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EXTREMES AND LIMIT THEOREMS FOR DIFFERENCE OF CHI-TYPE PROCESSES

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Abstract: Let $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}, \kappa > 0$ be random processes defined as the differences of two independent stationary chi-type processes with m and k degrees of freedom. In applications such as physical sciences and engineering dealing with structure reliability, of interest is the approximation of the probability that the random process $\zeta_{m,k}^{(\kappa)}$ stays in some safety region up to a fixed time T. In this paper, utilizing Albin's methodology we derive the asymptotics of $\mathbb{P}\left\{\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\}, u \to \infty$ under some assumptions on the covariance structures of the underlying Gaussian processes. We establish further a Berman sojourn limit theorem and a Gumbel limit result.

Key Words: Stationary Gaussian process; stationary chi-type process; extremes; Berman sojourn limit theorem; Gumbel limit theorem; Berman's condition.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_{m+k}(t)), t \ge 0, m \ge 1, k \ge 0$ be a vector process with independent components which are centered stationary Gaussian processes with almost surely (a.s.) continuous sample paths and covariance functions satisfying

$$r_i(t) = 1 - C_i \left|t\right|^{\alpha} + o(\left|t\right|^{\alpha}), \quad t \to 0 \quad \text{and} \quad r_i(t) < 1, \quad \forall t \neq 0 \tag{1}$$

where $\alpha \in (0, 2]$ and $\boldsymbol{C} := (C_1, \ldots, C_{m+k}) \in (0, \infty)^{m+k}$. We define the following stationary non-Gaussian processes $\left\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\right\}$, $\kappa > 0$ by

$$\zeta_{m,k}^{(\kappa)}(t) := \left(\sum_{i=1}^{m} X_i^2(t)\right)^{\kappa/2} - \left(\sum_{i=m+1}^{m+k} X_i^2(t)\right)^{\kappa/2} =: |\boldsymbol{X}^{(1)}(t)|^{\kappa} - |\boldsymbol{X}^{(2)}(t)|^{\kappa}, \quad t \ge 0. \tag{2}$$

In this paper we shall investigate for any $T > 0$ the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\}, \quad u \to \infty \tag{3}$$

by using Albin's method established in [?].

Our study of the tail asymptotics of $\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t)$ is motivated by the exit problem in engineering sciences; see e.g., [?, ?, ?] and the references therein. Specifically, of interest is the probability that the Gaussian vector process $\boldsymbol{X}$ exits a predefined safety region $\boldsymbol{S}_u \subset \mathbb{R}^{m+k}$ up to the time T, namely

$$\mathbb{P}\left\{\boldsymbol{X}(t) \notin \boldsymbol{S}_u, \quad \text{for some } t \in [0, T]\right\}.$$

Various types of safety regions $\boldsymbol{S}_u$ were considered for smooth Gaussian vector processes in the aforementioned papers. Particularly, a safety region given by a ball centered at 0 with radius $u > 0$

$$\boldsymbol{B}_u = \left\{(x_1, \ldots, x_{m+k}) \in \mathbb{R}^{m+k} : \left(\sum_{i=1}^{m+k} x_i^2\right)^{1/2} \le u\right\}$$

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has been extensively studied; see, e.g., [?, ?, ?, ?]. Referring to [?, ?], we know that for $k = 0$

$$\begin{aligned}\mathbb{P}\left\{\boldsymbol{X}(t) \notin \boldsymbol{B}_u, \quad \text{for some } t \in [0,T]\right\} &= \mathbb{P}\left\{\sup_{t\in[0,T]} \zeta^{(1)}_{m,0}(t) > u\right\} \\ &= T H^{m,0}_{\alpha,1}(\boldsymbol{C}) u^{\frac{2}{\alpha}} \mathbb{P}\left\{\zeta^{(1)}_{m,0}(0) > u\right\}(1+o(1)), \quad u \to \infty \end{aligned} \tag{4}$$

where $H^{m,0}_{\alpha,1}(\boldsymbol{C})$ is a positive constant (see (8) below for a precise definition). Very recently [?] obtained the tail asymptotics of $\sup_{t\in[0,T]} \zeta^{(2)}_{1,1}(t)$.

Our first result, which derives the exact asymptotics of (3), extends the findings of [?, ?] and suggests an asymptotic approximation for the exit probability of the Gaussian vector process $\boldsymbol{X}$ from the safety regions $\boldsymbol{S}^{(\kappa)}_u$ given, with the notation of (2), as

$$\boldsymbol{S}^{(\kappa)}_u = \left\{(x_1, \ldots, x_{m+k}) \in \mathbb{R}^{m+k} : |\boldsymbol{x}^{(1)}|^\kappa - |\boldsymbol{x}^{(2)}|^\kappa \le u\right\} \tag{5}$$

for large enough u.

Since chi-type processes appear naturally as limiting processes; see, e.g., [?, ?], when one considers two independent asymptotic models, the study of the supremum of the difference of the two chi-type processes is of some interest in mathematical statistics and its applications. Another motivation for considering the tail asymptotics of the supremum of the difference of chi-type processes is from ruin theory, where the tail asymptotics can be interpreted as the expansion of the ruin probability since the net loss of an insurance company can be modelled by the difference of two positive random processes; see, e.g., [?].

Although for $k \ge 1$ the random process $\zeta^{(\kappa)}_{m,k}$ is not Gaussian and the analysis of the supremum can not be transformed into the study of the supremum of a related Gaussian random field (which is the case for chi-type processes; see, e.g., [?, ?, ?, ?, ?, ?]), it turns out that it is possible to apply the techniques for dealing with extremes of stationary processes developed mainly in [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?].

Sojourn limit theorems, initiated by Berman [?, ?], have been proved to be significant results in the study of extreme values of stationary and self-similar processes; see, e.g., [?, ?]. In the second part of Section 2 we derive a sojourn limit theorem for $\zeta^{(\kappa)}_{m,k}$. Further, we show a Gumbel limit theorem for the supremum of $\zeta^{(\kappa)}_{m,k}$ over an increasing infinite interval. We refer to [?, ?, ?, ?, ?, ?] for results on the Gumbel limit theorem for Gaussian processes and chi-type processes.

Brief outline of the paper: Our main results are stated in Section 2. In Section 3 we present proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3 followed then by an Appendix containing the somewhat complicated proofs of three lemmas utilized in Section 3.

2. Main Results

We start by introducing some notation. Let $\{Z(t), t \ge 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0, 1]$, i.e., it is a centered Gaussian process with a.s. continuous sample paths and covariance function

$$\mathrm{Cov}(Z(s), Z(t)) = \frac{1}{2}\left(s^\alpha + t^\alpha - |s-t|^\alpha\right), \quad s, t \ge 0.$$

In the following, let $\{Z_i(t), t \ge 0\}, 1 \le i \le m+k$ be independent copies of Z and define $\mathcal{W}_\kappa$ to be a Gamma distributed random variable with parameter $(k/\kappa, 1)$. Further let $\boldsymbol{O}_1 = (O_1, \ldots, O_m), \boldsymbol{O}_2 = (O_{m+1}, \ldots, O_{m+k})$ denote two random vectors uniformly distributed on the unit sphere of $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Hereafter we shall suppose that $\boldsymbol{O}_1, \boldsymbol{O}_2, \mathcal{W}_\kappa$ and Z_i's are mutually independent. Define for $m \ge 1, \ k \ge 0, \ \kappa > 0$

$$\eta^{(\kappa)}_{m,k}(t) = \widetilde{Z}^{(\kappa)}_{m,k}(t) + E, \quad t \ge 0, \tag{6}$$

where E is a unit mean exponential random variable being independent of all the other random elements involved, and

$$\widetilde{Z}^{(\kappa)}_{m,k}(t) = \left(\sum_{i=1}^{m} \sqrt{2C_i}O_iZ_i(t) - L(t)\right)\mathbb{I}\{\kappa \ge 1\}$$
$$+ \left(\mathcal{W}_\kappa - \left(\mathcal{W}_\kappa^{2/\kappa} + 2(\mathcal{W}_\kappa/\kappa)^{1/\kappa}\sum_{i=m+1}^{m+k}\sqrt{2C_i}O_iZ_i(t) + 2\kappa^{-2/\kappa}\sum_{i=m+1}^{m+k}C_iZ_i^2(t)\right)^{\kappa/2}\right)\mathbb{I}\{\kappa \le 1\}$$

with $L(t) = \left(\sum_{i=1}^m C_iO_i^2\right)t^\alpha$, $\mathbb{I}\{\cdot\}$ the indicator function and the convention that $\sum_{i=m+1}^m = 0$. In addition, denote by $\Gamma(\cdot)$ the Euler Gamma function. We state next our main result.

