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A GEOMETRIC DESCRIPTION OF THE m-CLUSTER CATEGORIES OF TYPE D_n

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Abstract. We show that the m-cluster category of type D_n is equivalent to a certain geometrically-defined category of arcs in a punctured regular $nm-m+1$-gon. This generalises a result of Schiffler for $m=1$. We use the notion of the mth power of a translation quiver to realise the m-cluster category in terms of the cluster category.

Introduction

Let k be a field and Q a quiver of Dynkin type Δ. Let $D^b(kQ)$ denote the bounded derived category of finite dimensional kQ-modules. Let τ denote the Auslander-Reiten translate of $D^b(kQ)$ and let S denote the shift. For $m \in \mathbb{N}$ the *m-cluster category* associated to kQ is the orbit category

$$\mathcal{C}^m_\Delta := \frac{D^b(kQ)}{S^m\tau^{-1}}.$$

This category was introduced in [Kel] and has been studied by the authors [BaM], Thomas [Tho], Wralsen [Wra] and Zhu [Zhu]. It is known that $\mathcal{C}^m_\Delta$ is triangulated [Kel], Krull-Schmidt and has almost split triangles [BMRRT, 1.2,1.3].

The m-cluster category is a generalisation of the cluster category. The cluster category was introduced in [CCS1] (for type A) and [BMRRT] (general hereditary case), and can be regarded as the case $m=1$ of the m-cluster category. Keller has shown that the m-cluster category is Calabi-Yau of dimension $m+1$ [Kel]. We remark that such Calabi-Yau categories have also been studied in [KR]. One of the aims of the definition of the cluster category was to model the Fomin-Zelevinsky cluster algebra [FZ] representation-theoretically.

We show that $\mathcal{C}^m_{D_n}$ can be realised geometrically in terms of a category of arcs in a punctured polygon with $nm-m+1$ vertices. This generalises a result of Schiffler [Sch], who considered the case $m=1$. We remark that the punctured polygon model for the cluster algebra of type D_n appears in work of Fomin, Schapiro and Thurston [FST] as part of a more general set-up, building on [FG1, FG2, GSV1, GSV2] which consider links between cluster algebras and Teichmüller theory.

Also, such a geometric realisation of a cluster category first appeared (with a construction for type A_n in the case $m=1$) in [CCS1].

Our approach is based on the idea of the *mth power* of a translation quiver introduced in [BaM]. We show that, with a slight modification of the definition for $m=2$, the Auslander-Reiten quiver of $\mathcal{C}^m_{D_n}$ can be realised as a connected component of the mth power of the Auslander-Reiten quiver of $\mathcal{C}^1_{D_{nm-m+1}}$. In

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Section 4 we show that, if this modification is not made, the square of the Auslander-Reiten quiver of $\mathcal{C}^1_{D_4}$ has a connected component whose underlying topological space is a torus.

1. Notation and Definitions

Let Q be a quiver of underlying Dynkin type D_n. The vertices of Q are labelled $0, \overline{0}, 1, \ldots, n-2$ and the arrows are $i \to i-1$ ($i = 1, \ldots, n-2$) together with $1 \to \overline{0}$; see Figure 1.

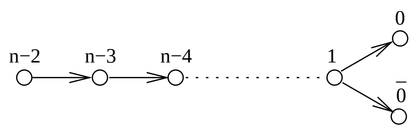

Figure 1. Quiver of type D_n

We now recall the Auslander-Reiten quiver of the cluster category $\mathcal{C}_{D_n}$ (see [BMRRT, §1], [Hap]). It is a stable translation quiver built from n copies of Q. We denote it by $\Gamma(D_n, 1)$. The vertices of $\Gamma(D_n, 1)$ are $V(D_n, 1) := \mathbb{Z}_n \times \{0, \overline{0}, 1, \ldots, n-2\}$. The arrows are

$$\left.\begin{array}{l}(i,j) \to (i,k)\\ (i,k) \to (i+1,k)\end{array}\right\} \quad \text{whenever there is an arrow } j \to k \text{ in } D_n.$$

Finally, the translation τ is given by

$$\tau(i,j) = \begin{cases} (i-1, \overline{j}), & \text{if } i = 0, j \in \{0, \overline{0}\} \text{ and } n \text{ is odd,} \\ (i-1, j), & \text{otherwise.} \end{cases}$$

We use the convention that $\overline{\overline{0}} = 0$. Note that the switch described here only occurs for odd n.

As an example, we draw the quivers $\Gamma(D_n, 1)$ for $n = 3$ and $n = 4$; see Figures 2 and 3. The translation τ is indicated by dotted lines (it is directed to the left).

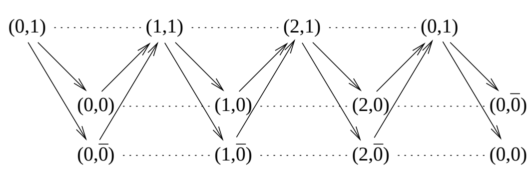

Figure 2. The quiver $\Gamma(D_3, 1)$

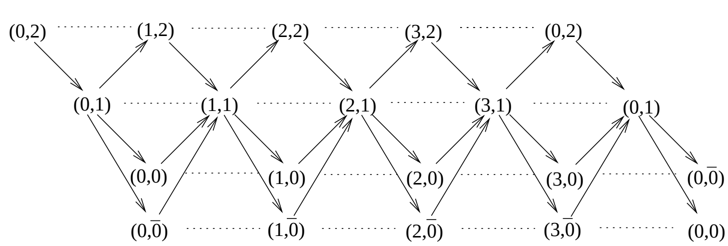

Figure 3. The quiver $\Gamma(D_4, 1)$

We recall the notion of the m-th power of a translation quiver (cf. [BaM]). If Γ is a translation quiver with translation τ, then the quiver Γ^m, the *m-th power of* Γ, is the quiver whose objects are the same as the objects of Γ and whose arrows are the sectional paths (in Γ) of length m. A path $x = x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m = y$ is said to be *sectional* if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \ldots, m-1$ (in the cases where τx_{i+1} is defined), cf. [Rin].

One of our goals is to realise the Auslander-Reiten quiver for the m-cluster category of type D_n in terms of the mth power of the Auslander-Reiten quiver of a cluster category of type D_{nm-m+1}. To be able to do this, we introduce a new class of sectional paths.

Definition 1.1. Let $\Gamma = \Gamma(D_n, 1)$ be the translation quiver defined above, with vertices $V(D_n, 1) = \mathbb{Z}_n \times \{0, \overline{0}, 1, \ldots, n-2\}$. We say that a sectional path $x = x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m = y$ (where $x_i \in V(D_n, 1)$) is *restricted* if there is no i such that $x_{i+1} = (r, 0)$ for some r while $x_{i-1} = (r-1, \overline{0})$ or such that $x_{i+1} = (r, \overline{0})$ for some r while $x_{i-1} = (r-1, 0)$.

