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Quasipotential for the ferromagnetic wire governed by the 1D Landau-Lifshitz-Gilbert Equations

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Abstract

We consider 1D Landau-Lifshitz-Gilbert equations with an external force. We first prove the existence and uniqueness of the strong solution, and then give a definition of the quasipotential and prove that the quasipotential is equal to the potential energy of the system.

Keywords: Landau-Lifshitz-Gilbert equations, strong solution, quasipotential

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1. Introduction

We consider the following Landau-Lifshitz-Gilbert (LLG) equations with the magnetisation field m depending on the time and the space variables $(t, x) \in \mathbb{R} \times \mathcal{O}$, where $\mathcal{O} := (0, 2\pi)$ and taking values in $\mathbb{R}^3$:

$$\frac{\partial m}{\partial t} = \lambda_1 m \times (\Delta m - \phi'(m) + h) - \lambda_2 m \times \big(m \times (\Delta m - \phi'(m) + h)\big), \quad \lambda_1 \in \mathbb{R},\ \lambda_2 > 0. \tag{1.1}$$

with periodic boundary conditions

$$m(t,0) = m(t,2\pi), \quad \nabla m(t,0) = \nabla m(t,2\pi), \quad \text{for a.e. } t \in [0,T], \tag{1.2}$$

and initial condition

$$m(0) = m_0, \tag{1.3}$$

where we assume that the data m_0 takes values in the two dimensional sphere $\mathbb{S}^2$. In the system (1.1)-(1.3), the function $\phi : \mathbb{R}^3 \to \mathbb{R}^+$ is a given map related to the anisotropy energy and h can be regarded as an external "force". To each configuration $m \in H^{1,2}(\mathcal{O}; \mathbb{R}^3)$, we associate the energy defined by

$$\mathcal{E}(m) = \frac{1}{2}\|\nabla m\|_{\mathrm{H}}^2 + \int_{\mathcal{O}} \phi(m(x))\,\mathrm{d}x. \tag{1.4}$$

We will show that the solution of the system (1.1)-(1.3) takes values in the two dimensional sphere $\mathbb{S}^2$. If, e.g. $h = 0$ and

$$\phi(m) = \frac{1}{2}(m_1^2 + m_2^2),$$

for $m = (m_1, m_2, m_3)$, the constant configurations $\zeta_\pm$ equal respectively to the north and south poles on the sphere $\mathbb{S}^2$ are asymptotically stable equilibria of the system (1.1)-(1.3). We are interested in how stable each equilibrium is, i.e. how much "action" is necessary to move the solution m from ζ_+ to another configuration a taking values in $\mathbb{S}^2$ following the trajectory of the system (1.1)-(1.3). To find out an answer to this question, we use a notion of the so called *quasipotential* defined to be equal to, modulo a multiplicative constant, the infimum of the L^2-space time norms of the external forces h and prove that it is equal to the energy $\mathcal{E}(a)$ at the configuration a. This is our main result, see Theorem 5.2.

To the best of our knowledge, the quasipotential of LLG system has not been studied before. The "quasipotential" has been an object studied only in the context of stochastic dynamical systems, such as in the monograph [10] by Freidlin and Wentzell, where it has been defined via the so-called "action functional", which is the rate function of the large deviation property of the solution process. The quasipotential is defined as an appropriate infimum of the action functional. But in some special cases, of which could be called the gradient systems, there is an essential property (where the name "quasipotential" comes from) of the equipotential, see [10, Theorem 4.3.1] and [8, Theorem 3.7] , according to which the quasipotential differs from the potential of the system only by a multiplicative constant. This property is not necessary to be stochastic, and it is the motivation of our definition (Definition 5.1). A similar result for the 2-D Navier-Stokes Equations with periodic boundary conditions has recently been proved by the first named author et all [4]. The LLGEs, contrary to those studied before but similarly to the NSEs, are not of the gradient type. In this sense, the main result of our paper, i.e. Theorem 5.2, shows that our definition of quasipotential is well defined. In view of our deterministic definition, i.e. in view of Definition 5.1, we can also expect that in classical results, such as the exit point is close to the minimum point of the quasipotential on the boundary, the principal term of the asymptotics of the mean time before reaching the boundary can be expressed in terms of the quasipotential and the behaviour of the invariant measure of the solution can also be described in terms of the quasipotential, hold if we change the external force h in (1.1) to be stochastic. These properties will be studied in the future.

In section 2, we introduce the notation used throughout the whole paper and the definition of a solution of the system of LLG system (1.1)-(1.3).

In section 3, we show the existence and uniqueness of the strong solution of the system (1.1)-(1.3). To do this, we first prove the existence of weak solutions by the Galerkin approximation method. In the next section, we show that the weak solutions have sufficient regularity to be strong solutions. It is worth mentioning that to show the regularity of weak solutions of LLG system; one can write the equation in a mild form and then use the ultracontractivity and the maximal regularity properties of the heat semigroup, see the paper [6]. This method works here as well but would need quite a lot of calculations. In the current paper, we prove the regularity of weak solutions easily by using a certain geometric property of the LLG system without any complicated calculations. In the last part of this section, we state the uniqueness result which can be proved by the same method as in [6].

In section 4, we show that with a suitably defined function $\phi \in C_0^2(\mathbb{R}^3; \mathbb{R}^+)$ which relates to the anisotropy energy, the constant function ζ_+ which equals to the north pole of the unit ball is an asymptotically stable equilibrium of the system (1.1)-(1.3). If the initial data m_0 is close enough in the H^1 metric to ζ_+, then the solution will converge to it exponentially. The particular ϕ we choose is different from the one in [6]. With our choice of ϕ, we have the following property

$$\frac{1}{2}\|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 = \mathcal{E}(m(t)), \qquad t \geq 0.$$

This equality is somehow the reason for the exponential convergence.

Inspired by this, we use the energy as control of $\|m(t)-\zeta_+\|^2_{\mathbb{H}^1}$ and prove a better convergence result with the function ϕ as in [6]. We postpone the proof in the first part of Appendix B. In the second part of Appendix B, we give another choice of ϕ, such that there are infinitely many stationary solutions of the system (1.1)-(1.3) and the poles are no longer attractors. In section 5, we present a definition of quasipotential of the deterministic LLG system and prove that the quasipotential is just the energy as in (1.4) which is our main result.

In the last part of this introduction, we would like to summarise the novelties of this paper. Firstly, this is the first time one proved the existence and uniqueness of a strong solution of this version of Landau-Lifshitz-Gilbert equations, i.e. the system (2.7)-(2.9). The difficulty in proving existence and uniqueness is that our assumptions are quite weak. In particular, we assume only that the external “force” h belongs to space $L^2_{\mathrm{loc}}([0,\infty);\mathbb{L}^2)$. Secondly, as mentioned before, we use a new method to show the regularity of the solution. Thirdly, we have improved the convergence result in [6]. Finally, we define a quasipotential for this infinite dimensional non-gradient system and prove that this is well defined.

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2. Notation and preliminaries

Notation 2.1. $\mathcal{O} := (0, 2\pi)$, $\mathbb{W}^{m,p} := W^{m,p}(\mathcal{O};\mathbb{R}^3)$, $\mathbb{H}^m := H^m(\mathcal{O};\mathbb{R}^3)$. Moreover, we will use $\mathcal{S}^2$ to denote the unit sphere, use ∇ to denote the weak derivative with respect to the spatial variable x.

We define the linear operators A and A_1 by the following definition:

Definition 2.2. Let A be a linear operator acting on $\mathbb{L}^2(\mathcal{O})$ with the domain defined by

$$D(A) := \left\{u \in \mathbb{H}^2 : u(0) = u(2\pi) \text{ and } \nabla u(0) = \nabla u(2\pi)\right\}, \quad Au := -\Delta u,\ u \in D(A).$$

$$D(A_1) := D(A), \qquad A_1 := I + A.$$

It is known that, A_1 is positive self-adjoint operator in the Hilbert space $\mathrm{H} = L^2(\mathcal{O}, \mathbb{R}^3)$ and that

$$D(A_1^{\frac{1}{2}}) = \left\{u \in \mathbb{H}^1 : u(0) = u(2\pi)\right\}. \tag{2.1}$$

Notation 2.3. We define $\mathrm{V} := D(A_1^{\frac{1}{2}})$. Since V is a dense subset of H and the embedding $\mathrm{V} \hookrightarrow \mathrm{H}$ is continuous, we have the following Gelfand triple

$$\mathrm{V} \subset \mathrm{H} \equiv \mathrm{H}^* \subset \mathrm{V}^*$$

where V^* is the dual space of V.

To denote the norm and the inner product in the space H, resp. V, we will use the subscript $_\mathrm{H}$, resp. $_\mathrm{V}$.

Definition 2.4. For $m \in \mathrm{V}$ we define Δm, $m \times \Delta m$ and $m \times (m \times \Delta m)$ as elements of the dual space V^* by the following formulae

$$_\mathrm{V}\langle u, \Delta m\rangle_{\mathrm{V}^*} := -\left\langle \nabla u, \nabla m\right\rangle_\mathrm{H}, \qquad u \in \mathrm{V}. \tag{2.2}$$

$$_\mathrm{V}\langle u, m \times \Delta m\rangle_{\mathrm{V}^*} := \left\langle m \times \nabla u, \nabla m\right\rangle_\mathrm{H}, \qquad u \in \mathrm{V}. \tag{2.3}$$

and

$$_\mathrm{V}\langle u, m \times (m \times \Delta m)\rangle_{\mathrm{V}^*} := -\left\langle \nabla\big(m \times (m \times u)\big), \nabla m\right\rangle_\mathrm{H}, \qquad u \in \mathrm{V}. \tag{2.4}$$

Remark 2.5. An attentive reader would notice that in order the definitions above are correct, the RHS on the three equalities in Definition 2.4 should be continuous linear w.r.t. u from V to $\mathbb{R}$. Since $m \in \mathrm{V}$ and V is an algebra, this is the case.

To describe the problem we are going to deal with, we also need the following notations:

Notation 2.6.

$$\mathcal{M} \quad := \quad \left\{a \in \mathbb{H}^1 : a(x) \in \mathcal{S}^2 \text{ for all } x \in \mathcal{O}\right\}, \tag{2.5}$$

$$\mathcal{M}^+ \quad := \quad \{a \in \mathcal{M} : a_3(x) > 0 \text{ for all } x \in \mathcal{O}\}. \tag{2.6}$$

Let us fix a function

$$\phi \in C_0^2(\mathbb{R}^3; \mathbb{R}^+),$$

an external "force"

$$h \in L^2_{\text{loc}}([0, \infty); \mathrm{H})$$

and the initial data

$$m_0 \in \mathrm{V} \cap \mathcal{M}.$$

Let us consider the following initial value problem for the 1-dimensional Landau-Lifshitz-Gilbert (LLG) system with periodic boundary conditions

$$\frac{\partial m}{\partial t} = \lambda_1 m \times (\Delta m - \phi'(m) + h) - \lambda_2 m \times \big(m \times (\Delta m - \phi'(m) + h)\big), \quad \lambda_1 \in \mathbb{R},\ \lambda_2 > 0. \tag{2.7}$$

$$m(t, 0) = m(t, 2\pi), \quad \text{for a.e. } t \in [0, \infty), \tag{2.8}$$

$$m(0, \cdot) = m_0. \tag{2.9}$$

Definition 2.7. We say that $m \in L^2_{\text{loc}}([0, \infty); \mathrm{V}) \cap H^1_{\text{loc}}([0, \infty); \mathrm{V}^*)$ is a weak solution of the LLG system (2.7)-(2.9) iff for all $u \in \mathrm{V}$, m satisfies

$$\begin{aligned} & \int_{\mathcal{O}} \langle m(t) - m_0, u \rangle \,\mathrm{d}x \\ = & \int_0^t \int_{\mathcal{O}} \langle \lambda_1 m \times (-\phi'(m) + h) - \lambda_2 m \times (m \times (-\phi'(m) + h)), u \rangle \,\mathrm{d}x \,\mathrm{d}s \\ & + \int_0^t \int_{\mathcal{O}} \lambda_1 \langle \nabla m, \nabla(m \times u) \rangle + \lambda_2 \langle \nabla m, \nabla(m \times (m \times u)) \rangle \,\mathrm{d}x \,\mathrm{d}s, \end{aligned} \tag{2.10}$$

for all $t \in [0, \infty)$.

In what follows, we will look for solutions on finite time intervals of type $[0, T], T > 0$. Since we will prove uniqueness, proving existence on such time intervals is sufficient to proving the existence on infinite interval $[0, \infty)$.

Remark 2.8.(i) Since by Lions-Magenes [13],

$$L^2(0, T; \mathrm{V}) \cap H^1(0, T; \mathrm{V}^*) \hookrightarrow C([0, T]; \mathrm{H}) \text{ continuously,}$$

every weak solution m of equations (2.7)-(2.9) belongs to the space $C([0, T]; \mathrm{H})$.

(ii) By Definition 2.4, the equation (2.10) can be written in the following form

$$\frac{\partial m}{\partial t} = \lambda_1 m \times (\Delta m - \phi'(m) + h) - \lambda_2 m \times (m \times (\Delta m - \phi'(m) + h)) \tag{2.11}$$

with the initial condition (2.9).

We have the following proposition about the weak solution of equations (2.7)-(2.9):

Proposition 2.9. *If m is a weak solution of the LLG system* (2.7)-(2.9)*, then for all $t \in [0, T]$ and a.e. $x \in \mathcal{O}$ we have*

$$|m(t, x)| = |m_0(x)|. \tag{2.12}$$

Proof. Let us point out that the idea of this proof is borrowed from [5].

For any $\varphi \in C_0^\infty(\mathcal{O}; \mathbb{R})$, let us define a function

$$\psi : \mathrm{H} \ni m \longmapsto \langle m, \varphi m \rangle_{\mathrm{H}} \in \mathbb{R}.$$

It is known that ψ is of C^1 class and

$$\psi'(m) = 2\varphi m, \qquad m \in \mathrm{H}.$$

Since by Definition 2.4 and equation (2.11) and Lemma III.1.2 in [18], one has

$$\psi(m(t)) - \psi(m_0) = \int_0^t {}_{\mathrm{V}}\left\langle \psi'(m(s)), \frac{\mathrm{d}m}{\mathrm{d}s} \right\rangle_{\mathrm{V}^*} \mathrm{d}s$$
$$= \int_0^t {}_{\mathrm{V}}\left\langle 2\varphi m(s), \frac{\mathrm{d}m}{\mathrm{d}s} \right\rangle_{\mathrm{V}^*} \mathrm{d}s = 0, \qquad t \in [0, T].$$

We infer that for every $t \in [0,T]$ we have

$$\int_{\mathcal{O}} \varphi(x)\left(|m(t,x)|^2 - |m_0(x)|^2\right) \mathrm{d}x$$
$$= \int_{\mathcal{O}} \langle m(t,x), \varphi(x) m(t,x)\rangle \,\mathrm{d}x - \int_{\mathcal{O}} \langle m_0(x), \varphi(x) m_0(x)\rangle \,\mathrm{d}x = 0.$$

Since $\varphi \in C_0^\infty(\mathcal{O};\mathbb{R})$ is arbitrary, we deduce that the equality $|m(t,x)| = |m_0(x)|$ is valid for all $t \in [0,T]$ and a.e. $x \in \mathcal{O}$. Hence the proof is complete. □

Definition 2.10 (Energy)**.** For $a \in \mathbb{H}^1$, we define the energy $\mathcal{E}(a)$ by

$$\mathcal{E}(a) = \frac{1}{2}\|\nabla a\|_{\mathrm{H}}^2 + \int_{\mathcal{O}} \phi(a(x))\,\mathrm{d}x. \tag{2.13}$$

About the energy $\mathcal{E}$, we have the following dissipative proposition:

Proposition 2.11. *Let m be the weak solution of the system* (2.7)-(2.9) *with $h = 0$, then for all $t \geq s \geq 0$,*

$$\mathcal{E}(m(t)) + \lambda_2 \int_s^t \|m(r) \times (\Delta m(r) - \phi'(m(r)))\|_{\mathbb{H}}^2 \,\mathrm{d}r = \mathcal{E}(m(s)). \tag{2.14}$$

Proof. Let m be a weak solution of the system (2.7), (2.8) and (2.9) with $h = 0$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(m(t)) &= \langle \nabla_m \mathcal{E}(m(t)), m'(t)\rangle_{\mathrm{H}} \\ &= \langle -\Delta m(t) + \phi'(m), \lambda_1 m \times (\Delta m(t) - \phi'(m)) \\ &- \lambda_2 m \times (m \times (\Delta m(t) - \phi'(m)))\rangle_{\mathrm{H}} \\ &= -\lambda_2 \|m \times (\Delta m(t) - \phi'(m))\|_{\mathbb{H}}^2 \leq 0.\end{aligned}$$

So (2.14) is proved. □

Definition 2.12. We say that a weak solution m of the LLG system (2.7)-(2.9) is a strong solution iff

$$m \in L^2(0,T;D(A)) \cap H^1(0,T;\mathrm{H}).$$

3. Existence and uniqueness of solution

We will prove the existence and uniqueness of a strong global solution of the system (2.7)-(2.9). The results of this section are summarized in the Theorem 3.37.