Theorem 2.1. *Let $\{\zeta^{(\kappa)}_{m,k}(t), t \ge 0\}$ be given by (2) with the involved Gaussian processes X_i's satisfying (1). Then for any $T > 0$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t) > u\right\} = TH^{m,k}_{\alpha,\kappa}(\boldsymbol{C})u^{\frac{2\tau}{\alpha\kappa}}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\}(1+o(1)), \quad u\to\infty \tag{7}$$

where $\tau := \mathbb{I}\{\kappa \ge 1\} + (2/\kappa - 1)\mathbb{I}\{\kappa < 1\}$ and

$$H^{m,k}_{\alpha,\kappa}(\boldsymbol{C}) = \lim_{a\downarrow 0}\frac{1}{a}\mathbb{P}\left\{\sup_{j\ge1}\eta^{(\kappa)}_{m,k}(aj) \le 0\right\} \in (0,\infty) \tag{8}$$

with $\eta^{(\kappa)}_{m,k}$ given by (6).

Remarks: The tail asymptotics of the Gaussian chaos $\zeta^{(\kappa)}_{m,k}(0)$ and its density can be easily derived using Theorem 1 in [**?**]. We give a self-contained proof in Lemma 3.1 below.

b) Clearly, $H^{m,k}_{\alpha,\kappa}(\boldsymbol{C})$ in (8) is more involved than the classical Pickands constant

$$H_\alpha = \lim_{a\downarrow0}\frac{1}{a}\mathbb{P}\left\{\sup_{j\ge1}\left(\sqrt{2}Z(aj) - (aj)^\alpha\right) \le -E\right\} \in (0,\infty),$$

see, e.g., [**?**, **?**, **?**] for the above definition which is an alternative expression of the Pickands constant (cf. [**?**]).

c) Define exit times $\tau_\kappa(u) = \inf\{t > 0 : \boldsymbol{X}(t) \notin \boldsymbol{S}^{(\kappa)}_u\}, \kappa > 0$ with $\boldsymbol{S}^{(\kappa)}_u$ given by (5). By a direct application of Theorem 2.1 for any $T > 0$ we obtain

$$\lim_{u\to\infty}\mathbb{P}\{\tau_\kappa(u) \le t | \tau_\kappa(u) \le T\} = \frac{t}{T}, \quad \forall t\in[0,T],$$

which means that asymptotically $\tau_\kappa(u)|\{\tau_\kappa(u) \le T\}$ is uniformly distributed on $[0,T]$.

d) If $\kappa > 2$, then the claim of Theorem 2.1 implies

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,k}(t) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,0}(t) > u\right\}(1+o(1)), \quad u\to\infty, \tag{9}$$

hence $X_{m+1},\dots,X_{m+k}$ do not influence the tail asymptotics of $\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,k}(t)$. This is expected since $\sup_{t\in[0,T]}\zeta^{(\kappa)}_{m,k}(t)$ has a sub-exponential tail behaviour for any $\kappa > 2$.

Consider the sojourn time of the random process $\zeta^{(\kappa)}_{m,k}$ above a threshold $u > 0$ in the time interval $[0,t]$ defined by

$$L^{(\kappa)}_{m,k,t}(u) = \int_0^t \mathbb{I}\{\zeta^{(\kappa)}_{m,k}(s) > u\}\,ds, \quad t > 0. \tag{10}$$

Our second result below concerns a Berman sojourn limit theorem for $\zeta^{(\kappa)}_{m,k}$.

Theorem 2.2. *Under the assumptions and notation of Theorem 2.1 for any $t > 0$*

$$\int_x^\infty \mathbb{P}\left\{u^{\frac{2\tau}{\alpha\kappa}}L^{(\kappa)}_{m,k,t}(u) > y\right\}dy = u^{\frac{2\tau}{\alpha\kappa}}\mathbb{E}\left\{L^{(\kappa)}_{m,k,t}(u)\right\}\Upsilon_\kappa(x)(1+o(1)), \quad u\to\infty \tag{11}$$

holds for all continuity point $x > 0$ of $\Upsilon_\kappa(x) := \mathbb{P}\left\{\int_0^\infty \mathbb{I}\{\eta^{(\kappa)}_{m,k}(s) > 0\}\,ds > x\right\}$.

Remarks: a) It might be possible to allow X_i's to be dependent. Results for extremes of chi-type processes for such generalizations can be found in [?, ?].
b) Following the methodology in [?] one could consider X_i's to be self-similar Gaussian processes. Further extensions for random fields could also be possible by adopting the recent findings in [?, ?].

In the following, we derive a Gumbel limit theorem for $\sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t)$ under a linear normalization.

Theorem 2.3. *Under the assumptions and notation of Theorem 2.1, if further the following Berman's condition*

$$\lim_{t\to\infty} \max_{1\le l\le m+k} |r_l(t)| \ln t = 0 \tag{12}$$

holds, then

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ a^{(\kappa)}_T \Big(\sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t) - b^{(\kappa)}_T \Big) \le x \right\} - \exp\left(-e^{-x}\right) \right| = 0 \tag{13}$$

where for $T>0$

$$a^{(\kappa)}_T = \frac{(2\ln T)^{1-\kappa/2}}{\kappa}, \quad b^{(\kappa)}_T = (2\ln T)^{\kappa/2} + \frac{\kappa}{2(2\ln T)^{1-\kappa/2}} \left(K_0 \ln\ln T + \ln D_0\right) \tag{14}$$

with

$$\begin{aligned} D_0 &= 2^{\frac{2\tau}{\alpha}+2\left(1-\frac{k}{\kappa}\right)\mathbb{I}\{\kappa\le 2\}} \left(\frac{H^{m,k}_{\alpha,\kappa}(\boldsymbol{C})}{\Gamma(m/2)\Gamma(k/2)} \Gamma\left(\frac{k}{\kappa}\mathbb{I}\{\kappa\le 2\} + \frac{k}{2}\mathbb{I}\{\kappa>2\}\right) \kappa^{(k/\kappa-1)\mathbb{I}\{\kappa<2\}-\mathbb{I}\{\kappa=2\}} \right)^2 \\ K_0 &= m-2+\frac{2\tau}{\alpha} + k\left(1-\frac{2}{\kappa}\right)\mathbb{I}\{\kappa\le 2\}. \end{aligned}$$

Under the assumptions of Theorem 2.3, we have the following convergence in probability (denoted by $\xrightarrow{p}$)

$$\frac{\sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t)}{(2\ln T)^{\kappa/2}} \xrightarrow{p} 1, \quad T\to\infty$$

which follows from the fact that $\lim_{T\to\infty} b^{(\kappa)}_T/(2\ln T)^{\kappa/2} = 1$ and that $a^{(\kappa)}_T$ is bounded away from zero, together with elementary considerations. In several cases such a convergence in probability can be strengthened to the pth mean convergence which is referred to as the Seleznjev pth mean convergence since the idea was first suggested in [?]. In order to show the Seleznjev pth mean convergence of crucial importance is the Piterbarg inequality (see [?], Theorem 8.1). Since the Piterbarg inequality holds also for chi-square processes (see [?], Proposition 3.2), using further the fact that

$$\zeta^{(\kappa)}_{m,k}(t) \le |\boldsymbol{X}^{(1)}(t)|^{\kappa}, \quad t\ge 0$$

we immediately get the Piterbarg inequality for the difference of chi-type processes by simply applying the aforementioned proposition. Specifically, under the assumptions of Theorem 2.3 for any $T>0$ and all large u

$$\mathbb{P}\left\{ \sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t) > u \right\} \le KTu^{\beta} \exp\left(-\frac{1}{2}u^{2/\kappa}\right) \tag{15}$$

where K and β are two positive constants not depending on T and u. Note that the above result also follows immediately from Theorem 2.1 combined with Lemma 3.1 below. Hence utilizing Lemma 4.5 in [?] we arrive at our last result.

Corollary 2.4. *(Seleznjev pth mean theorem) Under the assumptions of Theorem 2.3 we have for any* $p>0$

$$\lim_{T\to\infty} \mathbb{E}\left\{ \left(\frac{\sup_{t\in[0,T]} \zeta^{(\kappa)}_{m,k}(t)}{(2\ln T)^{\kappa/2}} \right)^p \right\} = 1.$$

3. Further Results and Proofs

We shall first give some preliminary lemmas; hereafter we use the same notation and assumptions as in Section 1. By $\overset{d}{\to}$ and $\overset{d}{=}$ we shall denote the convergence in distribution (or the convergence of finite-dimensional distributions if both sides of it are random processes) and equality in distribution function, respectively. Further, we write $f_\xi(\cdot)$ for the pdf of a random variable ξ and write $h_1 \sim h_2$ if two functions $h_i(\cdot), i = 1,2$ are such that h_1/h_2 goes to 1 as the argument tends to some limit. For simplicity we shall denote for $\kappa > 0$ (recalling $\tau = \mathbb{I}\{\kappa \ge 1\} + (2/\kappa - 1)\mathbb{I}\{\kappa<1\}$)

$$q_\kappa = q_\kappa(u) = u^{-2\tau/(\alpha\kappa)}, \quad w_\kappa(u) = \frac{1}{\kappa}u^{2/\kappa-1}, \quad u > 0.$$

In the proofs of Lemmas 3.1–3.3, we denote $u_{\kappa,x} = u + x/w_\kappa(u)$ for all $u, x > 0$.