Remark 1.2. Note that unless $m = 2$, the restricted sectional paths of length m in Γ are exactly the sectional paths of length m. We can see this as follows. Firstly, it is clear that any sectional path of length 1 is necessarily restricted. Suppose that $m > 2$. Let $x = x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m = y$ be sectional, and suppose that there is an $i \in \{1, \ldots, m-1\}$ such that $x_{i+1} = (r, 0)$ and $x_{i-1} = (r-1, \overline{0})$. Then $x_i = (r, 1)$. In case $i = 1$ we have $x_{i+2} = (r+1, 1)$, and it follows that the original path is not sectional, a contradiction. Similarly, if $i > 1$, we have $x_{i-2} = (r-1, 1)$, and again the original path is not sectional.

Hence the only sectional paths that are not restricted are the paths of the form $(i, 0) \to (i, 1) \to (i+1, \overline{0})$ and $(i, \overline{0}) \to (i, 1) \to (i+1, 0)$ $(i \in \mathbb{Z}_n)$.

With this new notion we are now ready to introduce a restricted version of the translation quiver $((\Gamma(D_n, 1))^m, \tau^m)$. We define a translation quiver $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ as follows. The vertices of $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ are the same as the vertices of $(\Gamma(D_n, 1), \tau^m)$, i.e. $\mathbb{Z}_n \times \{0, \overline{0}, 1, \ldots, n-2\}$, the arrows are the restricted sectional paths of length m in $(\Gamma(D_n, 1), \tau^m)$ and the translation is τ^m.

Lemma 1.3. *For any m, the pair $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ is a stable translation quiver.*

Proof. We firstly note that the unrestricted version, $(\Gamma(D_n, 1)^m, \tau^m)$, is a stable translation quiver by [BaM, 6.2]. By Remark 1.2, the quiver $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ is the same as $(\Gamma(D_n, 1)^m, \tau^m)$ if $m \neq 2$, so the result follows in this case.

Now assume that $m = 2$ and fix a vertex x in $\Gamma(D_n, 1)$. To show that $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ is a translation quiver, we need to show that there is a restricted sectional path of length 2 from y to x if and only if there is a restricted sectional path of length 2 from $\tau^2(x)$ to y. Since the restricted sectional paths in $\Gamma(D_n, 1)$ of length 2 starting or ending at x are the same as the sectional paths provided x is not of the form $(i, 0)$ or $(i, \overline{0})$, we are reduced to this case. If $x = (i, 0)$, the sectional paths of length 2 ending in x are $(i, 2) \to (i, 1) \to (i, 0)$ and $(i-1, \overline{0}) \to (i, 1) \to (i, 0)$. The sectional paths of length 2 starting at $\tau^2(x) = (i-2, 0)$ are $(i-2, 0) \to (i-1, 1) \to (i, 2)$ and $(i-2, 0) \to (i-1, 1) \to (i-1, \overline{0})$. The second path only in each case is not restricted, so we see that there is a restricted sectional path of length 2 from y to x if and only if there is a restricted sectional path of length 2 from $\tau^2(x)$ to y. The argument in case $x = (i, \overline{0})$ is similar. Hence $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ is a translation quiver.

By construction, no vertex is projective and τ^m is defined on all vertices (since τ is). Therefore, $(\mu_m(\Gamma(D_n, 1)), \tau^m)$ is stable. □

2. The m-cluster category of type D_n as a component of a restricted mth power

Let $n, m \in \mathbb{N}$, with $n \geq 3$. We recall that [Hap] the derived category of a quiver of Dynkin type D_n has vertices $\mathbb{Z} \times \{0, \overline{0}, 1, 2, \ldots, n-2\}$ and arrows given by $(i, j) \to (i, j-1)$ and $(i, j-1) \to (i+1, j)$ for $1 \leq j \leq n-2$, and $(i, 1) \to (i, \overline{0})$ and $(i, \overline{0}) \to (i+1, 1)$, where $i \in \mathbb{Z}$ is arbitrary. We also have that

$$S^m(i,0) = \begin{cases} (i+m, 0), & nm \text{ even}, \\ (i+m, \overline{0}), & nm \text{ odd}. \end{cases},$$

while $S^m(i, j) = (i + m, j)$, otherwise.

Let $\Gamma(D_n, m)$ be the quiver with vertices

$$V(D_n, m) = \{(i, j) : i \in \mathbb{Z}_{nm-m+1},\ j \in \{0, \overline{0}, 1, 2, \ldots, n-2\}\}.$$

The arrows are given by $(i, j) \to (i, j-1)$ and $(i, j-1) \to (i+1, j)$ for $1 \leq j \leq n-2$, and $(i, 1) \to (i, \overline{0})$ and $(i, \overline{0}) \to (i+1, 1)$, where $i \in \mathbb{Z}_{nm-m+1}$ is arbitrary and the addition is modulo $nm - m + 1$. We also define

$$\widetilde{\tau}(i,j) = \begin{cases} (i-1, \overline{j}) & \text{if } i = 0,\ j \in \{0, \overline{0}\} \text{ and } nm \text{ is odd}, \\ (i-1, j) & \text{otherwise}. \end{cases}$$

It follows from the construction of $\mathcal{C}^m_{D_n}$ and the above description of the derived category that $(\Gamma(D_n, m), \widetilde{\tau})$ is the Auslander-Reiten quiver of $\mathcal{C}^m_{D_n}$ (and, in particular, is a stable translation quiver).

The vertices of the Auslander-Reiten quiver $\Gamma(D_{nm-m+1}, 1)$ of $\mathcal{C}^1_{D_{nm-m+1}}$ are

$$V(D_{nm-m+1}, 1) = \mathbb{Z}_{nm-m+1} \times \{0, \overline{0}, 1, 2, \ldots, nm - m - 1\}.$$

The arrows are given by $(i, j) \to (i, j-1)$ and $(i, j-1) \to (i+1, j)$ for $1 \leq j \leq nm - m - 1$, and $(i, 1) \to (i, \overline{0})$ and $(i, \overline{0}) \to (i+1, 1)$, where i is arbitrary and the addition is modulo $nm - m + 1$. We also have

$$\tau(i,j) = \begin{cases} (i-1, \overline{j}) & \text{if } i = 0,\ j \in \{0, \overline{0}\} \text{ and } nm-m+1 \text{ is odd}, \\ (i-1, j) & \text{otherwise}. \end{cases}$$

Definition 2.1. *We define a map σ' from $V(D_n, m)$ to $V(D_{nm-m+1}, 1)$ as follows. We set $\sigma'(i, j) = (im, jm)$ whenever $j \notin \{0, \overline{0}\}$ or $j \in \{0, \overline{0}\}$, m is odd and n is even. Otherwise, we have $j = 0$ or $\overline{0}$ and we set*

$$\sigma'(i,j) = \begin{cases} (im, jm), & \left\lfloor \frac{im}{nm-m+1} \right\rfloor \text{ even}, \\ (im, \overline{jm}), & \left\lfloor \frac{im}{nm-m+1} \right\rfloor \text{ odd}. \end{cases}$$

Here we restrict i to lie in the set $\{0, 1, 2, n-1\}$.