3.1. Existence of a weak solution

To prove Theorem 3.37, we first prove the following theorem about the existence of a weak solution.

Theorem 3.1. *For any $T > 0$, $h \in L^2(0,T;\mathrm{H})$ and $\phi \in C_0^2(\mathbb{R}^3;\mathbb{R}^+)$, there exists a weak solution $m \in L^\infty(0,T;\mathbb{H}^1) \cap H^1(0,T;\mathrm{H})$ of equation* (2.7).

We use 3 steps, which is similar as in [19] to prove Theorem 3.1.

Step 1: Galerkin Approximation. We first consider a n dimensional system. We have the following result similar as explained in ([9], p.355, Thm 6.5.1):

Proposition 3.2. *[9] There exists a countable ONB $\{e_n\}_{n=1}^\infty$ of H, consisting of eigenvectors of A_1 such that $e_n \in C^\infty(\mathcal{O}) \cap D(A)$. The corresponding eigenvalues $\{\gamma_n\}_{n=1}^\infty$ satisfies*

$$0 < \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n \le \gamma_{n+1} \le \cdots \qquad \textit{and} \qquad \gamma_n \to \infty \ \textit{as}\ n \to \infty.$$

Next, let us define the orthogonal projection

$$\pi_n : \mathrm{H} \to \mathrm{H}_n := \mathrm{linspan}\{e_1, \ldots, e_n\},$$

and let us consider the following finite dimensional problem in $\mathbb{H}^n$:

$$\begin{aligned} m_n' &= \lambda_1 \pi_n\big(m_n \times (\Delta m_n - \pi_n[\phi'(m_n)] + h_n)\big) \\ &\quad - \lambda_2 \pi_n\big(m_n \times (m_n \times (\Delta m_n - \pi_n[\phi'(m_n)] + h_n))\big), \end{aligned} \tag{3.1}$$

$$m_n(0) = m_{0,n} := \pi_n m_0, \tag{3.2}$$

where $h_n(t) := \pi_n h(t)$, for $t \in [0,T]$. About the finite dimensional system (3.1) and (3.2), we have the following result:

Lemma 3.3. *The system* (3.1) *and* (3.2) *has a unique global solution $m_n \in C^1([0,\infty);\mathrm{H}_n)$.*

Proposition 3.4. *There is a constant $C > 0$ such that for all $n \in \mathbb{N}$, we have*

$$|m_{0,n}(x)| \le C\|m_0\|_{\mathbb{H}^1}, \qquad \textit{for a.e. } x \in \mathcal{O}. \tag{3.3}$$

Proof. By the Sobolev imbedding and (2.1), it follows that there exists a constant $C > 0$ such that

$$\|m_{0,n}\|_{\mathbb{L}^\infty} \le C\|m_{0,n}\|_{\mathbb{H}^1} = C\|A_1^{\frac{1}{2}}\pi_n m_0\|_{\mathrm{H}} \le C\|A_1^{\frac{1}{2}} m_0\|_{\mathrm{H}} = C\|m_0\|_{\mathbb{H}^1}.$$

$\square$

Step 2: A'priori Estimates. About the solution of n-dimensional system m_n, we have the following results:

Proposition 3.5. *For all $n \in \mathbb{N}$, we have*

$$\|m_n(t)\|_{\mathrm{H}} = \|m_n(0)\|_{\mathrm{H}}, \qquad \forall t \in [0,T]. \tag{3.4}$$

Proof. By the equation (3.1) and since we have (A.3), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m_n(t)\|_{\mathrm{H}}^2 = 2\,\langle m_n'(t), m_n(t)\rangle_{\mathrm{H}} = 0.$$

Therefore the proof is complete. $\square$

Let us define the n-dimensional total energy $\mathcal{E}_n : \mathrm{H}_n \to \mathbb{R}$ by

$$\mathcal{E}_n(u) = \int_{\mathcal{O}} \phi(u(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla u\|_{\mathrm{H}}^2, \qquad u \in \mathrm{H}_n. \tag{3.5}$$

In addition, let us define for the solution m_n of the n-dimensional system (3.1) and (3.2),

$$\rho_n := -\nabla_m \mathcal{E}_n(m_n) = -\pi_n \phi'(m_n) + \Delta m_n \in \mathrm{H}_n. \tag{3.6}$$

Then the equation (3.1) can be written as follows

$$m_n' = \lambda_1 \pi_n\big(m_n \times (\rho_n + h_n)\big) - \lambda_2 \pi_n\big(m_n \times (m_n \times (\rho_n + h_n))\big). \tag{3.7}$$

Theorem 3.6. *There exists a constant $C > 0$ which may depend on $\|\phi\|_{L^\infty(\mathbb{R}^3)}$, $\|m_0\|_{\mathbb{H}^1}$ and $\|h\|_{L^2(0,T;\mathrm{H})}$, such that for all $n \in \mathbb{N}$, we have*

$$\|m_n\|_{L^\infty(0,T;\mathbb{H}^1)} \le C, \tag{3.8}$$

$$\|m_n \times \rho_n\|_{L^2(0,T;\mathrm{H})} \leq C, \tag{3.9}$$

and

$$\|m_n \times (m_n \times \rho_n)\|_{L^2(0,T;\mathrm{H})} \leq C. \tag{3.10}$$

Proof of Theorem 3.6. Firstly, we will show the proof of (3.8) and (3.9).

For simplicity, we may use C to denote different constants. By the n-dimensional equation (3.1), one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{E}_n(m_n)\big)(t) = \left\langle \mathrm{d}_{m_n}\mathcal{E}_n(m_n), \frac{\mathrm{d}m_n}{\mathrm{d}t} \right\rangle_{\mathrm{H}} \\
&= \big\langle -\rho_n, \lambda_1 \pi_n(m_n \times (\rho_n + h_n) - \lambda_2 \pi_n\big(m_n \times (m_n \times (\rho_n + h_n))\big)\big\rangle_{\mathrm{H}} \\
&= \lambda_1 \left\langle -\rho_n, \pi_n(m_n \times h_n)\right\rangle_{\mathrm{H}} - \lambda_2 \|m_n \times \rho_n\|^2_{\mathrm{H}} - \lambda_2 \big\langle -\rho_n, \pi_n\big(m_n \times (m_n \times h_n)\big)\big\rangle_{\mathrm{H}}.
\end{aligned}$$

Integrating both sides of the equation above over $[0,t]$, we get

$$\begin{aligned}
\mathcal{E}_n(t) - \mathcal{E}_n(0) \;=\; &\lambda_1 \int_0^t \left\langle -\rho_n(s), \pi_n(m_n(s) \times h_n(s))\right\rangle_{\mathrm{H}} \mathrm{d}s - \lambda_2 \int_0^t \|m_n(s) \times \rho_n(s)\|^2_{\mathrm{H}} \,\mathrm{d}s \\
&-\lambda_2 \int_0^t \big\langle -\rho_n(s), \pi_n\big(m_n(s) \times (m_n(s) \times h_n(s))\big)\big\rangle_{\mathrm{H}} \,\mathrm{d}s, \qquad t \in [0,T].
\end{aligned}$$

Note that

$$\left\langle \rho_n, \pi_n(m_n \times h_n)\right\rangle_{\mathrm{H}} = -\left\langle m_n \times \rho_n, h_n\right\rangle_{\mathrm{H}},$$

and

$$\left\langle \rho_n, \pi_n(m_n \times (m_n \times h_n))\right\rangle_{\mathrm{H}} = -\left\langle m_n \times \rho_n, m_n \times h_n\right\rangle_{\mathrm{H}}.$$

Therefore by the Young's inequality, we have for all $\varepsilon > 0$

$$\begin{aligned}
&\int_{\mathcal{O}} \phi(m_n(t))\,\mathrm{d}x + \frac{1}{2}\|\nabla m_n(t)\|^2_{\mathrm{H}} + \lambda_2 \int_0^t \|m_n(s) \times \rho_n(s)\|^2_{\mathrm{H}} \,\mathrm{d}s \qquad (3.11)\\
&= \int_{\mathcal{O}} \phi(m_{0,n}(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla m_{0,n}\|^2_{\mathrm{H}} + \lambda_1 \int_0^t \left\langle m_n(s) \times \rho_n(s), h_n(s)\right\rangle_{\mathrm{H}} \mathrm{d}s \\
&\quad + \lambda_2 \int_0^t \left\langle m_n(s) \times \rho_n(s), m_n(s) \times h_n(s)\right\rangle_{\mathrm{H}} \mathrm{d}s \\
&\leq \int_{\mathcal{O}} \phi(m_{0,n}(x))\,\mathrm{d}x + \frac{1}{2}\|m_0\|^2_{\mathbb{H}^1} + \frac{\lambda_1 \varepsilon}{2} \int_0^t \|m_n(s) \times \rho_n(s)\|^2_{\mathrm{H}} \,\mathrm{d}s + \frac{\lambda_1}{2\varepsilon} \int_0^t \|h_n(s)\|^2_{\mathrm{H}} \,\mathrm{d}s \\
&\quad + \frac{\lambda_2 \varepsilon}{2} \int_0^t \|m_n(s) \times \rho_n(s)\|^2_{\mathrm{H}} \,\mathrm{d}s + \frac{\lambda_2}{2\varepsilon} \int_0^t \|m_n(s) \times h_n(s)\|^2_{\mathrm{H}} \,\mathrm{d}s.
\end{aligned}$$

Now let us consider the last term of the right hand side of the inequality above.

$$\frac{\lambda_2}{2\varepsilon}\int_0^t \|m_n(s)\times h_n(s)\|_{\mathrm{H}}^2\,\mathrm{d}s \le \frac{\lambda_2}{2\varepsilon}\int_0^t \|m_n(s)\|_{\mathbb{L}^\infty}^2\|h_n(s)\|_{\mathrm{H}}^2\,\mathrm{d}s.$$

By the interpolation inequality,

$$\|u\|_{\mathbb{L}^\infty}^2 \le C\|u\|_{\mathrm{H}}\|u\|_{\mathbb{H}^1}, \qquad u\in\mathbb{H}^1, \tag{3.12}$$

and making use of (3.4), we have

$$\begin{aligned}\|m_n(s)\|_{\mathbb{L}^\infty}^2 &\le C\|m_n(s)\|_{\mathrm{H}}\|m_n(s)\|_{\mathbb{H}^1}\\ &\le C\|m_0\|_{\mathbb{H}^1}(\|m_0\|_{\mathrm{H}}+\|\nabla m_n(s)\|_{\mathrm{H}}) \le C + C\|\nabla m_n(s)\|_{\mathrm{H}}.\end{aligned}$$

So

$$\begin{aligned}&\frac{\lambda_2}{2\varepsilon}\int_0^t \|m_n(s)\times h_n(s)\|_{\mathrm{H}}^2\,\mathrm{d}s \le \frac{\lambda_2}{2\varepsilon}\int_0^t \|m_n(s)\|_{\mathbb{L}^\infty}^2\|h_n(s)\|_{\mathrm{H}}^2\,\mathrm{d}s\\ \le\ &\frac{\lambda_2}{2\varepsilon}\int_0^t (C+C\|\nabla m_n(s)\|_{\mathrm{H}})\,\|h_n(s)\|_{\mathrm{H}}^2\,\mathrm{d}s \le \frac{\lambda_2}{2\varepsilon}\left(C+C\|\nabla m_n(s)\|_{L^\infty(0,T;\mathrm{H})}\right)\|h\|_{L^2(0,T;\mathrm{H})}^2\\ \le\ &\frac{\lambda_2}{2\varepsilon}\left(\varepsilon^2 C+\varepsilon^2 C\|\nabla m_n\|_{L^\infty(0,T;\mathrm{H})}^2+\frac{1}{\varepsilon^2}\|h\|_{L^2(0,T;\mathrm{H})}^4\right).\end{aligned}$$

Hence, by the inequality (3.11), we have

$$\begin{aligned}&\left(\frac{1}{2}-\frac{\lambda_2\varepsilon C}{2}\right)\|\nabla m_n(t)\|_{L^\infty(0,T;\mathrm{H})}^2+\left(\lambda_2-\frac{\varepsilon(\lambda_1+\lambda_2)}{2}\right)\int_0^t \|m_n(s)\times\rho_n(s)\|_{\mathrm{H}}^2\,\mathrm{d}s\\ \le\ &C+\frac{1}{2}\|m_0\|_{\mathbb{H}^1}^2+\frac{\lambda_1}{2\varepsilon}\|h\|_{L^2(0,T;\mathrm{H})}^2+\frac{\lambda_2\varepsilon C}{2}+\frac{\lambda_2}{2\varepsilon^3}\|h\|_{L^2(0,T;\mathrm{H})}^4.\end{aligned}$$

Next we choose ε such that $\lambda_2-\frac{\varepsilon(\lambda_1+\lambda_2)}{2}>0$ and $1-\lambda_2\varepsilon C>0$. It is easy to see that such an $\varepsilon>0$ exists. Then the estimates (3.8) and (3.9) are following by Proposition 3.5.

Finally, we will prove (3.10).

Since $\mathbb{H}^1\hookrightarrow\mathbb{L}^\infty$, one has

$$\begin{aligned}\|m_n\times(m_n\times\rho_n)\|_{L^2(0,T;\mathrm{H})} &\le \|m_n\|_{L^\infty(0,T;\mathbb{L}^\infty)}\|m_n\times\rho_n\|_{L^2(0,T;\mathrm{H})}\\ &\le \|m_n\|_{L^\infty(0,T;\mathbb{H}^1)}\|m_n\times\rho_n\|_{L^2(0,T;\mathrm{H})}.\end{aligned}$$

By (3.8) and (3.9), we have proved (3.10). □

Theorem 3.7. *There is a constant $C > 0$ which may depend on $\|\phi\|_{L^\infty(\mathbb{R}^3)}$, $\|m_0\|_{\mathbb{H}^1}$ and $\|h\|_{L^2(0,T;\mathrm{H})}$, such that for all $n \in \mathbb{N}$,*

$$\|m_n\|_{H^1(0,T;\mathrm{H})} \leq C. \tag{3.13}$$

Proof. We have

$$\begin{aligned}
&\left\|\pi_n\big(m_n \times (m_n \times (\rho_n + h_n))\big)\right\|^2_{L^2(0,T;\mathrm{H})} \leq \int_0^T \|m_n \times (m_n \times (\rho_n + h_n))\|^2_{\mathbb{H}}\, \mathrm{d}t \\
&\leq 2\int_0^T \left(\|m_n \times (m_n \times \rho_n)\|^2_{\mathbb{H}} + \|m_n \times (m_n \times h_n)\|^2_{\mathbb{H}}\right) \mathrm{d}t \\
&\leq 2\int_0^T \int_{\mathcal{O}} \big(|m_n|^2 |m_n \times \rho_n|^2 + |m_n|^4 |h_n|^2\big)\, \mathrm{d}x\, \mathrm{d}t \\
&\leq 2\|m_n\|^2_{L^\infty(0,T;\mathbb{L}^\infty)} \|m_n \times \rho_n\|^2_{L^2(0,T;\mathrm{H})} + 2\|m_n\|^4_{L^\infty(0,T;\mathbb{L}^\infty)} \|h_n\|^2_{L^2(0,T;\mathrm{H})} \\
&\leq C\left(\|m_n\|^2_{L^\infty(0,T;\mathbb{H}^1)} \|m_n \times \rho_n\|^2_{L^2(0,T;\mathrm{H})} + \|m_n\|^4_{L^\infty(0,T;\mathbb{H}^1)} \|h\|^2_{L^2(0,T;\mathrm{H})}\right).
\end{aligned}$$

Similarly, we also have

$$\left\|\pi_n\big(m_n \times (\rho_n + h_n)\big)\right\|^2_{L^2(0,T;\mathrm{H})} \leq C\left(\|m_n \times \rho_n\|^2_{L^2(0,T;\mathrm{H})} + \|m_n\|^2_{L^\infty(0,T;\mathbb{H}^1)} \|h\|^2_{L^2(0,T;\mathrm{H})}\right).$$

Therefore the result follows from (3.7), (3.8), (3.9) and (3.10). $\square$

We have the following corollary of Theorem 3.6 and Theorem 3.7:

Corollary 3.8. *There exist $m \in L^\infty(0,T;\mathbb{H}^1) \cap H^1(0,T;\mathrm{H})$ and $P \in L^2(0,T;\mathrm{H})$ such that*

$$m_n \longrightarrow m \qquad \textit{weakly}^* \textit{ in } L^\infty(0,T;\mathbb{H}^1) \textit{ and weakly in } H^1(0,T;\mathrm{H}), \tag{3.14}$$

and

$$m_n \times \rho_n \longrightarrow P \qquad \textit{weakly in } L^2(0,T;\mathrm{H}). \tag{3.15}$$

From now on all the m below are the same one as in Corollary 3.8.