Lemma 3.1. *For all integers $m \ge 1, k \ge 0$ we have as $u \to \infty$*

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} \sim \frac{f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}{w_\kappa(u)} \sim \frac{2^{2-(m+k)/2}}{\kappa^2\Gamma(k/2)\Gamma(m/2)}\frac{1}{w_\kappa(u)}u^{m/\kappa-1}\exp\left(-\frac{1}{2}u^{2/\kappa}\right)\begin{cases}\frac{\Gamma(k/\kappa)}{(w_\kappa(u))^{k/\kappa}}, & \kappa < 2;\\ \Gamma(k/2), & \kappa = 2;\\ \kappa 2^{k/2-1}\Gamma(k/2), & \kappa > 2.\end{cases}$$

Proof of Lemma 3.1: For $k = 0$ the claim of the lemma is elementary (see, e.g., [**?**], p.117).
Note that for any $k \ge 1$

$$f_{|\mathbf{X}^{(2)}(0)|^\kappa}(y) = \frac{2^{1-k/2}}{\kappa\Gamma(k/2)}y^{k/\kappa-1}\exp\left(-\frac{1}{2}y^{2/\kappa}\right), \quad y \ge 0.$$

We have by e.g., [**?**], p.117 together with elementary consideration

$$\begin{aligned} f_{\zeta_{m,k}^{(\kappa)}(0)}(u) &= \frac{1}{w_\kappa(u)}\int_0^\infty f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})\, f_{|\mathbf{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)dy\\ &= \frac{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u)}{w_\kappa(u)}\int_0^\infty \frac{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})}{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u)}\frac{2^{1-k/2}}{\kappa\Gamma(k/2)}\left(\frac{y}{w_\kappa(u)}\right)^{k/\kappa-1}\exp\left(-\frac{1}{2}\left(\frac{y}{w_\kappa(u)}\right)^{2/\kappa}\right)dy\\ &\sim \frac{2^{1-k/2}}{\kappa\Gamma(k/2)}\frac{f_{|\mathbf{X}^{(1)}(0)|^\kappa}(u)}{w_\kappa(u)}\int_0^\infty\left(\frac{y}{w_\kappa(u)}\right)^{k/\kappa-1}\exp\left(-\frac{1}{2}\left(\frac{y}{w_\kappa(u)}\right)^{2/\kappa} - y\right)dy, \quad u \to \infty. \end{aligned}$$

Recalling that $w_\kappa(u) \to \infty, = 1/2, \to 0$ correspond to $\kappa <, =, > 2$, respectively, we conclude the second claimed asymptotic relation of the lemma. The first claimed asymptotic relation then follows similarly as

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} = \frac{f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}{w_\kappa(u)}\int_0^\infty \frac{f_{\zeta_{m,k}^{(\kappa)}(0)}(u_{\kappa,x})}{f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}\,dx \sim \frac{f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}{w_\kappa(u)}\int_0^\infty e^{-x}\,dx.$$

□

Lemma 3.2. *If $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}$ is as in Theorem 2.1, then*

$$\left\{w_\kappa(u)(\zeta_{m,k}^{(\kappa)}(q_\kappa t) - u)|(\zeta_{m,k}^{(\kappa)}(0) > u),\ t \ge 0\right\} \overset{d}{\to} \left\{\eta_{m,k}^{(\kappa)}(t),\ t \ge 0\right\}, \quad u \to \infty$$

with $\eta_{m,k}^{(\kappa)}$ given by (6).

Proof of Lemma 3.2: We henceforth adapt the notation introduced in Section 2. By Lemma 3.1

$$w_\kappa(u)(\zeta_{m,k}^{(\kappa)}(0) - u)\Big|(\zeta_{m,k}^{(\kappa)}(0) - u > 0) \overset{d}{\to} E, \quad u \to \infty. \tag{16}$$

Thus, in view of Theorem 5.1 in [?], it suffices to show that, for any $0 < t_1 < t_2 < \cdots < t_n < \infty$ and $z_j \in \mathbb{R}, 1 \le j \le n$, $n \in \mathbb{N}$,

$$\begin{aligned} p_k(u) &:= \mathbb{P}\left\{\cap_{j=1}^n \{\zeta_{m,k}^{(\kappa)}(q_\kappa t_j) \le u_{\kappa,z_j}\}\Big|\zeta_{m,k}^{(\kappa)}(0) = u_{\kappa,x}\right\} \\ &\to \mathbb{P}\left\{\cap_{j=1}^n \{\widetilde{Z}_{m,k}^{(\kappa)}(t_j)+x \le z_j\}\right\}, \quad u \to \infty \end{aligned} \tag{17}$$

holds for all $x > 0$ and $z_j \in \mathbb{R}, 1 \le j \le n$. Define below

$$\Delta_{iu}(t_j) = X_i(q_\kappa t_j) - r_i(q_\kappa t_j)X_i(0), \quad 1 \le i \le m+k,\ 1 \le j \le n.$$

By (1) we have

$$\begin{aligned} u^{2\tau/\kappa}\mathrm{Cov}(\Delta_{iu}(s), \Delta_{iu}(t)) &\to C_i(s^\alpha + t^\alpha - |s-t|^\alpha) \\ &= 2C_i\mathrm{Cov}(Z_i(s), Z_i(t)), \quad u \to \infty,\ s,t > 0,\ 1 \le i \le m+k. \end{aligned}$$

Therefore

$$\{u^{\tau/\kappa}\Delta_{iu}(t), t \ge 0\} \overset{d}{\to} \{\sqrt{2C_i}Z_i(t), t \ge 0\}, \quad u \to \infty, \quad 1 \le i \le m+k.$$

Furthermore, by the independence of $\Delta_{iu}(t_j)$'s and $X_i(0)$'s, the random processes Z_i's can be chosen such that they are independent of $\zeta_{m,k}^{(\kappa)}(0)$. Note that $\boldsymbol{X}^{(1)}(0) \overset{d}{=} R_1\boldsymbol{O}_1$ holds for some $R_1 > 0$ which is independent of $\boldsymbol{O}_1$. Then similar arguments as in [?] yield that, for any $z_j \in \mathbb{R}$, $1 \le j \le n$

$$\begin{aligned} p_0(u) &= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{|\boldsymbol{X}^{(1)}(q_\kappa t_j)|^\kappa \le u_{\kappa,z_j}\right\} \Big| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x}\right\} \\ &= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{w_\kappa(u)\left(R_1^\kappa\left[1+\frac{1}{R_1^2}V_u(t_j)\right]^{\kappa/2} - R_1^\kappa\right) \le z_j - x\right\}\Big| R_1^\kappa = u_{\kappa,x}\right\} \\ &= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{\frac{\kappa}{2}w_\kappa(u)R_1^{\kappa-2}V_u(t_j)(1+o_p(1)) \le z_j - x\right\} \Big| R_1^\kappa = u_{\kappa,x}\right\} \\ &= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{\sum_{i=1}^m \frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1+o_p(1)) - \left(\sum_{i=1}^m \frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1+o_p(1)) + x \le z_j\right\}\right\}, \quad u \to \infty \end{aligned} \tag{18}$$

where $V_u(t_j) := \sum_{i=1}^m \Delta_{iu}^2(t_j) + 2\sum_{i=1}^m \Delta_{iu}(t_j)r_i(q_\kappa t_j)X_i(0) - \sum_{i=1}^m(1-r_i^2(q_\kappa t_j))X_i^2(0)$. Consequently, the claim for $k = 0$ follows. Next, for $k \ge 1$, we rewrite $p_k(u)$ as

$$\begin{aligned} p_k(u) &= \int_0^\infty \mathbb{P}\left\{\bigcap_{j=1}^n \left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t_j) \le u_{\kappa,z_j}\right\} \Big| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\right\} \\ &\quad\times \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u)f_{\zeta_{m,k}^{(\kappa)}(0)}(u_{\kappa,x})}\,dy \\ &= \int_0^\infty \mathbb{P}\left\{\bigcap_{j=1}^n \left\{|\boldsymbol{X}^{(1)}(q_\kappa t_j)|^\kappa \le u_{\kappa,z_j+w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t_j)|^\kappa}\right\} \Big| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\right\} h_{\kappa,u}(y)\,dy \end{aligned} \tag{19}$$

where by Lemma 3.1

$$
\begin{aligned}
h_{\kappa,u}(y) &:= \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u) f_{\zeta^{(\kappa)}_{m,k}(0)}(u_{\kappa,x})} \\
&= \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})}{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u)} \frac{f_{\zeta^{(\kappa)}_{m,k}(0)}(u)}{f_{\zeta^{(\kappa)}_{m,k}(0)}(u_{\kappa,x})} \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u) f_{\zeta^{(\kappa)}_{m,k}(0)}(u)} \\
&\sim \frac{f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u)) e^{-y}}{\int_0^\infty f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u)) e^{-y}\, dy}, \quad u \to \infty
\end{aligned} \tag{20}
$$