We use the usual convention that, for a real number x, $\lfloor x \rfloor$ denotes the largest integer k such that $k \leq x$.

Let $V := \{(r, s) \in V(D_{nm-m+1}, 1) : m|s\}$. Here we adopt the convention that $m|\overline{0}$.

Lemma 2.2. *With σ' defined as above, we have that im$(\sigma') = V$.*

Proof. First we note that it is clear from the definition of σ' that $\mathrm{im}(\sigma') \subseteq V$. Let $(r, s) \in V(D_{nm-m+1}, 1)$ and suppose that $m|s$. Suppose first that $s \neq 0, \overline{0}$. Write $s = km$ for $k \in \mathbb{Z}$. We have $r = r(nm-m+1-(n-1)m) = -r(n-1)m$ in $\mathbb{Z}_{nm-m+1}$, and it follows that $\sigma'(-r(n-1), k) = (-r(n-1)m, km) = (r, s)$ so $(r, s) \in \mathrm{im}(\sigma')$. If $s = 0$ or $\overline{0}$ then $\{\sigma'(-r(n-1), 0), \sigma'(-r(n-1), \overline{0})\} = \{(r, s), (r, \overline{s})\}$ and we are done. □

Let Γ denote the full subquiver of $\mu_m(\Gamma(D_{nm-m+1},1))$ induced by V, and let σ be the (surjective) map obtained by restricting the codomain of σ' to V. We will show that σ is an isomorphism from $\Gamma(D_n,m)$ to Γ and that Γ is a connected component of $\mu_m(\Gamma(D_{nm-m+1},1))$.

Lemma 2.3. *Let $x := (r,s) \in V$ and suppose that*

$$x = x_0 \to x_1 \to x_2 \to \cdots \to x_m = y$$

is a restricted sectional path of length m in $\Gamma(D_{nm-m+1},1)$. Then $y \in V$.

Proof. If $s \neq 0, \overline{0}$ we can argue as in [BaM, 7.1] to see that y is either $(r, s-m)$, $(r+m, s+m)$ or $(r, \overline{s-m})$, where the last possibility only arises if $s = m$. If $s = 0$ or $\overline{0}$ a similar argument shows that y must be $(r+m, m)$. If $m = 2$ we have the sectional paths $(r,0) \to (r+1,1) \to (r+2,\overline{0})$ and $(r,\overline{0}) \to (r+1,1) \to (r+2,0)$, but, by definition, these are not restricted sectional paths. □

Lemma 2.4. *The map $\sigma : \Gamma(D_n,m) \to \Gamma$ is an isomorphism of quivers.*

Proof. Since $|V| = |V(D_n,m)|$ and σ is surjective, it follows that σ is bijective. The arrows in $\Gamma(D_n,m)$ are of the form $(i,j) \to (i,j-1)$, $(i,j) \to (i+1,j+1)$, $(i,1) \to (i,\overline{0})$ or $(i,\overline{0}) \to (i+1,1)$. The arrows in $V \subseteq \mu_m(\Gamma(D_{nm-m+1},1))$ are of the form $(r,s) \to (r,s-m)$, $(r,s) \to (r+m,s+m)$, $(r,m) \to (r,\overline{0})$ or $(r,\overline{0}) \to (r+m,m)$ (see the proof of Lemma 2.3). It follows that σ is an isomorphism of quivers. □

Proposition 2.5. *The map $\sigma : \Gamma(D_n,m) \to \Gamma$ is an isomorphism of translation quivers. Its image, Γ, is a connected component of $\mu_m(\Gamma(D_{nm-m+1},1))$.*

Proof. By Lemma 2.3, both statements of the theorem will follow if we can show that, for all $(i,j) \in V(D_n,m)$, $\sigma(\widetilde{\tau}(i,j)) = \tau^m(\sigma(i,j))$, since this will also imply that the image of σ is closed under τ^m. We firstly note that if $j \neq 0, \overline{0}$ then $\widetilde{\tau}(i,j) = (i-1,j)$ while $\tau^m(im,jm) = ((i-1)m,jm)$. Since $\sigma(i,j) = (im,jm)$ and $\sigma(i-1,j) = ((i-1)m,jm)$, the result holds. So we are left with the case where $j = 0$ or $\overline{0}$. We break this down into cases, considering first the case where $j = 0$.
Case (a): m odd and n even.
In this case we have that $nm-m+1$ is even, and nm is even, so for any $(i,0) \in V(D_n,m)$, $\tau^m(im,0) = ((i-1)m,0)$ while $\widetilde{\tau}(i,0) = (i-1,0)$. Since $\sigma(i-1,0) = ((i-1)m,0)$ and $\sigma(i,0) = (im,0)$, we are done.
Case (b): m is even.
In this case we have that $nm-m+1$ is odd, so for $l = 0$ or $\overline{0}$, we have:

$$\tau^m(im,l) = \begin{cases} ((i-1)m,\overline{l}), & im \bmod nm-m+1 \in \{0,1,\ldots,m-1\}, \\ ((i-1)m,l), & \text{otherwise.} \end{cases}$$

Since m is even, nm is even, so $\widetilde{\tau}(i,0) = (i-1,0)$ for all i.

(i) Suppose first that $im \notin \{0,1,\ldots,m-1\}$. Then $\left\lfloor \frac{im}{nm-m+1} \right\rfloor = \left\lfloor \frac{(i-1)m}{nm-m+1} \right\rfloor$. It follows that either $\sigma(i-1,0) = ((i-1)m,0)$ and $\sigma(i,0) = (im,0)$ or $\sigma(i-1,0) = ((i-1)m,\overline{0})$ and $\sigma(i,0) = (im,\overline{0})$. In either case we see that $\sigma(\widetilde{\tau}(i,0)) = \tau^m(\sigma(i,0))$.

(ii) Suppose next that $im \in \{1,\ldots,m-1\}$. Then $\left\lfloor \frac{im}{nm-m+1} \right\rfloor - 1 = \left\lfloor \frac{(i-1)m}{nm-m+1} \right\rfloor$. It follows that either $\sigma(i-1,0) = ((i-1)m,\overline{0})$ and $\sigma(i,0) = (im,0)$ or $\sigma(i-1,0) = ((i-1)m,0)$ and $\sigma(i,0) = (im,\overline{0})$. In either case we see that $\sigma(\widetilde{\tau}(i,0)) = \tau^m(\sigma(i,0))$.

