$$P_t(\mathfrak{M}_{\tau,d}) = P_t(B\mathcal{G}) - \mathbf{I}_a^{d/2}(t) - \sum_{j=\lfloor d/2+1 \rfloor}^{d-1} \mathbf{I}_a^j(t) - \sum_{j=d}^{\infty} \mathbf{I}_b^j(t) - \mathbf{II}_0^-(t) - \sum_{j=1}^{d-1} \mathbf{II}_j^+(t)$$

Now notice that the term $\mathbf{II}_0^-(t)$ is cancelled by the second term in $\mathbf{I}_b^d$. We combine the remaining terms in the sum of $\mathbf{I}_b^j$ with the sum of $\mathbf{I}_a^j$. We have

$$\begin{aligned}
P_t(\mathfrak{M}_{\tau,d}) = P_t(B\mathcal{G}) - \mathbf{I}_a^{d/2}(t) - &\sum_{j=\lfloor d/2+1\rfloor}^{\infty} \frac{t^{2(2j-d+g-1)}}{(1-t^2)^2} P_t(J_j(M)\times J_{d-j}(M)) \\
&+ \sum_{j=\lfloor d/2+1\rfloor}^{d-1} \frac{t^{2j}}{(1-t^2)} P_t(S^jM\times J_{d-j}(M)) \\
&+ \sum_{j=\lfloor d/2+1\rfloor}^{d-1} \frac{t^{2(2j-d+g-1)}}{(1-t^2)} P_t(S^{d-j}M\times J_j(M)) \\
&- \sum_{j=1}^{d-1} \frac{t^{2j}}{(1-t^2)} P_t(S^jM\times J_{d-j}(M)) \\
= P_t(B\mathcal{G}) - \mathbf{I}_a^{d/2}(t) - &\sum_{j=\lfloor d/2+1\rfloor}^{\infty} \frac{t^{2(2j-d+g-1)}}{(1-t^2)^2} P_t(J_j(M)\times J_{d-j}(M)) \\
&+ \sum_{j=\lfloor d/2+1\rfloor}^{d-1} \frac{t^{2(2j-d+g-1)}}{(1-t^2)} P_t(S^{d-j}M\times J_j(M)) \\
&- \sum_{j=1}^{\lfloor d/2\rfloor} \frac{t^{2j}}{(1-t^2)} P_t(S^jM\times J_{d-j}(M))
\end{aligned}$$

Now make the substitution $j \mapsto d-j$ in the second to the last sum, using

$$d - \lfloor d/2+1\rfloor = \begin{cases} d/2-1 = \lfloor d/2\rfloor - 1 & \text{if } d \text{ even} \\ d/2-1/2 = \lfloor d/2\rfloor & \text{if } d \text{ odd} \end{cases}$$

The result now follows from this, [1, Thm. 7.14], and the fact that $P_t(\mathfrak{M}_{\tau,d})$ is equal to the $j=0$ term in the sum. □

4.2. **Comparison with the results of Thaddeus.** In [20], Thaddeus computed the Poincaré polynomial of the moduli space using different methods to those of this paper. The idea is to show that when the parameter τ passes a critical value, then the moduli space $\mathfrak{M}_{\tau,d}$ undergoes a birational transformation consisting of a blow-down along a submanifold and a blow-up along a different submanifold (these transformations are known as "flips"). By computing the change in Poincaré polynomial caused by the flips as the parameter crosses the critical values, and also observing that the moduli space is a projective space for one extreme value of τ, Thaddeus computed the Poincaré polynomial of the moduli space for any value of the parameter. In this section we recover this result from Theorem 4.1. In the Morse theory picture we see that the critical point structure changes: As τ increases past a critical value then a new critical set appears, and the index may change at existing critical points.