implying that $h_{\kappa,u}(\cdot)$ is asymptotically equal to some pdf $h_\kappa(\cdot)$ as $u \to \infty$.
Next we consider the limit distribution of $w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t)|^\kappa \Big| \{w_\kappa(u)|\boldsymbol{X}^{(2)}(0)|^\kappa = y\}$. Noting that $\boldsymbol{X}^{(2)}(0) \overset{d}{=} R_2 \boldsymbol{O}_2$ holds for some $R_2 > 0$ which is independent of $\boldsymbol{O}_2$, we have by similar arguments as in (18) that, for any $t \geq 0$

$$
\begin{aligned}
&\left(w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t)|^\kappa\right)^{2/\kappa} \Big| \{w_\kappa(u)|\boldsymbol{X}^{(2)}(0)|^\kappa = y\} \\
&= (w_\kappa(u))^{2/\kappa} \Bigg[\sum_{i=m+1}^{m+k} X_i^2(0) + 2 \sum_{i=m+1}^{m+k} r_i(q_\kappa t) X_i(0) \Delta_{iu}(t) + \sum_{i=m+1}^{m+k} \Delta_{iu}^2(t) \\
&\quad - \sum_{i=m+1}^{m+k} (1 - r_i(q_\kappa t)^2) X_i^2(0) \Bigg] \Bigg| \left\{ R_2^\kappa = \frac{y}{(w_\kappa(u))^{1/\kappa}} \right\} \\
&= (w_\kappa(u))^{2/\kappa} \Bigg[R_2^2 + 2 \frac{R_2}{u^{\tau/\kappa}} \sum_{i=m+1}^{m+k} \sqrt{2C_i} O_i Z_i(t)(1 + o_p(1)) + \frac{2}{u^{2\tau/\kappa}} \sum_{i=m+1}^{m+k} C_i Z_i^2(t)(1 + o_p(1)) \\
&\quad - 2 \left(\frac{R_2}{u^{\tau/\kappa}} \right)^2 \sum_{i=m+1}^{m+k} C_i O_i^2 t^\alpha (1 + o_p(1)) \Bigg] \Bigg| \left\{ R_2^\kappa = \frac{y}{(w_\kappa(u))^{1/\kappa}} \right\} \\
&= y^{2/\kappa} + 2 y^{1/\kappa} \left(\frac{w_\kappa(u)}{u^\tau} \right)^{1/\kappa} \sum_{i=m+1}^{m+k} \sqrt{2C_i} O_i Z_i(t)(1 + o_p(1)) + 2 \left(\frac{w_\kappa(u)}{u^\tau} \right)^{2/\kappa} \sum_{i=m+1}^{m+k} C_i Z_i^2(t)(1 + o_p(1)).
\end{aligned}
$$

This together with (18) and (19) implies that

$$
\begin{aligned}
p_k(u) &= \int_0^\infty \mathbb{P} \Bigg\{ \bigcap_{j=1}^n \Bigg\{ \sum_{i=1}^m \frac{\sqrt{2C_i} O_i Z_i(t_j)}{u^{(\tau-1)/\kappa}} (1 + o_p(1)) - \left(\sum_{i=1}^m \frac{C_i O_i^2}{u^{2(\tau-1)/\kappa}} \right) t_j^\alpha (1 + o_p(1)) + x + y \\
&\leq z_j + w_\kappa(u) |\boldsymbol{X}^{(2)}(q_\kappa t_j)|^\kappa \Bigg\} \Bigg| |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)} \Bigg\} h_{\kappa,u}(y)\, dy \\
&= \int_0^\infty \mathbb{P} \Bigg\{ \bigcap_{j=1}^n \Bigg\{ \sum_{i=1}^m \frac{\sqrt{2C_i} O_i Z_i(t_j)}{u^{(\tau-1)/\kappa}} (1 + o_p(1)) - \left(\sum_{i=1}^m \frac{C_i O_i^2}{u^{2(\tau-1)/\kappa}} \right) t_j^\alpha (1 + o_p(1)) + x + y \leq z_j \; + \Big(y^{\frac{2}{\kappa}} + 2 y^{\frac{1}{\kappa}} \\
&\times \left(\frac{w_\kappa(u)}{u^\tau} \right)^{\frac{1}{\kappa}} \sum_{i=m+1}^{m+k} \sqrt{2C_i} O_i Z_i(t_j)(1 + o_p(1)) + 2 \left(\frac{w_\kappa(u)}{u^\tau} \right)^{\frac{2}{\kappa}} \sum_{i=m+1}^{m+k} C_i Z_i^2(t_j)(1 + o_p(1)) \Big)^{\frac{\kappa}{2}} \Bigg\} \Bigg\} h_{\kappa,u}(y)\, dy.
\end{aligned}
$$

Recall that $\tau = \mathbb{I}\{\kappa \geq 1\} + (2/\kappa - 1)\mathbb{I}\{\kappa < 1\}$. It follows by (20) and Lemma 3.1 that, for $\kappa \leq 1$

$$
h_\kappa(y) := \lim_{u \to \infty} h_{\kappa,u}(y) = \frac{1}{\Gamma(k/\kappa)} y^{k/\kappa - 1} e^{-y}, \quad y > 0
$$

which is the pdf of a Gamma distributed rv with parameter $(k/\kappa, 1)$. Consequently, the desired result follows. □

The next lemma corresponds to Condition B in [?]; see also [?, ?]. We note in passing that this condition, motivated by [?], is often referred to as the "short-lasting-exceedance" condition. As shown in Chapter 5 in [?] this condition is crucial. Denote in the following by $[x]$ the integer part of $x \in \mathbb{R}$.

Lemma 3.3. *Let $\{\zeta^{(\kappa)}_{m,k}(t), t\ge 0\}$ be given as in Theorem 2.1. For any $T, a>0$, we have*

$$\limsup_{u\to\infty}\sum_{j=N}^{[T/(aq_\kappa)]}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(aq_\kappa j)>u\Big|\zeta^{(\kappa)}_{m,k}(0)>u\right\}\to 0,\quad N\to\infty.$$

Proof of Lemma 3.3: Note first that the case $k=0$ is treated in [?], p.119. Using the fact that the standard bivariate Gaussian distribution is exchangeable, we have for $u>0$

$$\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t)>u\Big|\zeta^{(\kappa)}_{m,k}(0)>u\right\}=2\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t)>u, |\boldsymbol{X}^{(1)}(q_\kappa t)|>|\boldsymbol{X}^{(1)}(0)|\Big|\zeta^{(\kappa)}_{m,k}(0)>u\right\}=:2\Theta(u).$$

Further, it follows from Lemma 3.1 that for any $k\ge 1$

$$\begin{aligned}\Theta(u)&=\int_0^\infty\int_0^\infty\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t)>u, |\boldsymbol{X}^{(1)}(q_\kappa t)|>|\boldsymbol{X}^{(1)}(0)|\Big||\boldsymbol{X}^{(1)}(0)|^\kappa=u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa=\frac{y}{w_\kappa(u)}\right\}\\
&\quad\times\frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)}{w_\kappa^2(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}}\,dxdy\\
&\le\int_0^\infty\int_0^\infty\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa>u_{\kappa,y}\Big||\boldsymbol{X}^{(1)}(0)|^\kappa=u_{\kappa,x+y}\right\}\frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})\,f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)}{w_\kappa^2(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}}\,dxdy\\
&=\int_0^\infty\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa>u_{\kappa,y}\Big||\boldsymbol{X}^{(1)}(0)|^\kappa>u_{\kappa,y}\right\}\frac{\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|^\kappa>u_{\kappa,y}\right\}}{w_\kappa(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}}f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)dy.\end{aligned}$$

Moreover, in view of the treatment of the case $k=0$ in [?], p.119 we readily see that, for any $p\ge 1$, with $R(t):=\max_{1\le i\le m}r_i(t), r(t):=\min_{1\le i\le m}r_i(t)$ and $\Phi(\cdot)$ denoting the $N(0,1)$ distribution function,

$$\begin{aligned}\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa>u_{\kappa,y}\Big||\boldsymbol{X}^{(1)}(0)|^\kappa>u_{\kappa,y}\right\}&\le 4m\left(1-\Phi\left(\frac{(1-R(q_\kappa t))u^{1/\kappa}}{\sqrt{m(1-r^2(q_\kappa t))}}\right)\right)\\
&\le K_pt^{-\alpha p/2},\quad\forall q_\kappa t\in(0,T]\end{aligned}$$

holds for some $K_p>0$ not depending on u,t and y. Consequently

$$\begin{aligned}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t)>u\Big|\zeta^{(\kappa)}_{m,k}(0)>u\right\}&\le 2K_pt^{-\alpha p/2}\int_0^\infty\frac{\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|^\kappa>u_{\kappa,y}\right\}}{w_\kappa(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}}f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)dy\\
&=2K_pt^{-\alpha p/2},\quad\forall q_\kappa t\in(0,T].\end{aligned}\tag{21}$$

Therefore, letting $p=4/\alpha$

$$\begin{aligned}&\limsup_{u\to\infty}\sum_{j=N}^{[T/(aq_\kappa)]}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(aq_\kappa j)>u\Big|\zeta^{(\kappa)}_{m,k}(0)>u\right\}\\
&\le 2K_p\int_{aN}^\infty x^{-2}\,dx=\frac{2K_p}{aN}\to 0,\quad N\to\infty\end{aligned}$$

establishing the proof. □

The lemma below concerns the accuracy of the discrete approximation to the continuous process, which is related to Condition C in [?]. As shown in [?] (see Eq. (7) therein), in order to verify Condition C the following lemma is sufficient. The technical proof of it is relegated to the Appendix.