Proposition 3.9.

$$\nabla m_n \longrightarrow \nabla m \qquad \textit{weakly}^* \textit{ star in } L^\infty(0,T;\mathrm{H}). \tag{3.16}$$

Proof. By the estimate (3.8), there exists some $\mathrm{v} \in L^\infty(0,T;\mathrm{H})$ such that

$$\nabla m_n \longrightarrow \mathrm{v} \qquad \text{weakly}^* \text{ star in } L^\infty(0,T;\mathrm{H}).$$

Note that by (3.14), for any $u \in L^1(0,T;\mathrm{H})$, one has

$$\begin{aligned} & {}_{L^\infty(0,T;\mathrm{H})}\langle \mathrm{v} - \nabla m, u\rangle_{L^1(0,T;\mathrm{H})} = \lim_{n\to\infty} {}_{L^\infty(0,T;\mathrm{H})}\langle \nabla m_n - \nabla m, u\rangle_{L^1(0,T;\mathrm{H})} \\ = & -\lim_{n\to\infty} {}_{L^\infty(0,T;\mathbb{H}^1)}\langle m_n - m, \nabla u\rangle_{L^1(0,T;(\mathbb{H}^1)^*)} = 0. \end{aligned}$$

So $\mathrm{v} = \nabla m$ in $L^\infty(0,T;\mathrm{H})$.

□

Step 3: Proof that the limit function m is a weak solution.

Lemma 3.10 (See [18],Th 3.2.1)**.** *Let X_0, X, X_1 be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, where the embeddings are continuous. In addition X_0, X_1 are reflexive and the embedding $X_0 \hookrightarrow X$ is compact. Let $T > 0$ be a fixed finite number, and let α_0, α_1 be two finite numbers such that $\alpha_i > 1$, $i = 0, 1$. We consider the space*

$$Y = \left\{ v \in L^{\alpha_0}(0,T;X_0),\ v' = \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{\alpha_1}(0,T;X_1) \right\},$$

with the norm

$$\|v\|_Y = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v'\|_{L^{\alpha_1}(0,T;X_1)}.$$

Then $Y \subset L^{\alpha_0}(0,T;X)$ and the embedding $Y \hookrightarrow L^{\alpha_0}(0,T;X)$ is compact.

Theorem 3.11. *For all $\alpha_0 \in (1,\infty)$, we have*

$$\lim_{n\to\infty} \|m - m_n\|_{L^{\alpha_0}\left(0,T;C(\bar{\mathcal{O}})\right)} = 0. \tag{3.17}$$

Proof. Let $X_0 = \mathbb{H}^1$, $X = C(\bar{\mathcal{O}})$ and $X_1 = \mathrm{H}$. Then by [1] Theorem 6.3, it follows that $X_0 \hookrightarrow X$ compactly and $X \hookrightarrow X_1$ continuously. By the estimates (3.8) and (3.13), m_n is bounded in the space Y as in the Lemma 3.10 for any $\alpha_0 > 1$ and $\alpha_1 = 2$. By Lemma 3.10, $Y \hookrightarrow L^{\alpha_0}(0,T;C(\bar{\mathcal{O}}))$ is compact. So there exists an element $\tilde{m} \in L^{\alpha_0}(0,T;C(\bar{\mathcal{O}}))$ such that there is a subsequence of $\{m_n\}$ (still denoted by $\{m_n\}$) converges to $\tilde{m}$ in $L^{\alpha_0}(0,T;C(\bar{\mathcal{O}}))$.

Note that

$$\begin{aligned}
&\|m-\tilde{m}\|^2_{\mathbb{L}^2(0,T;\mathrm{H})} = \langle m-\tilde{m}, m-\tilde{m}\rangle_{\mathbb{L}^2(0,T;\mathrm{H})}\\
&= \lim_{n\to\infty} \langle m-m_n, m-\tilde{m}\rangle_{\mathbb{L}^2(0,T;\mathrm{H})}\\
&= \lim_{n\to\infty} {}_{L^\infty(0,T;\mathbb{H}^1)}\langle m-m_n, m-\tilde{m}\rangle_{L^1(0,T;(\mathbb{H}^1)^*)} = 0.
\end{aligned}$$

Therefore, we have $m=\tilde{m}$ a.e. and they are both in $L^{\alpha_0}\big(0,T;C(\bar{\mathcal{O}})\big)$, so $m=\tilde{m}$ in $L^{\alpha_0}\big(0,T;C(\bar{\mathcal{O}})\big)$. This completes the proof. $\square$

We have the following corollary of Theorem 3.11,

Corollary 3.12. *For all* $\alpha_0>1$*,*

$$m(\cdot,0)=m(\cdot,2\pi) \text{ in } L^{\alpha_0}(0,T;\mathbb{R}^3), \tag{3.18}$$

and

$$m(t,0)=m(t,2\pi) \text{ for a.e. } t\in[0,T]\ . \tag{3.19}$$

Proof. By the convergence (3.17), one has

$$\begin{aligned}
&\int_0^T |m(t,0)-m(t,2\pi)|^{\alpha_0}_{\mathbb{R}^3}\,\mathrm{d}t\\
&\le C\int_0^T |m(t,0)-m_n(t,0)|^{\alpha_0}+|m_n(t,2\pi)-m(t,2\pi)|^{\alpha_0}\,\mathrm{d}t\\
&\le C\int_0^T \sup_{x\in\bar{\mathcal{O}}}|m(t,x)-m_n(t,x)|^{\alpha_0}\,\mathrm{d}t = C\int_0^T \|m(t)-m_n(t)\|^{\alpha_0}_{C(\bar{\mathcal{O}})}\,\mathrm{d}t\to 0,
\end{aligned}$$

as $n\to\infty$.

Since $\alpha_0>1$, the equality (3.19) follows from (3.18) immediately. $\square$

Proposition 3.13.

$$m_n\times(m_n\times\rho_n)\longrightarrow m\times P \qquad \textit{weakly in } L^2(0,T;\mathrm{H}). \tag{3.20}$$

Proof. Let $u\in C_0^\infty([0,T]\times\mathcal{O})$, then we have

$$\begin{aligned}
&\langle m_n\times(m_n\times\rho_n)-m\times P, u\rangle_{L^2(0,T;\mathrm{H})}\\
&=\langle (m_n-m)\times(m_n\times\rho_n), u\rangle_{L^2(0,T;\mathrm{H})} + \langle m\times(m_n\times\rho_n-P), u\rangle_{L^2(0,T;\mathrm{H})}\\
&=\langle m_n\times\rho_n, u\times(m_n-m)\rangle_{L^2(0,T;\mathrm{H})} + \langle m_n\times\rho_n-P, u\times m\rangle_{L^2(0,T;\mathrm{H})}\\
&\le \|m_n\times\rho_n\|_{L^2(0,T;\mathrm{H})}\|u\|_{L^\infty(0,T;\mathbb{L}^\infty)}\|m_n-m\|_{L^2(0,T;\mathrm{H})} + \langle m_n\times\rho_n-P, u\times m\rangle_{L^2(0,T;\mathrm{H})}\,.
\end{aligned}$$

By (3.17) and (3.15) and since $u \times m \in L^2(0,T;\mathrm{H})$, we infer that

$$\langle m_n \times (m_n \times \rho_n) - m \times P, u\rangle_{L^2(0,T;\mathrm{H})} \to 0, \qquad n \to \infty.$$

Since $C_0^\infty([0,T]\times\mathcal{O})$ is dense in $L^2(0,T;\mathrm{H})$ and by (3.10) $\|m_n\times(m_n\times\rho_n)\|_{L^2(0,T;\mathrm{H})}$ is uniformly bounded, $m_n \times (m_n \times \rho_n) \longrightarrow m \times P$ weakly in $L^2(0,T;\mathrm{H})$. $\square$

Proposition 3.14.

$$\pi_n\big(m_n \times (m_n \times \rho_n)\big) \longrightarrow m \times P \qquad \textit{weakly in } L^2(0,T;\mathrm{H}). \tag{3.21}$$

Proof. Let $u \in L^2(0,T;\mathrm{H})$,

$$\begin{aligned}
&\big\langle \pi_n\big(m_n \times (m_n \times \rho_n)\big) - m \times P, u\big\rangle_{L^2(0,T;\mathrm{H})}\\
=\ &\langle m_n \times (m_n \times \rho_n) - m \times P, \pi_n u - u\rangle_{L^2(0,T;\mathrm{H})}\\
&+ \langle m_n \times (m_n \times \rho_n) - m \times P, u\rangle_{L^2(0,T;\mathrm{H})} + \langle \pi_n(m \times P) - m \times P, u\rangle_{L^2(0,T;\mathrm{H})}.
\end{aligned}$$

By (3.10) and since $\pi_n u \to u$ strongly in $L^2(0,T;\mathrm{H})$, the first term on the right hand side of the equation above tends to 0, by (3.20) the second term tends to 0, and since $\pi_n(m \times P) \to m \times P$ strongly, the third term also tends to 0. The proof is complete. $\square$

Similarly we can prove:

Proposition 3.15.

$$\pi_n(m_n \times h_n) \longrightarrow m \times h \qquad \textit{weakly in } L^2(0,T;\mathrm{H}). \tag{3.22}$$

Corollary 3.16. *For $t \in [0,T]$, $u \in \mathrm{H}$,*

$$\int_{\mathcal{O}} \langle m(t)-m_0, u\rangle \,\mathrm{d}x = \lambda_1 \int_0^t \int_{\mathcal{O}} \langle m\times h-P, u\rangle \,\mathrm{d}x\,\mathrm{d}s - \lambda_2 \int_0^t \int_{\mathcal{O}} \langle m\times(m\times h-P), u\rangle \,\mathrm{d}x\,\mathrm{d}s. \tag{3.23}$$

Proof. By (3.14), (3.7), (3.15), (3.21) and (3.22), we have

$$\begin{aligned}
&\int_{\mathcal{O}} \langle m(t) - m_0, u\rangle \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathcal{O}} \langle m_n(t) - m_{n,0}, u\rangle \,\mathrm{d}x\\
=\ &\lim_{n\to\infty} \int_0^t \int_{\mathcal{O}} \big\langle \lambda_1 \pi_n(m_n \times (h_n + \rho_n)) - \lambda_2 \pi_n\big(m_n \times (m_n \times (h_n + \rho_n))\big), u\big\rangle \,\mathrm{d}x\,\mathrm{d}s\\
=\ &\lambda_1 \int_0^t \int_{\mathcal{O}} \langle m \times h - P, u\rangle \,\mathrm{d}x\,\mathrm{d}s - \lambda_2 \int_0^t \int_{\mathcal{O}} \langle m \times (m \times h - P), u\rangle \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

The proof is complete. $\square$

Theorem 3.17. *m satisfies the equation* (2.10).

Proof. By (3.17), $m_n(t,x)$ converges to $m(t,x)$ almost everywhere in $[0,T]\times\mathcal{O}$. By our assumption, $\phi' : \mathbb{R}^3 \to \mathbb{R}$ is continuous, so $\phi'(m_n(t,x))$ also converges to $\phi'(m(t,x))$ almost everywhere in $[0,T]\times\mathcal{O}$. Since ϕ' and the integral domain $[0,T]\times\mathcal{O}$ are both bounded, by the Lebesgue's dominated convergence theorem, one has

$$\phi'(m_n) \longrightarrow \phi'(m) \quad \text{strongly in } L^p(0,T;\mathbb{L}^p) \text{ for all } p \geq 1. \tag{3.24}$$

So one has

$$\pi_n[\phi'(m_n)] \longrightarrow \phi'(m) \quad \text{strongly in } L^2(0,T;\mathrm{H}). \tag{3.25}$$

Therefore, since

$$\begin{aligned}
&\langle m_n \times \pi_n[\phi'(m_n)] - m \times \phi'(m), u\rangle_{L^2(0,T;\mathrm{H})} \\
&= \left\langle m_n \times \left(\pi_n[\phi'(m_n)] - \phi'(m)\right), u\right\rangle_{L^2(0,T;\mathrm{H})} + \langle (m_n - m)\times \phi'(m), u\rangle_{L^2(0,T;\mathrm{H})} \\
&\leq \|\pi_n[\phi'(m_n)] - \phi'(m)\|_{L^2(0,T;\mathrm{H})}\|u\|_{L^2(0,T;\mathrm{H})}\|m_n\|_{L^\infty(0,T;\mathbb{L}^\infty)} \\
&\qquad + \|m_n - m\|_{L^2(0,T;\mathrm{H})}\|u\|_{L^2(0,T;\mathrm{H})}\|\phi'(m)\|_{L^\infty(0,T;\mathbb{L}^\infty)} \longrightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle m_n \times h_n - m \times h, u\rangle_{L^2(0,T;\mathrm{H})} \\
&= \langle m_n \times (h_n - h), u\rangle_{L^2(0,T;\mathrm{H})} + \langle (m_n - m)\times h, u\rangle_{L^2(0,T;\mathrm{H})} \\
&\leq \|h_n - h\|_{L^2(0,T;\mathrm{H})}\|m_n\|_{L^\infty(0,T;\mathbb{L}^\infty)}\|u\|_{L^2(0,T;\mathrm{H})} \\
&\qquad + \|m_n - m\|_{L^2(0,T;\mathbb{L}^\infty)}\|h\|_{L^2(0,T;\mathrm{H})}\|u\|_{L^\infty(0,T;\mathrm{H})} \longrightarrow 0,
\end{aligned}$$

we have

$$m_n \times (-\pi_n[\phi'(m_n)] + h_n) \longrightarrow m \times (-\phi'(m) + h) \quad \text{weakly in } L^2(0,T;\mathrm{H}). \tag{3.26}$$

So by (3.15),

$$m_n \times \Delta m_n \longrightarrow P - m \times (-\phi'(m)) \quad \text{weakly in } L^2(0,T;\mathrm{H}). \tag{3.27}$$

Therefore for $u \in L^2(0,T;\mathrm{V})$, one has

$$\begin{aligned}\lim_{n\to\infty} \langle \nabla m_n \times m_n, \nabla u \rangle_{L^2(0,T;\mathrm{H})} &= \lim_{n\to\infty} \langle m_n \times \Delta m_n, u \rangle_{L^2(0,T;\mathrm{H})} \\ &= \langle P - m \times (-\phi'(m)), u \rangle_{L^2(0,T;\mathrm{H})} . \quad (3.28)\end{aligned}$$

On the other hand, for $u \in L^2(0,T;\mathrm{H})$ one has

$$\begin{aligned}&\langle \nabla m_n \times m_n - \nabla m \times m, u \rangle_{L^2(0,T;\mathrm{H})} \\ &\quad = \langle \nabla m_n \times (m_n - m), u \rangle_{L^2(0,T;\mathrm{H})} + \langle (\nabla m_n - \nabla m) \times m, u \rangle_{L^2(0,T;\mathrm{H})} \quad (3.29) \\ &\quad \le \|\nabla m_n\|_{L^\infty(0,T;\mathrm{H})} \|m_n - m\|_{L^2(0,T;\mathbb{L}^\infty)} \|u\|_{L^2(0,T;\mathrm{H})} \\ &\quad + {}_{L^\infty(0,T;\mathrm{H})}\langle \nabla m_n - \nabla m, m \times u \rangle_{L^1(0,T;\mathrm{H})} \to 0, \qquad \text{as } n \to \infty.\end{aligned}$$

The convergence to 0 above follows from (3.8), (3.17) and (3.16). Therefore

$$\lim_{n\to\infty} \langle \nabla m_n \times m_n, u \rangle_{L^2(0,T;\mathrm{H})} = \langle \nabla m \times m, u \rangle_{L^2(0,T;\mathrm{H})} . \tag{3.30}$$

By (3.28) and (3.30), for $u \in L^2(0,T;\mathrm{V})$, one has

$$\langle P, u \rangle_{L^2(0,T;\mathrm{H})} = \langle \nabla m \times m, u \rangle_{L^2(0,T;\mathrm{H})} + \langle m \times (-\phi'(m)), u \rangle_{L^2(0,T;\mathrm{H})} . \tag{3.31}$$

By (3.14) and (3.18), for $u \in L^2(0,T;\mathrm{V})$, one has $u \times m \in L^2(0,T;\mathrm{V})$. Therefore by (3.31), one also gets

$$\langle P, u \times m \rangle_{L^2(0,T;\mathrm{H})} = \langle \nabla m \times m, \nabla (u \times m) \rangle_{L^2(0,T;\mathrm{H})} + \big\langle m \times \big(m \times (-\phi'(m))\big), u \big\rangle_{L^2(0,T;\mathrm{H})} . \tag{3.32}$$

Hence by (3.31), (3.32) and (3.23), we deduce (2.10). So the proof is complete. □

Remark 3.18. So far we have proved that there exists an element $m \in L^\infty(0,T;\mathbb{H}^1) \cap H^1(0,T;\mathrm{H})$ satisfying the equation (2.10). So Theorem 3.1 has been proved. Hence from now on we can use (2.12). Later on we will prove that our weak solution is in fact a strong solution.

