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Limit Shape of Minimal Difference Partitions and Fractional Statistics

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Abstract: The class of *minimal difference partitions* MDP(q) (with gap q) is defined by the condition that successive parts in an integer partition differ from one another by at least $q \geq 0$. In a recent series of papers by A. Comtet and collaborators, the MDP(q) ensemble with uniform measure was interpreted as a combinatorial model for quantum systems with *fractional statistics*, that is, interpolating between the classical Bose–Einstein ($q = 0$) and Fermi–Dirac ($q = 1$) cases. This was done by formally allowing values $q \in (0, 1)$ using an analytic continuation of the limit shape of the corresponding Young diagrams calculated for integer q. To justify this "replica-trick", we introduce a more general model based on a variable MDP-type condition encoded by an integer sequence $\mathfrak{q} = (q_i)$, whereby the (limiting) gap q is naturally interpreted as the Cesàro mean of $\mathfrak{q}$. In this model, we find the family of limit shapes parameterized by $q \in [0, \infty)$ confirming the earlier answer, and also obtain the asymptotics of the number of parts.

1. Introduction

1.1. Integer partitions and the limit shape. An *integer partition* is a decomposition of a given natural number into an *unordered sum* of integers; for example, $35 = 8 + 6 + 6 + 5 + 4 + 2 + 2 + 1 + 1$. That is to say, a non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\lambda_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ is a partition of $n \in \mathbb{N}_0$ if $n = \lambda_1 + \lambda_2 + \cdots$, which is expressed as $\lambda \vdash n$. Zero terms are added as a matter of convenience, without causing any confusion. The non-zero terms $\lambda_i \in \lambda$ are called the *parts* of the partition λ. We formally allow the case $n = 0$ represented by the "empty" partition $\varnothing = (0, 0, \ldots)$, with no parts. The set of all partitions $\lambda \vdash n$ is denoted by $\Lambda(n)$, and $\Lambda := \cup_{n\in\mathbb{N}_0} \Lambda(n)$ is the collection of *all* integer partitions. For a partition $\lambda = (\lambda_i) \in \Lambda$, the sum $N(\lambda) := \lambda_1 + \lambda_2 + \cdots$ is referred to as its *weight* (i.e., $\lambda \vdash N(\lambda)$), and the number of its parts $K(\lambda) := \#\{\lambda_i \in \lambda : \lambda_i > 0\}$ is called the *length* of λ. Thus, for $\lambda \in \Lambda(n)$, we have $N(\lambda) = n$ but $K(\lambda) \leq n$.

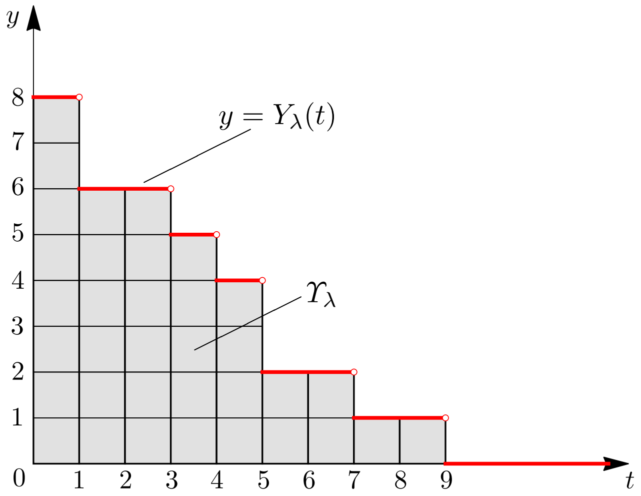

Fig. 1. The Young diagram $\varUpsilon_\lambda$ (shaded) of a partition $\lambda = (8, 6, 6, 5, 4, 2, 2, 1, 1, 0, \dots)$, with weight $N(\lambda) = 35$ and length $K(\lambda) = 9$. Note that the parts $\lambda_i > 0$ are represented by the successive columns of the diagram. The graph of the step function $t \mapsto Y_\lambda(t)$ (shown in red in the online version) gives the upper boundary of $\varUpsilon_\lambda$

A partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \varLambda$ is succinctly visualized by its *Young diagram* $\varUpsilon_\lambda$ formed by left- and bottom-aligned column blocks with $\lambda_1, \lambda_2, \dots$ unit square cells, respectively. In particular, the area of the Young diagram $\varUpsilon_\lambda$ equals the partition weight $N(\lambda)$. The upper boundary of $\varUpsilon_\lambda$ is a non-increasing step function $Y_\lambda \colon [0, \infty) \to \mathbb{N}_0$ (see Fig. 1 for illustration). Note that $\inf\{t \ge 0 \colon Y_\lambda(t) = 0\}$ coincides with the length $K(\lambda)$.

Theory of integer partitions is a classical branch of discrete mathematics and combinatorics dating back to Euler, with further fundamental contributions due to Hardy, Ramanujan, Rademacher and many more (see [3] for a general background). The study of asymptotic properties of *random* integer partitions (under the uniform distribution) was pioneered by Erdős and Lehner [13], followed by a host of research which in particular discovered a remarkable result that, under a suitable rescaling, the Young diagrams $\varUpsilon_\lambda$ of typical partitions λ of a large integer n are close to a certain deterministic *limit shape*. For *strict* partitions (i.e., with distinct parts) this result was (implicitly) contained already in [13]; for *plain* partitions (i.e., without any restrictions), the limit shape was first identified by Temperley [38] in relation to the equilibrium shape of a growing crystal, and obtained more rigorously much later by Vershik (as pointed out at the end of [42]) using some asymptotic estimates by Szalay and Turán [37]. An alternative proof in its modern form was outlined by Vershik [39] and elaborated by Pittel [31], both using the conditioning device[1] based on a suitable randomization of the integer n being partitioned.

Under the natural rescaling of Young diagrams $\varUpsilon_\lambda$ of partitions $\lambda \vdash n$ by $\sqrt{n}$ in each coordinate,[2] the limit shape for these two classical ensembles is determined, respectively, by the equations $\mathrm{e}^{-x\pi/\sqrt{6}} + \mathrm{e}^{-y\pi/\sqrt{6}} = 1$ (plain partitions) and $\mathrm{e}^{x\pi/\sqrt{12}} = \mathrm{e}^{-y\pi/\sqrt{12}} + 1$

[1] The randomization trick, often collectively called "Poissonization", is well known in the general enumerative combinatorics (see, e.g., Kolchin et al. [25]). In the context of integer partitions, it was introduced by Fristedt [16].

[2] See, however, Remark 1.6 below.

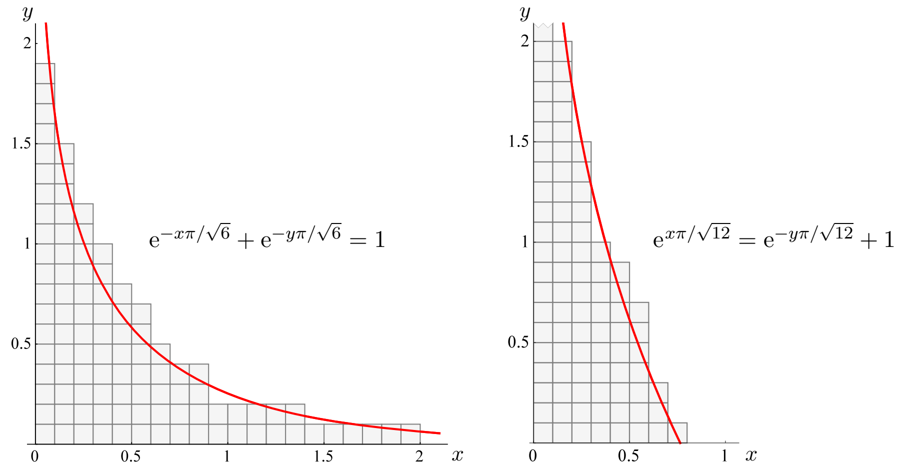

Fig. 2. The limit shape (shown in red in the online version) for plain partitions (left) and strict partitions (right), both under the scaling $Y_\lambda(t) \mapsto n^{-1/2}\,Y(n^{1/2}t)$ $(\lambda \vdash n)$ as $n \to \infty$. The scaled Young diagrams (shaded in grey) represent integer partitions uniformly sampled with $n = 100$. On the right picture, the largest part (depicted as the leftmost column) is only partially shown; in fact, here $\lambda_1 = 35$

(strict partitions); see Fig. 2. Note that in the latter case, the limit shape hits zero at $x = c_1 = \pi^{-1}\sqrt{12}\,\log 2 \doteq 0.764304$; this implies that the number of parts $K(\lambda)$ in a typical strict partition $\lambda \vdash n$ grows like $c_1\sqrt{n}$ as $n \to \infty$. In contrast, for plain partitions the number of parts grows faster than $\sqrt{n}$; more precisely, $K(\lambda) \sim c_0\sqrt{n}\,\log n$, where $c_0 = \sqrt{6}/(2\pi)$ [13].

To date, many limit shape results are known for integer partitions subject to various restrictions (see, e.g., Bogachev [6], Yakubovich [44], and also a review in DeSalvo and Pak [12]). Deep connections between statistical properties of quantum systems (where discrete random structures naturally arise due to quantization) and asymptotic theory of random integer partitions are discussed in a series of papers by Vershik [39,40]. Note that the idea of conditioning in problems of quantum statistical mechanics was earlier promoted by Khinchin [24] who advocated systematic use of local limit theorems of probability theory as a tool to prove the equivalence of various statistical ensembles in the thermodynamic limit.

From the point of view of statistical mechanics, it is conventional[3] to interpret the integer partition $\lambda = (\lambda_i) \in \Lambda$ as the energy spectrum in a sample configuration (state) of quantum gas, with $K(\lambda) = \#(\lambda_i > 0)$ particles and the total energy $\sum_i \lambda_i = N(\lambda)$. Note that decomposition into a sum of integers is due to the quantization of energy in quantum mechanics, while using *unordered* partitions corresponds to the fact that quantum particles are indistinguishable. In this context, the limit shape of Young diagrams associated with random partitions (for instance, under the uniform measure) is of physical interest as it describes the asymptotic distribution of particles in such ensembles over the energy domain.

[3] For a historic background, see older papers by Auluck and Kothari [2] and Temperley [38], and Vershik [40] for a modern exposition.

1.2. Minimal difference partitions. For a given $q \in \mathbb{N}_0$, the class of *minimal difference partitions with gap q*, denoted by MDP(q), is the set of integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ subject to the restriction $\lambda_i - \lambda_{i+1} \ge q$ whenever $\lambda_i > 0$. Two important special cases of the MDP(q) are furnished by the values $q = 0$ corresponding to *plain partitions* (i.e., with no restrictions), and $q = 1$ leading to *strict partitions* (i.e., with different parts).

In this paper, we propose a natural generalization of the MDP property as follows.

Definition 1.1. For a given sequence $\mathfrak{q} = (q_i)_{i\in\mathbb{N}_0}$ of non-negative integers (with the convention that $q_0 \ge 1$), we define $\Lambda_{\mathfrak{q}} \equiv \mathrm{MDP}(\mathfrak{q})$ to be the set of all integer partitions $\lambda = (\lambda_i)$ subject to the variable MDP-type condition

$$\lambda_i - \lambda_{i+1} \ge q_{k-i}, \qquad i = 1, \ldots, k, \tag{1.1}$$

where k is the number of (non-zero) parts in the partition λ. By convention, the empty partition $\varnothing$ satisfies (1.1). The sequence $\mathfrak{q}$ is referred to as the *gap sequence*.

Remark 1.1. For $i = k$, the inequality (1.1) specializes to $\lambda_k - \lambda_{k+1} \equiv \lambda_k \ge q_0$. That is to say, the smallest part of the partition $\lambda = (\lambda_i)$ is required to be not less than $q_0 \ge 1$ (which really poses a restriction only if $q_0 > 1$).

Remark 1.2. The partition model (1.1) appeared earlier (without any name) in a paper by Bessenrodt and Pak [4, § 4] devoted to partition bijections, in connection with *generalized Sylvester's transformation* $\lambda_{k-i} \mapsto \lambda_{k-i} + \sum_{j=0}^{i} q_i$ $(i = 0, \ldots, k-1)$, extending the classical case $q_i \equiv 1$.

Remark 1.3. Alternatively, one could consider partitions subject to similar restrictions as (1.1) but in the reverse order relative to the sequence $\mathfrak{q}$,

$$\lambda_i - \lambda_{i+1} \ge q_i, \qquad i = 1, \ldots, k.$$

However, the model (1.1) is preferable in view of the physical interpretation of parts λ_i as successive energy levels in a configuration (state) of a quantum system [40], which makes it more natural to enumerate the energy gaps starting from the minimal level $\lambda_k = \min\{\lambda_i : \lambda_i > 0\}$.

Throughout the paper, we impose the following

Assumption 1.1. The gap sequence $\mathfrak{q} = (q_i)$ satisfies the asymptotic regularity condition

$$Q_k := \sum_{i=0}^{k-1} q_i = qk + O(k^{\beta}) \qquad (k \to \infty), \tag{1.2}$$

with some $q \ge 0$ and $0 \le \beta < 1$.

Note that under Assumption 1.1 the sequence $\mathfrak{q} = (q_i)$ has a well-defined Cesàro mean, referred to as the *limiting gap*,

$$\lim_{k\to\infty} k^{-1} Q_k = q \ge 0. \tag{1.3}$$

Remark 1.4. In the case $q = 0$, the asymptotic relation (1.2) accommodates sequences (Q_k) that are irregularly growing (provided the growth is sublinear) or even bounded $(\beta = 0)$, including the case $Q_k \equiv 1$ corresponding to plain (unrestricted) integer partitions.

For $q = 0$ (when the leading term in (1.2) vanishes), it is still possible to derive the limit shape results under our standard Assumption 1.1. However, to obtain the asymptotics of the typical MDP length $K(\lambda)$, more regularity should be assumed by specifying the behaviour of the remainder term $O(k^{\beta})$.

Assumption 1.2 ($q = 0$). The gap sequence $\mathfrak{q} = (q_i)$ satisfies the asymptotic regularity condition

$$Q_k := \sum_{i=0}^{k-1} q_i = \tilde{q}\, k^{\beta} + O(k^{\tilde{\beta}}) \qquad (k \to \infty), \tag{1.4}$$

with some $\tilde{q} \ge 0$ and $0 \le \tilde{\beta} < \beta < 1$.

Remark 1.5. The utterly degenerate case $\tilde{q} = 0$ and $\tilde{\beta} = 0$ in Assumption 1.2 is equivalent to Assumption 1.1 with $q = 0$ and $\beta = 0$. In this case, we have $Q_k = O(1)$ as $k \to \infty$, and since $q_i \in \mathbb{N}_0$, this implies that $q_i = 0$ for all sufficiently large i. Clearly, the first few non-zero terms in the sequence $\mathfrak{q} = (q_i)$ (i.e., in the MDP conditions (1.1)) do not affect any limiting results, and so effectively such a model is identical with the classical case of plain partitions ($q_0 = 1$ and $q_i \equiv 0$ for $i \in \mathbb{N}$).

1.3. Main result. For $n \in \mathbb{N}_0$, consider the subset $\Lambda_{\mathfrak{q}}(n) = \Lambda_{\mathfrak{q}} \cap \Lambda(n)$ comprising MDP($\mathfrak{q}$) partitions of weight $N(\lambda) = n$. For example, the partition $\lambda = (8, 6, 6, 5, 4, 2, 2, 1, 1, 0, 0, \dots)$ used in Fig. 1 fits into the MDP-space $\Lambda_{\mathfrak{q}}(35)$ with the alternating sequence $\mathfrak{q} = (1, 0, 1, 0, 1, 0, \dots)$. Suppose that each (non-empty) space $\Lambda_{\mathfrak{q}}(n)$ is endowed with uniform probability measure denoted by $\nu_n^{\mathfrak{q}}$. We are interested in asymptotic properties (as $n \to \infty$) of this and similar measures on MDP spaces; in particular, we find the limit shape of properly scaled Young diagrams associated with partitions $\lambda \in \Lambda_{\mathfrak{q}}(n)$ and prove exponential bounds for deviations from the limit shape.

Let us state one of our main results, slightly simplifying the notation as compared to the more general case treated in Sect. 4. For every $q \ge 0$, define the function

$$\varphi(t; q) := \max\{0, -qt - \log(1 - \mathrm{e}^{-t})\}, \qquad t > 0, \tag{1.5}$$

and let $T_q := \inf\{t > 0\colon \varphi(t; q) = 0\}$; that is, T_q is the unique root of the equation

$$q = -T_q^{-1} \log(1 - \mathrm{e}^{-T_q}) \tag{1.6}$$

(with the convention $T_0 := +\infty$). The area under the graph of $\varphi(t; q)$ is computed as

$$\vartheta_q^2 := \int_0^{T_q} \varphi(t; q)\, \mathrm{d}t = -\frac{q T_q^2}{2} + \mathrm{Li}_2(1) - \mathrm{Li}_2(\mathrm{e}^{-T_q}), \tag{1.7}$$

where $\mathrm{Li}_2(\cdot)$ denotes the *dilogarithm* function (see, e.g., [27, p. 1]),

$$\mathrm{Li}_2(x) := -\int_0^x \frac{\log(1-u)}{u}\, \mathrm{d}u \equiv \sum_{k=1}^{\infty} \frac{x^k}{k^2}, \qquad 0 \le x \le 1. \tag{1.8}$$

Note that $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$. It is easy to check from (1.6) that $\lim_{q\downarrow 0} q T_q^2 = 0$, so using (1.7) we obtain

$$\vartheta_0 = \lim_{q\downarrow 0} \vartheta_q = \sqrt{\mathrm{Li}_2(1)} = \frac{\pi}{\sqrt{6}}. \tag{1.9}$$

Finally, observe that, setting $x = \mathrm{e}^{-T_q}$ in the well-known identity[4] [27, Eq. (1.11), p. 5]

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \mathrm{Li}_2(1) - \log x \cdot \log(1-x), \tag{1.10}$$

and using Eq. (1.6), the expression (1.7) is rewritten in a more appealing form,

$$\vartheta_q^2 = \frac{q\,T_q^2}{2} + \mathrm{Li}_2(1-\mathrm{e}^{-T_q}), \tag{1.11}$$

where the terms on the right-hand side can be given a meaningful geometric interpretation (see details in Sect. 4.4).

Theorem 1.1 (Limit shape in $\Lambda_{\mathfrak{q}}(n)$). *Let the sequence $\mathfrak{q} = (q_i)$ satisfy Assumption 1.1, with $q \geq 0$. Then, for every $t_0 > 0$ and any $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \nu_n^{\mathfrak{q}}\left\{\lambda \in \Lambda_{\mathfrak{q}}(n)\colon \sup_{t\geq t_0} \left|n^{-1/2} Y_\lambda(t n^{1/2}) - \vartheta_q^{-1}\varphi(t\vartheta_q; q)\right| > \varepsilon\right\} = 0, \tag{1.12}$$

where $Y_\lambda(\cdot)$ denotes the upper boundary of the Young diagram Υ_λ and ϑ_q is given by (1.11).

In view of formula (1.5), in the Cartesian coordinates

$$x = t\vartheta_q, \qquad y = \varphi(t\vartheta_q; q) \tag{1.13}$$

the limit shape (1.12) is given by the equation

$$\mathrm{e}^{-y} = \mathrm{e}^{qx}(1-\mathrm{e}^{-x}). \tag{1.14}$$

Clearly, $y = y(x)$ is a continuous decreasing function (as long as $y(x) > 0$), hitting zero at $x = T_q$ for $q > 0$ (see Eq. (1.6)) and with $\lim_{x\to\infty} y(x) = 0$ for $q = 0$.

Remark 1.6. It is common to scale Young diagrams via reducing their area n to 1 [40]. In our case, this leads to the additional rescaling in the expression of the limit shape (see (1.12)). Instead, it is more natural to work with the intrinsic scaling (1.13) to produce a simpler equation for the limit shape (1.14) but where the limiting area ϑ_q^2 varies with q (see (1.11)). See the precise corresponding assertions in Sect. 4.

Example 1.1. Let us specialize the notation introduced before Theorem 1.1 for a few simple values of $q \geq 0$, including all cases where closed expressions for T_q and ϑ_q in elementary functions are available.

- $q = 0$: here $T_0 = \infty$, $\vartheta_0 = \sqrt{\mathrm{Li}_2(1)} = \pi/\sqrt{6} \doteq 1.282550$, and the limit shape (1.14) specializes to (cf. Vershik [39, p. 99])

$$\mathrm{e}^{-x} + \mathrm{e}^{-y} = 1.$$

[4] This identity can be obtained from the definition (1.8) by integration by parts.