Lemma 3.4. *Let $\{\zeta^{(\kappa)}_{m,k}(t), t\ge 0\}$ be as in Theorem 2.1. Then, there exist some constants $C,p>0, d>1$ and $\lambda_0, u_0>0$ such that*

$$\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t)>u+\frac{\lambda}{w_\kappa(u)}, \zeta^{(\kappa)}_{m,k}(0)\le u\right\}\le Ct^d\lambda^{-p}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}$$

for $0<t^\varpi<\lambda<\lambda_0$ and $u>u_0$. Here $\varpi:=\alpha/2\mathbb{I}\{\kappa\ge 1\}+\alpha/2\min(\kappa/(4(1-\kappa)),1)\mathbb{I}\{\kappa<1\}$.

Proof of Theorem 2.1: It follows from Lemmas 3.1–3.4 that all the assumptions of Theorem 1 in [?] are satisfied by the process $\zeta^{(\kappa)}_{m,k}$, which immediately establishes the proof. □

Proof of Theorem 2.2: In view of (21) with $p = 4/\alpha$ and letting $v_\kappa = v_\kappa(u) = 1/q_\kappa(u) = u^{2\tau/(\alpha\kappa)}$ we obtain

$$v_\kappa \int_{N/v_\kappa}^{T} \mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(s) > u \Big| \zeta^{(\kappa)}_{m,k}(0) > u\right\} ds = \int_{N}^{v_\kappa T} \mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(s/v_\kappa) > u \Big| \zeta^{(\kappa)}_{m,k}(0) > u\right\} ds$$
$$\leq K_{4/\alpha} \int_{N}^{v_\kappa T} s^{-2}\, ds \leq \frac{K_{4/\alpha}}{N}, \quad u \to \infty.$$

Hence

$$\lim_{N\to\infty} \limsup_{u\to\infty} v_\kappa \int_{N/v_\kappa}^{T} \mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(s) > u \Big| \zeta^{(\kappa)}_{m,k}(0) > u\right\} ds = 0.$$

Since further Lemma 3.2 holds, the claim follows by Theorem 3.1 in [?]. □

As shown by Theorem 10 in [?], in order to derive the Gumbel limit theorem for the random process $\zeta^{(\kappa)}_{m,k}$ two additional conditions, which were first addressed by the seminal paper [?] (see Lemma 3.5 therein), need to be checked, namely the mixing Condition D and the Condition D' therein. These two conditions will be followed from Lemma 3.5 and Lemma 3.6 below; the technical proof of them will be displayed in the Appendix.

Lemma 3.5. *Let $\{\zeta^{(\kappa)}_{m,k}(t), t \geq 0\}$ be defined as in Theorem 2.3. Let T, a be any given positive constants and $M \in (0, T)$, then for any $0 \leq s_1 < \cdots < s_p < t_1 < \cdots < t_{p'}$ in $\{aq_\kappa j : j \in \mathbb{Z}, 0 \leq aq_\kappa j \leq T\}$ with $t_1 - s_p \geq M$, we have for $u > 0$*

$$\left|\mathbb{P}\left\{\bigcap_{i=1}^{p}\{\zeta^{(\kappa)}_{m,k}(s_i) \leq u\}, \bigcap_{j=1}^{p'}\{\zeta^{(\kappa)}_{m,k}(t_j) \leq u\}\right\} - \mathbb{P}\left\{\bigcap_{i=1}^{p}\{\zeta^{(\kappa)}_{m,k}(s_i) \leq u\}\right\} \mathbb{P}\left\{\bigcap_{j=1}^{p'}\{\zeta^{(\kappa)}_{m,k}(t_j) \leq u\}\right\}\right|$$
$$\leq K u^{\varsigma} \sum_{1\leq i\leq p, 1\leq j\leq p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)}\right) \tag{22}$$

and

$$\left|\mathbb{P}\left\{\bigcap_{i=1}^{p}\{\zeta^{(\kappa)}_{m,k}(s_i) > u\}, \bigcap_{j=1}^{p'}\{\zeta^{(\kappa)}_{m,k}(t_j) > u\}\right\} - \mathbb{P}\left\{\bigcap_{i=1}^{p}\{\zeta^{(\kappa)}_{m,k}(s_i) > u\}\right\} \mathbb{P}\left\{\bigcap_{j=1}^{p'}\{\zeta^{(\kappa)}_{m,k}(t_j) > u\}\right\}\right|$$
$$\leq K u^{\varsigma} \sum_{1\leq i\leq p, 1\leq j\leq p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)}\right) \tag{23}$$

with some positive constant K, where $\varsigma = 2/\kappa(m - k - 1 + \max(0, 2(1-\kappa)))$ and $\widetilde{r}(t) := \max_{1\leq l\leq m+k} |r_l(t)|$.

Lemma 3.6. *Under the assumptions of Theorem 2.3, we have, for ς, $\widetilde{r}(\cdot)$ as in Lemma 3.5 and T_κ given by*

$$T_\kappa = T_\kappa(u) = \frac{1}{H^{m,k}_{\alpha,\kappa}(\boldsymbol{C})} \frac{q_\kappa(u)}{\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\}} \tag{24}$$

and any given constant $\varepsilon \in (0, T_\kappa)$

$$u^{\varsigma} \frac{T_\kappa}{q_\kappa} \sum_{\varepsilon\leq aq_\kappa j\leq T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right) \to 0, \quad u \to \infty. \tag{25}$$

Proof of Theorem 2.3: To establish the Conditions D and D' in [?], we shall make use of Lemma 3.5 with $T = T_\kappa$ taken as in (24) and $M = \varepsilon \in (0, T_\kappa)$. First note that the right-hand side of (22) is bounded from above by

$$K u^{\varsigma} \frac{T_\kappa}{aq_\kappa} \sum_{\varepsilon\leq aq_\kappa j\leq T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right)$$

which by an application of (25) implies that the mixing Condition D in [?] holds for the random process $\zeta^{(\kappa)}_{m,k}$.

Next, we prove Condition D' in [?], i.e., for any given positive constants a and $\widetilde{T}$

$$\limsup_{u\to\infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \to 0, \quad \varepsilon \downarrow 0. \tag{26}$$

Indeed, by (23) for some large $\widetilde{M} > \widetilde{T}$ and a positive constant K

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \le \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} + Ku^{\varsigma}\frac{T_\kappa}{q_\kappa}\widetilde{r}(aq_\kappa j)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right)$$

holds for sufficiently large u and $aq_\kappa j > \widetilde{M}$. Consequently

$$\begin{aligned}
&\limsup_{u\to\infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \\
&\le \limsup_{u\to\infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{[\widetilde{M}/(aq_\kappa)]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} + \varepsilon \\
&+ \limsup_{u\to\infty} Ku^{\varsigma}\frac{T_\kappa}{q_\kappa} \sum_{j=[\widetilde{M}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \widetilde{r}(aq_\kappa j)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right)
\end{aligned}$$

where the first term and the last term on the right-hand side equal 0 by an application of Lemma 3.3 and (25), respectively. It follows then that (26) holds. Consequently, in view of Theorem 10 in [?] we have for T_κ given by (24)

$$\lim_{u\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T_\kappa]} \zeta_{m,k}^{(\kappa)}(t) \le u + \frac{x}{w_\kappa(u)}\right\} = \exp\left(-e^{-x}\right), \quad x \in \mathbb{R}.$$

Expressing u in terms of T_κ using (24) (see also (38)) we obtain the required claim with $a_T^{(\kappa)}, b_T^{(\kappa)}$ given by (14) for any $x \in \mathbb{R}$; the uniform convergence in x follows since all functions (with respect to x) are continuous, bounded and increasing. □

4. Appendix

Proof of Lemma 3.4: By (1), for any small $\epsilon \in (0,1)$ there exists some positive constant B such that

$$r_i(t) \ge \frac{1}{2} \quad \text{and} \quad 1 - r_i(t) \le Bt^\alpha, \quad \forall t \in (0,\epsilon],\ 1 \le i \le m+k.$$