(iii) Finally, suppose that $i = 0$. Then $i - 1 \equiv (n-1)m \bmod nm - m + 1$, and $\left\lfloor \frac{im}{nm-m+1} \right\rfloor = 0$ is even while

$$\begin{aligned} \left\lfloor \frac{(n-1)m}{nm-m+1} \right\rfloor &= \left\lfloor \frac{(n-1)m^2}{nm-m+1} \right\rfloor \\ &= \left\lfloor \frac{(nm-m+1)m}{nm-m+1} - \frac{m}{nm-m+1} \right\rfloor \\ &= \left\lfloor m - \frac{m}{nm-m+1} \right\rfloor = m-1, \end{aligned}$$

is odd (using here the fact that $m < (n-1)m + 1 = nm - m + 1$). It follows that $\sigma(i-1, \overline{0}) = ((i-1)m, 0)$ and $\sigma(i, 0) = (im, 0)$ and thus that $\sigma(\widetilde{\tau}(i, 0)) = \tau^m(\sigma(i, 0))$.

Case (c): n, m both odd.

In this case we have that $nm - m + 1$ is odd, so for $l = 0$ or $\overline{0}$, we have:

$$\tau^m(im, l) = \begin{cases} ((i-1)m, \overline{l}) & im \in \{0, 1, \ldots, m-1\}, \\ ((i-1)m, l) & \text{otherwise.} \end{cases}$$

Since n and m are both odd, nm is odd, so

$$\widetilde{\tau}(i, 0) = \begin{cases} ((i-1), \overline{0}) & i = 0, \\ (i-1, 0) & \text{otherwise.} \end{cases}$$

(i) Suppose first that $im \notin \{0, 1, \ldots, m-1\}$. Then $\left\lfloor \frac{im}{nm-m+1} \right\rfloor = \left\lfloor \frac{(i-1)m}{nm-m+1} \right\rfloor$. It follows that either $\sigma(i-1, 0) = ((i-1)m, 0)$ and $\sigma(i, 0) = (im, 0)$ or $\sigma(i-1, 0) = ((i-1)m, \overline{0})$ and $\sigma(i, 0) = (im, \overline{0})$. In either case we see that $\sigma(\widetilde{\tau}(i, 0)) = \tau^m(\sigma(i, 0))$.

(ii) Suppose next that $im \in \{1, \ldots, m-1\}$. Then $\left\lfloor \frac{im}{nm-m+1} \right\rfloor - 1 = \left\lfloor \frac{(i-1)m}{nm-m+1} \right\rfloor$. It follows that either $\sigma(i-1, 0) = ((i-1)m, \overline{0})$ and $\sigma(i, 0) = (im, 0)$ or $\sigma(i-1, 0) = ((i-1)m, 0)$ and $\sigma(i, 0) = (im, \overline{0})$. In either case we see that $\sigma(\widetilde{\tau}(i, 0)) = \tau^m(\sigma(i, 0))$.

(iii) Finally, suppose that $i = 0$. Then $i - 1 \equiv (n-1)m \bmod nm - m + 1$, and, as in Case (b)(i), $\left\lfloor \frac{im}{nm-m+1} \right\rfloor = 0$ is even and $\left\lfloor \frac{(n-1)m}{nm-m+1} \right\rfloor = m - 1$, which means in this case that it is also even. It follows that $\sigma(i-1, \overline{0}) = ((i-1)m, \overline{0})$ and $\sigma(i, 0) = (im, 0)$ and thus that $\sigma(\widetilde{\tau}(i, 0)) = \tau^m(\sigma(i, 0))$. □

We therefore have:

Theorem 2.6. *The translation quiver $\Gamma(D_n, m)$ can be realised as a connected component of the restricted mth power of the translation quiver $\Gamma(D_{nm-m+1}, 1)$.*

Since $\mathcal{C}^m_{D_n}$ is equivalent to the additive hull of the mesh category of $\Gamma(D_n, m)$, we obtain the following corollary.

Corollary 2.7. *The m-cluster category of type D_n can be realised as the additive hull of the mesh category of a connected component of the restricted mth power of the Auslander-Reiten quiver of the cluster category of type D_{nm-m+1}. For $m > 2$ it is enough to take the usual mth power.*

Example 2.8. We give an example of the theorem in the case where $n = 4$ and $m = 2$. The theorem tells us that $\Gamma(D_4, 2)$ is isomorphic to a connected component of $\mu_2(\Gamma(D_7, 1))$. In Figure 4 we show the translation quiver $\Gamma(D_7, 1)$ with the vertices of $V = \mathrm{im}(\sigma')$ shown in circles. In Figure 5 we isolate the connected component Γ of $\mu_2(\Gamma(D_7, 1))$ induced by V, and in Figure 6 we indicate the translation quiver $\Gamma(D_4, 2)$ with the usual labelling of its vertices.