3.2. Further regularity and existence of a strong solution

Remark 3.19. By the proof of Theorem 3.17, we also have following three results:

There exists a constant C independent of n, such that

$$\|m_n \times \Delta m_n\|_{L^2(0,T;\mathrm{H})} \leq C, \tag{3.33}$$

and hence

$$\|m_n \times \nabla m_n\|_{L^2(0,T;\mathbb{H}^1)} \leq C. \tag{3.34}$$

We also have

$$\nabla m_n \times m_n \longrightarrow \nabla m \times m \qquad \text{weakly in } L^2(0,T;\mathrm{H}). \tag{3.35}$$

Proof of (3.34). Note that $m_n(t) \in C^\infty(\mathcal{O})$ for every $t \in [0,\infty)$, so we have

$$\begin{aligned} \nabla(m_n(t) \times \nabla m_n(t)) &= \nabla m_n(t) \times \nabla m_n(t) + m_n(t) \times \Delta m_n(t) \\ &= m_n(t) \times \Delta m_n(t). \end{aligned}$$

Hence

$$\|\nabla(m_n \times \nabla m_n)\|^2_{L^2(0,T;\mathrm{H})} = \int_0^T \|\nabla(m_n(t) \times \nabla m_n(t))\|^2_{\mathrm{H}}\,\mathrm{d}t$$
$$= \int_0^T \|m_n(t) \times \Delta m_n(t)\|^2_{\mathrm{H}}\,\mathrm{d}t = \|m_n \times \Delta m_n\|^2_{L^2(0,T;\mathrm{H})}.$$

By (3.33), it follows that there exists some $C > 0$ such that

$$\|\nabla(m_n \times \nabla m_n)\|_{L^2(0,T;\mathrm{H})} \leq C, \qquad n \in \mathbb{N}$$

On the other hand, by (3.8), one has

$$\|m_n \times \nabla m_n\|_{L^2(0,T;\mathrm{H})} \leq \|m_n\|_{L^\infty(0,T;\mathbb{L}^\infty)} \|\nabla m_n\|_{L^2(0,T;\mathrm{H})} \leq C.$$

Therefore $\|m_n \times \nabla m_n\|_{L^2(0,T;\mathbb{H}^1)}$ is bounded by some constant independent of n. □

Proof of (3.35). It follows from (3.29) in the proof of Theorem 3.17. □

Corollary 3.20. *$\nabla m \times m$ belongs to $L^2(0,T;\mathbb{H}^1)$ and*

$$\nabla m_n \times m_n \longrightarrow \nabla m \times m \qquad \textit{weakly in } L^2(0,T;\mathbb{H}^1). \tag{3.36}$$

Proof. By (3.34) and the Banach-Alaoglu Theorem, there exists some $Q \in L^2(0,T;\mathbb{H}^1)$ such that

$$\nabla m_n \times m_n \longrightarrow Q \qquad \text{weakly in } L^2(0,T;\mathbb{H}^1). \tag{3.37}$$

By (3.35) and (3.37), one has

$$\begin{aligned} &\|\nabla m \times m - Q\|^2_{L^2(0,T;\mathrm{H})} = \langle \nabla m \times m - Q, \nabla m \times m - Q\rangle_{L^2(0,T;\mathrm{H})} \\ = &\lim_{n\to\infty} \langle \nabla m_n \times m_n - Q, \nabla m \times m - Q\rangle_{L^2(0,T;\mathrm{H})} \\ = &\lim_{n\to\infty} {}_{L^2(0,T;\mathbb{H}^1)}\langle \nabla m_n \times m_n - Q, \nabla m \times m - Q\rangle_{L^2(0,T;(\mathbb{H}^1)^*)} = 0. \end{aligned}$$

The proof is complete. □

Corollary 3.21. *For almost every $t \in [0,T]$,*

$$\nabla m_n(t) \times m_n(t) \longrightarrow \nabla m(t) \times m(t) \qquad \text{in } C(\bar{\mathcal{O}}). \tag{3.38}$$

Proof. By (3.34), for almost every $t \in [0,T]$, there exists $C(t)$ independent of n, such that

$$\|\nabla m_n(t) \times m_n(t)\|_{\mathbb{H}^1} \leq C(t).$$

By (3.36) and since the embedding $\mathbb{H}^1 \hookrightarrow C(\bar{\mathcal{O}})$ is compact, we get (3.38). □

Corollary 3.22.

$$\nabla m(t,x) \perp m(t,x), \qquad a.e.\ (t,x). \tag{3.39}$$

Proof. By (2.12) and by the chain rule of weak derivative we have

$$0 = \frac{1}{2}\nabla |m(t,x)|^2 = \langle \nabla m(t,x), m(t,x)\rangle, \qquad a.e.\ (t,x).$$

The proof is complete. □

Lemma 3.23.

$$(\nabla m \times m) \times m = -\nabla m \in L^2(0,T;\mathrm{H}). \tag{3.40}$$

Proof. Let us consider the following train of equalities:

$$\begin{aligned} &\|(\nabla m \times m) \times m + \nabla m\|^2_{L^2(0,T;\mathrm{H})} \\ = &\|(\nabla m \times m) \times m\|^2_{L^2(0,T;\mathrm{H})} + 2\,\langle \nabla m, (\nabla m \times m) \times m\rangle_{L^2(0,T;\mathrm{H})} + \|\nabla m\|^2_{L^2(0,T;\mathrm{H})} \\ = &\|(\nabla m \times m) \times m\|^2_{L^2(0,T;\mathrm{H})} - 2\|\nabla m \times m\|^2_{L^2(0,T;\mathrm{H})} + \|\nabla m\|^2_{L^2(0,T;\mathrm{H})}, \end{aligned}$$

where we used (A.2) for the second "=".

Since by the property of cross product, $\nabla m(t,x) \times m(t,x) \perp m(t,x)$ and by (2.12) $|m(t,x)| = 1$, one has $|\big(\nabla m(t,x) \times m(t,x)\big) \times m(t,x)| = |\nabla m(t,x) \times m(t,x)|$. Therefore

$$\begin{aligned}
&\|(\nabla m \times m) \times m + \nabla m\|^2_{L^2(0,T;\mathrm{H})} \\
= \;& \|\nabla m \times m\|^2_{L^2(0,T;\mathrm{H})} - 2\|\nabla m \times m\|^2_{L^2(0,T;\mathrm{H})} + \|\nabla m\|^2_{L^2(0,T;\mathrm{H})} \\
= \;& \|\nabla m\|^2_{L^2(0,T;\mathrm{H})} - \|\nabla m \times m\|^2_{L^2(0,T;\mathrm{H})}.
\end{aligned}$$

Similar as before, since by (3.39) $\nabla m(t,x) \perp m(t,x)$ and $|m(t,x)| = 1$, one has $|\nabla m(t,x) \times m(t,x)| = |\nabla m(t,x)|$. Hence

$$\|(\nabla m \times m) \times m + \nabla m\|^2_{L^2(0,T;\mathrm{H})} = 0.$$

The proof is complete. $\square$

Proposition 3.24.

$$m \in L^2(0,T;\mathbb{H}^2) \tag{3.41}$$

Proof. By (3.36) and (3.14), $\nabla m \times m \in L^2(0,T;\mathbb{H}^1)$ and $m \in L^\infty(0,T;\mathbb{H}^1)$. Hence

$$(\nabla m \times m) \times m \in L^2(0,T;\mathbb{H}^1).$$

Indeed, since $\mathbb{H}^1$ is an algebra, we get

$$(\nabla m(t) \times m(t)) \times m(t) \in \mathbb{H}^1, \quad \text{for a.e. } t \in [0,T].$$

Thus by (3.8) we also have $(\nabla m \times m) \times m \in L^2(0,T;\mathrm{H})$, hence $(\nabla m \times m) \times m \in L^2(0,T;\mathbb{H}^1)$.

On the other hand, we have (3.40), i.e.

$$(\nabla m \times m) \times m = -\nabla m \in L^2(0,T;\mathrm{H}).$$

Therefore $\nabla m \in L^2(0,T;\mathbb{H}^1)$, and so since $m \in L^2(0,T;\mathrm{H})$, $m \in L^2(0,T;\mathbb{H}^2)$ and the proof is complete. $\square$

Corollary 3.25. *m satisfies the following equation in $L^2(0,T;\mathrm{H})$ (i.e. m satisfies the following equation and all the terms in the following equation are in $L^2(0,T;\mathrm{H})$),*

$$\frac{\partial m}{\partial t} = \lambda_1 m \times (\Delta m - \phi'(m) + h) - \lambda_2 m \times \big(m \times (\Delta m - \phi'(m) + h)\big). \quad (3.42)$$

Corollary 3.26. *For almost every $t \in [0,T]$ and all $x \in \bar{\mathcal{O}}$, $\nabla m(t,x)$ exists (in strong sense) and*

$$\nabla m(t,x) \perp m(t,x) \qquad in\ \mathbb{R}^3. \quad (3.43)$$

Proof. By (3.41) and since $\mathbb{H}^2 \hookrightarrow C^1(\bar{\mathcal{O}})$, $\nabla m(t,x)$ exists in strong sense for almost every $t \in [0,T]$ and all $x \in \bar{\mathcal{O}}$.

Moreover, by (2.12),

$$\langle \nabla m(t,x), m(t,x) \rangle = \frac{1}{2} \nabla |m(t,x)|^2 = 0.$$

So the proof complete. □

Lemma 3.27. *For $a,b,c \in \mathbb{R}^3$ and $a \neq 0$, if $a \times b = a \times c$ and $a \perp b$ and $a \perp c$, then $b = c$.*

Theorem 3.28.

$$\nabla m(t,0) = \nabla m(t,2\pi), \qquad for\ a.e.\ t \in [0,T]. \quad (3.44)$$

Proof. Since $m_n(t) \in D(A)$, we infer that

$$\nabla m_n(t,0) \times m_n(t,0) = \nabla m_n(t,2\pi) \times m_n(t,2\pi),$$

for all $t \in [0,T]$. And by (3.38), one has

$$\nabla m_n(t,x) \times m_n(t,x) \longrightarrow \nabla m(t,x) \times m(t,x) \qquad \text{in } \mathbb{R}^3$$

for almost all $t \in [0,T]$ and all $x \in \bar{\mathcal{O}}$. Therefore

$$\nabla m(t,0) \times m(t,0) = \nabla m(t,2\pi) \times m(t,2\pi),$$

for almost all $t \in [0,T]$. Then by (3.43) and (3.19) and Lemma 3.27, we get (3.44). The proof is complete. □

Corollary 3.29.

$$m \in L^2(0, T; D(A)). \tag{3.45}$$

Proof. This is a direct result from (3.41), (3.19) and (3.44). □

Remark 3.30. By the following imbedding

$$L^2(0, T; D(A)) \cap H^1(0, T; \mathrm{H}) \hookrightarrow C([0, T]; \mathrm{V}),$$

(see [13], Theorem 3.1, p.19), we have

$$m \in C([0, T]; \mathrm{V}). \tag{3.46}$$

Proposition 3.31.

$$|m(t, x)| = 1, \qquad \textit{for all } (t, x) \in [0, T] \times \bar{\mathcal{O}}. \tag{3.47}$$

Proof. By (2.12), $|m(t, x)| = 1$ for a.e. $t \in [0, T]$ and $x \in \bar{\mathcal{O}}$, and by (3.46) and since $C([0, T]; \mathrm{V}) \hookrightarrow C([0, T] \times \bar{\mathcal{O}})$, we get (3.47). □

Proposition 3.32.

$$\langle m(t, x), \Delta m(t, x)\rangle_{\mathbb{R}^3} = -|\nabla m(t, x)|^2_{\mathbb{R}^3}, \qquad \textit{a.e. } (t, x). \tag{3.48}$$

Proof. By (2.12), $|m(t, x)|_{\mathbb{R}^3} = 1$. So by the chain rule of the weak derivative, we infer that

$$0 = \nabla |m(t, x)|^2_{\mathbb{R}^3} = 2 \langle \nabla m(t, x), m(t, x)\rangle_{\mathbb{R}^3}.$$

Therefore, by the chain rule used again

$$0 = \nabla \langle \nabla m(t, x), m(t, x)\rangle_{\mathbb{R}^3} = \langle \Delta m(t, x), m(t, x)\rangle_{\mathbb{R}^3} + |\nabla m(t, x)|^2_{\mathbb{R}^3}.$$

The proof is complete. □

Proposition 3.33.

$$m \in L^4(0, T; \mathbb{W}^{1,4}). \tag{3.49}$$

Proof. By (2.12) and (3.41),

$$\begin{aligned}&\int_0^T \int_{\mathcal{O}} |\langle m(t,x), \Delta m(t,x)\rangle_{\mathbb{R}^3}|^2 \, \mathrm{d}x \, \mathrm{d}t \\ \leq &\int_0^T \int_{\mathcal{O}} |m(t,x)|^2_{\mathbb{R}^3} |\Delta m(t,x)|^2_{\mathbb{R}^3} \, \mathrm{d}x \, \mathrm{d}t = \|\Delta m\|^2_{L^2(0,T;\mathrm{H})} < \infty.\end{aligned}$$

So the result followed by (3.48). □

Remark 3.34. Our proof of (3.41) and (3.49) is different from the proof of the one in a similar case given in ([6], Thm 5.2).

Proposition 3.35. *m satisfies the following equation in $L^2(0,T;\mathrm{H})$:*

$$\begin{aligned} m' \,=\, &\lambda_2 \Delta m + \lambda_1 m \times (\Delta m - \phi'(m) + h) + \lambda_2 m |\nabla m|^2 \\ &-\lambda_2 m \left\langle m, -\phi'(m) + h\right\rangle - \lambda_2 \phi'(m) + \lambda_2 h. \end{aligned} \tag{3.50}$$

Proof. This follows from (3.42), the equality $a \times (b \times c) = b\langle a, c\rangle - c\langle a, b\rangle$ and (3.48). □

3.3. Uniqueness of a weak solution

Theorem 3.36. *If m_1 and m_2 are both weak solutions of equation* (2.7)-(2.9) *with $m_0 \in \mathrm{V} \cap \mathcal{M}$ and if $m_i \in L^4(0,T;\mathbb{H}^1)$ for $i = 1,2$, then $m_1(t,x) = m_2(t,x)$ for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$.*

Proof. By Proposition 2.9, Theorem 3.36 can be proved similar as in the proof of Theorem 4.1 in [6] by using the formula (3.50). □

Since all the previous results hold for all $T > 0$, so the m in Corollary 3.8 is actually a global solution, we summarize the previous results by the following theroem:

Theorem 3.37. *For $h \in L^2_{\mathrm{loc}}(0,\infty;\mathrm{H})$ and $\phi \in C^2_0(\mathbb{R}^3;\mathbb{R}^+)$, there exists a unique strong global solution m of equation* (2.7) *satisfy* (3.47) *with the initial condition* (2.9) *and the periodic boundary condition* (2.8)*, such that*

$$m \in L^2(0,T;D(A)) \cap C([0,\infty);\mathrm{V}) \cap H^1(0,T;\mathrm{H}) \cap L^4(0,T;\mathbb{W}^{1,4}), \quad T > 0. \tag{3.51}$$

4. Convergence towards minimum of the energy

In this section, we only consider the case $h = 0$ and

$$\phi(u) = \frac{1}{2}|u - \zeta_+|^2_{\mathbb{R}^3}, \qquad u \in \mathbb{R}^3, \tag{4.1}$$

where $\zeta_+ = (0, 0, 1)$.

Let us observe that in this case we also have

$$\phi'(u) = u - \zeta_+, \qquad u \in \mathbb{R}^3. \tag{4.2}$$

In this case the equation (2.7) can be also written as:

$$\frac{\partial m}{\partial t} = \lambda_1 m \times \big(\Delta m - \phi'(m)\big) + \lambda_2 \Pi_m \big(\Delta m - \phi'(m)\big), \tag{4.3}$$

where $\Pi_m = \Pi_{m(t,x)}$ is the orthogonal projection from $\mathbb{R}^3$ to the tangent space at $m(t, x)$ for $(t, x) \in [0, T] \times \mathcal{O}$.