- $q=1$: from Eq. (1.6) we get $T_1=\log 2 \doteq 0.693147$. By virtue of Euler's result (see [27, Eq. (1.16), p. 6])

$$\mathrm{Li}_2\bigl(\tfrac{1}{2}\bigr)=\frac{\pi^2}{12}-\frac{(\log 2)^2}{2},$$

we obtain from (1.11)

$$\vartheta_1^2=\frac{T_1^2}{2}+\mathrm{Li}_2\bigl(\tfrac{1}{2}\bigr)=\frac{\pi^2}{12}.$$

Hence, $\vartheta_1=\pi/\sqrt{12}\doteq 0.906900$ and the limit shape (1.14) is reduced to (cf. Vershik [39, p. 100])

$$\mathrm{e}^{x}-\mathrm{e}^{-y}=1.$$

- $q=2$: the equation (1.6) (quadratic in $z=\mathrm{e}^{-T_2}$) solves to give $T_2=\log\left(\frac{1+\sqrt{5}}{2}\right)\doteq 0.481212$. Hence, we find $1-\mathrm{e}^{-T_2}=\frac{3-\sqrt{5}}{2}$. Using a known expression for the dilogarithm at this point (see [27, Eq. (1.20), p. 7]), we obtain from (1.11)

$$\vartheta_2^2=T_2^2+\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right)=\log^2\left(\frac{1+\sqrt{5}}{2}\right)+\frac{\pi^2}{15}-\frac{1}{4}\log^2\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15},$$

which gives $\vartheta_2=\pi/\sqrt{15}\doteq 0.811156$ (cf. Romik [33]).
- $q=\frac{1}{2}$: solving the equation (1.6) we get $T_{1/2}=\log\left(\frac{3+\sqrt{5}}{2}\right)\doteq 0.962424$. Hence, $1-\mathrm{e}^{-T_{1/2}}=\frac{\sqrt{5}-1}{2}$. Using another exact value of dilogarithm [27, Eq. (1.20), p. 7], formula (1.11) yields

$$\vartheta_{1/2}^2=\frac{T_{1/2}^2}{4}+\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right)=\frac{1}{4}\log^2\left(\frac{3+\sqrt{5}}{2}\right)+\frac{\pi^2}{10}-\log^2\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{10},$$

so that $\vartheta_{1/2}=\pi/\sqrt{10}\doteq 0.993459$.
- $q=3$: an exact value of T_3 can be found by solving the equation (1.6) (cubic in $z=\mathrm{e}^{-T_3}$), but no elementary expression is available for $\mathrm{Li}_2(1-\mathrm{e}^{-T_3})$ (cf. [27]). It is easy to find numerically $T_3\doteq 0.382245$ and $\vartheta_3\doteq 0.752618$ (cf. [9, Fig. 3, p. 8]).
- $q=\frac{1}{3}$: numerical values are given by $T_{1/3}\doteq 1.146735$ and $\vartheta_{1/3}\doteq 1.038508$.

1.4. MDP and fractional statistics. The special case of the MDP($\mathfrak{q}$) model with a *constant* gap sequence $q_i\equiv q\in\mathbb{N}_0$ in (1.1) was considered in a series of papers by Comtet et al. [8–10] in connection with fractional exclusion statistics of quantum particle systems (see [23,26] or [28] for a "physical" introduction to this area). These authors obtained the limit shape of MDP(q) using a physical argumentation. In particular, it was observed that the analytic continuation of the limit shape, as a function of $q\in\mathbb{N}_0$, into the range $q\in(0,1)$ (the so-called *replica trick*) may be interpreted as a quantum gas obeying fractional exclusion statistics, thus furnishing a family of probability measures "interpolating" between the Bose–Einstein statistics ($q=0$) and the Fermi–Dirac statistics ($q=1$).

In the present work,[5] we provide a combinatorial justification of this physical construction by working with a more general MDP($\mathfrak{q}$) model satisfying Assumption 1.1. In addition to many deterministic examples with such a property, the assumption (1.2) (and hence (1.3)) holds almost surely for sequences of independent random variables $\mathfrak{q} = (q_i)$ satisfying mild conditions, thus providing a stochastic version of the MDP($\mathfrak{q}$) model (see Sect. 6 below).

As was observed by Comtet et al. [8], another model of statistical physics leading to the MDP-type constraint is the one-dimensional *quantum Calogero model* with harmonic confinement (see [32] for a review and further references therein), defined by the Hamiltonian of a k-particle system with spatial positions $(x_i)_{i=1}^k$ on a line,

$$H_q := -\frac{1}{2}\sum_{i=1}^{k}\frac{\partial^2}{\partial x_i^2} + \sum_{1\le i<j\le k}\frac{q\,(q-1)}{(x_i - x_j)^2} + \frac{1}{2}\sum_{i=1}^{k} x_i^2.$$

This model is exactly solvable, and the solution can be expressed in terms of the pseudo-excitation numbers λ_i satisfying the condition $\lambda_i - \lambda_{i+1} \ge q$, with a positive real q.

As is common in such models (cf. [19]), an analogue of Pauli's exclusion principle is not strictly local for models MDP($\mathfrak{q}$) with sequences $\mathfrak{q} = (q_i)$ not degenerating to the trivial sequences $q_i \equiv 0$ or $q_i \equiv 1$ ($i \in \mathbb{N}$). Indeed, the occurrence of part $\lambda_i = j$ rules out a few adjacent values, that is, $\lambda_{i-1} \notin \{j, j+1, \dots, j+q_{k-i+1}-1\}$ if $q_{k-i+1} > 0$ or $\lambda_{i+1} \notin \{j, j-1, \dots, j-q_{k-i}+1\}$ if $q_{k-i} > 0$, but the actual index $k-i$ is determined by the entire partition $\lambda = (\lambda_i)$ through the rank of the part $\lambda_i = j$ among all (ordered) parts λ_i, together with the total number k of non-zero parts in λ.

Remark 1.7. Heuristically, the requirement $\lambda_i - \lambda_{i+1} \ge q$ with $q \in (0, 1]$ may be interpreted, at least for integer $m := q^{-1}$, as saying that $\lambda_i - \lambda_{i+m} \ge 1$ as long as $\lambda_i > 0$, that is, to prohibit more than $m = q^{-1}$ equal parts; in other words, no part counts bigger than q^{-1} are allowed. For $q = 1$ this indeed translates as only strict partitions being permissible. In the general case, this interpretation turns out to be true for the *expected* part counts (see [23, § 5.2]); however, literal restriction that the part counts do not exceed q^{-1} leads to a different model called *Gentile's statistics* [23, § 5.5]. The limit shape of partitions under Gentile's statistics was found in [29, § 9] (see also [44] where a rigorous proof is given).

The rest of the paper is organized as follows. In Sect. 2, several measures on minimal difference partitions are introduced, and certain relations between them are stated. Section 3 is devoted to finding the typical length of MDPs. In Sect. 4 the main results concerning the limit shape of MDPs, both with a restricted and unrestricted length growth, are proved. In fact, we obtain sharp exponential bounds for deviations from the limit shape. Section 5 describes an alternative approach to the limit shape based on a partition bijection that effectively removes the MDP-constraint. In Sect. 6, we extend our results to the case of random sequences $\mathfrak{q}$. Finally, the Appendix contains proof of the two technical propositions stated in Sect. 2, which establish the equivalence of ensembles.

2. Probability Measures on the MDP Spaces

2.1. Basic definitions and notation. In this paper, we shall use several probability measures on MDPs and other partition spaces. In the present section we describe them and

[5] A short announcement of our approach (in the case $q > 0$) appeared in [7].

establish some properties. First we introduce notation for some functionals on partitions we shall need. If one fixes a probability measure on partitions, these functionals become random variables.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ $(\lambda_1 \ge \lambda_2 \ge \dots \ge 0)$ be an integer partition, $\lambda \in \varLambda$. Recall that $N(\lambda) = \lambda_1 + \lambda_2 + \cdots$ and $K(\lambda) = \#\{\lambda_i \in \lambda \colon \lambda_i > 0\}$. An equivalent description of a partition λ can be given in terms of the consecutive differences $D_j(\lambda) = \lambda_j - \lambda_{j+1}$; obviously,

$$\lambda_i = \sum_{j \ge i} D_j(\lambda), \qquad N(\lambda) = \sum_{j \ge 1} j D_j(\lambda), \qquad K(\lambda) = \max\{j \colon D_j(\lambda) > 0\}. \tag{2.1}$$

Consider the function

$$Y_\lambda(t) := \sum_{j > t} D_j(\lambda), \qquad t \ge 0, \tag{2.2}$$

Clearly, the map $t \mapsto Y_\lambda(t)$ is non-increasing, piecewise constant, and right-continuous. From (2.1), it is also easy to see that $Y_\lambda(t) = \lambda_{\lfloor t \rfloor + 1}$ $(t \ge 0)$, with $\lfloor \cdot \rfloor$ denoting the floor function (i.e., integer part). The *Young diagram* $\varUpsilon_\lambda$ of a partition λ is defined as the closure of the planar set

$$\{(t, u) \in \mathbb{R}^2 \colon\ t \ge 0,\ 0 \le u \le Y_\lambda(t)\}.$$

That is to say, the Young diagram $\varUpsilon_\lambda$ is the union of (left- and bottom-aligned) column blocks with λ_1, λ_2, … unit squares, respectively; in particular, the function $t \mapsto Y_\lambda(t)$ defines its upper boundary (cf. Sect. 1.1). We shall often identify the Young diagram $\varUpsilon_\lambda$ with the (graph of the) function $Y_\lambda(t)$ (see Fig. 1).

The measure most important for us is the aforementioned uniform measure $\nu_n^{\mathfrak{q}}$ on the set $\varLambda_{\mathfrak{q}}(n)$:

$$\nu_n^{\mathfrak{q}}(\lambda) := \frac{1}{p_{\mathfrak{q}}(n)} \quad (\lambda \in \varLambda_{\mathfrak{q}}(n)), \qquad p_{\mathfrak{q}}(n) := \#\varLambda_{\mathfrak{q}}(n).$$

The space $\varLambda_{\mathfrak{q}}(n)$ can be further decomposed as a disjoint union of the sets $\varLambda_{\mathfrak{q}}(n, k) := \{\lambda \in \varLambda_{\mathfrak{q}}(n) \colon K(\lambda) = k\}$, and one can introduce the uniform measures on these spaces,

$$\nu_{n,k}^{\mathfrak{q}}(\lambda) := \frac{1}{p_{\mathfrak{q}}(n,k)} \quad (\lambda \in \varLambda_{\mathfrak{q}}(n,k)), \qquad p_{\mathfrak{q}}(n,k) := \#\varLambda_{\mathfrak{q}}(n,k).$$

Note that $\nu_{n,k}^{\mathfrak{q}}$ can be viewed as the measure $\nu_n^{\mathfrak{q}}$ conditioned on the event $\{K(\lambda) = k\}$; indeed, for any $\lambda \in \varLambda_{\mathfrak{q}}(n, k)$,

$$\begin{aligned}
\nu_n^{\mathfrak{q}}(\lambda \,|\, K(\lambda) = k) &= \frac{\nu_n^{\mathfrak{q}}(\lambda)}{\nu_n^{\mathfrak{q}}(K(\lambda) = k)} \\
&= \frac{1/p_{\mathfrak{q}}(n)}{\sum_{\lambda \in \varLambda_{\mathfrak{q}}(n,k)} 1/p_{\mathfrak{q}}(n)} = \frac{1}{p_{\mathfrak{q}}(n,k)} = \nu_{n,k}^{\mathfrak{q}}(\lambda).
\end{aligned}$$

This conditional measure is somewhat simpler than $\nu_n^{\mathfrak{q}}$ itself, since there exists a product expression for the Laplace generating function of $p_{\mathfrak{q}}(n, k)$ with respect to n (for any fixed k).

To establish such an expression, the following simple observation is useful. Define

$$\mathcal{D}_{\mathfrak{q}}(k) := \{(d_1, \dots, d_k) \in \mathbb{N}_0^k \colon\, d_j \ge q_{k-j},\ j = 1, \dots, k\}, \qquad k \in \mathbb{N}.$$

Then the MDP($\mathfrak{q}$) condition (1.1) implies that $\lambda \in \Lambda_{\mathfrak{q}}(\bullet, k) := \bigcup_{n\ge 0} \Lambda_{\mathfrak{q}}(n,k)$ if and only if $(D_1(\lambda), \dots, D_k(\lambda)) \in \mathcal{D}_{\mathfrak{q}}(k)$ and $D_j(\lambda) = 0$ for all $j > k$. Hence, the space $\Lambda_{\mathfrak{q}}(\bullet, k)$ is in one-to-one correspondence with the set $\mathcal{D}_{\mathfrak{q}}(k)$. Moreover, using the second of the formulas (2.1), the Laplace generating function $F_{\mathfrak{q}}(z,k)$ ($z \ge 0$) of the sequence $(p_{\mathfrak{q}}(n,k))_{n\ge 0}$ (with $k \ge 0$ fixed) is evaluated as $F_{\mathfrak{q}}(z,0) = 1$ and for $k \ge 1$

$$\begin{aligned}
F_{\mathfrak{q}}(z,k) := \sum_{n=0}^{\infty} p_{\mathfrak{q}}(n,k)\,\mathrm{e}^{-zn} &= \sum_{n=0}^{\infty} \sum_{\lambda\in\Lambda_{\mathfrak{q}}(\bullet,k)} \mathbb{1}_{\{N(\lambda)=n\}}\,\mathrm{e}^{-zN(\lambda)} \\
&= \sum_{\lambda\in\Lambda_{\mathfrak{q}}(\bullet,k)} \mathrm{e}^{-zN(\lambda)} = \prod_{j=1}^{k} \sum_{d_j=q_{k-j}}^{\infty} \mathrm{e}^{-zj\,d_j} \\
&= \prod_{j=1}^{k} \frac{\mathrm{e}^{-zj\,q_{k-j}}}{1-\mathrm{e}^{-zj}} = \frac{\mathrm{e}^{-zs_k}}{(1-\mathrm{e}^{-z})\cdots(1-\mathrm{e}^{-zk})}\,,
\end{aligned} \tag{2.3}$$

where we set

$$s_k := \sum_{j=1}^{k} j\,q_{k-j} \equiv \sum_{i=1}^{k} Q_i, \qquad k \in \mathbb{N}, \tag{2.4}$$

with Q_i defined in (1.2). In particular, $s_k \ge k$ for all $k \ge 1$ (because $Q_i \ge q_0 \ge 1$, see (1.2)); moreover, the asymptotic condition (1.2) implies that, for $q \ge 0$,

$$s_k = \frac{qk^2}{2} + O(k^{\beta+1}) \qquad (k \to \infty). \tag{2.5}$$

Remark 2.1. The product structure of $F_{\mathfrak{q}}(z,k)$ revealed in (2.3) is similar to that of multiplicative measures introduced by Vershik [39]. However, there are some distinctions from multiplicative measures. Firstly, the partition length $K(\lambda)$ must be fixed to obtain independence. Secondly, the role of the part counts which become independent after randomization of $N(\lambda) = n$ is played here by the differences $D_j(\lambda)$.

Let us define an auxiliary probability measure $\mu_{z,k}^{\mathfrak{q}}$ on the space $\Lambda_{\mathfrak{q}}(\bullet,k)$ (parameterized by $z > 0$) by setting

$$\mu_{z,k}^{\mathfrak{q}}(\lambda) := \frac{\mathrm{e}^{-zN(\lambda)}}{F_{\mathfrak{q}}(z,k)}, \qquad \lambda \in \Lambda_{\mathfrak{q}}(\bullet,k). \tag{2.6}$$

Note that, for every $z > 0$, the measure $\mu_{z,k}^{\mathfrak{q}}$ conditioned on the event $\{N(\lambda) = n\}$ coincides with the uniform measure $\nu_{n,k}^{\mathfrak{q}}$ on the space $\Lambda_{\mathfrak{q}}(n,k)$; indeed, according to (2.6) we have, for any $\lambda \in \Lambda_{\mathfrak{q}}(n,k)$,

$$\mu_{z,k}^{\mathfrak{q}}(\lambda \,|\, N(\lambda) = n) = \frac{\mu_{z,k}^{\mathfrak{q}}(\lambda)}{\mu_{z,k}^{\mathfrak{q}}\{N(\lambda) = n\}}$$

$$= \frac{\mathrm{e}^{-zn}/F_{\mathfrak{q}}(z,k)}{\sum_{\lambda\in\Lambda_{\mathfrak{q}}(n,k)}\mathrm{e}^{-zn}/F_{\mathfrak{q}}(z,k)} = \frac{1}{\#\Lambda_{\mathfrak{q}}(n,k)} = \nu^{\mathfrak{q}}_{n,k}(\lambda). \tag{2.7}$$

The following fact will be instrumental below.

Lemma 2.1. *Under the measure $\mu^{\mathfrak{q}}_{z,k}$, the differences $(D_j(\lambda))_{j=1}^k$ are independent random variables such that the marginal distribution of $D_j(\lambda) - q_{k-j} \in \mathbb{N}_0$ is geometric with parameter $1-\mathrm{e}^{-zj}$ $(j = 1,\dots,k)$; that is, for any $(d_1,\dots,d_k) \in \mathcal{D}_{\mathfrak{q}}(k)$,*

$$\mu^{\mathfrak{q}}_{z,k}\{\lambda \in \Lambda_{\mathfrak{q}}(\bullet,k)\colon D_j(\lambda) = d_j,\ j = 1,\dots,k\} = \prod_{j=1}^{k}(1-\mathrm{e}^{-zj})\,\mathrm{e}^{-zj(d_j-q_{k-j})}.$$

In particular, the expected values are given by

$$\mathbf{E}^{\mathfrak{q}}_{z,k}[D_j(\lambda)] = q_{k-j} + \frac{\mathrm{e}^{-zj}}{1-\mathrm{e}^{-zj}} \qquad (j = 1,\dots,k). \tag{2.8}$$

Proof. The claim easily follows from the representation of $N(\lambda)$ through $(D_j(\lambda))$ (see (2.1)) and the product structure of the Laplace generating function (2.3). □

Similarly, we can assign the weight $\mathrm{e}^{-zN(\lambda)}$ to each partition $\lambda \in \Lambda_{\mathfrak{q}} = \bigcup_{k=0}^{\infty}\Lambda_{\mathfrak{q}}(\bullet,k)$ normalized by

$$F_{\mathfrak{q}}(z) := \sum_{\lambda\in\Lambda_{\mathfrak{q}}}\mathrm{e}^{-zN(\lambda)} = 1 + \sum_{k=1}^{\infty}F_{\mathfrak{q}}(z,k) \tag{2.9}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{\mathrm{e}^{-zs_k}}{(1-\mathrm{e}^{-z})\cdots(1-\mathrm{e}^{-zk})}\,. \tag{2.10}$$

Note that the series (2.10) converges for all $z > 0$, since it is bounded by the convergent series $\sum_k \mathrm{e}^{-zk}(1-\mathrm{e}^{-z})^{-1}\cdots(1-\mathrm{e}^{-zk})^{-1} = \prod_j(1-\mathrm{e}^{-zj})^{-1}$. This way, we get the probability measure

$$\mu^{\mathfrak{q}}_{z}(\lambda) := \frac{\mathrm{e}^{-zN(\lambda)}}{F_{\mathfrak{q}}(z)}, \qquad \lambda \in \Lambda_{\mathfrak{q}}. \tag{2.11}$$

Similarly to (2.7), it is easy to check that the measure $\mu^{\mathfrak{q}}_z$ conditioned on $\{N(\lambda) = n\}$ coincides with the uniform measure $\nu^{\mathfrak{q}}_n$ on $\Lambda_{\mathfrak{q}}(n)$,

$$\mu^{\mathfrak{q}}_z(\lambda\,|\,N(\lambda) = n) = \frac{1}{p_{\mathfrak{q}}(n)} = \nu^{\mathfrak{q}}_n(\lambda), \qquad \lambda \in \Lambda_{\mathfrak{q}}(n).$$

Furthermore, the definition (2.11) implies

$$\mu^{\mathfrak{q}}_z\{\lambda \in \Lambda_{\mathfrak{q}}\colon K(\lambda) = k\} = \frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z)}, \qquad k \in \mathbb{N}. \tag{2.12}$$

We finish this subsection by a comment linking the above MDP spaces and probability measures on them with the general nomenclature of ensembles in statistical mechanics (see, e.g., the monographs by Huang [21] or Greiner et al. [18]). Under the quantum interpretation of integer partitions $\lambda = (\lambda_i) \in \Lambda$ briefly mentioned in Sect. 1.1, the

$$\begin{tikzcd}
\boxed{\begin{array}{c}\textit{Micro-canonical:}\\ \nu^{\mathfrak{q}}_{n,k} \text{ on } \Lambda_{\mathfrak{q}}(n,k)\end{array}} \arrow[r, "+\text{heat bath}"] \arrow[d, "+\text{particle bath}"'] & \boxed{\begin{array}{c}\textit{Canonical:}\\ \mu^{\mathfrak{q}}_{z,k} \text{ on } \Lambda_{\mathfrak{q}}(\bullet,k)\end{array}} \arrow[d, "+\text{particle bath}"] \\
\boxed{\begin{array}{c}\textit{Meso-canonical:}\\ \nu^{\mathfrak{q}}_{n} \text{ on } \Lambda_{\mathfrak{q}}(n)\end{array}} \arrow[r, "+\text{heat bath}"] & \boxed{\begin{array}{c}\textit{Grand-canonical:}\\ \mu^{\mathfrak{q}}_{z} \text{ on } \Lambda_{\mathfrak{q}}\end{array}}
\end{tikzcd}$$

Fig. 3. Schematic diagram illustrating the relation between different MDP-ensembles. The integer parameters n and k are interpreted as the total energy of the (quantum) system and the number of particles, respectively. The arrows "heat bath" and "particle bath" indicate that fixation of energy or the number of particles is lifted

MDP($\mathfrak{q}$) restriction determines the exclusion rules for permissible energy levels (λ_i). In general, the weight $N(\lambda)$ (total energy) and length $K(\lambda)$ (number of particles) are random. Fixing one or both of these parameters leads to different measures on the corresponding spaces, and therefore determines different ensembles. In particular, a completely isolated system, with fixed $N(\lambda) = n$ and $K(\lambda) = k$ and under uniform measure $\nu^{\mathfrak{q}}_{n,k}$ on the corresponding space $\Lambda_{\mathfrak{q}}(n,k)$, has the meaning of *micro-canonical* MDP ensemble. When, say, the fixation $N(\lambda) = n$ is lifted (which may be thought of as connecting the system to a *heat bath*, whereby thermal equilibrium is settled through exchange of energy with the bath), we get an enlarged space $\Lambda(\bullet, k)$ with the measure $\mu^{\mathfrak{q}}_{z,k}$, which is interpreted as the *canonical* ensemble, with a fixed number of particles k. Furthermore, removing the latter constraint (which, similarly, is achieved by putting the canonical ensemble into a *particle bath* allowing free exchange of particles) leads to the space $\Lambda_{\mathfrak{q}}$ with the measure $\mu^{\mathfrak{q}}_{z}$, which is referred to as the *grand canonical* ensemble (see the schematic diagram in Fig. 3).