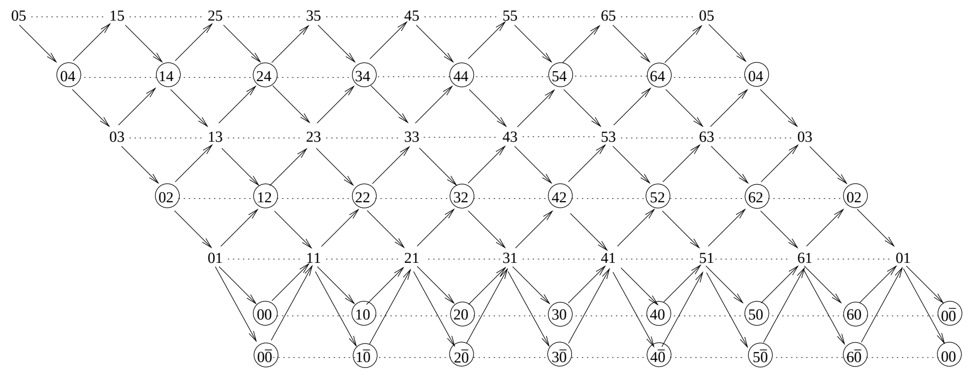

FIGURE 4. The translation quiver $\Gamma(D_7, 1)$

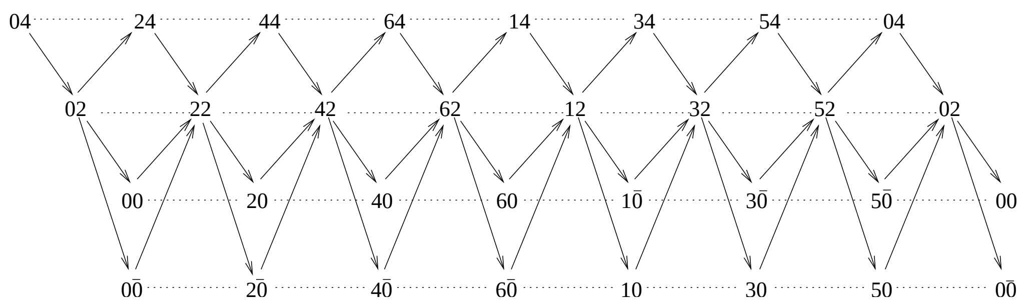

FIGURE 5. The connected component Γ of the translation quiver $\Gamma(D_7, 1)$

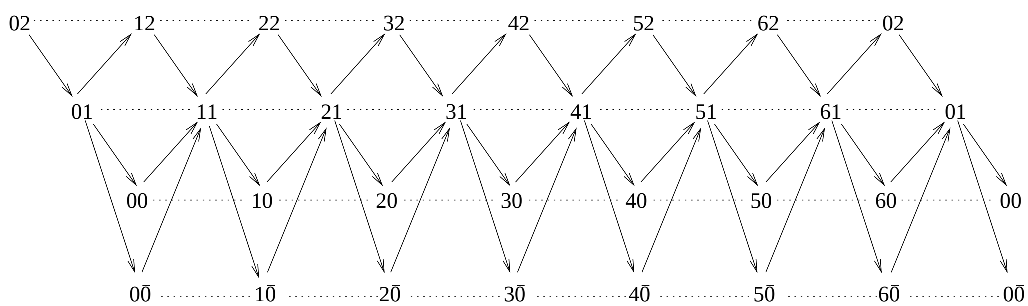

FIGURE 6. The translation quiver $\Gamma(D_4, 2)$

3. GEOMETRIC REALISATION

In this section, we give a geometric realisation of the m-cluster category of type D_n. To do so, we use certain m-arcs in a punctured $nm - m + 1$-gon. Thus we are generalising the notion of tagged edges of Schiffler [Sch] for the cluster category of type D_n and the notion of m-diagonals of our work on m-cluster categories of type A_n [BaM].

Let P be a punctured N-gon in the plane (later we shall specialise to the case where $N = nm - m + 1$). We label the vertices of P clockwise. For $i \neq j \in \{1, 2, \ldots, N\}$, we denote by B_{ij} the boundary path $i, i+1, \ldots, j-1, j$, going clockwise around the boundary (taking the vertices $\bmod N$). If $i = j$, we let B_{ii} be the whole boundary path $i, i+1, \ldots, i-1, i$ and $B_{ii}^{\bullet}$ denote the trivial path at i

consisting of the vertex i. The *length* $|B_{ij}|$ of the boundary path B_{ij} is the number of vertices it runs through. Here, we count both the starting and end point unless $B_{ij} = B_{ii}^{\bullet}$. In particular, $|B_{ii}| = N+1$ and $|B_{ii}^{\bullet}| = 1$.

As an example of a boundary path of length 4, we have indicated B_{62} inside a punctured 7-gon in Figure 7.

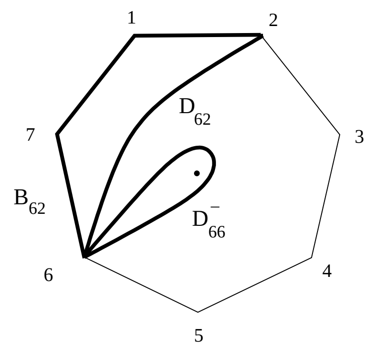

FIGURE 7. Punctured seven-gon with boundary path B_{62} and arcs D_{62} and D_{66}^{-}

For $i \neq j$, and j not the clockwise neighbour of i, an *arc* D_{ij} is a line from i to j that is homotopic to the boundary path B_{ij}. If j is the clockwise neighbour of i, there is no arc clockwise from i to j other than the boundary path B_{ij}. If $i = j$, we always tag the arc by $+$ or $-$, as in Schiffler's work [Sch]. Such arcs are denoted by D_{ii}^{+} and D_{ii}^{-}. We will occasionally write $D_{ij}^{\pm}$ to denote an arbitrary arc and call it a *tagged arc*. In that case, if $i \neq j$, then $D_{ij}^{\pm}$ will only stand for the arc D_{ij}. As an example, the arcs D_{62} and D_{66}^{-} of a punctured 7-gon are pictured in Figure 7. Addition of subscripts for the D_{ij} will always be modulo N.

In what follows, we will use a slightly generalised version of a polygon. We will allow arcs $D_{ij}^{\pm}$ and sides $B_{i,i+1}$ of the polygon P as sides of a polygon. We will say that such a (generalised) polygon is *degenerate* if it has more sides than vertices. Note that such polygons may or may not contain the puncture.

In the remainder, we will in particular be interested in the following types of generalised polygons and generalised degenerate polygons obtained from the regular N-gon P.

Type (i): A combination of an arc D_{ij} with the boundary path B_{ij}, or of an arc D_{ij} with the boundary path B_{ji}, $i \neq j$, where in the former case, $2 < |B_{ij}| \leq N$ and in the latter case, $1 < |B_{ji}| < N$. Such a polygon has $|B_{ij}|$ vertices (respectively, $|B_{ji}|$ vertices).

Type (ii): A combination of two arcs D_{ij}, D_{ik} with the boundary path B_{jk}, or of D_{ik}, D_{jk} with B_{ij}, where i, j, k are all distinct and lie in clockwise order on P. Furthermore, in the former case, $1 < |B_{jk}| < N-1$ and in the latter case, $1 < |B_{ij}| < N-1$. Such a polygon has $|B_{jk}|+1$ vertices (respectively, $|B_{ij}|+1$ vertices).

Type (iii): A combination of an arc $D_{ii}^{\pm}$ with the boundary path B_{ii}, or with $B_{ii}^{\bullet}$. In the first case, the polygon is has $N+1$ sides and N vertices. In the latter case, the polygon has one side and one vertex.

Type (iv): A combination of an arc $D_{ii}^{\pm}$ with an arc D_{ij} and the boundary path B_{ji}, or a combination of $D_{ii}^{\pm}$ with a boundary path B_{ij} and the arc D_{ji} (where we always have $i \neq j$). In the former case, $1 < |B_{ji}| < N$ and in the latter case, $1 < |B_{ij}| < N$. Such a polygon has $|B_{ji}|+1$ sides and $|B_{ji}|$ vertices (respectively, $|B_{ij}|+1$ sides and $|B_{ij}|$ vertices).

Note that we can view type (iii) as the limit $j \mapsto i$ of type (i) and type (iv) as the limit $k \mapsto i$ or $j \mapsto k$ of type (ii). We show each of these four types in Figure 8.