Theorem 4.5. *Let $N \in \mathbb{Z}$, $d/2 < N \leq d-1$. Then for $\tau \in (\max(d/2, N-1), N)$,*

$$P_t(\mathfrak{M}_{\tau+1,d}) - P_t(\mathfrak{M}_{\tau,d}) = \frac{t^{4N-2d+2g-2} - t^{2d-2N}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right) \tag{4.6}$$

As a consequence, the Poincaré polynomial of the moduli space has the form

$$P_t(\mathfrak{M}_{\tau,d}) = \frac{(1+t)^{2g}}{1-t^2} \mathrm{Coeff}_{x^N}\left(\frac{t^{2d+2g-2-4N}}{xt^4-1} - \frac{t^{2N+2}}{x-t^2}\right)\left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}\right) \tag{4.7}$$

Remark 4.6. Let $\mathfrak{M}^0_{\tau,d}$ denote the moduli space where the bundle has fixed determinant (see [20]). The analysis in this paper applies in this case as well. In particular, one obtains

$$P_t(\mathfrak{M}^0_{\tau+1,d}) - P_t(\mathfrak{M}^0_{\tau,d}) = \frac{t^{4N-2d+2g-2} - t^{2d-2N}}{1-t^2} P_t\left(S^{d-N}M\right) \tag{4.8}$$

This exactly corresponds to Thaddeus' results for $P_t(\mathbb{P}W_j^+) - P_t(\mathbb{P}W_j^-)$ [20, p. 21], where $j = d-N$.

Proof of Theorem 4.5. By Theorem 4.1,

$$P_t(\mathfrak{M}_{\tau+1,d}) - P_t(\mathfrak{M}_{\tau,d}) = -\mathbf{I}_a^N(t) + \mathbf{I}_b^N(t) + \mathbf{II}^-_{d-N}(t) - \mathbf{II}^+_{d-N}(t) - \mathbf{II}^+_N(t) \tag{4.9}$$

Substituting in the results of (4.2), (4.3), (4.5), and (4.4) gives

$$\begin{aligned}
P_t(\mathfrak{M}_{\tau+1,d}) - P_t(\mathfrak{M}_{\tau,d}) &= -\frac{t^{2(2N-d+g-1)}}{(1-t^2)^2} P_t\left(J_N(M) \times J_{d-N}(M)\right) + \frac{t^{2N}}{1-t^2} P_t\left(S^N M \times J_{d-N}(M)\right) \\
&\qquad + \frac{t^{2(2N-d+g-1)}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right) \\
&\quad + \frac{t^{2(2N-d+g-1)}}{(1-t^2)^2} P_t\left(J_N(M) \times J_{d-N}(M)\right) \\
&\qquad - \frac{t^{2(2N-d+g-1)}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right) \\
&\quad + \frac{t^{2(d-2(d-N)+g-1)}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right) \\
&\quad - \frac{t^{2(d-N)}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right) - \frac{t^{2N}}{1-t^2} P_t\left(S^N M \times J_{d-N}(M)\right) \\
&= \frac{1}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right)\left(t^{4N-2d+2g-2} - t^{2d-2N}\right)
\end{aligned}$$

as required. Using the results of [16] on the cohomology of the symmetric product, and the fact that $P_t(J_N(M)) = (1+t)^{2g}$, we see that the same method as for the proof of [20, (4.1)] gives equation (4.7). ∎

Remark 4.7. For τ as above, Theorem 4.2 shows that the difference

$$P_t^{\mathcal{G}}(\mathcal{B}^N_{ss}) - P_t(\mathfrak{M}_{\tau,d}) = \mathbf{II}^-_{d-N}(t) = \frac{t^{4N-2d+2g-2}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right)$$

comes from only one critical set; the type **II** critical set corresponding to a solution of the vortex equations when $\tau = N$. The rest of the terms in (4.9), corresponding to the difference

$$P_t(\mathfrak{M}_{\tau+1,d}) - P_t^{\mathcal{G}}(\mathcal{B}^N_{ss}) = -\mathbf{I}_a^N(t) + \mathbf{I}_b^N(t) - \mathbf{II}^+_{d-N}(t) - \mathbf{II}^+_N(t) = -\frac{t^{2N}}{1-t^2} P_t\left(S^N M \times J_{d-N}(M)\right)$$

come from a number of changes that occur in the structure of the critical sets as τ increases past N: the term $-\mathbf{II}^+_{d-N}(t)$ corresponds to the type **II** critical point that no longer is a solution to the vortex equations, the term $-\mathbf{II}^+_N(t)$ corresponds to the new critical point of type $\mathbf{II}^+$ that appears, and the term $-\mathbf{I}^N_a(t) + \mathbf{I}^N_b(t)$ corresponds to the critical point that changes type from $\mathbf{I}_b$ to $\mathbf{I}_a$.