Note however that the space $\Lambda_{\mathfrak{q}}(n)$ (i.e., with a fixed energy $N(\lambda) = n$ and endowed with uniform measure $\nu^{\mathfrak{q}}_{n}$), which is most natural from the combinatorial point of view, is missing in this picture; indeed, it may not be physically meaningful to talk about systems with fixed energy and free number of particles. But logically, it is perfectly possible to interchange the order of relaxations described above and first lift the condition $K(\lambda) = k$ by connecting the micro-canonical system to a particle bath; we take the liberty to call the resulting ensemble *meso-canonical*,[6] indicating an intermediately coarse partitioning of the phase space (cf. [14]). Finally, removing the remaining constraint $N(\lambda) = n$ (by connecting the system further to a heat bath) we again obtain the grand canonical ensemble.

2.2. Asymptotic equivalence of ensembles. For $q \ge 0$, define the function

$$\vartheta_q(t) := \sqrt{\tfrac{1}{2} q t^2 + \mathrm{Li}_2(1 - \mathrm{e}^{-t})}, \qquad t > 0, \tag{2.13}$$

where $\mathrm{Li}_2(\cdot)$ is the dilogarithm (see (1.8)). Recall that $T_q > 0$ is the unique solution of the equation (cf. (1.6))

$$\mathrm{e}^{-qT_q} = 1 - \mathrm{e}^{-T_q}. \tag{2.14}$$

Note that the value $\vartheta_q(T_q)$ coincides with the notation ϑ_q introduced in (1.11).

The following curious identity will be explained in Sect. 4.4.

[6] This is just a placeholder in lieu of an established physical term.

Lemma 2.2. *For all $q > 0$, we have*

$$T_{q^{-1}} = qT_q. \tag{2.15}$$

Proof. Rewriting Eq. (2.14) in the form $\mathrm{e}^{-q^{-1}(qT_q)} = 1 - \mathrm{e}^{-qT_q}$, we see that $\tau = qT_q$ satisfies (2.14) with q replaced by q^{-1}. By uniqueness, this implies (2.15). $\square$

The next proposition establishes an asymptotic link between the measures $\mu_z^{\mathfrak{q}}$ and $\nu_n^{\mathfrak{q}}$.

Proposition 2.3. *Suppose that the sequence $\mathfrak{q}$ satisfies the condition (1.2). Let $\{A_z\}_{z>0}$ be a family of subsets of the space $\Lambda_{\mathfrak{q}}$ such that, for some positive constant κ,*

$$\limsup_{z\downarrow 0} z^{\kappa} \log \mu_z^{\mathfrak{q}}(A_z) < 0. \tag{2.16}$$

Then there exists a sequence (z_n) such that

$$\lim_{n\to\infty} z_n\sqrt{n} = \vartheta_q \equiv \vartheta_q(T_q) \tag{2.17}$$

and

$$\limsup_{n\to\infty} n^{-\kappa/2} \log \nu_n^{\mathfrak{q}}(A_{z_n}) < 0. \tag{2.18}$$

There is a similar connection between the measures $\mu_{z,k}^{\mathfrak{q}}$ and $\nu_{n,k}^{\mathfrak{q}}$, provided that $z \downarrow 0$, $k \to \infty$ and $n \to \infty$ in a coordinated manner.

Proposition 2.4. *Let the sequence $\mathfrak{q} = (q_i)$ satisfy Assumption 1.1. Let a family of sets $A_{z,k} \subset \Lambda_{\mathfrak{q}}(\bullet, k)$ $(z > 0, k \in \mathbb{N})$ be such that, for some constant $\kappa > 0$,*

$$\limsup_{z\downarrow 0} z^{\kappa} \log \mu_{z,k(z)}^{\mathfrak{q}}(A_{z,k(z)}) < 0, \tag{2.19}$$

for any $k = k(z)$ such that $zk(z) \to T \in (0, \infty]$ as $z \downarrow 0$.

(a) *If $T < \infty$ then for any sequence (k_n) satisfying*

$$\lim_{n\to\infty} \frac{k_n}{\sqrt{n}} = \frac{T}{\vartheta_q(T)}, \tag{2.20}$$

there exists a sequence (z_n) such that

$$\lim_{n\to\infty} z_n\sqrt{n} = \vartheta_q(T) \tag{2.21}$$

and

$$\limsup_{n\to\infty} n^{-\kappa/2} \log \nu_{n,k_n}^{\mathfrak{q}}(A_{z_n,k_n}) < 0. \tag{2.22}$$

(b) *Let $T = \infty$ and $q = 0$, and assume in addition that $z^{2/(\beta+1)}k(z) \to 0$ as $z \downarrow 0$. Then for any sequence (k_n) satisfying*

$$\lim_{n\to\infty} \frac{k_n}{k(\pi/\sqrt{6n})} = 1, \tag{2.23}$$

there exists a sequence (z_n) such that the asymptotic relations (2.21) and (2.22) hold true, with the right-hand side of (2.21) reducing to $\vartheta_0(\infty) \equiv \vartheta_0 = \pi/\sqrt{6}$ (see (1.9)).

These two propositions are instrumental for our method; their proof, being rather technical, is postponed until Appendix A.

3. Number of Parts in a Typical MDP

In this section, our ultimate goal is to show that if Assumption 1.1 holds then, under the measures $\nu_n^{\mathfrak{q}}$ on the MDP-space $\Lambda_{\mathfrak{q}}(n)$, the typical length $K(\lambda)$ (i.e., the number of parts) of a partition $\lambda \in \Lambda_{\mathfrak{q}}(n)$ of large weight $N(\lambda) = n$ is concentrated around $c\sqrt{n}$ (with a suitable constant $c > 0$) if $q > 0$, or grows slightly faster than $\sqrt{n}$ if $q = 0$. To this end, we will first study the distribution of $K(\lambda)$ under the measure $\mu_z^{\mathfrak{q}}$ in the space $\Lambda_{\mathfrak{q}}$.

3.1. Preparatory lemmas. For $z > 0$, denote

$$\eta_k(z) := \frac{\mathrm{e}^{-zQ_k}}{1 - \mathrm{e}^{-zk}}, \qquad k \in \mathbb{N}, \tag{3.1}$$

where Q_k is given by (1.2). For every $z > 0$, the sequence $(\eta_k(z))_{k \ge 1}$ is decreasing, and in particular

$$\eta_k(z) \le \eta_1(z) = \frac{\mathrm{e}^{-zq_0}}{1 - \mathrm{e}^{-z}}, \qquad k \in \mathbb{N}.$$

Furthermore,

$$0 \le \lim_{k \to \infty} \eta_k(z) \le \mathrm{e}^{-zq_0} < 1.$$

Thus, the set $\{k \colon \eta_k(z) \ge 1\}$ is always finite (possibly empty). Define

$$k_* \equiv k_*(z) := \begin{cases} \max\{k \in \mathbb{N} \colon \eta_k(z) \ge 1\} & \text{if } \eta_1(z) \ge 1, \\ 1 & \text{if } \eta_1(z) < 1. \end{cases} \tag{3.2}$$

Remark 3.1. Note that $\lim_{z \downarrow 0} \eta_k(z) = +\infty$ for any fixed $k \in \mathbb{N}$, and so $k_*(z) > 1$ for all $z > 0$ small enough.

First, let us record a few auxiliary statements that do not depend on Assumption 1.1.

Lemma 3.1. (a) *For every $z > 0$, we have*

$$\max_{k \in \mathbb{N}} \mu_z^{\mathfrak{q}}\{K(\lambda) = k\} = \mu_z^{\mathfrak{q}}\{K(\lambda) = k_*\},$$

where $k_* = k_*(z)$ is defined in (3.2). Moreover,

$$\mu_z^{\mathfrak{q}}\{K(\lambda) = k_*\} > \mu_z^{\mathfrak{q}}\{K(\lambda) = k_* + 1\} > \mu_z^{\mathfrak{q}}\{K(\lambda) = k_* + 2\} > \cdots, \tag{3.3}$$

and, for $k_* \ge 2$,

$$\mu_z^{\mathfrak{q}}\{K(\lambda) = k_*\} \ge \mu_z^{\mathfrak{q}}\{K(\lambda) = k_* - 1\} > \cdots > \mu_z^{\mathfrak{q}}\{K(\lambda) = 1\}. \tag{3.4}$$

(b) *The function $z \mapsto k_*(z)$ is non-increasing and has no jumps larger than* 1. *Moreover, $k_*(z) \to +\infty$ as $z \downarrow 0$.*

Proof. (a) Using (2.3), (2.4) and (2.12), we can rewrite (3.1) (for $k \ge 2$) as

$$\eta_k(z) = \frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z,k-1)} = \frac{\mu_z^{\mathfrak{q}}\{K(\lambda)=k\}}{\mu_z^{\mathfrak{q}}\{K(\lambda)=k-1\}}. \tag{3.5}$$

The definition of $k_* = k_*(z)$ (see (3.2)) implies that $\eta_k(z) < 1$ for $k > k_*$, and (3.3) follows. Similarly, assuming that $k_* \ge 2$, we have $\eta_{k_*}(z) \ge 1$ and $\eta_k(z) > 1$ for $k < k_*$, which is the same as (3.4).

(b) For $k \in \mathbb{N}$, let $z = \zeta_k$ be the (unique) solution of the equation

$$\eta_k(z) = 1. \tag{3.6}$$

From the formulas (3.1) and (3.6), it is clear that the sequence $(\zeta_k)_{k\ge 1}$ is decreasing and, moreover, $\zeta_k \downarrow 0$ as $k \to \infty$. If $z = \zeta_k$ $(k \ge 2)$ then $\mu_{\zeta_k}^{\mathfrak{q}}\{K(\lambda)=k\} = \mu_{\zeta_k}^{\mathfrak{q}}\{K(\lambda)=k-1\}$ are the two maxima of the sequence $(\mu_z^{\mathfrak{q}}\{K(\lambda)=j\})_{j\ge 1}$, whereas for $z \in (\zeta_{k+1},\zeta_k)$ the unique maximum of this sequence is attained exactly at $j = k$. Hence, $k_*(z) \equiv k$ for $z \in (\zeta_{k+1},\zeta_k]$, that is, $z \mapsto k_*(z)$ is a non-increasing (left-continuous) step function with unit downward jumps at points ζ_k $(k \ge 2)$. Since $\lim_{k\to\infty}\zeta_k = 0$, it also follows that $\lim_{z\downarrow 0} k_*(z) = +\infty$. $\square$

Remark 3.2. Willing to use a "one-sided" version of the notation $f(z) = O(g(z))$ $(z \downarrow 0)$, in what follows we write $f(z) \le O(g(z))$ $(z \downarrow 0)$ if $\limsup_{z\downarrow 0} f(z)/g(z) < +\infty$.

Lemma 3.2. *Uniformly in* $k \in \mathbb{N}$, *as* $z \downarrow 0$,

$$\log \mu_z^{\mathfrak{q}}\{K(\lambda)=k\} \le z^{-1}\bigl(\mathrm{Li}_2(\mathrm{e}^{-zk_*}) - \mathrm{Li}_2(\mathrm{e}^{-zk})\bigr) + z(s_{k_*} - s_k) + O\bigl(\log \tfrac{1}{z}\bigr) \tag{3.7}$$

$$\le (k_* - k)\log(1-\mathrm{e}^{-zk_*}) + z(s_{k_*} - s_k) + O\bigl(\log \tfrac{1}{z}\bigr). \tag{3.8}$$

Proof. Recalling (2.12), for each $k \in \mathbb{N}$ we can write (see (2.9))

$$\log \mu_z^{\mathfrak{q}}\{K(\lambda)=k\} = \log \frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z)} \le \log F_{\mathfrak{q}}(z,k) - \log F_{\mathfrak{q}}(z,k_*), \tag{3.9}$$

where, according to (2.3),

$$\log F_{\mathfrak{q}}(z,k) = -zs_k - \sum_{j=1}^{k} \log(1-\mathrm{e}^{-zj}) \qquad (k \in \mathbb{N}). \tag{3.10}$$

By the well-known Euler–Maclaurin sum formula [1, 23.1.36, p. 806] applied to the function $x \mapsto \log(1-\mathrm{e}^{-zx})$, we get, uniformly in $k \in \mathbb{N}$ as $z \downarrow 0$,

$$\begin{aligned}\sum_{j=1}^{k}\log(1-\mathrm{e}^{-zj}) &= \int_1^k \log(1-\mathrm{e}^{-zx})\,\mathrm{d}x + O(1)\log(1-\mathrm{e}^{-z})\\ &\quad + O(1)\int_1^k \frac{z\,\mathrm{e}^{-zx}}{1-\mathrm{e}^{-zx}}\,\mathrm{d}x\\ &= z^{-1}\bigl(\mathrm{Li}_2(\mathrm{e}^{-zk}) - \mathrm{Li}_2(\mathrm{e}^{-z})\bigr) + O\bigl(\log \tfrac{1}{z}\bigr),\end{aligned} \tag{3.11}$$

where $\mathrm{Li}_2(\cdot)$ is the dilogarithm function (see (1.8)). Thus, substituting (3.11) into (3.10) and returning to (3.9), we obtain (3.7).

Furthermore, since the derivative $(\mathrm{Li}_2(\mathrm{e}^{-t}))' = \log(1-\mathrm{e}^{-t})$ is increasing in $t \in (0,\infty)$, the function $t \mapsto \mathrm{Li}_2(\mathrm{e}^{-t})$ is convex, hence

$$\mathrm{Li}_2(\mathrm{e}^{-zk_*}) - \mathrm{Li}_2(\mathrm{e}^{-zk}) \le z\,(k_* - k)\log(1-\mathrm{e}^{-zk_*}), \qquad k \in \mathbb{N}.$$

Combining this bound with (3.7) yields (3.8). $\square$

Lemma 3.3. *Suppose that Assumption 1.1 is in force, that is, the sequence* $\mathfrak{q} = (q_i)$ *satisfies* (1.2) *with* $q \ge 0$ *and* $0 \le \beta < 1$.

(a) *If* $q > 0$ *then*

$$k_*(z) = z^{-1}T_q + O(z^{-\beta}) \qquad (z \downarrow 0), \tag{3.12}$$

where T_q *is defined in* (2.14).

(b) *If* $q = 0$ *then*

$$1-\beta \le \liminf_{z\downarrow 0} \frac{zk_*(z)}{\log\frac{1}{z}} \le \limsup_{z\downarrow 0} \frac{zk_*(z)}{\log\frac{1}{z}} \le 1. \tag{3.13}$$

In particular, for all $q \ge 0$,

$$\lim_{z\downarrow 0} zk_*(z) = T_q. \tag{3.14}$$

Proof. (a) Like in the proof of Lemma 3.1(b), denote by ζ_k ($k \in \mathbb{N}$) the solution of the equation (3.6). Using the definition (3.1), Eq. (3.6) is expressed at $z = \zeta_k$ as

$$k^{-1}Q_k = -(k\zeta_k)^{-1}\log(1-\mathrm{e}^{-k\zeta_k}). \tag{3.15}$$

Comparing this with Eq. (1.6), observe that $k\zeta_k = T_{\tilde{q}_k}$, where $\tilde{q}_k := k^{-1}Q_k \to q > 0$ as $k \to \infty$, due to the limit (1.3), and therefore $\lim_{k\to\infty} T_{\tilde{q}_k} = T_q$, thanks to continuity of the mapping $q \mapsto T_q$.

To see why this implies (3.12), recall from the proof of Lemma 3.1(b) that $k_*(z) \equiv k$ for $z \in (\zeta_{k+1}, \zeta_k]$ ($k \in \mathbb{N}$) and the limit $z \downarrow 0$ is equivalent to $k \to \infty$. Hence,

$$k_*z = k\zeta_k - k(\zeta_k - z) \to T_q \qquad (z \downarrow 0),$$

because $k\zeta_k \to T_q$ and

$$0 \le k(\zeta_k - z) \le k\zeta_k - k\zeta_{k+1} \to 0 \qquad (k \to \infty).$$

Furthermore, by a standard perturbation analysis it is easy to estimate the corresponding remainder term in the limit (3.12). Indeed, setting $\delta_k := k\zeta_k - T_q \to 0$ and using the asymptotic relation (1.2), we can rewrite (3.15) in the form

$$(T_q + \delta_k)\Bigl(q + O(k^{\beta-1})\Bigr) = -\log\bigl(1-\mathrm{e}^{-T_q}\bigr) - \frac{\mathrm{e}^{-T_q}}{1-\mathrm{e}^{-T_q}}\,\delta_k + O(\delta_k^2),$$

which yields, in view of the identity (2.14), that $\delta_k = O(k^{\beta-1})$.

In turn, for $\zeta_{k+1} < z \le \zeta_k$ we get

$$\begin{aligned} k_*z - T_q &= (k\zeta_k - T_q) - k(\zeta_k - z) \\ &= \delta_k + O(1)\,(\delta_k + \delta_{k+1}) \\ &= O(k^{\beta-1}) = O(z^{1-\beta}) \qquad (z \downarrow 0), \end{aligned}$$

and the estimate (3.12) follows.

(b) Fix $\varepsilon \in (0, 1-\beta)$. For $z > 0$ small enough, $\eta_{k_*}(z) = \mathrm{e}^{-zQ_{k_*}}(1-\mathrm{e}^{-zk_*})^{-1} \ge 1$ by the definition (3.2), so

$$\mathrm{e}^{-zk_*} \ge 1 - \mathrm{e}^{-zQ_{k_*}} \ge 1 - \mathrm{e}^{-z} \ge z^{1+\varepsilon},$$

because $Q_{k_*} \ge q_0 \ge 1$. Thus,

$$zk_*(z) \le (1+\varepsilon)\log\tfrac{1}{z}, \tag{3.16}$$

which implies the last inequality in (3.13), since $\varepsilon > 0$ can be taken arbitrarily close to 0.

On the other hand, from (2.14) we also have $\eta_{k_*+1}(z) < 1$, that is,

$$zk_*(z) > \log\tfrac{1}{z} - z - \log Q_{k_*+1}. \tag{3.17}$$

Furthermore, using the asymptotic bound (1.2) for $k = k_*$ (with $q = 0$) and the estimate (3.16), we obtain

$$\log Q_{k_*+1} = O(1) + \beta\log\tfrac{1}{z} + \beta\log\log\tfrac{1}{z} \qquad (z \downarrow 0).$$

Substituting this into (3.17), it is easy to see that

$$\liminf_{z\downarrow 0}\frac{zk_*(z)}{\log\frac{1}{z}} \ge 1 - \lim_{z\downarrow 0}\frac{z}{\log\frac{1}{z}} - \lim_{z\downarrow 0}\frac{\log Q_{k_*+1}}{\log\frac{1}{z}} = 1-\beta,$$

and the first inequality in (3.13) is proved. $\square$

Remark 3.3. In the case $q = 0$, the asymptotic bounds in (3.13) are optimal in the following sense: under Assumption 1.2 (i.e., when $Q_k \sim \tilde{q}k^\beta$ as $k \to \infty$), one can show that $\lim_{z\downarrow 0} z\,(\log\frac{1}{z})^{-1}k_*(z) = 1-\beta > 0$.

3.2. Asymptotics of $K(\lambda)$ in the space $\varLambda_{\mathfrak{q}}$: case $q > 0$. We can now give exponential estimates on the asymptotic behaviour of the random variable $K = K(\lambda)$ (see (2.1)) under the measure $\mu_z^{\mathfrak{q}}$. We start with the case $q > 0$.

Theorem 3.4. *Let the sequence $\mathfrak{q} = (q_i)$ satisfy Assumption 1.1 with $q > 0$ and $0 \le \beta < 1$. Then, for every $\gamma \in \bigl(0, \frac{1}{2}(1-\beta)\bigr)$ and any constant $c > 0$, we have*

$$\limsup_{z\downarrow 0} z^{1-2\gamma}\log\mu_z^{\mathfrak{q}}\bigl\{\lambda \in \varLambda_{\mathfrak{q}}\colon |K(\lambda) - k_*| > cz^{\gamma-1}\bigr\} \le -\tfrac{1}{2}qc^2 < 0, \tag{3.18}$$

where $k_* = k_*(z)$ is defined in (3.2).