Furthermore, for any positive t satisfying (recall $\varpi = \alpha/2\mathbb{I}\{\kappa \ge 1\} + \alpha/2\min(\kappa/(4(1-\kappa)),1)\mathbb{I}\{\kappa < 1\}$)

$$0 < t^{\varpi} < \lambda < \lambda_0 := \min\left(\frac{1}{2^{\kappa+4}B}, \frac{\kappa}{2^{\kappa+2}}, \epsilon^{\varpi}\right)$$

and any $u > 2$

$$u^{2\tau/\kappa}\theta_\kappa(t) \le 2^\kappa \kappa B t^\alpha \le \frac{\kappa t^{\alpha/2}}{16} \quad \text{with} \quad \theta_\kappa(t) := \frac{1}{(r(q_\kappa t))^\kappa} - 1,\ r(t) := \min_{1\le i\le m+k} r_i(t). \tag{27}$$

Let $(\boldsymbol{X}_{1/r}^{(1)}(t), \boldsymbol{X}_{1/r}^{(2)}(t)) := \left(X_1(t) - r_1^{-1}(t)X_1(0), \ldots, X_{m+k}(t) - r_{m+k}^{-1}(t)X_{m+k}(0)\right)$ which by definition is independent of $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}$. For $j = 1, 2$

$$\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(j)}(q_\kappa t)| > x\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{x}{2\sqrt{2Bu^{-2\tau/\kappa}t^\alpha}}\right\}, \quad u\theta_\kappa(t) \le \frac{\lambda}{2w_\kappa(u)}. \tag{28}$$

In the following, the cases $\kappa = 1, \kappa \in (1,\infty)$ and $\kappa \in (0,1)$ will be considered in turn.

Case $\kappa = 1$: Note that by the triangular inequality

$$\zeta_{m,k}^{(1)}(q_1 t) \le |\boldsymbol{X}_{1/r}^{(1)}(q_1 t)| + |\boldsymbol{X}_{1/r}^{(2)}(q_1 t)| + \frac{1}{r(q_1 t)}\zeta_{m,k}^{(1)}(0) + \theta_1(t)|\boldsymbol{X}^{(2)}(0)|.$$

Consequently, from (28) we get

$$
\begin{aligned}
&\mathbb{P}\left\{\zeta^{(1)}_{m,k}(q_1 t) > u + \frac{\lambda}{u}, \zeta^{(1)}_{m,k}(0) \le u\right\}\\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| + |\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| + \theta_1(t)|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda}{u} - u\theta_1(t), \zeta^{(1)}_{m,k}(q_1 t) > u\right\}\\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| + |\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| > \frac{\lambda}{3u}\right\}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(q_1 t) > u\right\} + \mathbb{P}\left\{\theta_1(t)|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda}{6u}\right\}\\
&=: I_{1u} + I_{2u}.
\end{aligned}
$$

By (27) and (28), we have for any $p > 1$

$$
\mathbb{P}\left\{|\boldsymbol{X}^{(1)}_{1/r}(q_1 t)| > \frac{\lambda}{6u}\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > \frac{\lambda}{12\sqrt{2B}t^{\alpha/2}}\right\} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}
$$

holds with some $K > 0$ (the values of p and K might change from line to line below). Similarly

$$
\mathbb{P}\left\{|\boldsymbol{X}^{(2)}_{1/r}(q_1 t)| > \frac{\lambda}{6u}\right\} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}
$$

and hence

$$
I_{1u} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\}. \tag{29}
$$

Moreover, in view of Lemma 3.1 and (27) we have for sufficiently large u that

$$
I_{2u} \le \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| > \frac{2\lambda u}{t^{\alpha/2}}\right\}}{\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\}}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\} \le K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-(p-k+2)} u^{-(p+m-2k)}\mathbb{P}\left\{\zeta^{(1)}_{m,k}(0) > u\right\}. \tag{30}
$$

Hence, the claim for $\kappa = 1$ follows from (29) and (30) by choosing $p > \max(4/\alpha + k, 2k)$.
Case $\kappa \in (1, \infty)$: Denote below by $(\boldsymbol{Y}^{(1)}(t), \boldsymbol{Y}^{(2)}(t)) := \left(r_1^{-1}(t)X_1(0), \ldots, r_{m+k}^{-1}(t)X_{m+k}(0)\right)$. Note that $|\boldsymbol{Y}^{(1)}(t)| \le |\boldsymbol{X}^{(1)}(0)|/r(t)$ and $|\boldsymbol{X}^{(2)}(0)| \le |\boldsymbol{Y}^{(2)}(t)| \le |\boldsymbol{X}^{(2)}(0)|/r(t)$ for all $t < \varepsilon$, and for some constants $K_1, K_2 > 0$ whose values might change from line to line below

$$
|1 + x|^\kappa \ge 1 + \kappa x, \quad x \in \mathbb{R} \quad \text{and} \quad (1 + x)^\kappa \le 1 + K_1 x + K_2 x^\kappa, \quad x \ge 0. \tag{31}
$$

We have further by the triangle inequality

$$
\begin{aligned}
\zeta^{(\kappa)}_{m,k}(q_\kappa t) &\le \left(|\boldsymbol{Y}^{(1)}(q_\kappa t)| + |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|\right)^\kappa - \left||\boldsymbol{Y}^{(2)}(q_\kappa t)| - |\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|\right|^\kappa\\
&\le |\boldsymbol{Y}^{(1)}(q_\kappa t)|^\kappa + K_1|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{Y}^{(1)}(q_\kappa t)|^{\kappa-1} + K_2|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|^\kappa\\
&\quad -|\boldsymbol{Y}^{(2)}(q_\kappa t)|^\kappa + \kappa|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa-1}\\
&\le K_1|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1} + K_2|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|^\kappa\\
&\quad +K_3|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1} + \frac{\zeta^{(\kappa)}_{m,k}(0)}{(r(q_\kappa t))^\kappa} + \theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa
\end{aligned}
$$

holds for $q_\kappa t \le \epsilon$ and some constant $K_3 > 0$. Therefore, with $\mu = 1/(2(\kappa - 1))$ and $\varphi = \alpha/(4(\kappa - 1))$

$$
\begin{aligned}
&\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta^{(\kappa)}_{m,k}(0) \le u\right\}\\
&\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > \frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right\} + \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right\}\\
&\quad +\mathrm{P}\left\{K_1|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|\left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1} + K_2|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|^\kappa + K_3|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|\left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1}\right.\\
&\quad \left.+\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa \ge \frac{\lambda}{2w_\kappa(u)}, \zeta^{(\kappa)}_{m,k}(q_\kappa t) > u\right\}\\
&=: \tilde{I}_{1u} + \tilde{I}_{2u} + \tilde{I}_{3u}.
\end{aligned} \tag{32}
$$

Note by (27) that $\lambda^\mu/t^\varphi>1$. Similar arguments as in (30) yield that

$$\left\{\begin{array}{l}\tilde I_{1u}\le K\left(\frac{\lambda^\mu}{t^\varphi}\right)^{-(p-m+2)}u^{-(p-k(2/\kappa-1)\mathbb{I}\{\kappa\le2\})/\kappa}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}\\ \tilde I_{2u}\le K\left(\frac{\lambda^\mu}{t^\varphi}\right)^{-(p-k+2)}u^{-(p-k+m-k(2/\kappa-1)\mathbb{I}\{\kappa\le2\})/\kappa}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}\end{array}\right.$$

and

$$\begin{aligned}\tilde I_{3u}\ \le\ &\left(\mathbb{P}\left\{K_1|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|\left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1}>\frac{\lambda}{8w_\kappa(u)}\right\}+\mathbb{P}\left\{K_2|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|^\kappa>\frac{\lambda}{8w_\kappa(u)}\right\}\right.\\ &\left.+\mathbb{P}\left\{K_3|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|\left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1}>\frac{\lambda}{8w_\kappa(u)}\right\}\right)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t)>u\right\}\\ &+\mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa>\frac{\lambda}{8w_\kappa(u)}\right\}\\ =:\ &(I\!I_{1u}+I\!I_{2u}+I\!I_{3u})\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}+I\!I_{4u}.\end{aligned}$$

Furthermore

$$\begin{aligned}I\!I_{1u}\ &\le\ \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|>K_1\frac{\lambda^{1/2}u^{-1/\kappa}}{t^{-\alpha/4}(r^{-2}(q_\kappa t)-1)^{1/2}}\right\}\\ &\le\ \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|>K_1\frac{\lambda^{1/2}}{t^{\alpha/4}}\right\}\\ &\le\ K\left(\frac{\lambda^{1/2}}{t^{\alpha/4}}\right)^{-p}.\end{aligned}$$

Similarly

$$I\!I_{2u}\le K\left(\frac{\lambda u^{2(1-1/\kappa)}}{t^{\alpha\kappa/2}}\right)^{-p/\kappa},\quad I\!I_{3u}\le K\left(\frac{\lambda^{1/2}}{t^{\alpha/4}}\right)^{-p}.$$

Next, we deal with $I\!I_{4u}$. We have by (27) that $2^{\kappa+4}Bt^{\alpha/2}\le1$. Therefore, similar arguments as for (30) yield that

$$\begin{aligned}I\!I_{4u}\ &\le\ \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)|^\kappa>\frac{2\lambda u}{t^{\alpha/2}}\frac{1}{2^{\kappa+4}Bt^{\alpha/2}}\right\}\\ &\le\ K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-(p-k+2)}u^{-(p-k+m-k(2/\kappa-1)\mathbb{I}\{\kappa\le2\})/\kappa}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}.\end{aligned}\tag{33}$$

Therefore, the claim for $\kappa\in(1,\infty)$ follows from (32) and the inequalities for $\tilde I_{1u},\tilde I_{2u},I\!I_{1u},I\!I_{2u},I\!I_{3u}$ and $I\!I_{4u}$ by choosing $p>\max(8(\kappa-1)/\alpha+k+m,2k)$.