Suppose $m_0 \in \mathcal{M}^+$, i.e. $m_{0,3}(x) > 0$ for almost all $x \in \mathcal{O}$, where $m_0 = (m_{0,1}, m_{0,2}, m_{0,3})$, we want to prove that the solution $m = (m_1, m_2, m_3)$ of equations (2.7)-(2.9) satisfy $m(t) \to \zeta_+$ as $t \to \infty$ exponentially.

Remark 4.1. Note that the restriction of ϕ to $\mathcal{S}^2$ satisfies

$$\phi(u) = 1 - u_3, \qquad u \in \mathcal{S}^2. \tag{4.4}$$

Remark 4.2. Let us point out that if ϕ is defined in (4.1), then

$$\mathcal{E}(u) = \frac{1}{2}\|u - \zeta_+\|^2_{\mathbb{H}^1}, \qquad u \in \mathbb{H}^1. \tag{4.5}$$

Moreover, if m is a solution of of equations (2.7)-(2.9), then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t) - \zeta_+\|^2_{\mathbb{H}^1} = -\lambda_2 \|\Pi_m \nabla_m \mathcal{E}(m(t))\|^2_{\mathbb{H}}. \tag{4.6}$$

Proof of (4.6).

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t) - \zeta_+\|^2_{\mathbb{H}^1} &= -\lambda_1 \left\langle \zeta_+, m \times \big(\Delta m - \phi'(m)\big) \right\rangle_{\mathrm{H}} - \lambda_1 \left\langle \Delta m, m \times \big(\Delta m - \phi'(m)\big) \right\rangle_{\mathrm{H}} \\
&\quad -\lambda_2 \left\langle \Delta m, \Pi_m \big(\Delta m - \phi'(m)\big) \right\rangle_{\mathrm{H}} - \lambda_2 \left\langle \zeta_+, \Pi_m \big(\Delta m - \phi'(m)\big) \right\rangle_{\mathrm{H}} \\
&= -\lambda_1 \left\langle \Delta m - \phi'(m), m \times \big(\Delta m - \phi'(m)\big) \right\rangle_{\mathrm{H}} - \lambda_2 \left\langle \Delta m - \phi'(m), \Pi_m \big(\Delta m - \phi'(m)\big) \right\rangle_{\mathrm{H}} \\
&= -\lambda_2 \left\| \Pi_m \big(\Delta m - \phi'(m)\big) \right\|^2_{\mathrm{H}} \le 0.
\end{aligned}$$

□

Lemma 4.3. *If m is the solution of the system* (4.3), (2.8) *and* (2.9) *satisfies*

$$\inf_{x\in\mathcal{O}} m_3(t,x) > \delta, \tag{4.7}$$

for some $t \geq 0$, $\delta > 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|^2_{\mathbb{H}^1} \leq \frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|^2_{\mathrm{H}} \leq -\lambda_2 \min\left\{\delta, \frac{1}{2}\right\}\|m(t)-\zeta_+\|^2_{\mathbb{H}^1}. \tag{4.8}$$

Proof. Let m be the global solution to the system (4.3), (2.8) and (2.9), let us fix $t \geq 0$ and $\delta > 0$ such that (4.7) holds.

Firstly, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|^2_{\mathrm{H}} = \langle m(t)-\zeta_+, m'(t)\rangle_{\mathrm{H}} = -\langle \zeta_+, m'(t)\rangle_{\mathrm{H}}\\ &= -\lambda_1\left\langle \zeta_+, m(t)\times\big(\Delta m(t) - \phi'(m(t))\big)\right\rangle_{\mathrm{H}} - \lambda_2\left\langle \zeta_+, \Pi_m\big(\Delta m(t)-\phi'(m(t))\big)\right\rangle_{\mathrm{H}}.\end{aligned}$$

By (4.2), (A.2), (A.3), (A.4) and (A.6), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|^2_{\mathrm{H}} = -\lambda_1\left\langle \zeta_+, m(t)\times\big(\Delta m(t)+\zeta_+\big)\right\rangle_{\mathrm{H}} - \lambda_2\left\langle \zeta_+, \Pi_m\big(\Delta m(t)+\zeta_+\big)\right\rangle_{\mathrm{H}}\\ &= \lambda_1\langle m(t)\times\zeta_+, \Delta m(t)\rangle_{\mathrm{H}} - \lambda_2\langle \zeta_+, \Pi_m\Delta m(t)\rangle_{\mathrm{H}} - \langle \zeta_+, \Pi_m\zeta_+\rangle_{\mathrm{H}} =: \lambda_1 I_1 - \lambda_2 I_2 - I_3.\end{aligned}$$

By integration by parts, we have

$$I_1 = \langle m(t)\times\zeta_+, \Delta m(t)\rangle_{\mathrm{H}} = -\langle \nabla m(t)\times\zeta_+ + m(t)\times\nabla\zeta_+, \nabla m(t)\rangle_{\mathrm{H}} = 0,$$

By the equality (3.48) and (4.7), we have

$$\begin{aligned}I_2 &= \langle \zeta_+, \Pi_m\Delta m(t)\rangle_{\mathrm{H}} = \left\langle \zeta_+, \Delta m(t) + m(t)|\nabla m(t)|^2\right\rangle_{\mathrm{H}}\\ &= \langle \zeta_+, \Delta m(t)\rangle_{\mathrm{H}} + \left\langle \zeta_+, m(t)|\nabla m(t)|^2\right\rangle_{\mathrm{H}} = \int_{\mathcal{O}} m_3(t,x)|\nabla m(t,x)|^2\,\mathrm{d}x \geq \delta\|\nabla m\|^2_{\mathbb{H}},\end{aligned}$$

which is dissiptivity of projected Laplacian.

Since $m_3(t,x) \in (0,1]$, $m_3^2(t,x) \leq m_3(t,x)$ and therefore by the equality (4.4), we have

$$\begin{aligned}I_3 &= \langle \zeta_+, \Pi_m\zeta_+\rangle_{\mathrm{H}} = \langle \zeta_+, \zeta_+ - m\langle m, \zeta_+\rangle\rangle_{\mathrm{H}}\\ &= \int_{\mathcal{O}}(1-|m_3(x)|^2)\,\mathrm{d}x \geq \int_{\mathcal{O}}(1-m_3(x))\,\mathrm{d}x = \frac{1}{2}\|m-\zeta_+\|^2_{\mathbb{H}}.\end{aligned}$$

Therefore we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|^2_{\mathrm{H}} \leq -\lambda_2\delta\|\nabla m\|^2_{\mathbb{H}} - \lambda_2\frac{1}{2}\|m-\zeta_+\|^2_{\mathbb{H}} \leq -\lambda_2\min\left\{\delta, \frac{1}{2}\right\}\|m(t)-\zeta_+\|^2_{\mathbb{H}^1}.$$

So we get the second half of the inequality (4.8).

Secondly, by integration by parts, (4.2), (A.3) and since Π_m is a self-adjoint operator, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla m(t)\|^2_{\mathrm{H}} = -\langle \Delta m, m'(t)\rangle_{\mathrm{H}}\\ &= -\lambda_1\left\langle \Delta m, m\times\big(\Delta m - \phi'(m)\big)\right\rangle_{\mathrm{H}} - \lambda_2\left\langle \Delta m, \Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\\ &= -\lambda_1\left\langle \Delta m, m\times\zeta_+\right\rangle_{\mathrm{H}} - \lambda_2\|\Pi_m\Delta m\|^2_{\mathbb{H}} - \lambda_2\langle\Delta m, \Pi_m\zeta_+\rangle_{\mathrm{H}}\\ &= -\lambda_1 I_1 - \lambda_2\|\Pi_m\Delta m\|^2_{\mathbb{H}} - \lambda_2 I_2 \le 0.\end{aligned}$$

So we have proved the inequality (4.8). ☐

Theorem 4.4. *If m is the solution of the system* (4.3), (2.8) *and* (2.9) *with the initial $m_0 \in \mathrm{V}\cap\mathcal{M}$ satisfies*

$$\|m_0-\zeta_+\|^2_{\mathbb{H}^1} < \frac{2}{k^2}(2-2\delta) = 1-\delta, \tag{4.9}$$

for some $\delta\in(0,1)$, where

$$k = 2\max\left(1, \frac{1}{\sqrt{|\mathcal{O}|}}\right).$$

265 *then $m(t)\to\zeta_+$ in $\mathbb{H}^1$ exponentially as $t\to\infty$.*

Proof. Let us fix $\delta\in(0,1)$, $m_0\in\mathrm{V}\cap\mathcal{M}$ satisfy (4.9) and let m be the global solution of the system (4.3), (2.8) and (2.9).

Let us define:

$$\tau = \inf\left\{t\ge 0 : \|m(t)-\zeta_+\|_{\mathbb{H}^1} \ge \frac{2}{k^2}(2-2\delta)\right\}. \tag{4.10}$$

Note that by Theorem 3.37, $m\in C([0,\infty);\mathrm{V})$, so the set

$$\left\{t\ge 0 : \|m(t)-\zeta_+\|_{\mathbb{H}^1} \ge \frac{2}{k^2}(2-2\delta)\right\}$$

is closed, hence if it is not empty, the infimum will be the minimum.

Again since $m:[0,\infty)\to\mathbb{H}^1$ is continuous, by (4.9), we infer that $\tau>0$.

270 We will prove that $\tau=\infty$. Suppose by contradiction that $\tau<\infty$.

By the following interpolation inequality:

$$\|m\|_{\mathbb{L}^\infty}^2 \le k^2 \|m\|_{\mathrm{H}} \cdot \|\nabla m\|_{\mathrm{H}}, \quad \forall m \in \mathbb{H}^1, \tag{4.11}$$

we have, for all $t < \tau$,

$$\begin{aligned}
\sup_{x\in\mathcal{O}} |m(t,x) - \zeta_+|_{\mathbb{R}^3}^2 &\le k^2 \|m(t) - \zeta_+\|_{\mathrm{H}} \|\nabla m(t)\|_{\mathrm{H}} \\
&\le \frac{k^2}{2}\left(\|m(t) - \zeta_+\|_{\mathrm{H}}^2 + \|\nabla m(t)\|_{\mathrm{H}}^2\right) \\
&< \frac{k^2}{2}\frac{2}{k^2}(2-2\delta) = 2 - 2\delta.
\end{aligned}$$

We infer that

$$m_3(t,x) > \delta, \qquad x \in \mathcal{O},\ t < \tau. \tag{4.12}$$

Therefore by Lemma 4.3, we have

$$\|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 \le \|m_0 - \zeta_+\|_{\mathbb{H}^1}^2 < \frac{2}{k^2}(2-2\delta), \qquad t < \tau.$$

Since $m \in C([0,\infty);\mathbb{H}^1)$ and $\tau < \infty$, we infer that

$$\|m(\tau) - \zeta_+\|_{\mathbb{H}^1}^2 \le \|m_0 - \zeta_+\|_{\mathbb{H}^1}^2 < \frac{2}{k^2}(2-2\delta),$$

which contradicts the definition of τ. Therefore $\tau = \infty$. Hence we can use Lemma 4.3 for all time $t \ge 0$.

Next by Lemma 4.3, and by Gronwall inequality, we get

$$\|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 \le \|m_0 - \zeta_+\|_{\mathbb{H}^1}^2 \exp\left(-\lambda_2 \min\left\{\delta, \frac{1}{2}\right\} t\right), \qquad t \ge 0.$$

In particular, this implies that $\|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 \to 0$ as $t \to \infty$ exponentially. Hence the proof is complete. $\square$

Remark 4.5. If m_0 satisfies the condition (4.9), then we say that m_0 is in the basin of attraction of ζ_+. Moreover ζ_+ is an asymptotically stable equilibrium position of the system (4.3), (2.8) and (2.9).

5. The quasi-potential of LLG system

From now on we will only consider the function ϕ such that there exist a basin of attraction of ζ_+, if m_0 is in the basin of attraction, then the solution m of the system (4.3), (2.8) and (2.9) will converge to ζ_+ as $t \to \infty$, e.g. the ϕ as in (4.1).

Definition 5.1. For $a \in \mathrm{V} \cap \mathcal{M}^+$, we define the quasipotential $U(a)$ of system (2.7)-(2.9) as follows:

$$U(a) = \inf \left\{ \frac{\lambda_1^2 + \lambda_2^2}{4\lambda_2} \int_{-\infty}^{0} \|h(t)\|_{\mathrm{H}}^2 \,\mathrm{d}t : \right. \qquad (5.1)$$
$$\exists m \in C((-\infty, 0]; \mathrm{V}), \text{ a weak solution of}$$

$$\left. \begin{cases} \dfrac{\partial m}{\partial t} = \lambda_1 m \times (\Delta m - \phi'(m) + h) - \lambda_2 m \times \big(m \times (\Delta m - \phi'(m) + h)\big), \\ m(t, 0) = m(t, 2\pi), \quad \nabla m(t, 0) = \nabla m(t, 2\pi), \quad \text{a.e. } t \in (-\infty, 0], \\ m(-\infty, \cdot) = \zeta_+, \quad m(0, \cdot) = a \end{cases} \right\} \qquad (5.2)$$

The following theorem show that the quasipotential in Definition 5.1 is well defined in both mathematic sense and physical sense. Mathematically the qusipotential would not be infinite and physically it is just the potential energy of the system without external force.

Theorem 5.2. *If a is in the basin of attraction of ζ_+ (see Remark 4.5), then*

$$U(a) = \mathcal{E}(a),$$

where the energy $\mathcal{E}(a)$ is defined in (2.13).

Proof. Let us fix $a \in \mathrm{V} \cap \mathcal{M}$ and a be in the basin of attraction of ζ_+. We prove the theorem by three steps:

Step 1: We show that for some particular $h \in L^2(-T, 0; \mathrm{H})$ for all $T > 0$, there exists a solution $m \in C((-\infty, 0]; \mathrm{V})$ for the system (5.2).

To do this, let us consider the following system:

$$\begin{cases} m'(t) = \lambda_1 m(t) \times (\Delta m(t) - \phi'(m(t))) \\ \qquad\qquad + \lambda_2 m(t) \times \big(m(t) \times (\Delta m(t) - \phi'(m(t)))\big), \quad t \in (-\infty, 0) \\ m(t,0) = m(t,2\pi), \quad \nabla m(t,0) = \nabla m(t,2\pi), \quad \text{for a.e. } t \in (-\infty, 0], \\ m(0) = a. \end{cases} \tag{5.3}$$

Let us define $\mathrm{v}(t) = m(-t)$, then in formal way $\mathrm{v}'(t) = -m'(-t)$, hence solving the above system is equivalent to solve the following:

$$\begin{cases} \mathrm{v}'(t) = -\lambda_1 \mathrm{v}(t) \times (\Delta \mathrm{v}(t) - \phi'(\mathrm{v}(t))) \\ \qquad\qquad - \lambda_2 \mathrm{v}(t) \times \big(\mathrm{v}(t) \times (\Delta \mathrm{v}(t) - \phi'(\mathrm{v}(t)))\big), \quad t \in (0, \infty) \\ \mathrm{v}(t,0) = \mathrm{v}(t,2\pi), \quad \nabla \mathrm{v}(t,0) = \nabla \mathrm{v}(t,2\pi), \quad \text{for a.e. } t \in [0, \infty), \\ \mathrm{v}(0) = a. \end{cases} \tag{5.4}$$

This is LLGE for v with $\mathrm{v}(0) = a$ as in the system (2.7)-(2.9) with coefficients $-\lambda_1$, λ_2 and $h = 0$. By theorem 3.37, there exists a unique strong solution

$$\mathrm{v} \in L^2(0,T;D(A)) \cap C([0,\infty);\mathrm{V}) \cap H^1(0,T;\mathrm{H}) \cap L^4(0,T;\mathbb{W}^{1,4}), \qquad T > 0,$$

and v also satisfies the following property:

$$|\mathrm{v}(t,x)| = 1, \qquad \text{for all } (t,x) \in [0,\infty) \times \bar{\mathcal{O}}.$$

Since a satisfies the conditions in Theorem 4.4, we infer that $\mathrm{v}(\infty) = \zeta_+$.