Proof. From (2.12) we have

$$\mu_z^{\mathfrak{q}}\bigl\{|K(\lambda) - k_*| > cz^{\gamma-1}\bigr\} = \frac{1}{F_{\mathfrak{q}}(z)}\sum_{k\in\mathcal{I}_z} F_{\mathfrak{q}}(z,k), \tag{3.19}$$

where $\mathcal{I}_z := \{k \in \mathbb{N}\colon |k - k_*| > cz^{\gamma-1}\}$. Recalling (3.1) and (3.5), observe that for $k > 2k_*$

$$\frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z,k-1)} = \eta_k(z) \le \eta_{2k_*}(z) = \frac{\mathrm{e}^{-zQ_{2k_*}}}{1-\mathrm{e}^{-2zk_*}}. \tag{3.20}$$

By the asymptotic formulas (1.2) and (3.14), this gives

$$\begin{aligned}\limsup_{z\downarrow 0}\log\frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z,k-1)} &\le -\lim_{z\downarrow 0} z\,Q_{2k_*} - \lim_{z\downarrow 0}\log\left(1-\mathrm{e}^{-2zk_*}\right)\\ &= -2q\,T_q - \log\left(1-\mathrm{e}^{-2T_q}\right)\\ &< -2q\,T_q - \log\left(1-\mathrm{e}^{-T_q}\right) = -q\,T_q < 0,\end{aligned}\tag{3.21}$$

where the last equality in (3.21) is due to Eq. (2.14). Hence, the part of the sum (3.19) with $k > 2k_*$ is asymptotically dominated by a geometric series with ratio $\mathrm{e}^{-qT_q} < 1$, so that

$$\frac{1}{F_{\mathfrak{q}}(z)}\sum_{k>2k_*} F_{\mathfrak{q}}(z,k) \le \frac{F_{\mathfrak{q}}(z,2k_*)}{F_{\mathfrak{q}}(z)}\cdot\frac{\mathrm{e}^{-qT_q}}{1-\mathrm{e}^{-qT_q}}.\tag{3.22}$$

Furthermore, with the help of the asymptotic relations (2.5) and (3.12) and in view of Eq. (2.14), the estimate (3.8) specializes as follows

$$\begin{aligned}\log\frac{F_{\mathfrak{q}}(z,2k_*)}{F_{\mathfrak{q}}(z)} &\le -z^{-1}\left\{T_q\log\left(1-\mathrm{e}^{-T_q}\right)+\tfrac{3}{2}q\,T_q^2+O(z^{1-\beta})\right\}+O\left(\log\tfrac{1}{z}\right)\\ &= -\tfrac{1}{2}q\,T_q^2 z^{-1}+O(z^{-\beta}) \qquad (z\downarrow 0).\end{aligned}\tag{3.23}$$

Let us now turn to the case $k \le 2k_*$. Denote $k_- := \lfloor k_* - cz^{\gamma-1}\rfloor$, $k_+ := \lceil k_* + cz^{\gamma-1}\rceil$ (here and in what follows, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling functions, respectively). Observe from Lemma 3.3(a) that $k_\pm \to \infty$ as $z \downarrow 0$ and $k_- < k_* < k_+ < 2k_*$. Hence, the monotonicity properties (3.3) and (3.4) yield

$$\begin{aligned}\frac{1}{F_{\mathfrak{q}}(z)}\sum_{k\in\mathcal{I}_z,\,k\le 2k_*} F_{\mathfrak{q}}(z,k) &\le \frac{2k_*}{F_{\mathfrak{q}}(z)}\max_{k\in\mathcal{I}_z,\,k\le 2k_*} F_{\mathfrak{q}}(z,k)\\ &\le 2k_*\,\frac{F_{\mathfrak{q}}(z,k_-)+F_{\mathfrak{q}}(z,k_+)}{F_{\mathfrak{q}}(z)}.\end{aligned}\tag{3.24}$$

Similarly to (3.25), from (3.8) we obtain

$$\begin{aligned}\log\frac{F_{\mathfrak{q}}(z,k_\pm)}{F_{\mathfrak{q}}(z)} &\le \mp cz^{\gamma-1}\left\{\log\left(1-\mathrm{e}^{-T_q}\right)+q\,T_q\right\}-\tfrac{1}{2}q\,c^2z^{2\gamma-1}+O(z^{\gamma-\beta})+O\left(\log\tfrac{1}{z}\right)\\ &= -\tfrac{1}{2}q\,c^2z^{2\gamma-1}+O(z^{\gamma-\beta})+O\left(\log\tfrac{1}{z}\right) \qquad (z\downarrow 0),\end{aligned}\tag{3.25}$$

again by making use of Eq. (2.14).

Finally, returning to the expansion (3.19) and combining the estimates (3.24), (3.25), (3.24) and (3.25), with the help of the elementary inequality

$$\log(x+y) \le \log 2 + \max\{\log x, \log y\}, \qquad x, y > 0,\tag{3.26}$$

we obtain (3.18), which completes the proof. □

Theorem 3.4 combined with the asymptotic formula (3.12) implies the following law of large numbers for the number of parts $K(\lambda)$ under the measure $\mu_z^{\mathfrak{q}}$.

Corollary 3.5. *Let Assumption 1.1 hold with $q > 0$. Then, for any $\varepsilon > 0$,*

$$\lim_{z\downarrow 0}\mu_z^{\mathfrak{q}}\left\{\lambda\in\Lambda_{\mathfrak{q}}\colon |zK(\lambda)-T_q|>\varepsilon\right\}=0.$$

3.3. Asymptotics of $K(\lambda)$ in the space $\Lambda_{\mathfrak{q}}$: case $q = 0$. When Assumption 1.1 holds with $q = 0$, the asymptotics for $k_*(z)$ as $z \downarrow 0$ cannot be obtained, as was mentioned in Remark 3.3. So there is no hope to find exponential bounds for $K(\lambda)$ to fit into an interval of order smaller than z^{-1}, as in (3.18). Nevertheless we can still find an interval such that $K(\lambda)$ does not hit it with an exponentially small $\mu_z^{\mathfrak{q}}$-probability, as $z \downarrow 0$. To this end, we need some additional notation.

Fix $\gamma \in (0, 1)$ and define the function

$$z \mapsto k_\gamma \equiv k_\gamma(z) := \inf\{k \in \mathbb{N}: s_k \geq z^{-2(1-\gamma)}\}, \qquad z \in (0, 1). \tag{3.27}$$

Recalling that $s_k \geq k$ (see after formula (2.4)), from the definition (3.27) it follows that

$$k_\gamma(z) \leq \lceil z^{-2(1-\gamma)} \rceil. \tag{3.28}$$

On the other hand, it is clear that $k_\gamma(z) \to \infty$ as $z \downarrow 0$. Actually we can tell more.

Lemma 3.6. *Let Assumption 1.1 hold with $q = 0$ and some $\beta \in [0, 1)$. Then, for any $\gamma \in (0, 1)$,*

$$\lim_{z\downarrow 0} z^{2(1-\gamma)} s_{k_\gamma(z)} = 1, \tag{3.29}$$

$$\liminf_{z\downarrow 0} z^{2(1-\gamma)/(\beta+1)} k_\gamma(z) > 0. \tag{3.30}$$

Moreover, if $0 < \gamma < \frac{1}{2}$ then for any $t > 0$

$$\limsup_{z\downarrow 0} z^{1-2\gamma} Q_{k_\gamma(z)-\lfloor t/z \rfloor} \leq t^{-1}. \tag{3.31}$$

Proof. The definition (3.27) implies that $s_{k_\gamma - 1} < z^{-2(1-\gamma)} \leq s_{k_\gamma}$. Hence, recalling notation (2.4) and combining the asymptotics (1.2) (with $q = 0$) and the bound (3.31), we have

$$\begin{aligned} z^{-2(1-\gamma)} \leq s_{k_\gamma} &= s_{k_\gamma - 1} + Q_{k_\gamma} < z^{-2(1-\gamma)} + Q_{k_\gamma} \\ &= z^{-2(1-\gamma)} + O(z^{-2\beta(1-\gamma)}) \sim z^{-2(1-\gamma)}, \end{aligned} \tag{3.32}$$

since $\beta < 1$ and $1 - \gamma > 0$. Now, the limit (3.29) follows from the two-sided estimate (3.32). Similarly, using (2.5) (with $q = 0$), we obtain the asymptotic bound

$$z^{-2(1-\gamma)} \leq s_{k_\gamma} = O(k_\gamma^{\beta+1}) \qquad (z \downarrow 0),$$

which implies (3.30). Finally, since the sequence (Q_k) is non-decreasing (see (1.2)), for $t > 0$ we can write

$$s_{k_\gamma} \geq \sum_{k=k_\gamma - \lfloor t/z \rfloor}^{k_\gamma} Q_k \geq \lfloor t/z \rfloor \cdot Q_{k_\gamma - \lfloor t/z \rfloor},$$

and the claim (3.31) readily follows in view of (3.29). $\square$

The next result is a counterpart of Theorem 3.4 for the case $q = 0$.

Theorem 3.7. *Let Assumption 1.1 hold with $q = 0$. Then, for any $\gamma \in \bigl(0, \frac{1}{2}(1-\beta)\bigr)$,*

$$\limsup_{z\downarrow 0} z^{1-2\gamma} \log \mu_z^{\mathfrak{q}}\bigl\{\lambda \in \Lambda_{\mathfrak{q}} : K(\lambda) < z^{-1}\log\log \tfrac{1}{z}\bigr\} = -\infty, \tag{3.33}$$

$$\limsup_{z\downarrow 0} z^{1-2\gamma} \log \mu_z^{\mathfrak{q}}\{\lambda \in \Lambda_{\mathfrak{q}} : K(\lambda) > k_\gamma(z)\} \le -1. \tag{3.34}$$

Proof. Put $k^\dagger \equiv k^\dagger(z) := \lfloor z^{-1}\log\log \frac{1}{z}\rfloor$. In view of the lower bound in (3.13), it is clear that $k^\dagger(z)/k_*(z) \to 0$ as $z \downarrow 0$, and hence $k^\dagger(z) < k_*(z)$ for all $z > 0$ small enough. Then, using (2.12) and (3.4), we can write

$$\mu_z^{\mathfrak{q}}\{K(\lambda) < k^\dagger\} = \sum_{k<k^\dagger} \frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z)} \le k^\dagger \,\frac{F_{\mathfrak{q}}(z,k^\dagger)}{F_{\mathfrak{q}}(z)}. \tag{3.35}$$

Furthermore, for any $\varepsilon \in (0, 1-\beta)$ and all $z > 0$ small enough, according to (3.13) we have

$$(1-\beta-\varepsilon)\, z^{-1}\log \tfrac{1}{z} \le k_*(z) \le (1+\varepsilon)\, z^{-1}\log \tfrac{1}{z},$$

which also gives $s_{k_*} = O\bigl(z^{-\beta-1}(\log \frac{1}{z})^{\beta+1}\bigr)$ by (2.5). Then from (3.7) we get

$$\begin{aligned}
\log \frac{F_{\mathfrak{q}}(z,k^\dagger)}{F_{\mathfrak{q}}(z)} &\le z^{-1}\bigl\{\mathrm{Li}_2(z^{1-\beta-\varepsilon}) - \mathrm{Li}_2(\mathrm{e}^{-zk^\dagger})\bigr\} + O\bigl(z^{-\beta}(\log \tfrac{1}{z})^{\beta+1}\bigr)\\
&= -z^{-1}\,\mathrm{Li}_2\bigl((\log \tfrac{1}{z})^{-1}\bigr) + O(z^{-\beta-\varepsilon})\\
&\sim -z^{-1}\bigl(\log \tfrac{1}{z}\bigr)^{-1} \qquad (z \downarrow 0),
\end{aligned} \tag{3.36}$$

and (3.33) follows by combining (3.35) and (3.36).

Next, to estimate the probability

$$\mu_z^{\mathfrak{q}}\{K(\lambda) > k_\gamma\} = \frac{1}{F_{\mathfrak{q}}(z)} \sum_{k>k_\gamma} F_{\mathfrak{q}}(z,k), \tag{3.37}$$

observe (cf. (3.20)) that, for $k > k_\gamma$ and all $z > 0$ small enough, we have

$$\frac{F_{\mathfrak{q}}(z,k)}{F_{\mathfrak{q}}(z,k-1)} = \eta_k(z) = \frac{\mathrm{e}^{-zQ_k}}{1-\mathrm{e}^{-zk}} < \frac{\mathrm{e}^{-z}}{1-\mathrm{e}^{-zk_\gamma}} \le 1 - \tfrac{1}{2}z.$$

Indeed, if $2\gamma < 1-\beta$ then the asymptotic bound (3.30) implies $\lim_{z\downarrow 0} z^{-1}\mathrm{e}^{-zk_\gamma} = 0$, and therefore

$$\frac{1}{z}\left(\frac{\mathrm{e}^{-z}}{1-\mathrm{e}^{-zk_\gamma}} - 1\right) = \frac{\mathrm{e}^{-z}-1}{z(1-\mathrm{e}^{-zk_\gamma})} + \frac{\mathrm{e}^{-zk_\gamma}}{z(1-\mathrm{e}^{-zk_\gamma})} \to -1 \qquad (z \downarrow 0).$$

Thus, we can estimate the right-hand side of (3.37) by the sum of a geometric progression with ratio $1 - \frac{1}{2}z < 1$, that is,

$$\mu_z^{\mathfrak{q}}\{K(\lambda) > k_\gamma\} \le 2z^{-1}\,\frac{F_{\mathfrak{q}}(z,k_\gamma)}{F_{\mathfrak{q}}(z)}. \tag{3.38}$$

Next, using again the estimate (3.7) and also the asymptotics (3.29), we obtain (cf. (3.36))

$$\begin{aligned}\log \frac{F_{\mathfrak{q}}(z,k_\gamma)}{F_{\mathfrak{q}}(z)} &\le z^{-1}\operatorname{Li}_2(z^{1-\beta-\varepsilon}) - z s_{k_\gamma} + O\bigl(z^{-\beta}(\log \tfrac{1}{z})^{\beta+1}\bigr)\\ &= -z^{-1+2\gamma}\bigl(1+o(1)\bigr) + O(z^{-\beta-\varepsilon})\\ &\sim -z^{-1+2\gamma},\end{aligned} \tag{3.39}$$

where the asymptotic equivalence in (3.39) holds provided that $0<\varepsilon<1-\beta-2\gamma$. Now, the desired result (3.34) follows by combining (3.38) and (3.39). □

In the case $q=0$, under the refined Assumption 1.2 with $\tilde q>0$ (see (1.4)) one can prove the following analogue of the exponential bound (3.18): *for any $c>0$ and $\gamma\in\bigl(0,\frac12\beta\bigr)$,*

$$\limsup_{z\downarrow 0} z^{\beta-2\gamma}\log \mu_z^{\mathfrak{q}}\bigl\{\lambda\in\Lambda_{\mathfrak{q}}: |K(\lambda)-k_*|>cz^{\gamma-1}\bigr\}\le -2^{\beta-2}\tilde q\,\beta c^2<0. \tag{3.40}$$

Here $k_*=k_*(z)$ is again defined by (3.2) but now has the refined asymptotics (cf. (3.13))

$$k_*(z)=z^{-1}\Bigl((1-\beta)\log\tfrac{1}{z}-\beta\log\log\tfrac{1}{z}-\beta\log(1-\beta)-\log\tilde q+o(1)\Bigr). \tag{3.41}$$

The exponential bound (3.40) together with the asymptotic formula (3.41) immediately imply the law of large numbers for the number of parts (cf. Corollary 3.5): *for any $\varepsilon>0$,*

$$\lim_{z\downarrow 0}\mu_z^{\mathfrak{q}}\Bigl\{\lambda\in\Lambda_{\mathfrak{q}}: \bigl|z\bigl(\log\tfrac{1}{z}\bigr)^{-1}K(\lambda)-(1-\beta)\bigr|>\varepsilon\Bigr\}=0.$$

Formally, these results do not cover the utterly degenerate case $\tilde q=0$, $\tilde\beta=0$ in the asymptotic formula (1.4) of Assumption 1.2; however, as explained in Remark 1.5, it is equivalent to the classical case of unrestricted partitions, where the asymptotic behaviour of $K(\lambda)$ (under the measure μ_z on Λ) is described by the limit theorem [16]

$$\lim_{z\downarrow 0}\mu_z\bigl\{\lambda\in\Lambda: zK(\lambda)-\log\tfrac{1}{z}\le t\bigr\}=\exp(-\mathrm{e}^{-t}),\quad t\in\mathbb{R}. \tag{3.42}$$

The asymmetry of the limiting distribution (3.42) (i.e., exponential tail on the right and super-exponential tail on the left) explains the appearance of the two claims in Theorem 3.7.

3.4. Asymptotics of $K(\lambda)$ in the space $\Lambda_{\mathfrak{q}}(n)$. It is now easy to derive the analogues of Theorems 3.4 and 3.7 under the measures $\nu_n^{\mathfrak{q}}$.

Theorem 3.8. *Suppose that Assumption 1.1 holds, with $q\ge 0$ and $0\le\beta<1$, and let $\gamma\in\bigl(0,\frac12(1-\beta)\bigr)$.*

(a) *If $q>0$ then there exists a sequence (k_n) satisfying the asymptotic relation*

$$k_n\sim\frac{T_q\sqrt{n}}{\vartheta_q}\qquad(n\to\infty), \tag{3.43}$$

such that, for any $a>0$,

$$\limsup_{n\to\infty} n^{\gamma-1/2}\log\nu_n^{\mathfrak{q}}\bigl\{\lambda\in\Lambda_{\mathfrak{q}}(n): |K(\lambda)-k_n|>a\,n^{(1-\gamma)/2}\bigr\}<0. \tag{3.44}$$

(b) *If* $q = 0$ *then*

$$\limsup_{n\to\infty} n^{\gamma-1/2} \log \nu_n^{\mathfrak{q}}\left\{\lambda \in \Lambda_{\mathfrak{q}}(n) \colon K(\lambda) < \tfrac{1}{2}\sqrt{n}\log\log n \text{ or } K(\lambda) > n^{1-\gamma}\right\} < 0. \tag{3.45}$$

Proof. (a) Applying Theorem 3.4 to the set $A_z = \{|K(\lambda) - k_*| > cz^{\gamma-1}\} \subset \Lambda_{\mathfrak{q}}$ with $c := \frac{1}{2}a\vartheta_q^{1-\gamma}$, we see that A_z satisfies the condition (2.16) of Proposition 2.3 with $\kappa = 1-2\gamma > 0$. Hence, setting $k_n := k_*(z_n)$ and using (2.18) together with the property (2.17), we obtain (3.44), as claimed. Finally, relation (3.43) easily follows from (2.17) and (3.14).

(b) Consider the set $A_z = \{K(\lambda) < z^{-1}\log\log\frac{1}{z} \text{ or } K(\lambda) > k_\gamma(z)\}$. By Theorem 3.7, the set A_z satisfies the condition (2.16) of Proposition 2.3. Moreover, if the asymptotic relation (2.17) with $q = 0$ holds for a sequence z_n, then the set referred to in (3.45) is a subset of A_{z_n}, at least for n large enough, because

$$z_n^{-1}\log\log\frac{1}{z_n} > \tfrac{1}{2}\sqrt{n}\log\log n,$$

$$k_\gamma(z_n) \le \left\lceil z_n^{-2(1-\gamma)}\right\rceil \sim \left(\frac{6n}{\pi^2}\right)^{1-\gamma} < n^{1-\gamma} \quad (n\to\infty).$$

Thus, the required relation (3.45) follows from (2.18). □

Similarly as before, Theorem 3.8 with $q > 0$ implies the law of large numbers for $K(\lambda)$ under the measure $\nu_n^{\mathfrak{q}}$, analogous to Corollary 3.5.

Corollary 3.9. *Let Assumption 1.1 hold with* $q > 0$. *Then, for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \nu_n^{\mathfrak{q}}\left\{\lambda \in \Lambda_{\mathfrak{q}}(n) \colon \left|\frac{K(\lambda)}{\sqrt{n}} - \frac{T_q}{\vartheta_q}\right| > \varepsilon\right\} = 0.$$

If $q = 0$ then, under Assumption 1.2 with $\tilde{q} > 0$ (see (1.4)), one can deduce in a similar fashion the law of large numbers for $K(\lambda)$: *for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \nu_n^{\mathfrak{q}}\left\{\lambda \in \Lambda_{\mathfrak{q}}(n) \colon \left|\frac{K(\lambda)}{\sqrt{n}\log n} - \frac{\sqrt{6}(1-\beta)}{2\pi}\right| > \varepsilon\right\} = 0. \tag{3.46}$$

In fact, an exponential bound for large deviations of $K(\lambda)$ can be obtained by combining (3.40) with Theorem 3.8(b), but we omit technical details.

Finally, if $\tilde{q} = 0$ and $\tilde{\beta} = 0$ in (1.4), then the classical limit theorem (under the uniform measure ν_n on $\Lambda(n)$) states that [13,16]

$$\lim_{n\to\infty} \nu_n\left\{\lambda \in \Lambda(n) \colon \frac{\pi K(\lambda)}{\sqrt{6n}} - \log\frac{\sqrt{6n}}{\pi} \le t\right\} = \exp(-\mathrm{e}^{-t}), \qquad t \in \mathbb{R}. \tag{3.47}$$

Of course, this result implies the law of large numbers,

$$\lim_{n\to\infty} \nu_n\left\{\lambda \in \Lambda(n) \colon \left|\frac{K(\lambda)}{\sqrt{n}\log n} - \frac{\sqrt{6}}{2\pi}\right| > \varepsilon\right\} = 0,$$

which can be formally considered as the limiting case of (3.46) as $\beta \downarrow 0$.

Remark 3.4. To be more precise, the results by Erdős and Lehner [13] and Fristedt [16], quoted above as formulas (3.42) and (3.47), are technically about the maximal part λ_1, but due to the invariance of the measures μ_z and ν_n under *conjugation* of Young diagrams (whereby columns become rows and vice versa; see also Sect. 5), the random variable λ_1 has the same distribution as the number of parts $K(\lambda)$.

4. Limit Shape of the Minimal Difference Partitions

4.1. The parametric family of limit shapes. Mutual independence of the random variables $(D_j(\lambda))_{j=1}^k$ with respect to the measure $\mu_{z,k}^{\mathrm{q}}$ (see Lemma 2.1) provides an easy way to find the limit shape for MDPs as $z \downarrow 0$. It is natural to allow the maximal part k to grow to infinity as z approaches 0, where the correct growth rate, as suggested by Theorem 3.4, is of order z^{-1} when $q > 0$ and possibly faster, by a logarithmic factor, when $q = 0$. It turns out that if the condition (1.2) holds and $\lim_{z\downarrow 0} zk = T < \infty$ then $\mu_{z,k}^{\mathrm{q}}$-typical partitions $\lambda \in \Lambda_{\mathrm{q}}(\bullet, k)$ concentrate around the limit shape determined by the function

$$\varphi_T(t; q) := \begin{cases} q(T-t) + \log \dfrac{1-\mathrm{e}^{-T}}{1-\mathrm{e}^{-t}}, & 0 < t \le T, \\ 0, & t \ge T. \end{cases} \tag{4.1}$$

If $q = 0$ then the expression (4.1) is reduced to

$$\varphi_T(t; 0) = \begin{cases} \log \dfrac{1-\mathrm{e}^{-T}}{1-\mathrm{e}^{-t}}, & 0 < t \le T, \\ 0, & t \ge T, \end{cases} \tag{4.2}$$

which coincides, as one could expect, with the limit shape of plain (unrestricted) partitions subject to the condition $zk \to T$ (see [43]).