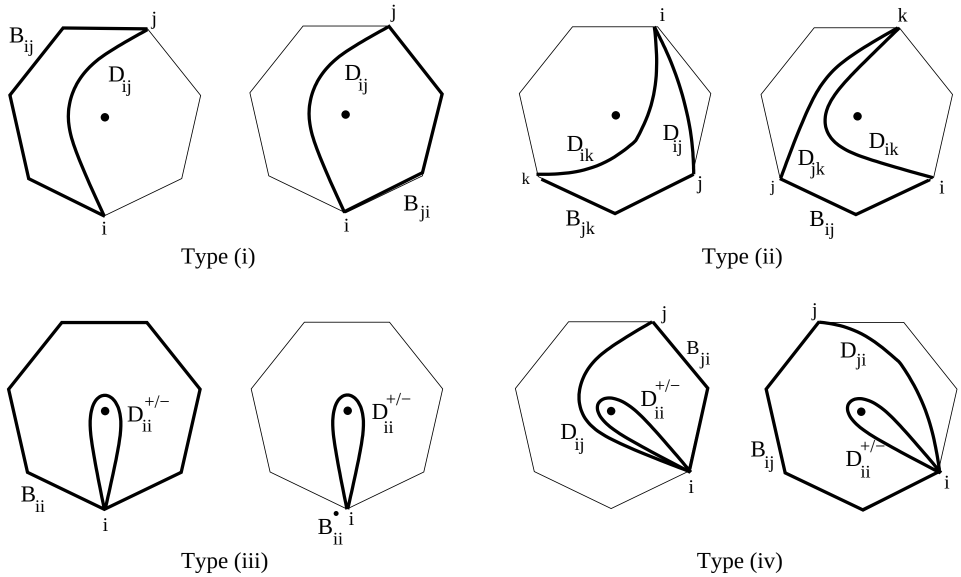

FIGURE 8. Generalised polygons (some degenerate)

Definition 3.1. *Let $D_{ij}^{\pm}$ be an arc of P. If $i \neq j$, we say that D_{ij} is an m-*arc *if the following hold:*

(i) D_{ij} and B_{ij} form a $km+2$-gon for some k,

(ii) D_{ij} and B_{ji} form a $lm+1$-gon for some l.

If $i = j$, $D_{ii}^{\pm}$ is a tagged m-arc *if $D_{ii}^{\pm}$ and B_{ii} form a degenerate $km+2$-gon for some k.*

The parts (i) and (ii) in the definition of an m-arc also apply to the case $i = j$ if we use the boundary paths B_{ii} and $B_{ii}^{\bullet}$. Namely, $D_{ii}^{\pm}$ is a tagged m-arc if $D_{ii}^{\pm}$ and B_{ii} form a degenerate $km+2$-gon for some k and if $D_{ii}^{\pm}$ and $B_{ii}^{\bullet}$ form a degenerate 1-gon.

Example 3.2. *Let P be a punctured 7-gon and $m = 2$. The arc D_{62} is a 2-arc (cf. Figure 7), since the arc D_{62} together with the boundary path B_{62} forms a 4-gon (i.e. $k = 1$) whereas D_{62} and B_{26} form a 5-gon (i.e. $l = 2$). Each of the arcs $D_{66}^{\pm}$ forms an 8-gon together with B_{66}, and thus also is a 2-arc.*

We now define m-moves generalising the m-rotation for type A_n of [BaM] and the elementary moves for type D_n of [Sch].

Definition 3.3. *Let P be a punctured N-gon. An* m-move *arises when there are two arcs in P with a common end-point such that the two arcs and a part of the boundary bound an unpunctured $m+2$-gon, possibly degenerate. If the angle from the first arc to the second at the common end-point is negative (i.e. clockwise), then we say that there is an m-move taking the first arc to the second. More precisely, it is a move of one of the following forms:*

(i) $D_{ij} \to D_{ik}$ if D_{ij}, B_{jk} and D_{ik} form an $m+2$-gon, $|B_{jk}| = m+1$.

(ii) $D_{ij} \to D_{kj}$ if D_{ij}, B_{ik} and D_{kj} form an $m+2$-gon, $|B_{ik}| = m+1$.

(iii) $D_{ij} \to D_{ii}^{\pm}$ if D_{ij}, $D_{ii}^{\pm}$ and B_{ji} form a degenerate $m+2$-gon, $|B_{ji}| = m+1$.

(iv) $D_{ii}^{\pm} \to D_{ji}$ if $D_{ii}^{\pm}$, D_{ji} and B_{ij} form a degenerate $m+2$-gon; $|B_{ij}| = m+1$.

In Figure 9, we illustrate the four types of m-moves inside a heptagon, i.e. $n = 4$, $m = 2$.

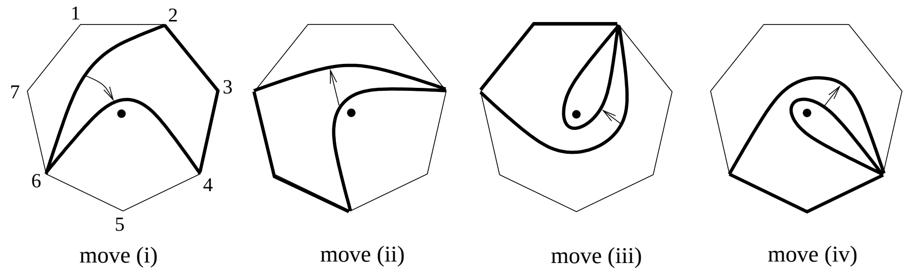

FIGURE 9. 2-moves inside a heptagon

Our goal is to model the m-cluster category $\mathcal{C}^m(D_n)$ geometrically. To do so, we will from now on assume that $N = nm - m + 1$, so the polygon P has $nm - m + 1$ vertices.

Remark 3.4. *Let P be a punctured polygon with $nm - m + 1$ vertices and let $i \neq j$. Then the two conditions of Definition 3.1 are equivalent, i.e. D_{ij} and B_{ij} form a $km + 2$-gon for some k if and only if D_{ij} and B_{ji} form an $lm + 1$-gon for some l.*

We are now ready to define a translation quiver using the punctured polygon, P.

Let $\Gamma_\odot = \Gamma_\odot(n, m)$ be the quiver whose vertices are the tagged m-arcs of P and whose arrows are given by m-moves. Let τ_m be the map sending an arc $D_{ij}^{\pm}$ to $D_{i-m,j-m}^{\pm}$ if $i \neq j$ or m is even. If $i = j$ and m is odd, we set $\tau_m(D_{ii}^{\pm}) = D_{i-m,i-m}^{\mp}$. In other words, if $i \neq j$ or m is even, τ_m rotates a tagged arc anti-clockwise around the center. In case $i = j$ and m is odd, τ_m rotates the tagged arc anti-clockwise around the center and changes its tag.

Figure 10 shows the example $\Gamma_\odot(4, 2)$ (we will see shortly, cf. Theorem 3.6, that $\Gamma_\odot(n, m)$ is a translation quiver).