Therefore there exists a unique solution m of the system (5.3) such that

$$\begin{cases} m \in L^2(-T,0;D(A)) \cap C((-\infty,0];\mathrm{V}) \cap H^1(-T,0;\mathrm{H}) \\ \qquad \cap L^4(-T,0;\mathbb{W}^{1,4}), \text{ for every} T > 0. \\ |m(t,x)| = 1, \qquad \text{for all } (t,x) \in (-\infty,0] \times \bar{\mathcal{O}}, \\ m(-\infty) = \zeta_+. \end{cases} \tag{5.5}$$

For such m, let us choose h by

$$h = \lambda_1 m \times (\Delta m - \phi'(m)) + \lambda_2 m \times \big(m \times (\Delta m - \phi'(m))\big) = m'. \tag{5.6}$$

Next we claim this m defined above is also the solution of equation (5.2) with the h just chosen in (5.6). Note that by (5.5) and the meaning of cross product $\times$, one has

$$m \times h = m \times \big(m \times (\Delta m - \phi'(m))\big) - m \times (\Delta m - \phi'(m)),$$

and

$$m \times (m \times h) = -m \times (\Delta m - \phi'(m)) - m \times \big(m \times (\Delta m - \phi'(m))\big).$$

So if we substitute this h into the right hand side of the first equation of (5.2), we get:

$$\begin{aligned} &\lambda_1 m \times (\Delta m - \phi'(m)) + \lambda_2 m \times \big(m \times (\Delta m - \phi'(m))\big) - \lambda_1 m \times (\Delta m - \phi'(m)) \\ &-\lambda_2 m \times \big(m \times (\Delta m - \phi'(m))\big) + \lambda_1 m \times (\Delta m - \phi'(m)) + \lambda_2 m \times \big(m \times (\Delta m - \phi'(m))\big) \\ = \ &\lambda_1 m \times (\Delta m - \phi'(m)) + \lambda_2 m \times \big(m \times (\Delta m - \phi'(m))\big) = h = m', \end{aligned}$$

where m' is the left hand side of the first equation of (5.2). Hence the claim follows.

Step 2: We show that if $m \in C((-\infty, 0]; \mathrm{V})$ is a weak solution of system (5.2) for some $h \in L^2(-T, 0; \mathrm{H})$ for all $T > 0$, then $U(a) \geq \mathcal{E}(a)$.

Let us assume $m \in C((-\infty, 0]; \mathrm{V})$ is a weak solution of system (5.2) for some $h \in L^2(-T, 0; \mathrm{H})$ for all $T > 0$. By the same way as in the proof of Proposition 2.9, one can show that $|m(t,x)| = 1$ for all $(t,x) \in (-\infty, 0] \times \mathcal{O}$. Hence by the meaning of the cross product "$\times$", one has

$$m(t,x) \times h(t,x) \perp m(t,x) \times (m(t,x) \times h(t,x)), \qquad (t,x) \in (-\infty, 0] \times \mathcal{O}$$

and

$$|m(t,x) \times h(t,x)| = |m(t,x) \times (m(t,x) \times h(t,x))|, \qquad (t,x) \in (-\infty, 0] \times \mathcal{O}.$$

Therefore one has

$$(\lambda_1^2 + \lambda_2^2)\|\Pi_m h(t)\|_{\mathrm{H}}^2 = \|\lambda_1 m(t) \times h(t) - \lambda_2 m(t) \times (m(t) \times h(t))\|_{\mathrm{H}}^2, \qquad t \leq 0,$$

where Π_m was explained at beginning of Section 4. Moreover, we can prove that m has the regularity of as in Theorem 3.37 by truncate a finite time and then make a change of time variable to positive by add a finite number on it.

Next by the first equation in (5.2) one has

$$
\begin{aligned}
&\|\lambda_1 m(t)\times h(t) - \lambda_2 m(t)\times (m(t)\times h(t))\|_{\mathrm{H}}^2 \\
=\ & \Big\| m'(t) - \lambda_1 m(t)\times(\Delta m(t) - \phi'(m(t))) + \lambda_2 m(t)\times \big(m(t)\times(\Delta m(t)-\phi'(m(t)))\big)\Big\|_{\mathrm{H}}^2 \\
=\ & \|m'(t) - \lambda_1 m(t)\times(\Delta m(t) - \phi'(m(t))) + \lambda_2 \Pi_m(-\Delta m(t) + \phi'(m(t)))\|_{\mathrm{H}}^2 \\
=\ & \|m'(t) - \lambda_1 m(t)\times(\Delta m(t) - \phi'(m(t))) - \lambda_2 \Pi_m(-\Delta m(t) + \phi'(m(t)))\|_{\mathrm{H}}^2 \\
& \qquad +4\left\langle m'(t) - \lambda_1 m(t)\times(\Delta m(t) - \phi'(m(t))), \lambda_2\Pi_m(-\Delta m(t) + \phi'(m(t)))\right\rangle_{\mathrm{H}}.
\end{aligned}
$$

Since $\langle a\times b, \Pi_a(b)\rangle = 0$ for $a, b\in\mathbb{R}^3$, one has

$$
\begin{aligned}
&\left\langle m'(t) - \lambda_1 m(t)\times(\Delta m(t) - \phi'(m(t))), \lambda_2\Pi_m(-\Delta m(t) + \phi'(m(t)))\right\rangle_{\mathrm{H}} \\
=\ & \left\langle m'(t), \lambda_2\Pi_m(-\Delta m(t) + \phi'(m(t)))\right\rangle_{\mathrm{H}}.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&(\lambda_1^2+\lambda_2^2)\|\Pi_m h(t)\|_{\mathrm{H}}^2 \\
=\ & \|m'(t) - \lambda_1 m(t)\times(\Delta m(t) - \phi'(m(t))) - \lambda_2 \Pi_m(-\Delta m(t) + \phi'(m(t)))\|_{\mathrm{H}}^2 \\
& \qquad +4\left\langle m'(t), \lambda_2\Pi_m(-\Delta m(t) + \phi'(m(t)))\right\rangle_{\mathrm{H}}.
\end{aligned}
\tag{5.7}
$$

Note that by integration by parts, one has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(m(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla m(t)\|_{\mathrm{H}}^2 + \int_{\mathcal{O}}\phi(m(t,x))\,\mathrm{d}x\right) \\
=\ & \langle m'(t), -\Delta m(t)\rangle_{\mathrm{H}} + \int_{\mathcal{O}}\langle \phi'(m(t,x)), m'(t,x)\rangle\,\mathrm{d}x \\
=\ & \langle m'(t), -\Delta m(t) + \phi'(m(t))\rangle_{\mathrm{H}} = \langle m'(t), \Pi_m(-\Delta m(t) + \phi'(m(t))\rangle_{\mathrm{H}}.
\end{aligned}
\tag{5.8}
$$

The last equality above is from $m'(t,x)\in T_{m(t,x)}\mathcal{S}^2$.

Hence by (5.7) and (5.8) we have

$$
\begin{aligned}
\int_{-\infty}^0 \|\Pi_m h(t)\|_{\mathrm{H}}^2\,\mathrm{d}t \geq\ & \frac{4\lambda_2}{\lambda_1^2+\lambda_2^2}\int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(m(s))\,\mathrm{d}s \\
=\ & \frac{4\lambda_2}{\lambda_1^2+\lambda_2^2}\big(\mathcal{E}(a) - \mathcal{E}(\zeta_+)\big) = \frac{4\lambda_2}{\lambda_1^2+\lambda_2^2}\mathcal{E}(a).
\end{aligned}
$$

Therefore $U(a)\geq \mathcal{E}(a)$.

Step 3: We show that h defined by (5.6) is in $L^2(-\infty, 0; \mathrm{H})$ and

$$\int_{-\infty}^{0} \|h(t)\|_{\mathrm{H}}^2 \, \mathrm{d}t = \frac{4\lambda_2}{\lambda_1^2 + \lambda_2^2} \mathcal{E}(a),$$

and therefore $U(a) = \mathcal{E}(a)$.

By (5.3), the solution m of (5.2) with h defined in (5.6) satisfies

$$\|m'(t) - \lambda_1 m(t) \times (\Delta m(t) - \phi'(m(t))) - \lambda_2 \Pi_m(-\Delta m(t) + \phi'(m(t)))\|_{\mathrm{H}}^2 = 0.$$

So by (5.7) and (5.8),

$$\int_{-\infty}^{0} \|\Pi_m h(t)\|_{\mathrm{H}}^2 \, \mathrm{d}t = \frac{4\lambda_2}{\lambda_1^2 + \lambda_2^2} \mathcal{E}(a).$$

Note that $h(t,x) = m'(t,x) \perp m(t,x)$, therefore $h(t,x) = \Pi_m h(t,x)$ for almost all $t < 0$ and $x \in \mathcal{O}$. Hence actually we have

$$\int_{-\infty}^{0} \|h(t)\|_{\mathrm{H}}^2 \, \mathrm{d}t = \frac{4\lambda_2}{\lambda_1^2 + \lambda_2^2} \mathcal{E}(a).$$

Therefore $h \in L^2(-\infty, 0; \mathrm{H})$ and by the result of Step 2,

$$U(a) = \mathcal{E}(a).$$

The proof of Theorem 5.2 is complete. □

Appendix A. Some frequently used equalities

For $a, b, c \in \mathbb{R}^3$, we have the following equalities

$$a \times (b \times c) = b\langle a, c\rangle - c\langle a, b\rangle \tag{A.1}$$

$$\langle a \times b, c\rangle_{\mathbb{R}^3} = \langle a, b \times c\rangle_{\mathbb{R}^3}. \tag{A.2}$$

$$\langle a \times b, a\rangle_{\mathbb{R}^3} = 0. \tag{A.3}$$

$$a \times a = 0. \tag{A.4}$$

For $|a| = 1$,

$$\Pi_a(b) = -a \times (a \times b), \qquad b \in \mathbb{R}^3. \tag{A.5}$$

$$\Pi_a a = 0. \tag{A.6}$$

Appendix B. Discussion on other ϕ

Appendix B.1. $\phi_1(m) = \frac{1}{2}(m_1^2 + m_2^2)$

Now let us consider the function ϕ used in [6], which we denote by:

$$\phi_1(m) = \frac{1}{2}(m_1^2 + m_2^2) = \frac{1}{2}(1 - m_3^2), \qquad m \in \mathcal{S}^2. \tag{B.1}$$

So

$$\phi_1'(m) = -m_3\zeta_+.$$

Some of our results in this section are formulated for $\phi = \phi_1$, but some for a general ϕ.

Let us formulate, for the future reference, three useful identities.

Lemma Appendix B.1. *If m is the solution of the system* (4.3), (2.8) *and* (2.9) *with $\phi = \phi_1$, then one has the following equalities:*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t) - \zeta_+\|_{\mathrm{H}}^2 = -\langle \zeta_+, m'(t)\rangle_{\mathrm{H}} \\ &= -\lambda_1 \left\langle \zeta_+, m \times \big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}} - \lambda_2 \left\langle \zeta_+, \Pi_m\big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}}, \\ &= -\lambda_2 \left\langle \zeta_+, \Pi_m\big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}}. \end{aligned} \tag{B.2}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla m(t)\|_{\mathrm{H}}^2 = -\langle \Delta m, m'(t)\rangle_{\mathrm{H}} \\ &= -\lambda_1 \left\langle \Delta m, m \times \big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}} - \lambda_2 \left\langle \Delta m, \Pi_m\big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}} \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 = &-\lambda_1 \left\langle \Delta m + \zeta_+, m \times \big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}} \\ &-\lambda_2 \left\langle \Delta m + \zeta_+, \Pi_m\big(\Delta m - \phi_1'(m)\big)\right\rangle_{\mathrm{H}}. \end{aligned} \tag{B.4}$$

Proof. The equality (B.2) follows from $\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathrm{H}}^2 = -2\langle \zeta_+, m'(t)\rangle_{\mathrm{H}}$. The equality (B.3) follows from $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla m(t)\|_{\mathrm{H}}^2 = -2\langle \Delta m, m'(t)\rangle_{\mathrm{H}}$. And so the equality (B.4) follows from the equalities (B.2) and (B.3). □

Lemma Appendix B.2. *If $\phi = \phi_1$ and $\inf_{x\in\mathcal{O}} m_3(x) > 0$, then for every $m \in \mathcal{M}$,*

$$4\mathcal{E}(m) \geq \|m-\zeta_+\|_{\mathbb{H}^1}^2, \tag{B.5}$$

$$\mathcal{E}(m) \leq \|m-\zeta_+\|_{\mathbb{H}^1}^2, \tag{B.6}$$

where $\mathcal{E}$ is defined in (2.13).

Proof. Let us fix $m \in \mathcal{M}$ with $\inf_{x\in\mathcal{O}} m_3(x) > 0$ and assume $\phi = \phi_1$.

Firstly, we prove (B.5). Since

$$|m-\zeta_+|^2 = 2(1-m_3) \leq 2(1-m_3^2) = 4\phi_1(m),$$

by (2.13), we have

$$4\mathcal{E}(m) \geq \|m-\zeta_+\|_{\mathbb{H}}^2 + 2\|\nabla m\|_{\mathbb{H}}^2 \geq \|m-\zeta_+\|_{\mathbb{H}^1}^2.$$

Secondly, we prove (B.6). Since

$$|m-\zeta_+|^2 = 2(1-m_3) \geq (1-m_3)(1+m_3) = 1-m_3^2 = 2\phi_1(m),$$

we have

$$2\mathcal{E}(m) \leq \|m-\zeta_+\|_{\mathbb{H}}^2 + \|\nabla m\|_{\mathbb{H}}^2 = \|m-\zeta_+\|_{\mathbb{H}^1}^2.$$

Hence the proof is complete. □

Lemma Appendix B.3. *If m is the solution of the system* (4.3), (2.8) *and* (2.9) *with $\phi = \phi_1$, and for some $\delta > 0$ and some $t \geq 0$, one has*

$$m_3(t,x) > \delta, \qquad x \in \mathcal{O}. \tag{B.7}$$

Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathrm{H}}^2 \leq -\frac{1}{2}\lambda_2\delta\|m(t)-\zeta_+\|_{\mathbb{H}^1}^2. \tag{B.8}$$

Moreover if $\lambda_1 = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathbb{H}^1}^2 \leq 0. \tag{B.9}$$

Proof. Let m be the global solution to the system (4.3), (2.8) and (2.9) with $\phi = \phi_1$, let us fix $t \geq 0$ and $\delta > 0$ such that (B.7) holds.

We first prove (B.8).

By (B.2) and (A.5) and integration by parts, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathrm{H}}^2 = -\lambda_2\left\langle \zeta_+, \Pi_m\big(\Delta m - \phi'(m)\big)\right\rangle_{\mathrm{H}} \\
&= -\lambda_2\left\langle \Pi_m\zeta_+, \Delta m\right\rangle_{\mathrm{H}} + \lambda_2\left\langle \Pi_m\zeta_+, \phi'(m)\right\rangle_{\mathrm{H}} \\
&= \lambda_2\left\langle m\times(m\times\zeta_+), \Delta m\right\rangle_{\mathrm{H}} - \lambda_2\left\langle m\times(m\times\zeta_+), \phi'(m)\right\rangle_{\mathrm{H}} \\
&= -\lambda_2\left\langle m\times(\nabla m\times\zeta_+), \nabla m\right\rangle_{\mathrm{H}} - \lambda_2\left\langle m\times(m\times\zeta_+), \phi'(m)\right\rangle_{\mathrm{H}}.
\end{aligned}$$

By the formula (A.1), one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathrm{H}}^2 = -\lambda_2\left\langle \zeta_+, \Pi_m\big(\Delta m - \phi'(m)\big)\right\rangle_{\mathrm{H}} \qquad \text{(B.10)}\\
&= -\lambda_2\int_{\mathcal{O}} |\nabla m(t,x)|^2 m_3(t,x) - \langle\zeta_+, \nabla m(t,x)\rangle\langle m(t,x), \nabla m(t,x)\rangle\,\mathrm{d}x \\
&\quad -\lambda_2\int_{\mathcal{O}} \langle m(t,x), \phi'(m(t,x))\rangle m_3(t,x) - \langle\zeta_+, \phi'(m(t,x))\rangle\,\mathrm{d}x \\
&= -\lambda_2\Bigg(\int_{\mathcal{O}} |\nabla m(t,x)|^2 m_3(t,x)\,\mathrm{d}x + \int_{\mathcal{O}} \langle m(t,x), \phi'(m(t,x))\rangle m_3(t,x) \\
&\qquad\qquad - \langle\zeta_+, \phi'(m(t,x))\rangle\,\mathrm{d}x\Bigg).
\end{aligned}$$

For $\phi = \phi_1$, one has

$$\phi_1'(m) = (m_1, m_2, 0),$$

so that

$$\langle m, \phi'(m)\rangle = m_1^2 + m_2^2 \quad \text{and} \quad \langle\zeta_+, \phi'(m)\rangle = 0.$$

Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathrm{H}}^2 = -\lambda_2\left(\int_{\mathcal{O}} |\nabla u(t,x)|^2 m_3(t,x)\,\mathrm{d}x + \int_{\mathcal{O}} (1 - m_3(t,x)^2) m_3(t,x)\,\mathrm{d}x\right).$$

We also have

$$1 - m_3^2 \geq 1 - m_3 = \frac{1}{2}|m - \zeta_+|_{\mathbb{R}^3}^2, \quad \text{for } m_3 \geq 0,\ m \in \mathcal{S}^2. \qquad \text{(B.11)}$$

Hence

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathrm{H}}^2\\
&\le -\lambda_2\left(\int_{\mathcal{O}}|\nabla m(t,x)|^2 m_3(t,x)\,\mathrm{d}x+\frac{1}{2}\int_{\mathcal{O}}|m(t,x)-\zeta_+|_{\mathbb{R}^3}^2 m_3(t,x)\,\mathrm{d}x\right)\\
&\le -\frac{1}{2}\lambda_2\delta\left(\int_{\mathcal{O}}|\nabla m(t,x)|^2\,\mathrm{d}x+\int_{\mathcal{O}}|m(t,x)-\zeta_+|_{\mathbb{R}^3}^2\,\mathrm{d}x\right)\\
&= -\frac{1}{2}\lambda_2\delta\|m(t)-\zeta_+\|_{\mathbb{H}^1}^2\le 0.
\end{aligned}
$$

So we get the inequality (B.8).