If $q = 0$, one can also allow zk to grow slowly to infinity as $z \downarrow 0$ (which is actually a typical behaviour), whereby the limit shape is given by the formula

$$\varphi_\infty(t; 0) = -\log(1-\mathrm{e}^{-t})$$

(which is formally consistent with (4.2) if we set $T = \infty$).

Another simplification of formula (4.1) worth mentioning occurs for $q > 0$ and $T = T_q$ (see (2.14)), which determines the typical behaviour of the number of parts in this case (see Theorem 3.4 and the asymptotic formula (3.14)); here, the limit shape (4.1) is reduced to

$$\varphi_{T_q}(t; q) = \begin{cases} -tq - \log(1-\mathrm{e}^{-t}), & 0 < t \le T_q, \\ 0, & t \ge T_q. \end{cases} \tag{4.3}$$

This coincides with the limit shape found by Comtet et al. [9, Eq. (19)], [10, Eq. (11)]. The limit shape (4.3) is illustrated in Fig. 4 for various values of parameter $q \ge 0$ using Cartesian coordinates $x = t$, $y = -tq - \log(1-\mathrm{e}^{-t})$, whereby (4.3) takes the form

$$\mathrm{e}^{-y} = \mathrm{e}^{qx}(1-\mathrm{e}^{-x}), \tag{4.4}$$

which was already mentioned in Sect. 1.3 (see (1.14)).

4.2. The limit shape in the spaces $\Lambda_{\mathrm{q}}(\bullet, k)$ and $\Lambda_{\mathrm{q}}(n, k_n)$. The exact statement is as follows. Recall that the notation $k_\gamma(z)$ is defined in (3.27).

Theorem 4.1. *Let Assumption 1.1 hold, with $q \ge 0$. Then for every $t > 0$ and any $\varepsilon > 0$, uniformly in $k = k(z) \in \mathbb{N}$ such that $\lim_{z\downarrow 0} zk(z) = T \in (0, \infty)$,*

$$\limsup_{z\downarrow 0} z \log \mu_{z,k}^{\mathrm{q}} \bigl\{\lambda \in \Lambda_{\mathrm{q}}(\bullet, k) \colon |z Y_\lambda(t/z) - \varphi_T(t; q)| > \varepsilon\bigr\} < 0. \tag{4.5}$$

Furthermore, if $q = 0$ and $\lim_{z\downarrow 0} zk(z) = \infty$ but $k(z) \le k_\gamma(z)$, with some $\gamma \in \bigl(0, \tfrac{1}{2}(1-\beta)\bigr)$, then (4.5) holds with $\varphi_\infty(t; 0)$ in place of $\varphi_T(t; q)$.

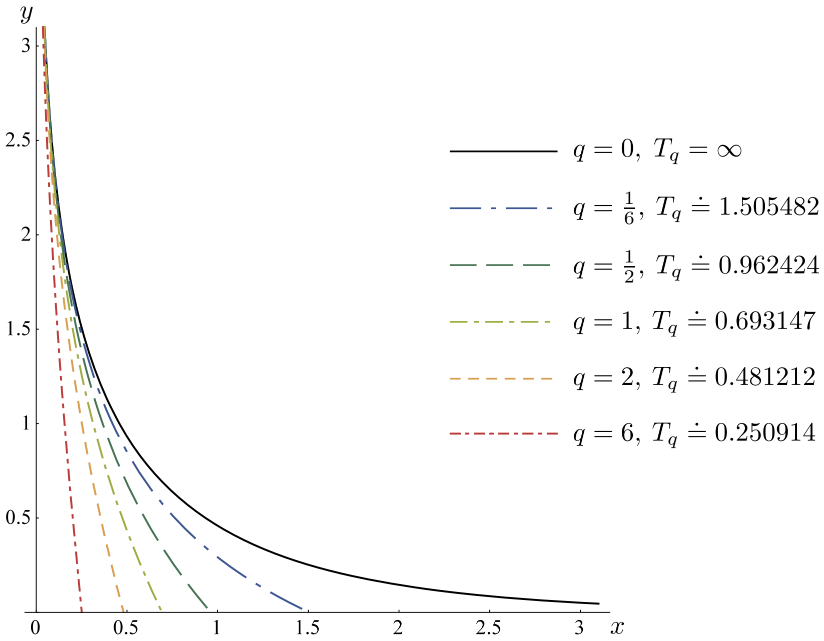

Fig. 4. The parametric family of the limit shapes (4.3) plotted in the Cartesian coordinates $x = t$ and $y = -tq - \log(1 - e^{-t})$ (see Eq. (4.4))

Proof. First, let us show that the curve $t \mapsto \varphi_T(t; q)$ is the limit of the $\mu^{\mathfrak{q}}_{z,k}$-mean of the scaled Young diagrams, that is, for every $t > 0$

$$\lim_{z\downarrow 0} z\,\mathbf{E}^{\mathfrak{q}}_{z,k}[Y_\lambda(t/z)] = \varphi_T(t; q). \tag{4.6}$$

To this end, using the definition (2.2) and the formula (2.8), we can write, for $0 < t < T$,

$$\mathbf{E}^{\mathfrak{q}}_{z,k}[Y_\lambda(t/z)] = \sum_{t/z<j\le k} \mathbf{E}^{\mathfrak{q}}_{z,k}[D_j(\lambda)] = \sum_{t/z<j\le k} q_{k-j} + \sum_{t/z<j\le k} \frac{e^{-jz}}{1-e^{-jz}}. \tag{4.7}$$

According to (1.2), for $T < \infty$ and $q \ge 0$ the first sum in (4.7) is asymptotically evaluated as follows

$$\sum_{t/z<j\le k} q_{k-j} = Q_{k-\lfloor t/z\rfloor} = q\,(k - \lfloor t/z\rfloor) + O\big((k-t/z)^{\beta}\big) = qz^{-1}(T-t) + o(z^{-1}) \tag{4.8}$$

since $zk \to T$ as $z \downarrow 0$. If $T = \infty$ and $q = 0$, then for $k \le k_\gamma(z)$ one has $Q_{k-\lfloor t/z\rfloor} \le Q_{k_\gamma(z)-\lfloor t/z\rfloor} = O(z^{-1+2\gamma}) = o(z^{-1})$ by Lemma 3.6 (see (3.31)).

For the second sum in (4.7), we get (e.g., via the Euler–Maclaurin sum formula) that

$$\begin{aligned}\sum_{t/z<j\le k}\frac{\mathrm{e}^{-jz}}{1-\mathrm{e}^{-jz}}&\sim\int_{t/z}^{k}\frac{\mathrm{e}^{-xz}}{1-\mathrm{e}^{-xz}}\,\mathrm{d}x\\&=z^{-1}\log(1-\mathrm{e}^{-zx})\Big|_{t/z}^{k}\\&=z^{-1}\log\frac{1-\mathrm{e}^{-zk}}{1-\mathrm{e}^{-t}}\\&\sim z^{-1}\log\frac{1-\mathrm{e}^{-T}}{1-\mathrm{e}^{-t}}\qquad(z\downarrow 0).\end{aligned}\tag{4.9}$$

The same calculation is valid when $zk\to\infty$, with the change of e^{-T} to 0. Thus, on substituting the estimates (4.8) and (4.9) into (4.7) we get (4.6).

To obtain the exponential bound (4.5), we use a standard technique often applied in similar problems (see, e.g., [11]). Suppose that $zk\to T\in(0,\infty]$, and fix $t\in(0,T)$ and $\varepsilon>0$. In what follows, we always assume that z is small enough so that $zk>t$ and

$$\left|z\,\mathbf{E}_{z,k}[Y_\lambda(t/z)]-\varphi_T(t;q)\right|<\tfrac{1}{2}\varepsilon.\tag{4.10}$$

Then for any $u\in(0,t)$

$$\begin{aligned}&\mu_{z,k}^{\mathfrak{q}}\big\{zY_\lambda(t/z)-\varphi_T(t;q)>\varepsilon\big\}\\&\quad\le\mu_{z,k}^{\mathfrak{q}}\big\{Y_\lambda(t/z)\ge\mathbf{E}_{z,k}^{\mathfrak{q}}[Y_\lambda(t/z)]+\tfrac{1}{2}z^{-1}\varepsilon\big\}\\&\quad\le\exp\big(-u\,\mathbf{E}_{z,k}^{\mathfrak{q}}[Y_\lambda(t/z)]-\tfrac{1}{2}uz^{-1}\varepsilon\big)\,\mathbf{E}_{z,k}^{\mathfrak{q}}\big[\exp(uY_\lambda(t/z))\big]\\&\quad=\exp\big(-\tfrac{1}{2}uz^{-1}\varepsilon\big)\prod_{t/z<j\le k}\mathbf{E}_{z,k}^{\mathfrak{q}}\big[\exp\big(uD_j-u\,\mathbf{E}_{z,k}^{\mathfrak{q}}(D_j)\big)\big],\end{aligned}\tag{4.11}$$

where the first inequality is a consequence of assumption (4.10), the second is the exponential Markov inequality, and the last line follows from the additive structure of $Y_\lambda(t)$ and independence of $(D_j)_{j=1}^{k}$.

Suppose that, for some $w\in(0,1)$ that will be specified later,

$$0<u\le\log\left(1+\frac{w}{h(t)}\right)=:v(w),\tag{4.12}$$

where we put for short

$$h(t):=\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}},\qquad t\in(0,\infty).\tag{4.13}$$

Then for $j\ge t/z$ we have

$$(\mathrm{e}^{u}-1)\,h(zj)\le(\mathrm{e}^{u}-1)\,h(t)\le w.$$

Applying the elementary inequalities

$$\begin{aligned}-\log(1-x)&\le -xw^{-1}\log(1-w)&&(0<x\le w),\\ \mathrm{e}^{u}-1&\le uv^{-1}(\mathrm{e}^{v}-1)&&(0<u\le v),\end{aligned}$$

with $x := (\mathrm{e}^u - 1)\, h(zj)$ and $v := v(w)$ (see (4.12)), we obtain

$$-\log\bigl(1 - (\mathrm{e}^u - 1)\, h(zj)\bigr) \le u\, y(w)\, h(zj), \qquad y(w) := \frac{-\log(1-w)}{h(t)\, v(w)}.$$

Hence, for $u \le \min\{v(w), t\} \le jz$

$$\begin{aligned}\log\Bigl(\mathbf{E}^{\mathfrak{q}}_{z,k}\bigl[\exp(uD_j - u\,\mathbf{E}^{\mathfrak{q}}_{z,k}D_j)\bigr]\Bigr) &= \log\frac{1-\mathrm{e}^{-zj}}{1-\mathrm{e}^{u-zj}} - u\, h(zj)\\ &= -\log\bigl[1 - (\mathrm{e}^u - 1)\, h(zj)\bigr] - u\, h(zj)\\ &\le u\,\bigl(y(w) - 1\bigr)\, h(zj).\end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.11) and recalling (4.9), we obtain

$$\begin{aligned}&z\log\mu^{\mathfrak{q}}_{z,k}\bigl\{\lambda\in\Lambda_{\mathfrak{q}}(\bullet,k)\colon zY_\lambda(t/z) - \varphi_T(t;q) > \varepsilon\bigr\}\\ &\qquad\le \frac{u}{2}\bigl(-\varepsilon + (y(w)-1)\,\varphi_T(t;0)\bigr)\\ &\qquad\le \frac{v(w)}{2}\bigl(-\varepsilon + (y(w)-1)\,\varphi_T(t;0)\bigr).\end{aligned} \tag{4.15}$$

Since $y(w) \to 1$ as $w \downarrow 0$, we can choose w small enough to make the right-hand side of (4.15) negative. This yields the desired bound for the probability of positive deviations in (4.5). The probability of negative deviations is estimated in the same fashion. □

We are now in a position to state and prove our first main result.

Theorem 4.2. *Suppose that Assumption 1.1 is satisfied with some $q \ge 0$, and let $k_n \to \infty$ so that $k_n/\sqrt{n} \to \tau$ as $n \to \infty$, for some $\tau \in (0, \sqrt{2/q})$, with the right bound understood as $+\infty$ when $q = 0$. Then, for every $t_0 > 0$ and any $\varepsilon > 0$,*

$$\limsup_{n\to\infty}\frac{1}{\sqrt{n}}\log\nu^{\mathfrak{q}}_{n,k_n}\left\{\lambda\in\Lambda_{\mathfrak{q}}(n,k_n)\colon \sup_{t\ge t_0}\bigl|z_nY_\lambda(t/z_n) - \varphi_{T_*}(t;q)\bigr| > \varepsilon\right\} < 0, \tag{4.16}$$

where $T_* = T_*(\tau;q) > 0$ is the (unique) solution of the equation

$$\tau\,\vartheta_q(T_*) = T_* \tag{4.17}$$

and

$$z_n := \frac{T_*}{k_n} \sim \frac{T_*}{\tau\sqrt{n}}. \tag{4.18}$$

Furthermore, if $q = 0$ then the result (4.16) is also valid in the case $k_n/\sqrt{n} \to \infty$ under the additional condition $\limsup_{n\to\infty} k_n^{\beta+1}/n^{1-\delta} < \infty$ for some $\delta \in (0,1)$, with $T = \infty$ and $\vartheta_0(\infty) = \pi/\sqrt{6}$.

Remark 4.1. The assumption $\tau^2 < 2/q$ in Theorem 4.2 arises naturally, because if $\lambda \in \Lambda_{\mathfrak{q}}(n,k)$ then, due to the MDP condition (1.1), we must have $n \ge s_k = \frac{1}{2}qk^2 + O(k^{1+\beta})$, which yields $\tau^2 \le 2/q$. The boundary case $\tau^2 = 2/q$ can in principle be realized, but both the formulation and analysis should be more accurate, so we do not consider it with the exception of the important special case $q = 0$ when additional difficulties can be treated without much effort.

Proof of Theorem 4.2. Note that Eq. (4.17) can be rewritten as

$$\frac{\mathrm{Li}_2(1-\mathrm{e}^{-T})}{T^2} = \frac{1}{\tau^2} - \frac{q}{2}$$

with the left-hand side decreasing from $+\infty$ to 0 as T grows from 0 to $+\infty$, so its positive solution $T = T_*$ always exists (and is unique) for any $\tau \in (0, \sqrt{2/q}\,)$.

For $0 < t_0 \le t < T \le \infty$ and $\varepsilon > 0$, denote

$$A_{z,k}(t,\varepsilon) := \{\lambda \in \Lambda(\bullet, k)\colon\ |zY_\lambda(t/z) - \varphi_T(t;q)| > \varepsilon\},$$
$$\widehat{A}_{z,k}(t_0,\varepsilon) := \left\{\lambda \in \Lambda(\bullet, k)\colon\ \sup\nolimits_{t\ge t_0} |zY_\lambda(t/z) - \varphi_T(t;q)| > \varepsilon\right\}.$$

Given $t_0 > 0$ and $\varepsilon > 0$, define t_i recursively by $\varphi_T(t_i;q) = \varphi_T(t_{i-1},q) - \varepsilon/2$ until $\varphi_T(t_{s-1}) - \varepsilon/2$ becomes negative for some s. By construction,

$$\bigcup_{i=0}^{s-1} A_{z,k}(t_i,\varepsilon/2) \supset \widehat{A}_{z,k}(t_0,\varepsilon), \tag{4.19}$$

because both $Y_\lambda(t)$ and $\varphi_T(t,q)$ decrease as functions of t.

Now, we aim to apply Theorem 4.1 and Proposition 2.4. To this end, in the case $T < \infty$ take $k(z)$ to be any integer-valued function such that $zk(z) \to T$; in the case $T = \infty$ (arising for $q = 0$, $\tau = \infty$) let $k(z) := k_n$ for $z \in (\pi/\sqrt{6(n+1)}, \pi/\sqrt{6n}]$, where the sequence (k_n) is referred to in the theorem. In the latter case one can write $k(z) = k_{\lfloor \pi^2/6z^2 \rfloor}$, and the additional requirement $\limsup_{n\to\infty} k_n^{\beta+1}/n^{1-\delta} < \infty$ combined with (2.5) implies $s_{k_n} = O(n^{1-\delta})$ which can be rewritten as $s_{k(z)} = O(z^{-2+2\delta})$. Thus, for $\gamma \in (0,\delta)$ and z small enough one has $s_{k(z)} < z^{-2+2\gamma}$, and thus $k(z) < k_\gamma(z)$ (see (3.27)).

Hence, Theorem 4.1 implies that for any $t \in (0, T)$ and $\varepsilon > 0$

$$\limsup_{z\downarrow 0} z \log \mu^{\mathfrak{q}}_{z,k(z)}\bigl(A_{z,k(z)}(t,\varepsilon)\bigr) < 0. \tag{4.20}$$

It follows from the asymptotic bound (4.20) (applied with $\varepsilon/2$ instead of ε) and the inclusion (4.19) that for any $t_0 > 0$

$$\limsup_{z\downarrow 0} z \log \mu^{\mathfrak{q}}_{z,k(z)}\bigl(\widehat{A}_{z,k(z)}(t_0,\varepsilon)\bigr) < 0.$$

Furthermore, $k_n/\sqrt{n} \to \tau = T/\vartheta_q(T)$ as $n \to \infty$; if $q = 0$ and $\tau = \infty$ then $k_n/k(\pi/\sqrt{6n}) = 1$ by construction and $z^{2/(\beta+1)}k(z) \to 0$ by the assumption $\limsup_{n\to\infty} k_n^{\beta+1}/n^{1-\delta} < \infty$ As a result, by Proposition 2.4 there exists a sequence $(\tilde{z}_n)$ such that for any $t_0 > 0$

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \nu^{\mathfrak{q}}_{n,k_n}\bigl(\widehat{A}_{\tilde{z}_n,k_n}(t_0,\varepsilon)\bigr) < 0.$$

Finally, it is easy to see that the sequence $(\tilde{z}_n)$ of Proposition 2.4 and the sequence (z_n) defined by (4.18) are asymptotically equivalent so can be interchanged in (4.16). □

4.3. The limit shape in the spaces $\varLambda_{\mathfrak{q}}$ and $\varLambda_{\mathfrak{q}}(n)$. Recall that the function $\varphi_{T_q}(t;q)$ is given by (4.3), where T_q is defined as the unique solution of the equation (2.14).

Theorem 4.3. *Let Assumption 1.1 hold, with $q \ge 0$. Then for every $t > 0$ and any $\varepsilon > 0$, $\delta > 0$*

$$\limsup_{z\downarrow 0} z^{1-\delta} \log \mu_z^{\mathfrak{q}}\bigl\{\lambda \in \varLambda_{\mathfrak{q}} \colon |zY_\lambda(t/z) - \varphi_{T_q}(t;q)| > \varepsilon\bigr\} < 0. \tag{4.21}$$

Proof. Let $A_z \subset \varLambda_{\mathfrak{q}}$ be the set on the left-hand side of (4.21). Then

$$\mu_z^{\mathfrak{q}}(A_z) = \sum_{k=0}^{\infty} \mu_z^{\mathfrak{q}}(A_z \cap \varLambda_{\mathfrak{q}}(\bullet, k)) = \sum_{k=0}^{\infty} \mu_{z,k}^{\mathfrak{q}}(A_z)\, \mu_z^{\mathfrak{q}}\{K(\lambda) = k\}\,.$$

Suppose that $q > 0$. Take $\gamma \in (0, \min\{\delta/2, (1-\beta)/2\})$ and set $\mathcal{I}_z := \{k \in \mathbb{N} \colon |k - k_*| > z^{\gamma-1}\}$, where $k_* = k_*(z)$ is defined in (3.2). Then

$$\begin{aligned}
\mu_z^{\mathfrak{q}}(A_z) &\le \Biggl(\sum_{k\in\mathcal{I}_z} + \sum_{k\notin\mathcal{I}_z}\Biggr) \mu_{z,k}^{\mathfrak{q}}(A_z)\, \mu_z^{\mathfrak{q}}\{K(\lambda) = k\} \\
&\le \mu_z^{\mathfrak{q}}\{K(\lambda) \in \mathcal{I}_z\} + \max_{k\notin\mathcal{I}_z} \mu_{z,k}^{\mathfrak{q}}(A_z).
\end{aligned} \tag{4.22}$$

Using the elementary inequality (3.26), we get from (4.22)

$$\log \mu_z^{\mathfrak{q}}(A_z) \le \log 2 + \max\Bigl\{\log \mu_z^{\mathfrak{q}}\{K(\lambda) \in \mathcal{I}_z\},\ \log \max_{k\notin\mathcal{I}_z} \mu_{z,k}^{\mathfrak{q}}(A_z)\Bigr\}.$$

Multiplying this by $z^{1-\delta}$ and applying Theorems 3.4 and 4.1, we obtain (4.21).

If $q = 0$ then we set $\mathcal{I}_z := \{k \in \mathbb{N} \colon k < z^{-1} \log\log \frac{1}{z}\} \cup \{k \in \mathbb{N} \colon k > k_\gamma(z)\}$ and repeat the above argumentation with a reference to Theorem 3.7 instead of Theorem 3.4. □

Our second main result describes the limit shape under the measure $\nu_n^{\mathfrak{q}}$, that is, without any restriction on the number of parts.

Theorem 4.4. *Let Assumption 1.1 be satisfied, with $q \ge 0$. Then for every $t_0 > 0$ and any $\varepsilon > 0$ and $\delta > 0$, we have*

$$\limsup_{n\to\infty} n^{\delta-1/2} \log \nu_n^{\mathfrak{q}}\Bigl\{\lambda \in \varLambda_{\mathfrak{q}}(n) \colon \sup_{t\ge t_0} \bigl|z_n Y_\lambda(t/z_n) - \varphi_{T_q}(t;q)\bigr| > \varepsilon\Bigr\} < 0,$$

where $z_n = \vartheta_q/\sqrt{n}$, with ϑ_q given by (1.11).