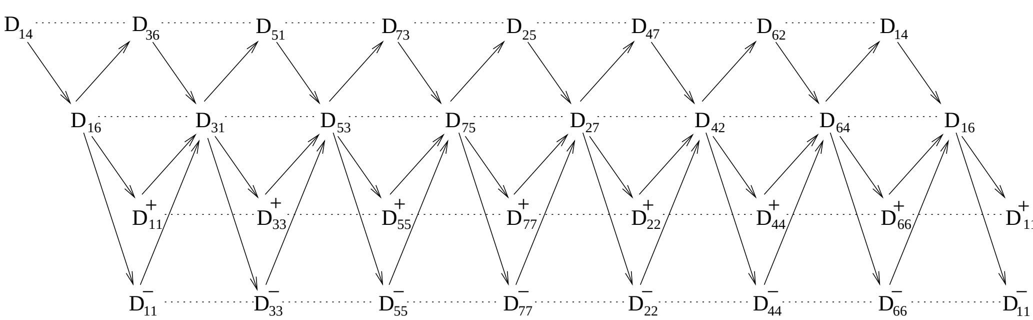

FIGURE 10. The translation quiver $\Gamma_\odot(4, 2)$

Lemma 3.5. *The sectional paths of length m in $\Gamma(D_{nm-m+1}, 1)$ are of the form*
(i) $(i, j) \to (i, j-1) \to \cdots \to (i, j-m)$ *if* $j - m > 0$,
(ii) $(i, j) \to (i, j-1) \to \cdots \to (i, 0)$ *and* $(i, j) \to \cdots \to (i, \overline{0})$ *if* $j = m$,
(iii) $(i, j) \to \cdots \to (i + m, j + m)$ *if* $j > 0$ *and* $(i + m, j + m)$ *exists,*
(iv) $(i, 0) \to \cdots \to (i + m, m)$ *and* $(i, \overline{0}) \to \cdots \to (i + m, m)$ *if* $(i + m, m)$ *exists.*

Proof. For (i) and (iii) we can argue as in [BaM, 7.1], using the vertices $\mathbb{Z}_{nm-m+1}\times\{0,\overline{0},1,\dots,n-2\}$ of $\Gamma(D_{nm-m+1},1)$ instead. The other cases follow with the same argument, using the assumption that the paths are restricted, i.e. excluding the sectional paths $(i,0)\to(i+1,1)\to(i+1,\overline{0})$ and $(i,\overline{0})\to(i+1,1)\to(i+1,0)$. □

Theorem 3.6. *The quiver $\Gamma_\odot$ is a translation quiver isomorphic to to the Auslander-Reiten quiver of $\mathcal{C}^m_{D_n}$.*

Proof. It is enough to show that $\Gamma_\odot$ is isomorphic to the image Γ of the map σ from Section 2 and that, under the isomorphism, the map τ_m on $\Gamma_\odot$ corresponds to τ^m on Γ.

Recall that the vertices of Γ are $V := \{(r,s)\in V(D_N) : m|s\}$ (using the convention that m divides $\overline{0}$), recalling that $N = nm-m+1$. In other words, V is the subset in $V(D_N)=\mathbb{Z}_N\times\{0,\overline{0},1,\dots,N\}$ of the vertices whose second coordinate is divisible by m.

We will now define a map $\rho: V(\Gamma_\odot)\to V$, where $V(\Gamma_\odot)$ denotes the set of vertices of $\Gamma_\odot$. Note that m-arcs in $V(\Gamma_\odot)$ going through two distinct vertices are always of form $D_{i,i+1+km}$. On such m-arcs, ρ is defined as follows:

$$\rho(D_{i,i+1+km}) = (lm,(n-1-k)m)\in\mathbb{Z}_N\times\{0,\overline{0},1,\dots,N-2\}$$

where $i\equiv lm+1$ modulo N and $k=1,\dots,n-2$.
On arcs $D^{\pm}_{ii}$, ρ is defined as follows.

$$\rho(D^+_{ii}) = \begin{cases}(lm,0), & \text{if } i \text{ is odd},\\ (lm,\overline{0}), & \text{if } i \text{ is even}.\end{cases}$$

$$\rho(D^-_{ii}) = \begin{cases}(lm,\overline{0}), & \text{if } i \text{ is odd},\\ (lm,0), & \text{if } i \text{ is even}.\end{cases}$$

(where $i\equiv lm+1$ modulo N).

To see that ρ is a bijection, we divide $V(\Gamma_\odot)$ up into n types of arcs. Let V_1 be the set of arcs of the form $D_{i,i+m+1}$ ($i=1,\dots,N$), i.e. the arcs homotopic to a boundary path of length $m+2$. Then ρ sends each element of V_1 to a vertex of the top row of V, $D_{i,i+m+1}\mapsto(lm,(n-2)m)$ (where $lm\equiv i-1 \bmod N$). It is straightforward to check that ρ induces a bijection from the set V_1 to the top row of V.

More generally, for $k=1,\dots,n-2$, let V_k be the set of arcs of the form $D_{i,i+km+1}$, i.e. the set of arcs homotopic to a boundary path of length $km+2$. Since $\rho(D_{i,i+km+1})=(lm,(n-1-k)m)$, ρ sends the arcs in V_k to the kth row (from the top) of V ($k\le n-2$). Clearly, this is also a bijection.

Furthermore, the arcs $D^{\pm}_{ii}$ are sent to the two last rows of V, also bijectively. Thus, we have that ρ is a bijection from $V(\Gamma_\odot)$ to V.

Next, we observe that the arrows given by the m-moves are the same as the arrows in Γ: for arcs in V_k with $1\le k<n-2$, an m-move sends $D_{i,i+1+km}$ to $D_{i,i+1+(k+1)m}$ or $D_{i,i+1+km}$ to $D_{i+m,i+1+km}$ whereas a restricted sectional path of length m sends $(lm,(n-1-k)m)$ to $(lm,(n-2-k)m)$ (type (i) in Lemma 3.5) or to $((l+1)m,(n-k)m)$ (type (iii) in Lemma 3.5). For arcs in V_{n-2}, an m-move sends $D_{i,i+1+(n-1)m}$ to D^+_{ii}, to D^-_{ii} or to $D_{i+m,i+1+(n-1)m}$ whereas a restricted sectional path of length m sends (lm,m) to $(lm,0)$, to $(lm,\overline{0})$ or to $((l+1)m,2m)$ (types (ii) and (iii) in Lemma 3.5). Finally, arcs $D^{\pm}_{ii}$ are sent to $D_{i+m,i}$ by m-moves, and restricted sectional paths of length m send $(lm,0)$ to $((l+1)m,m)$ and $(lm,\overline{0})$ to $((l+1)m,m)$ (type (iv) in Lemma 3.5).