Therefore we see that the changes in the critical set structure as τ crosses the critical value N are localized to two regions of $\mathcal{B}$. The first corresponds to interchanging critical sets of type $\mathbf{II}^-$ and type $\mathbf{II}^+$. This is the phenomenon illustrated in Figure 1. The second corresponds to critical sets of type $\mathbf{I}_a$ and $\mathbf{II}^+$ that merge to form a single component of type $\mathbf{I}_b$. The terms from the first change exactly correspond to those in (4.6), i.e.

$$\begin{aligned}\mathbf{II}^-_{d-N}(t) - \mathbf{II}^+_{d-N}(t) &= \frac{t^{4N-2d+2g-2} - t^{2d-2N}}{1-t^2} P_t\left(S^{d-N}M \times J_N(M)\right) \\ &= P_t(\mathfrak{M}_{\tau+1,d}) - P_t(\mathfrak{M}_{\tau,d})\end{aligned}$$

and the terms from the second change cancel each other, i.e. $\mathbf{I}^N_b(t) - \mathbf{I}^N_a(t) - \mathbf{II}^+_N(t) = 0$.

References

[1] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A, 308 (1983), 523–615.

[2] R. Bott. Nondegenerate critical manifolds. Ann. of Math., 60 (1954), no. 2, 248–261.

[3] S. Bradlow. Special metrics and stability for holomorphic bundles with global sections. J. Differential Geom. 33 (1991), no. 1, 169–213.

[4] S. Bradlow and G.D. Daskalopoulos. Moduli of stable pairs for holomorphic bundles over Riemann surfaces. Internat. J. Math. 2 (1991), no. 5, 477–513.

[5] S. Bradlow, G.D. Daskalopoulos, and R.A. Wentworth. Birational equivalences of vortex moduli. Topology 35 (1996), no. 3, 731–748.

[6] G.D. Daskalopoulos. The topology of the space of stable bundles on a compact Riemann surface. J. Differential Geom. 36 (1992), no. 3, 699–746.

[7] G.D. Daskalopoulos and R.A. Wentworth. Convergence properties of the Yang-Mills flow on Kähler surfaces. J. Reine Angew. Math. 575 (2004), 69–99.

[8] G.D. Daskalopoulos, R.A. Wentworth, J. Weitsman, and G. Wilkin. Morse theory and hyperkähler Kirwan surjectivity for Higgs bundles. Preprint.

[9] S. K. Donaldson. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proc. London Math. Soc. (3) 50(1) (1985), 1–26.

[10] M.-C. Hong. Heat flow for the Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric. Ann. Global Anal. Geom. 20(1) (2001), 23–46.

[11] M.-C. Hong and G. Tian. Asymptotical behaviour of the Yang-Mills flow and singular Yang-Mills connections. Math. Ann. 330(3) (2004), 441–472.

[12] A. King and P. Newstead. Moduli of Brill-Noether pairs on algebraic curves. Internat. J. Math. 6 (1995), no. 5, 733–748.

[13] F. C. Kirwan. "Cohomology of quotients in symplectic and algebraic geometry", volume 31 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1984.

[14] S. Kobayashi. "Differential geometry of complex vector bundles", volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1987. , Kano Memorial Lectures, 5.

[15] J. Le Potier. Systèmes cohérents et structures de niveau. Astérisque No. 214 (1993), 143 pp.

[16] I. G. Macdonald. Symmetric products of an algebraic curve. Topology 1 (1962), 319–343.

[17] J. Råde. On the Yang-Mills heat equation in two and three dimensions. J. Reine Angew. Math. 431 (1992), 123–163.

[18] N. Raghavendra and P. Vishwanath. Moduli of pairs and generalized theta divisors. Tohoku Math. J. (2) 46 (1994), no. 3, 321–340.
[19] L. Simon. Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. Ann. of Math. (2) 118(3) (1983), 525–571.
[20] M. Thaddeus. Stable pairs, linear systems and the Verlinde formula. Invent. Math. 117, no. 2 (1994), 317–353.
[21] G. Wilkin. Morse theory for the space of Higgs bundles. Comm. Anal.Geom. 16, no. 2 (2008), 283–332.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, and

Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address: raw@umd.edu

Department of Mathematics, University of Colorado, Boulder, CO 80309
E-mail address: graeme.wilkin@colorado.edu