Proof. The claim follows from Theorem 4.3 and Proposition 2.3 by the same argumentation as that used to derive Theorem 4.2 from Theorem 4.1 and Proposition 2.4. □

4.4. Ground state. Observe that, for $q > 0$, the area beneath the limit shape $t \mapsto \varphi_{T_q}(t; q)$ featured in Theorems 4.3 and 4.4 contains a right-angled triangle Δ_q (shaded in Fig. 5) obtained in the limit from the (rescaled) partitions in $\Lambda_{\mathfrak{q}}(n)$ satisfying the *hard version* of the MDP restrictions (1.1), that is, when all inequalities are replaced by equalities. Thus, we can say that the triangle Δ_q represents the *ground state* of the MDP($\mathfrak{q}$) system, while the remaining part of the limit shape corresponds to additional degrees of freedom in a $\nu_n^{\mathfrak{q}}$-typical partition. Note that, according to the $\nu_n^{\mathfrak{q}}$-typical asymptotic behaviour of $K(\lambda)$ described in Corollary 3.9, under the scaling of Theorem 4.4 the horizontal leg of the triangle Δ_q is identified as T_q. On the other hand, by the condition (1.3) the slope of the hypotenuse of the triangle is given by q, therefore the vertical leg of Δ_q is found to be qT_q; in particular, the area of Δ_q is $\frac{1}{2}qT_q^2$. Since the total area of the limit shape is ϑ_q^2 (see (1.11)), the area of the "free" part is given by

$$\vartheta_q^2 - \tfrac{1}{2}qT_q^2 = \mathrm{Li}_2(1 - \mathrm{e}^{-T_q}). \tag{4.23}$$

This remark helps to clarify the duality identity (2.15) of Lemma 2.2. To this end, consider the triangle $\tilde{\Delta}_q$ obtained from Δ_q by reflection about the principal coordinate diagonal, that is, with legs qT_q (horizontal) and T_q (vertical). This triangle may serve as the ground state of a suitable MDP($\tilde{\mathfrak{q}}$) ensemble. The slope of the hypotenuse of $\tilde{\Delta}_q$ is $1/q$, which therefore gives the limiting gap of the space MDP($\tilde{\mathfrak{q}}$). But according to the previous considerations, the legs of the triangle $\tilde{\Delta}_q$ must have the lengths $T_{1/q}$ (horizontal) and $(1/q)\,T_{1/q}$ (vertical). Comparing these values, we arrive at the identity (2.15) (see Fig. 5).

Finally, despite the limit shape of the ensemble MDP($\tilde{\mathfrak{q}}$) contains the triangle $\tilde{\Delta}_q = \Delta_{1/q}$ of the same area as Δ_q, the "free" area changes to (cf. (4.23))

$$\mathrm{Li}_2(1 - \mathrm{e}^{-T_{1/q}}) = \mathrm{Li}_2(1 - \mathrm{e}^{-qT_q}) = \mathrm{Li}_2(\mathrm{e}^{-T_q}).$$

Moreover, according to the identity (1.10), the total area of the free parts in the limit shapes with q and $1/q$ is given by $\frac{1}{6}\pi^2 - qT_q^2$, which in turn implies that the total area of both limit shapes including the ground state triangles equals $\frac{1}{6}\pi^2$,

$$\vartheta_q^2 + \vartheta_{1/q}^2 = \frac{\pi^2}{6}, \tag{4.24}$$

which may be interpreted as the (asymptotic) law of conservation of total energy in *dual systems*, that is, with limiting gaps q and $1/q$. It would be interesting to find a physical explanation of this identity.

5. Alternative Approach to the Limit Shape

Iterating the MDP condition (1.1), for any partition $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda_{\mathfrak{q}}(n, k)$ we get the explicit constraints on its parts,

$$\lambda_i \ge q_0 + \cdots + q_{k-i} \qquad (i = 1, \dots, k). \tag{5.1}$$

Note that equalities in (5.1) correspond to what was called the "ground state" in the discussion in Sect. 4.4. Now, it is natural to "subtract" the ground state by shifting the

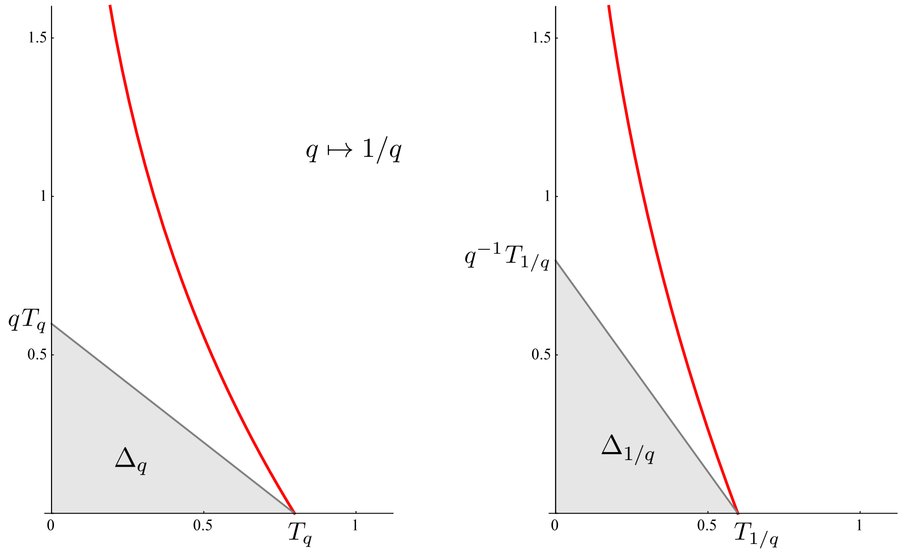

Fig. 5. The duality under the transformation $q \mapsto 1/q$ illustrated for $q = \frac{4}{3}$, where $T_{4/3} \doteq 0.598382$ and $T_{3/4} \doteq 0.797842$. The ground state triangles Δ_q and $\Delta_{1/q}$ (shaded in grey) are obtained from one another by reflection about the main coordinate diagonal. Thus, in line with Lemma 2.2, $T_{1/q} = q\,T_q$ and, equivalently, $T_q = q^{-1}\,T_{1/q}$; in particular, $T_{3/4} = \frac{4}{3}\,T_{4/3}$. The solid curves (red in the online version) show the limit shape graphs. According to formula (4.24), the areas under the limit shapes sum up to $\zeta(2) = \frac{1}{6}\pi^2$

parts of $\lambda \in \Lambda_{\mathfrak{q}}(n,k)$ so as to lift the constraints (5.1) (apart from the default condition that all parts are not smaller than 1). Specifically, consider the mapping

$$\mathfrak{I}\colon \lambda = (\lambda_1, \dots, \lambda_k) \to \rho = (\rho_1, \dots, \rho_k) \tag{5.2}$$

defined by

$$\rho_i := \lambda_i + 1 - q_0 - \cdots - q_{k-i} \geq 1 \qquad (i = 1, \dots, k). \tag{5.3}$$

Remark 5.1. The mapping (5.3) is the (shifted) inverse of the generalized Sylvester transformation mentioned in Remark 1.2.

Using (5.3) and (1.1), note that

$$\rho_i - \rho_{i+1} = \lambda_i - \lambda_{i+1} - q_{k-i} \geq 0, \qquad \rho_k = \lambda_k + 1 - q_0 \geq 1,$$

and, recalling the notation (2.4),

$$r := \sum_{i=1}^{k} \rho_i = \sum_{i=1}^{k} \lambda_i + k - \sum_{i=1}^{k} i\,q_{k-i} = n + k - s_k \geq k,$$

where $n \geq s_k$ as long as the set $\Lambda_{\mathfrak{q}}(n,k)$ is not empty. Hence, $\rho = \mathfrak{I}(\lambda)$ is a partition of the same length k and the new weight $r = n + k - s_k$, but with *no constraints* on its parts; that is, $\rho \in \Lambda(r,k)$. Moreover, it is evident that the mapping (5.2) is a bijection of $\Lambda_{\mathfrak{q}}(n,k)$ onto $\Lambda(r,k)$, for each $k \in \mathbb{N}$ and any $n \geq s_k$. In particular, if $\nu_{n,k}^{\mathfrak{q}}$ is the uniform measure on $\Lambda_{\mathfrak{q}}(n,k)$ then the push-forward $\mathfrak{J}^*\nu_{n,k}^{\mathfrak{q}} = \nu_{n,k}^{\mathfrak{q}} \circ \mathfrak{J}^{-1}$ is the uniform measure on $\Lambda(r,k)$.

This observation furnishes a more straightforward way to finding the limit shape of partitions in the MDP spaces $\Lambda_{\mathfrak{q}}(n,k)$ and $\Lambda_{\mathfrak{q}}(n)$. The heuristic idea is as follows. Consider a partition $\lambda \in \Lambda_{\mathfrak{q}}(n,k_n)$, where $k_n \sim \tau\sqrt{n}$ with $0 < \tau < \sqrt{2/q}$ (cf. the hypothesis in Theorem 4.2). On account of the asymptotics (2.5), for the weight of $\rho = \mathfrak{J}(\lambda)$ this gives

$$r = n + k_n - s_{k_n} \sim \left(1 - \tfrac{1}{2}q\tau^2\right)n = b^2 n, \tag{5.4}$$

where

$$b = b(q;\tau) := \sqrt{1 - \tfrac{1}{2}q\tau^2} > 0. \tag{5.5}$$

In particular, $k_n \sim (\tau/b)\sqrt{r}$. Suppose now that the limit shape of $\rho \in \Lambda(r,k_n)$ exists under the usual $\sqrt{r}$-scaling, so that for $x > 0$ and $r \to \infty$ we have approximately

$$\frac{\rho_{x\sqrt{r}}}{\sqrt{r}} \approx \phi(x).$$

By the relation (5.3) and the asymptotic formulas (1.2) and (5.4), this implies

$$\begin{aligned}\frac{\lambda_{x\sqrt{n}}}{\sqrt{n}} = \frac{\rho_{x\sqrt{n}}}{\sqrt{n}} - \frac{1}{\sqrt{n}} + \frac{Q_{k_n - x\sqrt{n}}}{\sqrt{n}} &\approx b\,\phi(x/b) + \frac{q\left(k_n - x\sqrt{n}\right)}{\sqrt{n}} \\ &\approx b\,\phi(x/b) + q\,(\tau - x),\end{aligned} \tag{5.6}$$

which yields the limit shape for $\lambda \in \Lambda_{\mathfrak{q}}(n,k_n)$ as $n \to \infty$. Note that the last term in (5.6) corresponds to the ground state discussed earlier, whereas the first term indicates the contribution from the "free part" of the partition $\lambda \in \Lambda_{\mathfrak{q}}(n,k_n)$.

Likewise, for partitions $\lambda \in \Lambda(n)$, assuming that their length follows the typical behaviour $K(\lambda) \approx T_q\vartheta_q^{-1}\sqrt{n}$ (see Corollary 3.9), formula (5.6) yields the limit shape

$$\frac{\lambda_{x\sqrt{n}}}{\sqrt{n}} \approx b_q\,\phi(x/b_q) + q\left(\frac{T_q}{\vartheta_q} - x\right),$$

where $b_q := \sqrt{1 - \frac{1}{2}qT_q^2/\vartheta_q^2}$ (cf. (5.5)).

Let us now give a more rigorous argumentation. We confine ourselves to the case $q > 0$ and prove a weaker statement than in the previous section (i.e., just convergence in probability instead of exponential bounds on deviations), since known results can be applied in this case. A similar approach was used by Romik [33] to find the limit shape of MDP(q) with $q = 2$, and by DeSalvo and Pak [12] for any positive integer q. The same technique can be worked out in the case $q = 0$, but this requires a more detailed analysis.

The limit shape for partitions under the uniform measure $\nu_{r,k}$ on the space $\Lambda(r,k)$ has been found by Vershik and Yakubovich [43] (see also Vershik [39]). Adapted to our notation, this result is formulated as follows. Recall that a partition ρ' is said to be *conjugate* to partition $\rho \in \Lambda(r)$ if their Young diagrams Υ_ρ and $\Upsilon_{\rho'}$ are symmetric to one another with respect to reflection about the main diagonal of the coordinate plane. In other words, column blocks of the diagram Υ_ρ become row blocks of the diagram $\Upsilon_{\rho'}$, and vice versa. Clearly, ρ' has the same weight as ρ, that is, $\rho' \in \Lambda(r)$. The next result refers to the conjugate Young diagrams $\Upsilon_{\rho'}$, but it easily translates to the original diagrams Υ_ρ.

Theorem 5.1 ([43, Theorem 1]). *Let $r, k \to \infty$ so that $k = c\sqrt{r} + O(1)$ with some $c > 0$, then for any $\varepsilon > 0$*

$$\nu_{r,k}\Big\{\rho \in \varLambda(r,k)\colon\ \sup\nolimits_{u\ge 0} |k^{-1} Y_{\rho'}(ru/k) - \psi_c(u)| > \varepsilon\Big\} \to 0, \tag{5.7}$$

where[7]

$$\psi_c(u) := \frac{\log\bigl(1 - y_c\,(1-y_c)^{u/c^2}\bigr)}{\log\,(1-y_c)}, \qquad u \ge 0, \tag{5.8}$$

and $y_c \in (0,1)$ is the (unique) solution of the equation

$$c^2\,\mathrm{Li}_2(y_c) = \log^2(1-y_c). \tag{5.9}$$

Equivalently, the statement of Theorem 5.1 can be rewritten as follows: *for any $s_0 \in (0,1]$ and $\varepsilon > 0$,*

$$\nu_{r,k}\big\{\rho \in \varLambda(r,k)\colon\ \sup\nolimits_{s\in[s_0,1]} |kr^{-1} Y_\rho(sk) - \phi_c(s)| > \varepsilon\big\} \to 0, \tag{5.10}$$

where $\phi_c(s)$ is the inverse function,

$$\phi_c(s) := \psi_c^{-1}(s) = \frac{c^2}{\log\,(1-y_c)}\,\log\left(\frac{1-(1-y_c)^s}{y_c}\right), \qquad s\in(0,1]. \tag{5.11}$$

Note that the scalings used here along the two axes are both proportional to $\sqrt{r}$ but different (unless $c = 1$). Unfortunately, the condition $k = c\sqrt{r} + O(1)$ is too strong for our purposes. However, tracking the proof given in [43] and using the continuity of the expression (5.8) with respect to c, one can verify that the limits (5.7) and (5.10) hold true provided only that $k \sim c\sqrt{r}$.

Returning to the limit shape problem for partitions $\lambda \in \varLambda(n, k_n)$, with $k_n \sim \tau\sqrt{n}$, put

$$c = \frac{\tau}{b} = \frac{\tau}{\sqrt{1 - \frac{1}{2}q\tau^2}}, \tag{5.12}$$

so that $k_n \sim \tau\sqrt{n} \sim c\sqrt{r}$ (see (5.4)). Let T_* be the solution of the equation (4.17). Using the definition (2.13), it is straightforward to check that $y_c = 1 - \mathrm{e}^{-T_*}$ solves the equation (5.9). Furthermore, expressing τ from (4.17) and using (2.13), formula (5.12) can be rewritten as

$$c^2 = \frac{T_*^2}{\vartheta_q^2(T_*) - \frac{1}{2}qT_*^2} = \frac{T_*^2}{\mathrm{Li}_2(1-\mathrm{e}^{-T_*})}.$$

Hence, the expression (5.11) takes the form

$$\phi_c(s) = \frac{T_*}{\mathrm{Li}_2(1-\mathrm{e}^{-T_*})}\,\log\frac{1-\mathrm{e}^{-T_*}}{1-\mathrm{e}^{-sT_*}}, \qquad s\in(0,1],$$

[7] There is a misprint in [43, Eq. (5), p. 459], where the variable u should be replaced with $-uc^{-2}\log\,(1-y_c)$.

and the asymptotic result (5.10), restated in the new variable $t = sT^*/\vartheta_q(T_*)$, readily yields

$$\lim_{n\to\infty} \nu^{\mathfrak{q}}_{n,k_n}\left\{\sup_{t\in[t_0,T_*/\vartheta_q(T_*)]}\left|\frac{1}{\sqrt{n}}\,Y_{\mathfrak{I}(\lambda)}(t\sqrt{n}) - \frac{1}{\vartheta_q(T_*)}\log\frac{1-\mathrm{e}^{-T_*}}{1-\mathrm{e}^{-t\,\vartheta_q(T_*)}}\right| > \varepsilon\right\} = 0. \tag{5.13}$$

Finally, to see how (5.13) produces the expression for the limit shape $\varphi_{T_*}(t;q)$ already obtained in Theorem 4.2, it remains to notice, using (1.2) and (5.3), that (cf. (5.6))

$$\frac{Y_\lambda(t\sqrt{n}) - Y_{\mathfrak{I}(\lambda)}(t\sqrt{n})}{\sqrt{n}} = \frac{Q_{k_n-\lfloor t\sqrt{n}\rfloor}}{\sqrt{n}} \to q\left(\frac{T_*}{\vartheta_q(T_*)} - t\right),$$

for all $\lambda \in \Lambda_{\mathfrak{q}}(n,k_n)$ and uniformly in $t \in [0, T_*/\vartheta_q(T_*)]$.

In a similar fashion, one can prove Theorem 4.4. More specifically, by Corollary 3.9 $K(\lambda)/k_n \to 1$ in $\nu^{\mathfrak{q}}_n$-probability, where $k_n = (T_q/\vartheta_q)\sqrt{n}$. The push-forward $\mathfrak{I}^*\nu^{\mathfrak{q}}_n = \nu^{\mathfrak{q}}_n \circ \mathfrak{I}^{-1}$ under the bijection $\mathfrak{I}$ defined in (5.2) is a measure on partitions $\rho \in \Lambda$ such that (random) $r = N(\rho)$ and $k = K(\rho)$ satisfy the relation $r = n + k - s_k$. Since $K(\lambda) = K(\rho)$, it follows that $K(\rho)/k_n \to 1$ in $(\mathfrak{I}^*\nu^{\mathfrak{q}}_n)$-probability. Hence, using (1.11) and (2.5), we obtain, in $(\mathfrak{I}^*\nu^{\mathfrak{q}}_n)$-probability as $n \to \infty$,

$$\frac{r}{k^2} = \frac{n}{k^2} + \frac{1}{k} - \frac{s_k}{k^2} \to \frac{\vartheta_q^2}{T_q^2} - \frac{q}{2} = \frac{\mathrm{Li}_2(1-\mathrm{e}^{-T_q})}{T_q^2} > 0.$$

Thus, taking $c = T_q/\sqrt{\mathrm{Li}_2(1-\mathrm{e}^{-T_q})}$ it is easy to see that $y_c = 1 - \mathrm{e}^{-T_q}$ solves the equation (5.9). Furthermore, using (2.14) the expression (5.11) is reduced to

$$\varphi(t) = \frac{T_q}{\mathrm{Li}_2(1-\mathrm{e}^{-T_q})}\left(-qT_q - \log(1-\mathrm{e}^{-tT_q})\right), \qquad t \in (0,1],$$

and (5.10) implies that

$$\lim_{n\to\infty} \nu^{\mathfrak{q}}_{n}\left\{\sup_{t\in[t_0,T_q/\vartheta_q]}\left|\frac{1}{\sqrt{n}}\,Y_{\mathfrak{I}(\lambda)}(t\sqrt{n}) - \frac{-qT_q - \log(1-\mathrm{e}^{-t\vartheta_q})}{\vartheta_q}\right| > \varepsilon\right\} = 0. \tag{5.14}$$

It remains to notice, using condition (1.2), that in $\nu^{\mathfrak{q}}_n$-probability

$$\sup_{t\in[t_0,T_q/\vartheta_q]}\left|\frac{Y_\lambda(t\sqrt{n}) - Y_{\mathfrak{I}(\lambda)}(t\sqrt{n})}{\sqrt{n}} - q\left(\frac{T_q}{\vartheta_q} - t\right)\right| \to 0,$$

which, together with (5.14), yields the expression $\varphi_{T_q}(t;q)$ for the limit shape already obtained in Theorem 4.4.

6. Minimal Difference Partitions with Random Gaps

The basic assumption (1.2), that the partial sums Q_k of the gap sequence $\mathfrak{q} = (q_i)$ asymptotically grow linearly ($q > 0$) or sub-linearly ($q = 0$), may be satisfied not only for fixed sequences but also for those obtained via some stochastic procedure. Without attempting to investigate this issue in full generality, we provide sufficient conditions for the asymptotics (1.2) under two simple models for random gaps:

(i) $\mathfrak{q} = (q_i)$ is a *sequence of independent random variables*;
(ii) $\mathfrak{q} = (q_i)$ is generated using a *random walk in random environment (RWRE)*, that is, a (nearest-neighbour) random walk with random transition probabilities.

In what follows, abbreviation "a.s." stands for "almost surely" with respect to the suitable probability measure (i.e., law of the sequence $\mathfrak{q}$).

6.1. Random gaps modelled as an independent sequence. Suppose that $\mathfrak{q} = (q_i)$ is a sequence of independent (not necessarily identically distributed) random variables (such that $q_i \ge 0$, $q_0 \ge 1$), defined on an auxiliary probability space with probability measure $\mathbb{P}$; we denote by $\mathbb{E}$ the corresponding expectation.

We will need the following standard result about the strong law of large numbers for independent sequences.

Lemma 6.1 ([30, Theorem 6.6, p. 209]). *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent random variables, and let constants $a_i > 0$ be such that $a_i \uparrow \infty$. If, for some $p \in [0, 1]$,*

$$\sum_{i=1}^{\infty} \frac{\mathbb{E}(|X_i|^p)}{a_i^p} < \infty, \tag{6.1}$$

or if $\mathbb{E}(X_i) = 0$ for all $i \in \mathbb{N}$ and (6.1) holds for some $p \in (1, 2]$, then

$$\frac{X_1 + \cdots + X_k}{a_k} \to 0 \qquad (\mathbb{P}\text{-}a.s.). \tag{6.2}$$

Theorem 6.2. *Suppose that, for some $p \in (0, 1]$ and $\delta \in [0, p)$,*

$$\sum_{i=0}^{k-1} \mathbb{E}(q_i^p) = O(k^\delta) \qquad (k \to \infty). \tag{6.3}$$

Then the asymptotic relation (1.2) holds $\mathbb{P}$-a.s. with $q = 0$ and any $\beta \in (\delta/p, 1)$.