Case $\kappa\in(0,1)$: Note that

$$(1+x)^\kappa\le1+x,\quad x\ge0\quad\text{and}\quad-|1-x|^\kappa\le-(1-x),\quad x\in[0,\infty).$$

We have further by the triangle inequality

$$\begin{aligned}\zeta^{(\kappa)}_{m,k}(q_\kappa t)\ &\le\ \left(|\boldsymbol{Y}^{(1)}(q_\kappa t)|+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|\right)^\kappa-\left||\boldsymbol{Y}^{(2)}(q_\kappa t)|-|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)|\right|^\kappa\\ &\le\ |\boldsymbol{Y}^{(1)}(q_\kappa t)|^\kappa+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1}-|\boldsymbol{Y}^{(2)}(q_\kappa t)|^\kappa+|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa-1}\\ &\le\ \frac{|\boldsymbol{X}^{(1)}(0)|^\kappa}{(r(q_\kappa t))^\kappa}+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1}-|\boldsymbol{X}^{(2)}(0)|^\kappa+|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1}\\ &=\ \frac{\zeta^{(\kappa)}_{m,k}(0)}{(r(q_\kappa t))^\kappa}+\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa+|\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1}+|\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1}.\end{aligned}$$

Therefore, we have by (28), with $\psi = \alpha/(4(1-\kappa))$

$$
\begin{aligned}
&\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(0) \le u, \right\}\\
&\quad \le \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa + \frac{|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|}{(u^{-\tau/\kappa}t^\psi)^{1-\kappa}} + \frac{|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)|}{(u^{-\tau/\kappa}t^\psi)^{1-\kappa}} > \frac{\lambda}{2w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}\\
&\quad +\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| \le u^{-\tau/\kappa}t^\psi, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\} + \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| \le u^{-\tau/\kappa}t^\psi, \zeta_{m,k}^{(\kappa)}(0) \le u, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}\right\}\\
&\quad =: I_{1u}^* + I_{2u}^* + I_{3u}^*.
\end{aligned}
$$

Now we deal with the three terms one by one. Clearly, for any $u > 2$

$$
\begin{aligned}
I_{1u}^* \quad &\le \quad \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa > \frac{\lambda}{6w_\kappa(u)}\right\} + \mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)| > \frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}\\
&\quad +\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)| > \frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\},
\end{aligned}
$$

where the first term can be treated as II_{4u}, see (33). For the rest two terms, using (28)

$$
\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(j)}(q_\kappa t)| > \frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{\kappa}{12\sqrt{2B}}\frac{\lambda}{t^{\alpha/4}}\right\} \le K\left(\frac{\lambda}{t^{\alpha/4}}\right)^{-p}, \quad j = 1, 2. \tag{34}
$$

In order to deal with I_{2u}^* and I_{3u}^*, set below $(\boldsymbol{X}_r^{(1)}(t), \boldsymbol{X}_r^{(2)}(t)) := \left(X_1(0) - r_1(t)X_1(t), \ldots, X_{m+k}(0) - r_{m+k}(t)X_{m+k}(t)\right)$ which by definition is independent of $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}$. For $j = 1, 2$

$$
\mathbb{P}\left\{|\boldsymbol{X}_r^{(j)}(q_\kappa t)| > x\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{2\sqrt{\lambda}x}{\sqrt{u^{-2\tau/\kappa}t^\alpha}}\right\}. \tag{35}
$$

Using further the triangle inequality $|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa \ge (r(q_\kappa t))^\kappa|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa - |\boldsymbol{X}^{(1)}(0)|^\kappa$ and (27) (recalling $|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa \ge \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u$)

$$
\begin{aligned}
I_{2u}^* \quad &\le \quad \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u\left((r(q_\kappa t))^\kappa - \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}\\
&\le \quad \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > \frac{(1-2^{-\kappa})u}{2^\kappa}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}\\
&\le \quad \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > (1-2^{-\kappa})^{1/\kappa}\frac{\sqrt{\lambda}}{t^{\alpha/2}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}\\
&\le \quad K\left(\frac{\lambda}{t^\alpha}\right)^{-p/2}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}.
\end{aligned}
\tag{36}
$$

For I_{3u}^*, using instead $|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u + \lambda/w_\kappa(u)$ and

$$
|\boldsymbol{X}^{(1)}(0)|^\kappa = \zeta_{m,k}^{(\kappa)}(0) + |\boldsymbol{X}^{(2)}(0)|^\kappa \le u\left(1 + \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)
$$

we have

$$
\begin{aligned}
I_{3u}^* \quad &\le \quad \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u\left((r(q_\kappa t))^\kappa\left(1 + \frac{\lambda}{uw_\kappa(u)}\right) - \left(1 + \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}\\
&= \quad \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u^{-\tau}\left(\lambda\kappa(r(q_\kappa t))^\kappa - u^{1+\tau}(1-(r(q_\kappa t))^\kappa) - t^{\psi\kappa}\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\},
\end{aligned}
$$

where by (27)

$$
\lambda\kappa(r(q_\kappa t))^\kappa - u^{1+\tau}(1-(r(q_\kappa t))^\kappa) - t^{\psi\kappa} \ge \frac{\lambda\kappa}{2^{\kappa+1}} - t^{\psi\kappa} \ge \frac{\lambda\kappa}{2^{\kappa+2}}.
$$

Consequently by (35)

$$\begin{aligned} I^*_{3u} &\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > 2^{-2/\kappa}\kappa^{1/\kappa}\frac{\lambda^{1/\kappa+1/2}}{t^{\alpha/2}}\right\}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}\\ &\le K\left(\frac{\lambda^{1/\kappa+1/2}}{t^{\alpha/2}}\right)^{-p}\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0)>u\right\}, \end{aligned}$$

which together with (33), (34) and (36) completes the proof for $\kappa\in(0,1)$ by taking $p>4/\alpha+k$. Consequently, the desired claim of Lemma 3.4 holds for all $\kappa\in(0,\infty)$. □

Proof of Lemma 3.5: We give only the proof for (22) since (23) follows by similar arguments. Since the claims for $k=0$ are already shown in [?], we only consider that $k\ge 1$ below. Define, for $j=1,2$, independent random vectors $\Big(|\boldsymbol{Y}^{(j)}(s_1)|,\ldots,|\boldsymbol{Y}^{(j)}(s_p)|\Big)$ and $\Big(|\widetilde{\boldsymbol{Y}}^{(j)}(t_1)|,\ldots,|\widetilde{\boldsymbol{Y}}^{(j)}(t_{p'})|\Big)$, which are independent of the process $\zeta^{(\kappa)}_{m,k}$ and have the same distributions as those of $\Big(|\boldsymbol{X}^{(j)}(s_1)|,\ldots,|\boldsymbol{X}^{(j)}(s_p)|\Big)$ and $\Big(|\boldsymbol{X}^{(j)}(t_1)|,\ldots,|\boldsymbol{X}^{(j)}(t_{p'})|\Big)$, respectively. Note that, for any $u>0$, the left-hand side of (22) is clearly bounded from above by

$$\begin{aligned} &\Bigg|\mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{X}^{(2)}(s_i)|^\kappa\ge|\boldsymbol{X}^{(1)}(s_i)|^\kappa-u\right\},\bigcap_{j=1}^{p'}\left\{|\boldsymbol{X}^{(2)}(t_j)|^\kappa\ge|\boldsymbol{X}^{(1)}(t_j)|^\kappa-u\right\}\right\}\\ &\quad-\mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{Y}^{(2)}(s_i)|^\kappa\ge|\boldsymbol{X}^{(1)}(s_i)|^\kappa-u\right\},\bigcap_{j=1}^{p'}\left\{|\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa\ge|\boldsymbol{X}^{(1)}(t_j)|^\kappa-u\right\}\right\}\Bigg|\\ &+\Bigg|\mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{X}^{(1)}(s_i)|^\kappa\le|\boldsymbol{Y}^{(2)}(s_i)|^\kappa+u\right\},\bigcap_{j=1}^{p'}\left\{|\boldsymbol{X}^{(1)}(t_j)|^\kappa\le|\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa+u\right\}\right\}\\ &\quad-\mathbb{P}\left\{\bigcap_{i=1}^{p}\left\{|\boldsymbol{Y}^{(1)}(s_i)|^\kappa\le|\boldsymbol{Y}^{(2)}(s_i)|^\kappa+u\right\},\bigcap_{j=1}^{p'}\left\{|\widetilde{\boldsymbol{Y}}^{(1)}(t_j)|^\kappa\le|\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa+u\right\}\right\}\Bigg|. \end{aligned} \tag{37}$$