Furthermore, the translation maps correspond: on V_k (with $1\le k\le n-2$) $\tau_m(D_{i,i+1+km})=D_{i-m,i+1+(k-1)m}$ (subscripts taken $\bmod N$) and on the $n-2$ first rows from the top, $\tau^m(lm,(n-1-k)m) = ((l-1)m,(n-1-k)m)$ (first entries taken mod N).

If $i>1$ then $\tau_1(D_{ii}^{\pm}) = D_{i-1,i-1}^{\mp}$ while $\tau(i-1,0)=(i-2,0)$ and $\tau(i-1,\overline{0}) = (i-2,\overline{0})$. If $i=1$ then $\tau_1(D_{11}^{+}) = D_{N,N}^{-}$ while

$$\tau(0,0) = \begin{cases} (N-1,\overline{0}), & \text{if } N \text{ is odd,} \\ (N-1,0), & \text{if } N \text{ is even.} \end{cases}$$

It follows that $\tau(\rho(D_{ii}^{+})) = \rho(\tau_1(D_{ii}^{+}))$ for all i. A similar argument applies to the tagged arcs D_{ii}^{-}. Since $\tau_m = \tau_1^m$, we see that $\tau^m(\rho(D_{ii}^{\pm})) = \rho(\tau_m(D_{ii}^{\pm}))$ for all i. We have seen that ρ induces an isomorphism of quivers and $\tau^m\rho(D_{ij}^{\pm}) = \rho(\tau_m(D_{ij}^{pm}))$ for all arcs $D_{ij}^{\pm}$. It follows that $\Gamma_{\odot}$ is a translation quiver and that ρ is an isomorphism of translation quivers. □

4. A toral translation quiver

In this section we give an example of a toral translation quiver arising from the cluster category $\mathcal{C}_{D_4}^1$ of type D_4. The Auslander-Reiten quiver of $\mathcal{C}_{D_4}^1$, $\Gamma(D_4,1)$, is shown in Figure 3. A connected component of its (unrestricted) square, $\Gamma(D_4,1)^2$ is shown in Figure 11. The underlying topological space $|\Gamma(D_4)^2|$ (in the sense of Gabriel and Riedtmann; see [Rin, p51]) is a torus.

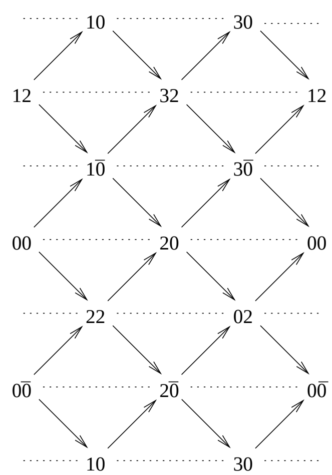

Figure 11. A connected component of the translation quiver $\Gamma(D_4,1)^2$

5. The components of $\mu_m(\Gamma(D_n,1))$

We have seen in Theorem 2.6 that the Auslander-Reiten quiver of the m-cluster category of type D_n is a connected component of the restricted m-th power $\mu_m(\Gamma(D_{nm-m+1},1))$. In this section, we describe the other components arising in the restricted m-th power of the translation quiver $(\Gamma(D_{nm-m+1},1),\tau)$.

Proposition 5.1. *The quiver $\mu_m(\Gamma(D_{nm-m+1},1))$ has $m-1$ connected components isomorphic to the Auslander-Reiten quiver of $D^b(A_{n-1})/\tau^{nm-m+1}$.*

Proof. We consider the following subset of the vertices of the quiver $\mu_m(\Gamma(D_{nm-m+1},1))$:

$$\begin{aligned} X_k &:= \{(i,j) \mid i \in \mathbb{Z}_{nm-m+1}, j \equiv k \bmod m\} \\ &= \mathbb{Z}_{nm-m+1} \times \{k, m+k, \ldots, (n-2)m+k\}. \end{aligned}$$

Such a set X_k is a union of rows in the quiver $\mu_m(\Gamma(D_{nm-m+1},1))$. We show that for each $1 \le k \le m-1$, the translation quiver generated by X_k (i.e. the full subquiver induced by X_k, together with τ^m) is a connected component of $\mu_m(\Gamma(D_{nm-m+1},1))$. This is done by first showing that X_k is closed under τ^m and under taking restricted sectional paths of length m. This tells us that X_k is a union of connected components of $\mu_m(\Gamma(D_{nm-m+1},1))$. Then we show that X_k is connected, hence is a single component.

1) The set X_k is closed under the translation τ^m, since, by definition, X_k is the union of all vertices of certain rows and τ^m shifts vertices along a row.

2) The set X_k is closed under restricted sectional paths of length m: we have seen that these paths are of the form $(ij) \to \cdots \to (i, j-m)$ or $(ij) \to \cdots \to (i+m, j-m)$, cf. Lemma 3.5. In particular, the new second entry is still congruent to k modulo m.

3) The subset X_k is connected: note that m is coprime to $nm-m+1$. Hence, the τ^m-orbit of any vertex (i,j) ($i \in \mathbb{Z}_{nm-m+1}, j \equiv k \bmod m$) is the same as the τ-orbit of (i,j). In other words, we can use τ^m to get everywhere in any given row of X_k, in particular in the row through $(0,k)$. Using the arrows and starting at $(0,k)$, we can get to any other row of X_k. Now by definition, X_k is the union of $n-1$ rows, namely the rows $(\cdot,k)$, $(\cdot, m+k)$ up to $(\cdot,(n-2)m+k)$. Each row is of length $nm-m+1$. It is clear from the arrows in X_k that X_k is isomorphic to the Auslander-Reiten quiver of $D^b(A_{n-1})/\tau^{nm-m+1}$. □

Thus we obtain a complete description of the restricted m-th power of $\Gamma(D_{nm-m+1},1)$.

Theorem 5.2. *The restricted m-th power $\mu_m(\Gamma(D_{nm-m+1},1),\tau^m)$ is the union of the following connected components:*

$$\mu_m(\Gamma(D_{nm-m+1},1),\tau^m) = \Gamma_\odot(n,m) \cup \bigcup_{k=i}^{m-1} \Gamma(D^b(A_{n-1})/\tau^{nm-m+1}),$$

where $\Gamma(D^b(A_{n-1})/\tau^{nm-m+1})$ denotes the Auslander-Reiten quiver of $D_b(A_{n-1})/\tau^{nm-m+1}$.

Proof. The statement follows from Theorem 2.6, Proposition 5.1 and the observation that the vertices of $(\mu_m(\Gamma(D_{nm-m+1},1)),\tau^m)$ are exhausted by the subsets X_k $(k = 1,\ldots,m-1)$ together with the vertices of $\Gamma_\odot(n,m)$. □

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