Proof. We wish to apply the first part of Lemma 6.1 with $X_i = q_{i-1}$ and $a_i = i^\beta$ ($\beta > \delta/p$). Denoting $S_k^{(p)} := \sum_{i=1}^k \mathbb{E}(X_i^p)$ ($S_0^{(p)} := 0$) and using summation by parts, we obtain

$$\begin{aligned}\sum_{i=1}^{k} \frac{\mathbb{E}(X_i^p)}{a_i^p} &= \sum_{i=1}^{k} \frac{S_i^{(p)} - S_{i-1}^{(p)}}{i^{\beta p}} \\ &= \frac{S_k^{(p)}}{k^{\beta p}} + \sum_{i=1}^{k-1} \left(\frac{1}{i^{\beta p}} - \frac{1}{(i+1)^{\beta p}}\right) S_i^{(p)}.\end{aligned} \tag{6.4}$$

Furthermore, note that

$$\frac{1}{i^{\beta p}} - \frac{1}{(i+1)^{\beta p}} = \frac{1}{i^{\beta p}}\left(1-\left(1+\frac{1}{i}\right)^{-\beta p}\right) \le \frac{\beta p}{i^{\beta p+1}},$$

by the elementary inequality $(1+x)^{-\gamma} \ge 1-\gamma x$ (see [20, Theorem 41, Eq. (2.15.1), p. 39]) with $x = 1/i$ and $\gamma = \beta p$. Hence, returning to (6.4), we get

$$\sum_{i=1}^{k} \frac{\mathbb{E}(X_i^p)}{a_i^p} \le \frac{S_k^{(p)}}{k^{\beta p}} + \beta p \sum_{i=1}^{k-1} \frac{S_i^{(p)}}{i^{\beta p+1}}. \tag{6.5}$$

From the hypothesis (6.3), we know that $S_k^{(p)} = O(k^\delta)$, and together with the assumption $\beta > \delta/p$ this implies that the right-hand side of (6.5) stays bounded as $k \to \infty$. Thus, the condition (6.1) is satisfied, and (1.2) follows due to (6.2). □

Similarly, we can treat the case where the random variables have finite expected values.

Theorem 6.3. *Suppose that the following two conditions are satisfied.*

(i) *For some $q \ge 0$ and $\beta_0 \in [0, 1)$,*

$$\sum_{i=0}^{k-1} \mathbb{E}(q_i) = qk + O(k^{\beta_0}) \qquad (k \to \infty). \tag{6.6}$$

(ii) *For some $p \in (1, 2]$ and $\delta \in [0, p)$,*

$$\sum_{i=0}^{k-1} \mathbb{E}\big(|q_i - \mathbb{E}(q_i)|^p\big) = O(k^\delta) \qquad (k \to \infty). \tag{6.7}$$

Then the asymptotic relation (1.2) holds $\mathbb{P}$-a.s. with $q \ge 0$ defined in (6.6) and $\beta = \beta_0$ if $\beta_0 > \delta/p$, or else with any $\beta \in (\delta/p, 1)$.

Proof. We can use the second part of Lemma 6.1 with $X_i = q_{i-1} - \mathbb{E}(q_{i-1})$ and $a_i = i^\beta$ ($\beta > \delta/p$). Indeed, repeating the argumentation in the proof of Theorem 6.2 and using the assumption (6.7), we see that (6.2) holds, that is, $\mathbb{P}$-a.s.

$$Q_k - \sum_{i=0}^{k-1} \mathbb{E}(q_i) = o(k^\beta) \qquad (k \to \infty).$$

Furthermore, on account of the assumption (6.6) this yields

$$Q_k = qk + O(k^{\beta_0}) + o(k^\beta) \qquad (k \to \infty). \tag{6.8}$$

It remains to notice that if $\beta_0 \le \delta/p$ then the combined error term on the right-hand side of (6.8) is $o(k^\beta)$ (with any $\beta > \delta/p$), while if $\beta_0 > \delta/p$ then this error term is $O(k^\beta)$ with $\beta = \beta_0$. This completes the proof of Theorem 6.3. □

Example 6.1. To illustrate Theorem 6.2, let q_i have a Bernoulli distribution,

$$\mathbb{P}\big(q_i = (i+1)^3\big) = (i+1)^{-2}, \qquad \mathbb{P}(q_i = 0) = 1 - (i+1)^{-2} \qquad (i \in \mathbb{N}_0).$$

Then for $p \in (0, 1]$ we have

$$\sum_{i=0}^{k-1} \mathbb{E}(q^p) = \sum_{i=1}^{k} (i+1)^{3p-2} = \begin{cases} O(k^{3p-1}), & p \in (\frac{1}{3}, 1], \\ O(\log k), & p = \frac{1}{3}, \\ O(1), & p \in (0, \frac{1}{3}). \end{cases}$$

Thus, the assumption (6.3) holds with $\delta = 3p - 1$ if $p > \frac{1}{3}$; any $\delta > 0$ if $p = \frac{1}{3}$; and $\delta = 0$ if $p < \frac{1}{3}$. Hence, the condition $\delta < p$ (as required in (6.3)) is satisfied for all $p \in (0, \frac{1}{2})$, and therefore Theorem 6.2 is applicable. In contrast, Theorem 6.3 cannot be used, because

$$\sum_{i=0}^{k-1} \mathbb{E}(q_i) = \sum_{i=0}^{k-1} (i+1) \sim \tfrac{1}{2} k^2 \qquad (k \to \infty),$$

so that the condition (6.6) is not fulfilled.

Example 6.2. Consider the particular case where the (independent) random variables $(q_i)_{i \geq 1}$ are *identically distributed*, and suppose that, for some $p \in (0, 2]$,

$$\mathbb{E}(q_1^p) < \infty. \tag{6.9}$$

If $p \leq 1$ then the condition (6.3) is satisfied only with $\delta \geq 1$, unless $\mathbb{E}(q_1^p) = 0$, that is, $q_1 = 0$ ($\mathbb{P}$-a.s.) when $\delta = 0$. Hence, in a non-degenerate case, we always have $\delta/p \geq 1$ and Theorem 6.2 cannot be used. However, the situation becomes more meaningful if $1 < p \leq 2$. Here, the conditions (6.6) and (6.7) are satisfied with $q = \mathbb{E}(q_1) \geq 0$, $\beta_0 = 0$ and $\delta \geq 1$ (assuming that $\mathbb{P}(q_1 > 0) > 0$). Hence, by Theorem 6.3, the asymptotic relation (1.2) holds with any $\beta \in (1/p, 1)$. Note that no moment assumption is required on q_0, because $q_0/k \to 0$ ($\mathbb{P}$-a.s.). If $p = 2$ (i.e., q_1 has finite variance), then the law of the iterated logarithm shows that one cannot take $\beta = \frac{1}{2}$; in the general case $p \in (1, 2)$, the optimality of the lower bound $\beta > 1/p$ follows from [30, § 7.5.16, p. 258].

6.2. Random gaps modelled via RWRE. RWRE is a random process $(X_k)_{k \geq 0}$ on $\mathbb{Z}$ constructed in two steps: (i) first, the environment $\omega \in \Omega$ is chosen at random (under some probability measure $\mathbb{P}$) and fixed; (ii) conditional on ω, (X_k) is a time-homogeneous random walk (Markov chain) with state-dependent transition probabilities determined by the environment. More precisely, let $p_j = p_j(\omega) \in (0, 1)$ ($j \in \mathbb{Z}$) be a family of *independent and identically distributed* random variables, defined on a sample space $\Omega = \{\omega\}$. Denoting by P_0^ω the *quenched* probability law of the random walk (X_k) conditioned on the environment $\omega \in \Omega$ (where the subscript 0 indicates the starting position of the walk, $X_0 = 0$), we have, for all $k \in \mathbb{N}$ and $j \in \mathbb{Z}$,

$$P_0^\omega(X_k = j+1 \,|\, X_{k-1} = j) = p_j(\omega), \qquad P_0^\omega(X_k = j-1 \,|\, X_{k-1} = j) = 1 - p_j(\omega).$$

By averaging the quenched measure P_0^ω with respect to the environment distribution $\mathbb{P}$, we obtain the *annealed* measure $\mathbf{P}_0 := \mathbb{P} \times P_0^\omega \equiv \mathbb{E} P_0^\omega$. For a general review of RWRE, with further details and references, see, for example, Bogachev [5] or Zeitouni [45].

Now, given RWRE (X_k), we can generate the gap sequence $\mathfrak{q} = (q_i)$ as follows:

$$q_i = a_i + b\,(X_{i+1} - X_i) \qquad (i \in \mathbb{N}_0), \tag{6.10}$$

where $b > 0$, $a_0 \geq b + 1$, $a_i \geq b$ for $i \geq 1$, and $q_i \in \mathbb{N}_0$. Hence, recalling that $X_0 = 0$, we get

$$Q_k = \sum_{i=0}^{k-1} q_i = A_k + bX_k, \tag{6.11}$$

where $A_k := \sum_{i=0}^{k-1} a_i$. To obtain asymptotics (1.2) for the sequence (6.11), it is natural to assume that the leading sequence (a_i) itself satisfies a similar condition,

$$A_k = ak + O(k^{\beta_0}) \qquad (k \to \infty), \tag{6.12}$$

with some $a \geq b$ and $\beta_0 \in [0, 1)$. In turn, the long-time behaviour of the RWRE (X_k) is described by the following results due to Solomon [36] (for a quick orientation, see also [5, Theorems 1 and 2, pp. 355–356]).

Lemma 6.4 ([36, Theorem (1.7), p. 4]). *Set $\rho_0 := (1 - p_0)/p_0$ and $\eta := \mathbb{E}(\log \rho_0)$.*

(a) *If $\eta < 0$ then $\lim_{k\to\infty} X_k = +\infty$, while if $\eta > 0$ then $\lim_{k\to\infty} X_k = -\infty$ ($\mathbf{P}_0$-a.s.).*
(b) *If $\eta = 0$ then $-\infty = \liminf_{k\to\infty} X_k < \limsup_{k\to\infty} X_k = +\infty$ ($\mathbf{P}_0$-a.s.).*

Note that, by Jensen's inequality, $\mathbb{E}(\log \rho_0) \leq \log \mathbb{E}(\rho_0)$ and $\{\mathbb{E}(\rho_0)\}^{-1} \leq \mathbb{E}(\rho_0^{-1})$, with all inequalities strict unless ρ_0 is a deterministic constant.

Lemma 6.5 ([36, Theorem (1.16), p. 7]). *The limit $v := \lim_{k\to\infty} X_k/k$ exists $\mathbf{P}_0$-a.s. and is given by*

$$v = \begin{cases} \dfrac{1 - \mathbb{E}(\rho_0)}{1 + \mathbb{E}(\rho_0)} & \text{if } \mathbb{E}(\rho_0) < 1, \\[2ex] -\dfrac{1 - \mathbb{E}(\rho_0^{-1})}{1 + \mathbb{E}(\rho_0^{-1})} & \text{if } \mathbb{E}(\rho_0^{-1}) < 1, \\[2ex] 0 & \text{if } \{\mathbb{E}(\rho_0)\}^{-1} \leq 1 \leq \mathbb{E}(\rho_0^{-1}). \end{cases} \tag{6.13}$$

Remark 6.1. Formula (6.13) implies that $|v| < 1$ in all cases.

From Lemma 6.5 and the condition (6.12), we immediately deduce a strong law of large numbers for the sequence Q_k (see (6.11)),

$$\frac{Q_k}{k} = \frac{A_k}{k} + \frac{bX_k}{k} \to a + bv, \qquad k \to \infty \qquad (\mathbf{P}_0\text{-a.s.}),$$

so that the limit (1.3) holds $\mathbf{P}_0$-a.s. with $q = a + bv$.

Remark 6.2. By the inequality $a \geq b$ and Remark 6.1, in the model (6.10) we always have $q > a - b \geq 0$.

To estimate the error term in a way similar to (1.2), we need information about the fluctuations of the RWRE (X_k) as $k \to \infty$. In the non-critical case (i.e., $\eta \neq 0$, see Lemma 6.4), this was investigated by Kesten, Kozlov and Spitzer [22] (see also discussion and commentary in [5, pp. 357–359]). We will state below (a corollary from) their results adapted to our purposes. A probability law on $\mathbb{R}$ is called *non-arithmetic* if it is not supported on a set $c\,\mathbb{Z}$. We write $Y_k = O_p(1)$ if (Y_k) is *stochastically bounded* (in $\mathbf{P}_0$), that is, if for any $\varepsilon > 0$ there is $M > 0$ such that $\limsup_{k\to\infty} \mathbf{P}_0(|Y_k| > M) \le \varepsilon$. The results from [22] are transcribed using that if Y_k weakly converges (to a proper distribution) then $Y_k = O_p(1)$. Recall the notation $\rho_0 = (1-p_0)/p_0$ and $\eta = \mathbb{E}(\log\rho_0)$.

Lemma 6.6 ([22, pp. 146–148]). *Assume that $-\infty \le \eta < 0$ and the distribution of $\log\rho_0$ (excluding a possible atom at $-\infty$) is non-arithmetic. Let $\varkappa \in (0,\infty)$ be such that*

$$\mathbb{E}(\rho_0^{\varkappa}) = 1 \quad \textit{and} \quad \mathbb{E}(\rho_0^{\varkappa}\log^{+}\rho_0) < \infty,$$

where $\log^{+}u := \max\{\log u, 0\}$. Then RWRE (X_k) has the following asymptotics as $k \to \infty$.

(a) *If $0 < \varkappa < 1$ then*

$$X_k = O_p(k^{\varkappa}).$$

(b) *If $\varkappa = 1$ then*

$$X_k = O_p\bigl(k/\log k\bigr).$$

(c) *If $\varkappa > 1$ then*

$$X_k = vk + O_p(k^{\beta_1}),$$

where v is defined in (6.13) and $\beta_1 := \max\{1/2, 1/\varkappa\}$.

Combining Lemma 6.6 and the assumption (6.12), we arrive at the following result. Recall that $\beta_0 \in (0,1)$ is defined in (6.12).

Theorem 6.7. *Under the hypotheses of Lemma 6.6, the following asymptotics hold for Q_k as $k \to \infty$.*

(a) *If $0 < \varkappa < 1$ then*

$$Q_k = ak + O_p(k^{\beta}),$$

where $\beta = \max\{\beta_0, \varkappa\}$.

(b) *If $\varkappa = 1$ then*

$$Q_k = ak + O_p\bigl(k/\log k\bigr).$$

(c) *If $\varkappa > 1$ then*

$$Q_k = (a+v)k + O_p(k^{\beta}),$$

where $\beta := \max\{\beta_0, 1/2, 1/\varkappa\}$.

Thus, in the RWRE model (6.10) the asymptotic formula (1.2) is valid in a $\mathbf{P}_0$-stochastic version, that is, with the error term estimated using $O_p(\cdot)$,

$$Q_k = qk + O_p(k^\beta) \qquad (k \to \infty), \tag{6.14}$$

where $q = a + v > 0$ and $0 \le \beta < 1$. To be more precise, formula (6.14) with $\beta < 1$ holds in all cases except for $\varkappa = 1$, where the error bound becomes logarithmically close to k.

Furthermore, a careful inspection of all the proofs shows that a stochastic version (6.14) of the asymptotics (1.2) is sufficient to guarantee convergence (in $\mathbf{P}_0$-probability) of the scaled Young boundary $Y_\lambda(t)$ to the limit shape, as described in Sect. 4. As for the special case $\varkappa = 1$, it is natural to expect that the error bound of order $k/\log k$ should be enough for the limit shape, but verification of the technical details is tedious, so this is left as a conjecture.

Finally, let us mention the critical case $\eta = 0$ not covered by Lemmas 6.4 and 6.5. Here, RWRE (X_k) is recurrent (see Lemma 6.4(b)), and its asymptotic behaviour is characterized by the so-called *Sinai's localization* [35] (see discussion and commentary in [5, pp. 359–360]). We state a corollary from this result adapted to our purposes.

Lemma 6.8 ([35]). *Suppose that* $\mathbb{P}(\rho_0 = 1) < 1$ *and* $c_1 \le \rho_0 \le c_2$ ($\mathbb{P}$-*a.s.*), *with some deterministic constants* $0 < c_1 < c_2 < \infty$. *If* $\eta = 0$ *then*

$$X_k = O_p(\log^2 k) \qquad (k \to \infty).$$

Combined with (6.12), this immediately implies asymptotics of Q_k (cf. (1.2)), which ensures the validity of our limit shape result.

Theorem 6.9. *Under the hypotheses of Lemma* 6.8,

$$Q_k = ak + O_p(k^{\beta_0}) \qquad (k \to \infty),$$

where $\beta_0 \in (0, 1)$ *is defined in* (6.12).

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A. Appendix: Proof of Propositions 2.3 and 2.4

A.1. Auxiliary lemmas. According to the representation (2.1) and independence of $\{D_j\}$ under the measure $\mu_{z,k}^{\mathrm{q}}$ (see Sect. 2), the weight $N(\lambda)$ of partition $\lambda \in \Lambda_{\mathrm{q}}$ is the sum of $k \to \infty$ independent random variables, so one may expect a local limit theorem to hold (cf. [6,16,17,41]). For our purposes, it suffices to obtain an asymptotic lower bound for the probability of the event $\{N(\lambda) = n\}$. To this end, we need some auxiliary technical results (for simplicity, we suppress the dependence on z in the notation of some functions introduced below).

Lemma A.1. *Let* $\chi_j(u) := \mathbf{E}^{\mathfrak{q}}_{z,k}\bigl[\mathrm{e}^{\mathrm{i}ujD_j}\bigr]$ $(u \in \mathbb{R})$ *be the characteristic function of the random variable* jD_j $(1 \le j \le k)$*. Then, as* $z \downarrow 0$*, uniformly in* $j \in \mathbb{N}$ *and* $u \in \mathbb{R}$

$$\log \chi_j(u) = \mathrm{i}\bigl(q_{k-j} + h(zj)\bigr)\, ju - \tfrac{1}{2} h(zj)\bigl(1 + h(zj)\bigr)\, j^2u^2 + R_j(u), \tag{A.1}$$

where $\log(\cdot)$ *denotes the principal branch of the logarithm,* $h(\cdot)$ *is given by* (4.13) *and*

$$R_j(u) = \bigl(h(zj) + h(zj)^2 + h(zj)^4 \log \tfrac{1}{z}\bigr)\, O(j^3u^3). \tag{A.2}$$

Proof. An easy computation shows that

$$\chi_j(u) = \sum_{r=0}^{\infty} \mathrm{e}^{\mathrm{i}uj(r+q_{k-j})}\mathrm{e}^{-zjr}(1 - \mathrm{e}^{-zj}) = \mathrm{e}^{\mathrm{i}ujq_{k-j}} \frac{1 - \mathrm{e}^{-zj}}{1 - \mathrm{e}^{-zj+\mathrm{i}uj}}.$$

Hence

$$\begin{aligned} \log \chi_j(u) &= \mathrm{i}ujq_{k-j} - \log \frac{1 - \mathrm{e}^{-zj} - \mathrm{e}^{-zj}(\mathrm{e}^{\mathrm{i}uj} - 1)}{1 - \mathrm{e}^{-zj}} \\ &= \mathrm{i}ujq_{k-j} - \log\bigl(1 - h(zj)(\mathrm{e}^{\mathrm{i}uj} - 1)\bigr). \end{aligned}$$

It is easy to check that the function

$$\zeta \mapsto g_j(\zeta) := -\log\bigl\{1 - h(zj)(\zeta - 1)\bigr\} \tag{A.3}$$

is analytic in the half-plane $\Re\zeta < 1 + 1/h(zj)$. Hence, Taylor's formula for complex-analytic functions (see, e.g., [34, § 5.2, p. 244]) gives for $|\zeta| < 1 + 1/h(zj)$

$$g_j(\zeta) = g_j(1) + g_j'(1)(\zeta - 1) + \frac{g_j''(1)}{2}(\zeta - 1)^2 + \frac{(\zeta - 1)^3}{2\pi\mathrm{i}} \oint_{\Gamma_j} \frac{g_j(\xi)}{(\xi - 1)^3(\xi - \zeta)}\,\mathrm{d}\xi, \tag{A.4}$$

where Γ_j is the circle of radius $1 + 1/(2h(zj))$ about the origin, positively oriented.

Note from (A.3) that for $\xi \in \Gamma_j$ we have

$$|\mathrm{e}^{-g_j(\xi)}| = \bigl|1 + h(zj) - h(zj)\xi\bigr| \le \tfrac{3}{2} + 2h(zj), \qquad |\arg \mathrm{e}^{-g_j(\xi)}| \le \frac{\pi}{2}.$$

Using that $|\log(r\mathrm{e}^{\mathrm{i}\theta})| \le |\log r| + \pi/2$ $(r > 0, |\theta| \le \pi/2)$, this yields

$$\begin{aligned} |g_j(\xi)| = \bigl|\log(\mathrm{e}^{-g_j(\xi)})\bigr| &\le \log\bigl(\tfrac{3}{2} + 2h(zj)\bigr) + \frac{\pi}{2} \\ &\le \log\bigl(1 + h(z)\bigr) + \log 2 + \frac{\pi}{2} \\ &\le \log \frac{z+1}{z} + \log 2 + \frac{\pi}{2}, \end{aligned} \tag{A.5}$$

by virtue of monotonicity of $h(\cdot)$ and the elementary bound

$$1 + h(z) = \frac{1}{1 - \mathrm{e}^{-z}} \le \frac{z+1}{z}.$$

Furthermore, for any $\xi \in \Gamma_j$ and $|\zeta| = 1$ (in particular, $\zeta = 1$) we have $|\xi - \zeta|^{-1} \le 2h(zj)$. Thus, computing the derivatives of $g_j(\cdot)$ at 1 and substituting (A.5) into (A.4), we get

$$g_j(\zeta) = h(zj)(\zeta - 1) + \frac{h(zj)^2}{2}(\zeta - 1)^2 + (\zeta - 1)^3 h(zj)^4 O(\log \tfrac{1}{z}) \qquad (z \downarrow 0), \tag{A.6}$$

where the estimate $O(\cdot)$ is uniform in $j \in \mathbb{N}$ and ζ such that $|\zeta| = 1$.