Next we assume $\lambda_1=0$ and prove (B.9).

Now let us consider $\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathbb{H}^1}^2$. By (B.4), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|_{\mathbb{H}^1}^2=-\lambda_2\left\langle \Delta m,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}-\lambda_2\left\langle \zeta_+,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\\
&=-\lambda_2\Big(\left\langle \Delta m,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}-\left\langle \phi'(m),\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\\
&\qquad+\left\langle \phi'(m),\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}+\left\langle \zeta_+,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\Big)\\
&=-\lambda_2\left(\|\Pi_m(\Delta m-\phi'(m))\|_{\mathrm{H}}^2+\left\langle \phi'(m)+\zeta_+,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\right).
\end{aligned}
$$

Now we only need to consider $\left\langle \phi'(m)+\zeta_+,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}$.

Using the equality $a\times(b\times c)=b\langle a,c\rangle-c\langle a,b\rangle$ and $|m(t,x)|=1$, one has

$$
\begin{aligned}
&\left\langle \phi'(m),\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\\
&=-\langle m\times(m\times(\phi'(m))),\Delta m\rangle_{\mathrm{H}}+\langle m\times(m\times(\phi'(m))),\phi'(m)\rangle_{\mathrm{H}}\\
&=\langle m\times(\nabla m\times(\phi'(m))),\nabla m\rangle_{\mathrm{H}}+\langle m\times(m\times\nabla\phi'(m)),\nabla m\rangle_{\mathrm{H}}\\
&\quad+\int_{\mathcal{O}}\langle m,\phi'(m)\rangle\langle m,\phi'(m)\rangle\,\mathrm{d}x-\langle\phi'(m),\phi'(m)\rangle_{\mathrm{H}}\\
&=\int_{\mathcal{O}}|\nabla m|^2\langle m,\phi'(m)\rangle\,\mathrm{d}x-\int_{\mathcal{O}}\langle\nabla\phi'(m),\nabla m\rangle\,\mathrm{d}x\\
&\quad+\int_{\mathcal{O}}\langle m,\phi'(m)\rangle\langle m,\phi'(m)\rangle\,\mathrm{d}x-\int_{\mathcal{O}}\langle\phi'(m),\phi'(m)\rangle\,\mathrm{d}x.
\end{aligned}
$$

For $\phi=\phi_1$ and by (B.10), one has

$$
\begin{aligned}
&\left\langle \phi'(m)+\zeta_+,\Pi_m\big(\Delta m-\phi'(m)\big)\right\rangle_{\mathrm{H}}\\
&=\int_{\mathcal{O}}|\nabla m|^2 m_3(1-m_3)\,\mathrm{d}x+\int_{\mathcal{O}}|\nabla m_3|^2\,\mathrm{d}x+\int_{\mathcal{O}}m_3(1-m_3)(1-m_3^2)\,\mathrm{d}x\ge 0.
\end{aligned}
$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m(t)-\zeta_+\|^2_{\mathbb{H}^1}\le 0.$$

This completes the proof. $\square$

Lemma Appendix B.4. *Let m be the solution of the system* (4.3), (2.8) *and* (2.9) *with $\phi=\phi_1$. If for some $t\ge 0$, one has*

$$\|m(t)-\zeta_+\|_{\mathbb{H}^1}<\frac{1}{2k^2\sqrt{|\mathcal{O}|}}\frac{\lambda_2}{\lambda_1+2\lambda_2}, \tag{B.12}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla m(t)\|^2_{\mathbb{H}}\le 0. \tag{B.13}$$

Moreover, in the particular case $\lambda_1=-1$ and $\lambda_2=1$, we have

$$m_3(t,x)\ge\frac{1}{2},\qquad x\in\mathcal{O}.$$

Proof. Let m be the solution of the system (4.3), (2.8) and (2.9) with $\phi=\phi_1$. Let us fix $t\ge 0$ such that (B.12) holds.

Let us first figure out if $\|\nabla m(t)\|_{\mathrm{H}}$ is decreasing. (The following calculation corresponds to equation (7.9) in [6].)

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla m(t)\|_{\mathrm{H}}=-\lambda_1\left\langle\Delta m(t),m(t)\times(-\phi_1'(m(t))\right\rangle_{\mathrm{H}}-\lambda_2\left\langle\Delta m(t),\Pi_m(\Delta m(t)-\phi_1'(m(t))\right\rangle_{\mathrm{H}}\\
&=-\lambda_1\int_{\mathcal{O}}(m_1(t)\nabla m_2(t,x)-m_2(t,x)\nabla m_1(t,x))\nabla m_3(t,x)\,\mathrm{d}x\\
&\qquad-\lambda_2\left\langle m(t)\times\Delta m(t),m(t)\times(\Delta m(t)-\phi_1'(m(t)))\right\rangle_{\mathrm{H}}\\
&=-\lambda_1\int_{\mathcal{O}}(m_1(t)\nabla m_2(t,x)-m_2(t,x)\nabla m_1(t,x))\nabla m_3(t,x)\,\mathrm{d}x-\lambda_2\|m(t)\times\Delta m(t)\|^2_{\mathbb{H}}\\
&\qquad-\lambda_2\left\langle m(t)\times\Delta m(t),m(t)\times m_3(t)\zeta_+\right\rangle_{\mathrm{H}}\\
&=-\lambda_1\int_{\mathcal{O}}(m_1(t)\nabla m_2(t,x)-m_2(t,x)\nabla m_1(t,x))\nabla m_3(t,x)\,\mathrm{d}x-\lambda_2\|m(t)\times\Delta m(t)\|^2_{\mathbb{H}}\\
&\qquad-\lambda_2\int_{\mathcal{O}}m_3^2(t,x)|\nabla m(t,x)|^2\,\mathrm{d}x+\lambda_2\int_{\mathcal{O}}|\nabla m_3(t,x)|^2\,\mathrm{d}x\\
&=-\lambda_2\|m(t)\times\Delta m(t)\|^2_{\mathbb{H}}+\int_{\mathcal{O}}R(t,x)\,\mathrm{d}x,
\end{aligned}$$

where we define

$$R:=-\lambda_1(m_1\nabla m_2-m_2\nabla m_1)\nabla m_3+\lambda_2|\nabla m_3|^2-\lambda_2 m_3^2|\nabla m|^2.$$

Therefore by using $m_1\nabla m_1 + m_2\nabla m_2 + m_3\nabla m_3 = 0$, we have (corresponds to (7.10) in [6])

$$\begin{aligned} R \, = \, & \lambda_1(m_1\nabla m_2 - m_2\nabla m_1)\frac{m_1\nabla m_1 + m_2\nabla m_2}{m_3} \\ & +\lambda_2(1-m_3^2)\left(\frac{m_1\nabla m_1 + m_2\nabla m_2}{m_3}\right)^2 - \lambda_2 m_3^2\left((\nabla m_1)^2 + (\nabla m_2)^2\right) \end{aligned} \tag{B.14}$$

By the Cauchy-Schwartz inequality, we obtain (corresponds to (7.11) in [6])

$$R \le \left(\lambda_1\frac{1-m_3^2}{m_3^2} + \lambda_2\frac{(1-m_3^2)^2}{m_3^2} - \lambda_2 m_3^2\right)\left((\nabla m_1)^2 + (\nabla m_2)^2\right). \tag{B.15}$$

By (B.12), we have (the following corresponds to (7.5) in [6])

$$\begin{aligned} \|m(t)-\zeta_+\|^2_{\mathbb{L}^\infty} \, & \le \, k^2\|m(t)-\zeta_+\|_{\mathrm{H}}\|m(t)-\zeta_+\|_{\mathbb{H}^1} \\ & \le k^2 2\sqrt{|\mathcal{O}|}\frac{1}{2k^2\sqrt{|\mathcal{O}|}}\frac{\lambda_2}{\lambda_1+2\lambda_2} = \frac{\lambda_2}{\lambda_1+2\lambda_2}. \end{aligned} \tag{B.16}$$

So (corresponding to (7.6) in [6])

$$m_3(t,x)^2 = 1-(m_1(t,x)^2 + m_2(t,x)^2) \ge 1-|m(t,x)-\zeta_+|^2 \ge \frac{\lambda_1+\lambda_2}{\lambda_1+2\lambda_2}, \quad x\in\mathcal{O}. \tag{B.17}$$

Therefore

$$\begin{aligned} & \lambda_1\frac{1-m_3^2}{m_3^2} + \lambda_2\frac{(1-m_3^2)^2}{m_3^2} - \lambda_2 m_3^2 \\ = \; & \frac{1}{m_3^2}\left(\lambda_1(1-m_3^2) + \lambda_2(1-2m_3^2+m_3^4) - \lambda_2 m_3^4\right) \\ = \; & \frac{1}{m_3^2}\left(\lambda_1 - \lambda_1 m_3^2 + \lambda_2 - 2\lambda_2 m_3^2\right) = \frac{1}{m_3^2}\left(-(\lambda_1+2\lambda_2)m_3^2 + \lambda_1+\lambda_2\right) \le 0. \end{aligned}$$

Hence by (B.15), we have (corresponding to (7.11) in [6])

$$R(t,x) \le 0 \qquad x\in\mathcal{O}.$$

Therefore we proved that $\|\nabla m(t)\|_{\mathrm{H}}$ is nonincreasing.

Moreover, by (B.16), in the particular case $\lambda_1 = -1$ and $\lambda_2 = 1$, we have

$$\sup_{x\in\mathcal{O}}(2-2m_3(t,x)) = \|m(t)-\zeta_+\|^2_{\mathbb{L}^\infty} \le 1.$$

So we have

$$m_3(t,x) \ge \frac{1}{2}, \qquad x\in\mathcal{O}.$$

□

Theorem Appendix B.5. *Let m be the solution of the system* (4.3), (2.8) *and* (2.9) *with $\phi = \phi_1$. If for some $0 < \delta < 1$, we have*

$$\|m_0 - \zeta_+\|^2_{\mathbb{H}^1} < \frac{2}{k^2}(2 - 2\delta) = 1 - \delta, \tag{B.18}$$

where

$$k = 2\max\left(1, \frac{1}{\sqrt{|\mathcal{O}|}}\right).$$

Then

(i) $m(t) \to \zeta_+$ *in* H *as* $t \to \infty$ *exponentially.*

(ii) $\int_0^\infty \mathcal{E}(m(t))\,\mathrm{d}t < \infty$.

(iii) $\mathcal{E}(m(t)) \to 0$ *as* $t \to \infty$.

(iv) $m(t) \to \zeta_+$ *in* $\mathbb{H}^1$ *as* $t \to \infty$.

Furthermore, if $\lambda_1 = -1$, $\lambda_2 = 1$ and

$$\|m_0 - \zeta_+\|_{\mathbb{H}^1} < \frac{1}{2k^2\sqrt{|\mathcal{O}|}}\frac{\lambda_2}{\lambda_1 + 2\lambda_2} = \frac{1}{2k^2\sqrt{|\mathcal{O}|}}, \tag{B.19}$$

then one has $m(t) \to \zeta_+$ in $\mathbb{H}^1$ as $t \to \infty$ exponentially.

Proof. Let m be the solution of the system (4.3), (2.8) and (2.9) with $\phi = \phi_1$. By the same method as in the proof of Theorem 4.4, we can see (B.18) implies that (B.7) is true for all $t \geq 0$.

We will prove the general result for $\lambda_1 \in \mathbb{R}$ and $\lambda_2 > 0$ in four steps:

(i) By (B.8), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|m(t) - \zeta_+\|^2_{\mathrm{H}} \leq -\frac{1}{2}\lambda_2\delta\|m(t) - \zeta_+\|^2_{\mathrm{H}}.$$

So by Gronwall's inequality, we have

$$\|m(t) - \zeta_+\|^2_{\mathrm{H}} \leq \|m_0 - \zeta_+\|^2_{\mathrm{H}} \exp\left(-\frac{1}{2}\lambda_2\delta t\right), \qquad t \geq 0.$$

So $m(t) \to \zeta_+$ in H as $t \to \infty$ exponentially.

(ii) By (B.8) and (B.6), we have

$$\mathcal{E}(m(t)) \le -\frac{1}{\lambda_2 \delta} \frac{\mathrm{d}}{\mathrm{d}t} \|m(t) - \zeta_+\|_{\mathrm{H}}^2, \qquad t \ge 0.$$

So by (i),

$$\int_0^\infty \mathcal{E}(m(t))\,\mathrm{d}t \le -\frac{1}{\lambda_2 \delta} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \|m(t) - \zeta_+\|_{\mathrm{H}}^2\,\mathrm{d}t = \frac{1}{\lambda_2 \delta} \|m_0 - \zeta_+\|_{\mathrm{H}}^2 < \infty.$$

(iii) Hence (ii) and (2.14) implies that $\mathcal{E}(m(t)) \to 0$ as $t \to \infty$.

(iv) Moreover, by (B.5) and (iii), we also have $m(t) \to \zeta_+$ in $\mathbb{H}^1$ as $t \to \infty$.

Now let us assume (B.19) is true and assume $\lambda_1 = -1$ and $\lambda_2 = 1$.

By Lemma Appendix B.4 and by the same method as in the proof of Proposition 7.2 in [6], we can see $\|\nabla m(t)\|_{\mathrm{H}}$ is decreasing and $m_3(t,x) \ge \frac{1}{2}$ for all $t \ge 0$ and $x \in \mathcal{O}$. Hence

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 \le \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|m(t) - \zeta_+\|_{\mathrm{H}}^2 = -\langle \zeta_+, m'(t) \rangle_{\mathrm{H}} \\
&= -\lambda_1 \langle \zeta_+, m(t) \times (\Delta m(t) + m_3(t)\zeta_+) \rangle_{\mathrm{H}} - \lambda_2 \langle \zeta_+, \Pi_m(\Delta m(t) + m_3(t)\zeta_+) \rangle_{\mathrm{H}} \\
&= -\lambda_2 \langle \zeta_+, \Pi_m(\Delta m(t) + m_3(t)\zeta_+) \rangle_{\mathrm{H}}.
\end{aligned}$$

By (B.10), we have

$$\begin{aligned}
&\|m(t) - \zeta_+\|_{\mathbb{H}^1}^2 \\
&\le \|m_0 - \zeta_+\|_{\mathbb{H}^1}^2 - 2\lambda_2 \int_0^t \int_{\mathcal{O}} |\nabla m(s,x)|^2 \langle m(s,x), \zeta_+ \rangle \,\mathrm{d}x\,\mathrm{d}s \\
&\qquad -2\lambda_2 \int_0^t \int_{\mathcal{O}} \left(\langle m(s,x), \phi'(m(s,x)) \rangle \langle m(s,x), \zeta_+ \rangle - \langle \zeta_+, \phi'(m(t,x)) \rangle \right) \mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

Note that

$$\begin{aligned}
&-2\lambda_2 \int_0^t \int_{\mathcal{O}} \left(\langle m(s,x), \phi'(m(s,x)) \rangle \langle m(s,x), \zeta_+ \rangle - \langle \zeta_+, \phi'(m(t,x)) \rangle \right) \mathrm{d}x\,\mathrm{d}s \\
&= -2\lambda_2 \int_0^t \int_{\mathcal{O}} (-m_3^3(s,x) + m_3(s,x))\,\mathrm{d}x\,\mathrm{d}s \\
&= -2\lambda_2 \int_0^t \int_{\mathcal{O}} m_3(s,x)(1 + m_3(s,x))(1 - m_3(s,x))\,\mathrm{d}x\,\mathrm{d}s \\
&\le -\lambda_2 \frac{1}{2} \int_0^t \int_{\mathcal{O}} (2 - 2m_3(s,x))\,\mathrm{d}x\,\mathrm{d}s = -\lambda_2 \frac{1}{2} \int_0^t \int_{\mathcal{O}} |m(s,x) - \zeta_+|_{\mathbb{R}^3}^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\|m(t)-\zeta_+\|^2_{\mathbb{H}^1}\\
&\le \|m_0-\zeta_+\|^2_{\mathbb{H}^1} - \lambda_2\frac{1}{2}\int_0^t\int_{\mathcal{O}}|\nabla m(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s - \lambda_2\frac{1}{2}\int_0^t\int_{\mathcal{O}}|m(s,x)-\zeta_+|^2_{\mathbb{R}^3}\,\mathrm{d}x\,\mathrm{d}s\\
&= \|m_0-\zeta_+\|^2_{\mathbb{H}^1} - \lambda_2\frac{1}{2}\int_0^t\|m(s)-\zeta_+\|^2_{\mathbb{H}^1}\,\mathrm{d}s \le \|m_0-\zeta_+\|^2_{\mathbb{H}^1}\exp\left\{-\frac{1}{2}\lambda_2 t\right\}.
\end{aligned}
$$

Hence $m(t)\to\zeta_+$ in $\mathbb{H}^1$ as $t\to\infty$ exponentially.