Next, note by Cauchy-Schwarz inequality that $u^2+v^2\le(u^2-2\rho uv+v^2)/(1-|\rho|)$ for all $\rho\in(-1,1)$ and $u,v\in\mathbb{R}$. It follows that $f_{ij}(\cdot,\cdot)$ the joint density function of $\Big(|\boldsymbol{X}^{(1)}(s_i)|,|\boldsymbol{X}^{(1)}(t_j)|\Big)$ satisfies that

$$\begin{aligned} f_{i,j}(x,y) &= \int_{|\boldsymbol{x}|=x,|\boldsymbol{y}|=y}\prod_{l=1}^{m}\frac{1}{2\pi\sqrt{1-r_l^2(t_j-s_i)}}\exp\left(-\frac{x_l^2-2r_l(t_j-s_i)x_ly_l+y_l^2}{2(1-r_l^2(t_j-s_i))}\right)d\boldsymbol{x}d\boldsymbol{y}\\ &\le \frac{1}{(2\pi)^m(1-(\widetilde{r}(t_j-s_i))^2)^{m/2}}\int_{|\boldsymbol{x}|=x,|\boldsymbol{y}|=y}\prod_{l=1}^{m}\exp\left(-\frac{x_l^2+y_l^2}{2(1+|r_l(t_j-s_i)|)}\right)d\boldsymbol{x}d\boldsymbol{y}\\ &\le \frac{1}{(2\pi)^m(1-(\widetilde{r}(t_j-s_i))^2)^{m/2}}\exp\left(-\frac{x^2+y^2}{2(1+\widetilde{r}(t_j-s_i))}\right)\int_{|\boldsymbol{x}|=x,|\boldsymbol{y}|=y}d\boldsymbol{x}d\boldsymbol{y}\\ &= \frac{(xy)^{m-1}}{2^{m-2}(\Gamma(m/2))^2(1-(\widetilde{r}(t_j-s_i))^2)^{m/2}}\exp\left(-\frac{x^2+y^2}{2(1+\widetilde{r}(t_j-s_i))}\right),\quad x,y>0. \end{aligned}$$

Therefore, in view of Lemma 2 in [?], with K a constant whose value might change from line to line, the first absolute value in (37) is bounded from above by

$$\begin{aligned} &K\sum_{i=1}^{p}\sum_{j=1}^{p'}\int_{x^\kappa>u}\int_{y^\kappa>u}\widetilde{r}(t_j-s_i)\Big((x^\kappa-u)(y^\kappa-u)\Big)^{(k-1)/\kappa}\exp\left(-\frac{(x^\kappa-u)^{2/\kappa}+(y^\kappa-u)^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right)f_{ij}(x,y)\,dxdy\\ &\le K\sum_{i=1}^{p}\sum_{j=1}^{p'}\widetilde{r}(t_j-s_i)\Bigg(\int_u^\infty(x-u)^{(k-1)/\kappa}x^{m/\kappa-1}\exp\left(-\frac{x^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right)dx\Bigg)^2\\ &\le Ku^{2((m-k+1)/\kappa-2)}\sum_{i=1}^{p}\sum_{j=1}^{p'}\widetilde{r}(t_j-s_i)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(t_j-s_i)}\right) \end{aligned}$$

where the last inequality follows by a change of variable $x' = u(x-u)$. Similarly, denoting by $g(\cdot)$ the pdf of $|\boldsymbol{X}^{(2)}(0)|$ we obtain that the second absolute value in (37) is bounded from above by

$$K\sum_{i=1}^{p}\sum_{j=1}^{p'}\widetilde{r}(t_j-s_i)\int_0^\infty\int_0^\infty\Big((x^\kappa+u)(y^\kappa+u)\Big)^{(m-1)/\kappa}\exp\left(-\frac{(x^\kappa+u)^{2/\kappa}+(y^\kappa+u)^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right)g(x)g(y)\,dxdy$$
$$\le K\sum_{i=1}^{p}\sum_{j=1}^{p'}\widetilde{r}(t_j-s_i)\left(\int_0^\infty (x+u)^{(m-1)/\kappa}x^{k/\kappa-1}\exp\left(-\frac{(x+u)^{2/\kappa}}{2(1+\widetilde{r}(t_j-s_i))}\right)dx\right)^2$$
$$\le Ku^{2(m-k-1)/\kappa}\sum_{i=1}^{p}\sum_{j=1}^{p'}\widetilde{r}(t_j-s_i)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(t_j-s_i)}\right)$$

where the last step follows by a change of variable $x' = ux$. Hence the proof of (22) is established by noting that

$$\frac{m-k-1}{\kappa}-\left(\frac{m-k+1}{\kappa}-2\right)=-2\left(\frac{1}{\kappa}-1\right).$$

This completes the proof. □

Proof of Lemma 3.6: The proof follows by the same arguments as for Lemma 12.3.1 in [?], using alternatively the following asymptotic relation (recall (24) and Lemma 3.1)

$$u^{2/\kappa}=2\ln T_\kappa+K_0\ln\ln T_\kappa+\ln D_0(1+o(1)),\quad T_\kappa\to\infty \tag{38}$$

with D_0, K_0 defined in Theorem 2.3. We split the sum in (25) at T_κ^β, where β is a constant such that $0<\beta<(1-\delta)/(1+\delta)$ and $\delta=\sup\{\widetilde{r}(t):t\ge\epsilon\}<1$ (cf. Lemma 8.1.1 (i) in [?]). Below K is again a positive constant which might change from line to line. From (38) we conclude that $\exp\left(-u^{2/\kappa}/2\right)\le K/T_\kappa$ and $u^{2/\kappa}=2\ln T_\kappa(1+o(1))$. Further (recall $\varsigma := 2/\kappa(m-k-1+\max(0,2(1-\kappa))))$)

$$u^\varsigma\frac{T_\kappa}{q_\kappa}\sum_{\varepsilon\le aq_\kappa j\le T_\kappa^\beta}\widetilde{r}(aq_\kappa j)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right)$$
$$\le u^{\varsigma+\frac{4\tau}{\alpha\kappa}}T_\kappa^{\beta+1}\exp\left(-\frac{u^{2/\kappa}}{1+\delta}\right)$$
$$\le K(\ln T_\kappa)^{\frac{\kappa\varsigma}{2}+\frac{2\tau}{\alpha}}T_\kappa^{\beta+1-\frac{2}{1+\delta}},$$

which tends to 0 as $T_\kappa\to\infty$ since $\beta+1-2/(1+\delta)<0$. For the remaining sum, denoting $\delta(t)=\sup\{|\widetilde{r}(s)\ln s|\,;s\ge t\}$, $t>0$, we have $\widetilde{r}(t)\le\delta(t)/\ln t$ as $t\to\infty$, and thus in view of (38) for $aq_\kappa j\ge T_\kappa^\beta$

$$\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right)\le\exp\left(-u^{2/\kappa}\left(1-\frac{\delta(T_\kappa^\beta)}{\ln T_\kappa^\beta}\right)\right)$$
$$\le K\exp(-u^{2/\kappa})\le KT_\kappa^{-2}(\ln T_\kappa)^{-K_0}.$$

Consequently

$$u^\varsigma\frac{T_\kappa}{q_\kappa}\sum_{T_\kappa^\beta\le aq_\kappa j\le T_\kappa}\widetilde{r}(aq_\kappa j)\exp\left(-\frac{u^{2/\kappa}}{1+\widetilde{r}(aq_\kappa j)}\right)$$
$$\le u^\varsigma\left(\frac{T_\kappa}{q_\kappa}\right)^2T_\kappa^{-2}(\ln T_\kappa)^{-K_0}\frac{1}{\ln T_\kappa^\beta}\frac{1}{T_\kappa/q_\kappa}\sum_{T_\kappa^\beta\le aq_\kappa j\le T_\kappa}\widetilde{r}(aq_\kappa j)\ln(aq_\kappa j)$$
$$\le K(\ln T_\kappa)^{\frac{\kappa\varsigma}{2}+\frac{2\tau}{\alpha}-K_0-1}\frac{1}{T_\kappa/q_\kappa}\sum_{T_\kappa^\beta\le aq_\kappa j\le T_\kappa}\widetilde{r}(aq_\kappa j)\ln(aq_\kappa j). \tag{39}$$

Since the Berman condition $\lim_{t\to\infty}\widetilde{r}(t)\ln t=0$ holds and further $\beta<1$ and $K_0=m-2+2\tau/\alpha+k(1-2/\kappa)\mathbb{I}\{\kappa\le 2\}$, the right-hand side of (39) tends to 0 as $u\to\infty$. Thus the proof is complete. □

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