Now, using the Taylor expansion

$$e^{ix} = \sum_{\ell=0}^{m-1} \frac{(ix)^\ell}{\ell!} + R_m(x), \qquad |R_m(x)| \le \frac{|x|^m}{m!},$$

which is valid for all $m \in \mathbb{N}$ and any real x (see, e.g., [15, § XV.4, Lemma 1, p. 512]), we substitute $\zeta = e^{iju}$ into (A.6) to obtain, as $z \downarrow 0$,

$$\begin{aligned} g_j(e^{iju}) &= h(zj)\Big(iju - \tfrac{1}{2} j^2u^2 + O(j^3u^3)\Big) - \tfrac{1}{2} h(zj)^2 \Big(j^2u^2 + O(j^3u^3)\Big) \\ &\quad + O(j^3u^3)\, h(zj)^4 \log \tfrac{1}{z}, \end{aligned}$$

where all O-estimates are uniform in $j \in \mathbb{N}$, $u \in \mathbb{R}$. [Note that it is convenient to use the representation $(\zeta - 1)^2 = (\zeta^2 - 1) - 2(\zeta - 1)$.] Finally, rearranging the terms we obtain (A.1) and (A.2). $\square$

Lemma A.2. *For $r, \ell \in \mathbb{N}$, denote*

$$\Sigma_{z,k}(r, \ell) := \sum_{j=1}^{k} j^r h(zj)^\ell. \tag{A.7}$$

Then, uniformly in $k \ge t_1/z$ (for any $t_1 > 0$), as $z \downarrow 0$,

$$\Sigma_{z,k}(1, 1) = z^{-2} \operatorname{Li}_2(1 - e^{-zk}) + O(z^{-1}), \tag{A.8}$$

$$\Sigma_{z,k}(2, 2) > \tfrac{1}{2} z^{-3}(1 - e^{-2t_1}), \tag{A.9}$$

$$\Sigma_{z,k}(3, \ell) = O(z^{-4}) \quad (\ell = 1, 2, 3), \qquad \Sigma_{z,k}(3, 4) = O(z^{-4} \log \tfrac{1}{z}). \tag{A.10}$$

Proof. Using the Euler–Maclaurin sum formula like in the proof of Lemma 3.2, we obtain

$$\begin{aligned} \sum_{j=1}^{k} j\, h(zj) &= \int_1^k \frac{x\, e^{-zx}}{1 - e^{-zx}}\, dx + O(1) \frac{e^{-z}}{1 - e^{-z}} + O(1) \int_1^k \frac{|(zx - 1)e^{-zx} + e^{-2zx}|}{(1 - e^{-zx})^2}\, dx \\ &= z^{-2} \int_0^{zk} \frac{y\, e^{-y}}{1 - e^{-y}}\, dy + O(z^{-1}) = z^{-2} \operatorname{Li}_2(1 - e^{-zk}) + O(z^{-1}), \end{aligned}$$

using the substitution $u = 1 - e^{-y}$ and formula (1.8). Hence, (A.8) is proved.

Similarly, (A.9) follows from the asymptotic estimate

$$\sum_{j=1}^{k} j^2 h(zj)^2 \sim z^{-3} \int_0^{zk} \frac{y^2 e^{-2y}}{(1 - e^{-y})^2}\, dy > z^{-3} \int_0^{t_1} e^{-2y}\, dy = \tfrac{1}{2} z^{-3}(1 - e^{-2t_1}).$$

Finally, noting that $y(1-\mathrm{e}^{-y})^{-1} \le \mathrm{e}^{y/2}$ for all $y>0$, we obtain

$$\sum_{j=1}^{k} j^3\, h(zj)^{\ell} \sim z^{-4} \int_z^{zk} \frac{y^3\,\mathrm{e}^{-\ell y}}{(1-\mathrm{e}^{-y})^{\ell}}\,\mathrm{d}y < z^{-4} \int_z^{\infty} y^{3-\ell}\,\mathrm{e}^{-\ell y/2}\,\mathrm{d}y,$$

which is $O(z^{-4})$ for $\ell < 4$ and $O(z^{-4}\log\frac{1}{z})$ for $\ell = 4$, and (A.10) follows. □

Remark A.1. Formula (A.8) may be obtained from (3.11) by formal differentiation with respect to z, using the dilogarithm identity (1.10).

Lemma A.3. *Let $\upsilon > \frac{3}{4}$ and $t_1 > 0$ be some constants. Then there exists $\delta > 0$ such that, for any $z \in (0,\delta)$ and all $k \ge t_1/z$, the inequality*

$$\mu^{\mathfrak{q}}_{z,k}\{N(\lambda)=n\} \ge n^{-\upsilon}$$

holds for all $n \in \mathbb{N}$ satisfying the bound

$$\Big| n - s_k - z^{-2}\,\mathrm{Li}_2(1-\mathrm{e}^{-zk})\Big| \le z^{-4/3}. \tag{A.11}$$

Proof. Let us start by pointing out that, for z sufficiently small, the inequality (A.11) has many integer solutions n. Moreover, since $\mathrm{Li}_2(1-\mathrm{e}^{-t})$ increases in t, it follows from (A.11) that for all $z>0$ small enough and for every $k \ge t_1/z$,

$$n \ge z^{-2}\,\mathrm{Li}_2(1-\mathrm{e}^{-zk}) + s_k - z^{-4/3} \ge \tfrac{1}{2} z^{-2}\,\mathrm{Li}_2(1-\mathrm{e}^{-t_1}) > 0. \tag{A.12}$$

Now, using the decomposition $N(\lambda) = \sum_j j D_j(\lambda)$ and independence of $D_j(\lambda)$ for different j (see Lemma 2.1), by the Fourier inversion formula we have

$$\begin{aligned}
\mu^{\mathfrak{q}}_{z,k}\{N(\lambda)=n\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_{j=1}^{k} \chi_j(s)\,\mathrm{e}^{-\mathrm{i}sn}\,\mathrm{d}s = \frac{1}{\pi}\int_0^{\pi} \Re \prod_{j=1}^{k} \chi_j(s)\,\mathrm{e}^{-\mathrm{i}sn}\,\mathrm{d}s \\
&= \frac{1}{\pi}\int_0^{z^{7/5}} \Re \prod_{j=1}^{k} \chi_j(s)\,\mathrm{e}^{-\mathrm{i}sn}\,\mathrm{d}s + \frac{1}{\pi}\int_{z^{7/5}}^{\pi} \Re \prod_{j=1}^{k} \chi_j(s)\,\mathrm{e}^{-\mathrm{i}sn}\,\mathrm{d}s \\
&=: I_1 + I_2.
\end{aligned} \tag{A.13}$$

First, we shall obtain a suitable lower bound for I_1 and then show that I_2 is small.

Using Lemma A.1 and recalling the notation (A.7), we have

$$\begin{aligned}
I_1 &= \frac{1}{\pi}\int_0^{z^{7/5}} \Re \exp\left\{ -\mathrm{i}un + \sum_{j=1}^{k} \log \chi_j(u) \right\} \mathrm{d}u \\
&= \frac{1}{\pi}\int_0^{z^{7/5}} \Re \exp\Big\{ -\mathrm{i}u\big(n - s_k - \Sigma_{z,k}(1,1)\big) - \tfrac{1}{2}u^2\big(\Sigma_{z,k}(2,1) + \Sigma_{z,k}(2,2)\big) \\
&\quad + O(u^3)\big(\Sigma_{z,k}(3,1) + \Sigma_{z,k}(3,2) + \Sigma_{z,k}(3,4)\log\tfrac{1}{z}\big)\Big\}\,\mathrm{d}u.
\end{aligned} \tag{A.14}$$

Due to the estimate (A.8) and the assumption (A.11),

$$n - s_k - \Sigma_{z,k}(1,1) = O(z^{-4/3}) \qquad (z \downarrow 0).$$

Next, using (A.9) we get

$$\Sigma_{z,k}(2,1)+\Sigma_{z,k}(2,2) > \Sigma_{z,k}(2,2) \ge \tfrac{1}{2} z^{-3}(1-\mathrm{e}^{-2t_1}) \qquad (z \downarrow 0).$$

Finally, by virtue of (A.10)

$$\Sigma_{z,k}(3,1)+\Sigma_{z,k}(3,2)+\Sigma_{z,k}(3,4)\log\tfrac{1}{z} = O\big(z^{-4}(\log\tfrac{1}{z})^2\big) \qquad (z \downarrow 0).$$

Substituting these three estimates into (A.14) and changing the variable $u = z^{3/2}v$, we obtain, after some simple calculations,

$$\begin{aligned} I_1 &\ge \frac{z^{3/2}}{\pi}\int_0^{z^{-1/10}} \Re \exp\Big\{-\mathrm{i}v\, O(z^{1/6}) - \tfrac{1}{4}v^2(1-\mathrm{e}^{-2t_1}) + O(v^3)\big(z^{1/2}(\log\tfrac{1}{z})^2\big)\Big\}\,\mathrm{d}v \\ &\sim \frac{z^{3/2}}{\pi}\int_0^{\infty} \exp\Big\{-\tfrac{1}{4}v^2(1-\mathrm{e}^{-2t_1})\Big\}\,\mathrm{d}v = \frac{z^{3/2}}{\sqrt{\pi(1-\mathrm{e}^{-2t_1})}} \qquad (z\downarrow 0). \end{aligned} \tag{A.15}$$

Estimation of I_2 is based on the inequality

$$\begin{aligned} |\chi_j(s)|^2 &= \frac{(1-\mathrm{e}^{-zj})^2}{|1-\mathrm{e}^{-zj+\mathrm{i}sj}|^2} = 1 - \frac{|1-\mathrm{e}^{-zj+\mathrm{i}sj}|^2-(1-\mathrm{e}^{-zj})^2}{|1-\mathrm{e}^{-zj+\mathrm{i}sj}|^2} \\ &= 1-\frac{2\mathrm{e}^{-zj}(1-\cos sj)}{|1-\mathrm{e}^{-zj+\mathrm{i}sj}|^2} \le 1-\frac{2\mathrm{e}^{-zj}(1-\cos sj)}{(1+\mathrm{e}^{-zj})^2} \le 1-\frac{\mathrm{e}^{-zj}(1-\cos sj)}{2}. \end{aligned}$$

This implies, for $k > k_1 := \lfloor t_1/z \rfloor$ as in the statement of the lemma, that

$$\begin{aligned} |I_2| \le \frac{1}{\pi}\int_{z^{7/5}}^{\pi} \prod_{j=1}^{k} |\chi_j(s)|\,\mathrm{d}s &= \frac{1}{\pi}\int_{z^{7/5}}^{\pi} \exp\bigg\{\frac{1}{2}\sum_{j=1}^{k}\log|\chi_j(s)|^2\bigg\}\,\mathrm{d}s \\ &\le \frac{1}{\pi}\int_{z^{7/5}}^{\pi} \exp\bigg\{\frac{1}{2}\sum_{j=1}^{k_1}\log\Big(1-\frac{\mathrm{e}^{-zj}}{2}(1-\cos sj)\Big)\bigg\}\,\mathrm{d}s \\ &\le \frac{2}{\pi}\int_{z^{7/5}/2}^{\pi/2} \exp\bigg\{-\frac{\mathrm{e}^{-t_1}}{4}\sum_{j=0}^{k_1}(1-\cos 2ju)\bigg\}\,\mathrm{d}u, \end{aligned} \tag{A.16}$$

where the substitution $s = 2u$ is made in the last line. The last sum in (A.16) can be easily estimated: for $u \in [0, \frac{1}{2}\pi]$

$$\sum_{j=0}^{k_1}(1-\cos 2ju) = \frac{2k_1+1}{2} - \frac{\sin((2k_1+1)u)}{2\sin u} \ge \min\left\{\frac{k_1^3u^2}{3}, \frac{2k_1+1}{4}\right\}, \tag{A.17}$$

where for $u \in [0, \pi/(2k_1+1)]$ the inequality (A.17) follows from the elementary inequalities $u - u^3/6 \le \sin u \le u$ and $\sin x \le x - x^3/12$, $x \in [0,\pi]$ (applied with $x = (2k_1+1)u$), while for $u \in [\pi/(2k_1+1), \pi/2]$ (A.17) follows from the inequalities $|\sin x| \le 1$ and $\sin u \ge 2u/\pi \ge 2/(2k_1+1)$. Hence, for $u \in [\frac{1}{2}z^{7/5}, \frac{1}{2}\pi]$ and small $z > 0$, the sum (A.17) is bounded below by $t_1^3 z^{-1/5}/12$, and this estimate combined with (A.16) yields

$$|I_2| \le \exp\{-t_1^3\mathrm{e}^{-t_1}z^{-1/5}/48\}. \tag{A.18}$$

Plugging (A.15) and (A.18) in to (A.13) and using (A.12) to reformulate the obtained estimate in terms of n yields the result. $\square$

A.2. Proof of Proposition 2.4. Consider case (a). Substituting (2.20) into (2.5), we obtain

$$s_{k_n} = \frac{q\,T^2}{2\,\vartheta_q(T)^2}\,n + O(n^{\beta+1}) \qquad (n\to\infty), \tag{A.19}$$

where the first term disappears for $q=0$. From (2.13) and (A.19) it follows, for $q\ge 0$,

$$n - s_{k_n} \sim n\left(1-\frac{q\,T^2}{2\vartheta_q(T)^2}\right) = n\,\frac{\mathrm{Li}_2(1-\mathrm{e}^{-T})}{\vartheta_q(T)^2} \qquad (n\to\infty). \tag{A.20}$$

Let $z_n>0$ be the unique solution of the equation

$$(n-s_{k_n})z^2 = \mathrm{Li}_2(1-\mathrm{e}^{-k_n z}). \tag{A.21}$$

Using the asymptotic Eqs. (A.20) and (2.20) one can verify that the limit

$$\xi := \lim_{n\to\infty}\frac{z_n\sqrt{n}}{\vartheta_q(T)} \tag{A.22}$$

must satisfy the equation

$$\xi^2\,\mathrm{Li}_2(1-\mathrm{e}^{-T}) = \mathrm{Li}_2(1-\mathrm{e}^{-T\xi}),$$

which has the unique root $\xi=1$. As a result, the relation (2.21) holds for such z_n; it also follows that $z_n k_n\to T$ as $n\to\infty$.

On the other hand, by Lemma A.3 we obtain, for $v>\frac{3}{4}$ and large enough n,

$$\mu^{\mathfrak{q}}_{z_n,k_n}\{N(\lambda)=n\}\ge n^{-v}. \tag{A.23}$$

Let the event $A_{z,k}$ be as given in Proposition 2.4, then

$$\nu^{\mathfrak{q}}_{n,k_n}(A_{z_n,k_n}) = \frac{\mu^{\mathfrak{q}}_{z_n,k_n}(A_{z_n,k_n}\cap\{N(\lambda)=n\})}{\mu^{\mathfrak{q}}_{z_n,k_n}\{N(\lambda)=n\}} \le \frac{\mu^{\mathfrak{q}}_{z_n,k_n}(A_{z_n,k_n})}{\mu^{\mathfrak{q}}_{z_n,k_n}\{N(\lambda)=n\}}$$

and an application of (A.23) and (2.19) with $z=z_n$ and $k(z_n)=k_n$ readily gives (2.22).

Case (b) is considered in a similar manner. The assumption $k(z)=o(z^{-2/(\beta+1)})$ (with $\beta<1$) and (2.23) imply that $k_n\sim\pi k/\sqrt{6n}=o(n^{1/(\beta+1)})$ as $n\to\infty$. In turn, it follows from (2.5) that $s_{k_n}=o(n)$. Hence, if $z_n>0$ is the solution of (A.21) then $\xi:=\lim_{n\to\infty} z_n\sqrt{n}/\vartheta_0$ with $\vartheta_0\equiv\vartheta_0(\infty)=\pi/\sqrt{6}$ (see (1.9)) satisfies

$$\xi^2\,\vartheta_0^2 = \mathrm{Li}_2(1) = \frac{\pi}{\sqrt{6}},$$

which readily implies that $\xi=1$. The rest of the proof is the same as for case (a) above.

A.3. Proof of Proposition 2.3. For any $z > 0$,

$$\nu_n^{\mathfrak{q}}(A_z) = \frac{\mu_z^{\mathfrak{q}}\bigl(A_z \cap \{N(\lambda) = n\}\bigr)}{\mu_z^{\mathfrak{q}}\{N(\lambda) = n\}} \le \frac{\mu_z^{\mathfrak{q}}(A_z)}{\mu_z^{\mathfrak{q}}\{N(\lambda) = n\}}\,. \tag{A.24}$$

The upper bound for the numerator on the right-hand side of (A.24) is guaranteed by condition (2.16), and the denominator can be bounded below as follows. Recall that the measure $\mu_{z,k}^{\mathfrak{q}}$ is the probability measure $\mu_z^{\mathfrak{q}}$ conditioned on the event $\{K(\lambda) = k\}$; hence, by the total probability formula we have

$$\mu_z^{\mathfrak{q}}\{N(\lambda) = n\} = \sum_{k=0}^{\infty} \mu_{z,k}^{\mathfrak{q}}\{N(\lambda) = n\} \cdot \mu_z^{\mathfrak{q}}\{K(\lambda) = k\}, \qquad n \in \mathbb{N}_0. \tag{A.25}$$

By virtue of Lemma 3.1(a), $k = k_* \equiv k_*(z)$ defined in (3.2) maximizes $\mu_z^{\mathfrak{q}}\{K(\lambda) = k\}$, and if $q > 0$, Theorem 3.4 applied with $c = \frac{1}{4}$ guarantees that, for any $\gamma \in \bigl(0, \frac{1}{2}(1-\beta)\bigr)$ and for $z > 0$ small enough,

$$\mu_z^{\mathfrak{q}}\{K(\lambda) = k_*\} \ge \frac{1 - \mu_z^{\mathfrak{q}}\{|K(\lambda) - k_*| > c z^{\gamma-1}\}}{1 + 2c z^{\gamma-1}} \sim \tfrac{1}{2} c^{-1} z^{1-\gamma} = 2 z^{1-\gamma}. \tag{A.26}$$

If $q = 0$ we refer to Theorem 3.7 instead, which gives, for any $\gamma \in \bigl(0, \frac{1}{2}(1-\beta)\bigr)$ and $z > 0$ small enough,

$$\mu_z^{\mathfrak{q}}\{K(\lambda) = k_*\} \ge \frac{1 - \mu_z^{\mathfrak{q}}\{K(\lambda) > k_\gamma\}}{k_\gamma} \ge \tfrac{1}{2} z^{2(1-\gamma)}, \tag{A.27}$$

because $k_\gamma(z) \le \lceil z^{-2(1-\gamma)} \rceil$ (see (3.28)).

Let (z_n) be a positive sequence satisfying, for large enough $n \in \mathbb{N}$, the inequality

$$\Bigl| n - s_{k_*(z)} - z^{-2}\,\mathrm{Li}_2(1 - \mathrm{e}^{-z k_*(z)}) \Bigr| \le z^{-4/3}. \tag{A.28}$$

It is easy to see that z_n must vanish in the limit as $n \to \infty$. Solutions of (A.28) exist despite the discontinuities of the function $z \mapsto s_{k_*(z)}$, because $k_*(z)$ has unit jumps and, consequently, the condition (1.2) and the asymptotic formula (3.14) imply that the jumps of $s_{k_*(z)}$ are bounded by $Q_{k_*(z)} = O(z^{-1})$ for $q > 0$, while for $q = 0$ the upper bound in (3.13) gives $Q_{k_*(z)} = O\bigl(z^{-\beta}(\log \frac{1}{z})^{\beta}\bigr)$. Thus, the left-hand side of (A.28) has discontinuities of order $O(z^{-1})$ as $z \downarrow 0$, which is much smaller than the term $z^{-4/3}$ on the right-hand side. Furthermore, note that $z_n k_*(z_n) \to T_q$ (see (3.14)). Hence, in the same fashion as in the proof of Proposition 2.4, we obtain that due to (A.28) the limit $\xi := \lim_{n\to\infty} z_n \sqrt{n}/\vartheta_q$ satisfies the equation

$$\xi^2 \vartheta_q^2 - \tfrac{1}{2} q T_q^2 = \mathrm{Li}_2(1 - \mathrm{e}^{-T_q}).$$

[For $q = 0$, use the values $T_0 = \infty$, $qT_q^2|_{q=0} = 0$ and $\vartheta_0 = \sqrt{\mathrm{Li}_2(1)} = \pi/\sqrt{6}$ (see (1.9)).] Comparing this with Eq. (2.13), we conclude that $\xi = 1$, and (2.17) readily follows.

With $z = z_n$ and $k = k_*(z_n)$, the conditions of Lemma A.3 are satisfied, so (A.25) and (A.26) (or (A.27) for $q = 0$) yield that, for any $v > \frac{3}{4}$ and for n large enough,

$$\mu_{z_n}^{\mathfrak{q}}\{N(\lambda) = n\} \ge \mu_{z_n, k_*(z_n)}^{\mathfrak{q}}\{N(\lambda) = n\} \cdot \mu_{z_n}^{\mathfrak{q}}\{K(\lambda) = k_*(z_n)\} \ge \tfrac{1}{2} n^{-v} z_n^{\sigma}, \tag{A.29}$$

where $\sigma = 1 - \gamma$ when $q > 0$ and $\sigma = 2(1-\gamma)$ when $q = 0$. But $z_n \sim \mathrm{const} \cdot n^{-1/2}$, so (A.29) provides a lower bound which is polynomial in $n \to \infty$. The claim of the proposition now follows from the estimates (A.29) and (A.24).

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