This completes the proof. $\square$

Appendix B.2. $\phi_2 = \frac{1}{2}-\phi_1$

Now let us consider $\phi_2=\frac{1}{2}-\phi_1$ in the system (4.3), (2.8) and (2.9), in this example the poles are no longer attractors.

Theorem Appendix B.6. *The function*

$$
m(t,x)=\bar{m}(x)=\big(\bar{m}_1(x),\bar{m}_2(x),\bar{m}_3(x)\big)=\Big(\sqrt{1-m_{0,3}^2}\cos x,\sqrt{1-m_{0,3}^2}\sin x, m_{0,3}\Big) \tag{B.20}
$$

is the unique global solution of the system (4.3), (2.8) *and* (2.9) *with* $\phi=\phi_2$ *and* $m_0=\bar{m}$, *where* $m_{0,3}$ *is a constant.*

Proof. Let us fix $m_{0,3}\in(-1,1)$ as a constant. Then by (B.20),

$$
\Delta m(t,x)=-\Big(\sqrt{1-m_{0,3}^2}\cos x,\sqrt{1-m_{0,3}^2}\sin x,0\Big)=-(m_1(t,x),m_2(t,x),0)=\phi_2'(m(t,x)),
$$

so $\Delta m(t,x)-\phi_2'(m(t,x))=0$. Hence the right hand side of (4.3) is 0. On the other hand, since m does not depend on t, so $m'(t)=0$. Therefore (4.3) is satisfied. Since $\cos(2\pi)=\cos(0)$ and $\sin(2\pi)=\sin(0)$, m also satisfy the periodic boundary condition (2.8). Therefore the proof is complete. $\square$

Corollary Appendix B.7. *For all* $\rho>0$, *there exist* $m_0\in\mathrm{V}\cap\mathcal{M}$ *such that*

$$
\|m_0-\zeta_+\|^2_{\mathbb{H}^1}<\rho, \tag{B.21}
$$

but there exists $\varepsilon>0$ *such that the solution* m *of the system* (4.3), (2.8) *and* (2.9) *with* $\phi=\phi_2$ *satisfies*

$$
\|m(t)-\zeta_+\|^2_{\mathrm{H}}>\varepsilon,\qquad t\ge 0. \tag{B.22}
$$

Proof. Let us fix $\rho > 0$. For any $\bar{m}$ defined as in (B.20), we have

$$\|\bar{m} - \zeta_+\|_{\mathbb{H}^1}^2 = \|\bar{m} - \zeta_+\|_{\mathrm{H}}^2 + \|\nabla \bar{m}\|_{\mathbb{H}}^2 = 4\pi(1 - m_{0,3}) + 2\pi(1 - m_{0,3}^2).$$

So we can choose $m_0 = \bar{m}$ as defined in (B.20) such that the constant $m_{0,3} \in (0,1)$ satisfies

$$4\pi(1 - m_{0,3}) + 2\pi(1 - m_{0,3}^2) < \rho,$$

then (B.21) would be satisfied. Let

$$\varepsilon = 2\pi(1 - m_{0,3}),$$

then by Theorem Appendix B.6,

$$\|m(t) - \zeta_+\|_{\mathrm{H}}^2 = \int_{\mathcal{O}} |2 - 2m_3(t,x)|^2 \, \mathrm{d}x = 4\pi(1 - m_{0,3}) > \varepsilon, \qquad t \geq 0.$$

So (B.22) is satisfied. Hence the proof is complete. $\square$

Appendix C. The connection between Definition 5.1 and the notions of quasipotential used in stochastic problems

According to the monograph [10] by Freidlin and Wentzell (from line 11 at page 90), given a system

$$\frac{\partial m}{\partial t} = b(m), \qquad u(0) = u_0, \tag{C.1}$$

We have the following definition of *quasipotential*.

Definition Appendix C.1. [10] The quasipotential of the system C.1 with respect to a point ζ was defined as

$$V(\zeta, a) = \inf \left\{ S_{T_1 T_2}(\varphi) : \varphi \in C([T_1, T_2]), \varphi(T_1) = \zeta, \varphi(T_2) = a, T_1 \leq T_2 \right\}, \tag{C.2}$$

where

$$S_{T_1 T_2}(\varphi) = \frac{1}{2} \int_{T_1}^{T_2} |\varphi'(s) - b(\varphi(s))|^2 \, \mathrm{d}s.$$

The connection of Definition 5.1 and the version of quasipotential now commonly used in stochastic problems is that they are both closely connected with the Definition Appendix C.1. Since the connection between the stochastic version of quasipotential and Definition Appendix C.1 is well known, we will only explain the connection between Definition 5.1 and Definition Appendix C.1.

Appendix C.1. The connection between Definition 5.1 and Definition Appendix C.1

The system (C.1) corresponds to our system (2.7) but with $h = 0$ and (2.9) with $m_0 = \zeta_+$ and (2.8), thus in our case

$$b(m) = \lambda_1 m \times (\Delta m - \phi'(m)) - \lambda_2 m \times (m \times (\Delta m - \phi'(m))).$$

Now let us consider $m' - b(m)$. As explained in Section 4, if $m' - b(m) = 0$, then m will stay at ζ_+ and will never reach $a \neq \zeta_+$. So we can assume (with (t, x) omitted) $m' - b(m) = u \neq 0$ for some $u \in T_m\mathcal{S}^2$, it is easy to check that for

$$h = \frac{1}{\lambda_2^2 + \lambda_1^2}(\lambda_1 m \times u - \lambda_2 u) \in T_m\mathcal{S}^2,$$

m is the solution of our system of (2.7), (2.9) with $m_0 = \zeta_+$ and (2.8). Therefore

$$\begin{aligned} |m' - b(m)|^2 &= |\lambda_1 m \times h - \lambda_2 m \times (m \times h)|^2 = |\lambda_1 m \times h + \lambda_2 h|^2 \\ &= \lambda_1^2 |m \times h|^2 + 2\lambda_1\lambda_2 \langle m \times h, h\rangle + \lambda_2^2 |h|^2 = (\lambda_1^2 + \lambda_2^2)|h|^2. \end{aligned}$$

Hence we can see that the Definition Appendix C.1 in our case can be written in the following way:

Definition Appendix C.2. The quasipotential with respect to the point ζ_+ is defined as

$$\begin{aligned} V(a) = \inf \Bigg\{ &\frac{\lambda_1^2 + \lambda_2^2}{4\lambda_2} \int_0^T \|h(t)\|_{\mathrm{H}}^2 \, \mathrm{d}t : T > 0, \\ &\exists m \in C([0, T]; \mathrm{V}), \text{ a weak solution of} \end{aligned} \tag{C.3}$$

$$\left\{\begin{aligned} &\frac{\partial m}{\partial t} = \lambda_1 m \times (\Delta m - \phi'(m) + h) - \lambda_2 m \times \big(m \times (\Delta m - \phi'(m) + h)\big), \\ &m(t,0) = m(t,2\pi), \quad \nabla m(t,0) = \nabla m(t,2\pi), \quad \text{a.e. } t \in [0,T], \\ &m(0,\cdot) = \zeta_+, \quad m(T,\cdot) = a \end{aligned}\right\}. \tag{C.4}$$

By setting $\bar{m}(t) = m(t+T)$, $\bar{h}(t) = h(t+T)$ for $T > 0$, we have the following result:

Proposition Appendix C.3.

$$V(a) = \inf\left\{\frac{\lambda_1^2 + \lambda_2^2}{4\lambda_2}\int_{-T}^{0} \|\bar{h}(t)\|_{\mathrm{H}}^2 \,\mathrm{d}t : T > 0, \right. \tag{C.5}$$

$$\exists \bar{m} \in C([-T,0];\mathrm{V}), \ \textit{a weak solution of}$$

$$\left\{\begin{aligned} &\frac{\partial \bar{m}}{\partial t} = \lambda_1 \bar{m} \times (\Delta \bar{m} - \phi'(\bar{m}) + \bar{h}) - \lambda_2 \bar{m} \times \big(\bar{m} \times (\Delta \bar{m} - \phi'(\bar{m}) + \bar{h})\big), \\ &\bar{m}(t,0) = m(t,2\pi), \quad \nabla \bar{m}(t,0) = \nabla m(t,2\pi), \quad \textit{a.e. } t \in [-T,0], \\ &\bar{m}(-T,\cdot) = \zeta_+, \quad \bar{m}(0,\cdot) = a \end{aligned}\right\}. \tag{C.6}$$

Next we will use a similar procedure as in [4] to show that $V(a) = U(a)$, which means that Definition Appendix C.2 and Definition 5.1 are equivalent.

Notation Appendix C.4. For simplicity, we denote

$$\Phi(m) := m' - m \times (\Delta m - \phi'(m)) - \Pi_m(\Delta m - \phi'(m)) \in L^2(0,T;\mathbb{H}). \tag{C.7}$$

Remark. ζ_+ is an asymptotically stable equilibrium position of the system (4.3), (2.8) and (2.9). So if $m(0,\cdot) = \zeta_+$, then $m = \zeta_+$ would be the solution of the system (4.3), (2.8) and (2.9). Hence we have

$$\Phi(\zeta_+) = 0. \tag{C.8}$$

Proposition Appendix C.5. *For $a \in \mathrm{V} \cap \mathcal{M}^+$, we have $V(a) < \infty$.*

Proof. Let us fix $T > 0$. We construct a continuous map

$$\begin{aligned} R : \mathrm{V} &\longrightarrow L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H}) \\ a &\longmapsto m \end{aligned}$$

such that $m(T) = a$, $m(0) = \zeta_+$ and $R(\zeta_+) = \zeta_+$.

As explain in §1.2.3 of [13], we have the Hilbert space

$$\mathcal{H} = \left\{ f : \int_{\lambda_0}^{\infty} \|f(\lambda)\|^2_{\mathcal{H}(\lambda)} \, \mathrm{d}\mu(\lambda) < \infty \right\},$$

and an unitary operator $U : \mathbb{H} \longrightarrow \mathcal{H}$ which is an isomorphism of $[D(A), \mathbb{H}]_\theta$ onto $\mathcal{H}_{1-\theta}$ for $\theta \in [0,1]$. For some $\varphi \in C^\infty([0,T];\mathbb{R}^+)$ such that $\varphi(0) = 1$, we construct a map $w(a) : [0,T] \times [\lambda_0, \infty) \longrightarrow \mathbb{R}$ by

$$w(a)(t,\lambda) = \frac{T-t}{T} U(\zeta_+) + \frac{t}{T} U(a)(\lambda) \varphi\left(\|a - \zeta_+\|_{\mathrm{V}} (T-t)\lambda\right). \tag{C.9}$$

(Since we assumed that $a \in \mathcal{M}^+$, $w(a)(t,\lambda) \neq 0$ for all (t,λ).) We can see that $w(a) \in L^2(0,T;\mathcal{H}_1) \cap H^1(0,T;\mathcal{H})$, $w(\zeta_+) = U(\zeta_+)$, $w(a)(0) = U(\zeta_+)$, $w(a)(T) = U(a)$. Next we define

$$\tilde{R}(a) := U^{-1}(w(a)),$$

and

$$R(a) := \frac{\tilde{R}(a)}{|\tilde{R}(a)|}.$$

(Note that divided by the norm "increase the smoothness" of a function.) It is not difficult to see this map R is the one we need.

Then we define $m := R(a)$.

Hence $m(0) = \zeta_+$ and $m(T) = a$ and $m \in L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H})$. And so $m \in C([0,T];\mathrm{V})$ (see [13], Theorem 3.1, p.19). Hence we have

$$\Phi(m) \in L^2(0,T;\mathbb{H}).$$

Next we find h such that m is a weak solution of equation (C.4).

If such h exists, then

$$\Phi(m) = \lambda_1 m \times h + \lambda_2 \Pi_m(h).$$

Note that $m(t,x) \times h(t,x) \perp \Pi_m(h)(t,x)$ and $|m(t,x) \times h(t,x)| = |\Pi_m(h)(t,x)|$, we have

$$\Pi_m(h) = \frac{1}{\lambda_2 + \frac{\lambda_1^2}{\lambda_2}} \left(\Phi(m) + \frac{\sqrt{\lambda_1^2 + \frac{\lambda_1^4}{\lambda_2^2}}}{\sqrt{\lambda_1^2 + \lambda_2^2}} \Phi(m) \times m \right).$$

Therefore, let

$$h := \frac{1}{\lambda_2 + \frac{\lambda_1^2}{\lambda_2}} \left(\Phi(m) + \frac{\sqrt{\lambda_1^2 + \frac{\lambda_1^4}{\lambda_2^2}}}{\sqrt{\lambda_1^2 + \lambda_2^2}} \Phi(m) \times m \right) \in L^2(0,T;\mathbb{H}),$$

then our m satisfies the first equation in (C.4) in $L^2(0,T;\mathbb{H})$.

And since $m \in L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H})$, the other conditions in (C.4) are also satisfied. □

Proposition Appendix C.6. *If (m,h) satisfies the equation (2.7), then $(m, \Pi_m h)$ also satisfies (2.7). Moreover, if (m,h_1) and (m,h_2) both satisfy (2.7), then $\Pi_m h_1 = \Pi_m h_2$.*

Proof. We only prove the second statement here. If (m,h_1) and (m,h_2) are both satisfy (2.7), then by (2.7), one has

$$m \times (h_1 - h_2) = m \times (m \times (h_1 - h_2)).$$

But

$$m(t,x) \times (h_1(t,x) - h_2(t,x)) \perp m(t,x) \times (m(t,x) \times (h_1(t,x) - h_2(t,x))) \in \mathbb{R}^3, \quad \forall t,x.$$

therefore

$$m(t,x) \times (h_1(t,x) - h_2(t,x)) = 0.$$

Hence $\Pi_m h_1(t,x) = \Pi_m h_2(t,x)$, for all t,x. □

Notation Appendix C.7. For $T_1, T_2, T \in [0,\infty)$, $a \in \mathrm{V} \cap \mathcal{M}^+$, we denote

$$\mathcal{X}_T(a) := \Big\{ m \in L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H}) \cap C([0,T];\mathcal{M}) : \\ m(0,\cdot) = \zeta_+ \text{ and } m(T,\cdot) = a \Big\},$$

$$\mathcal{X}_{-T}(a) := \Bigg\{ m \in L^2(-T,0;D(A)) \cap H^1(-T,0;\mathbb{H}) \cap C([-T,0];\mathcal{M}) : \\ m(-T,\cdot) = \zeta_+ \text{ and } m(0,\cdot) = a \Bigg\},$$

$$\mathcal{X}_{-\infty}(a) := \Bigg\{ m - \zeta_+ \in L^2(-\infty,0;D(A)) \cap H^1(-\infty,0;\mathbb{H}) \cap C((-\infty,0];\mathcal{M}) : \\ m(-\infty,\cdot) = \zeta_+ \text{ and } m(0,\cdot) = a \Bigg\},$$

$$\mathcal{X}_{-T_1,-T_2}(a) := \Bigg\{ m \in L^2(-T_1,-T_2;D(A)) \cap H^1(-T_1,-T_2;\mathbb{H}) \cap C([-T_1,-T_2];\mathcal{M}) : \\ m(T_1,\cdot) = \zeta_+ \text{ and } m(T_2,\cdot) = a \Bigg\}.$$

$$S_T(m) := \frac{1}{4}\int_0^T \|\Phi(m(t))\|^2_{\mathbb{H}}\,\mathrm{d}t, \quad S_{-T}(m) := \frac{1}{4}\int_{-T}^0 \|\Phi(m(t))\|^2_{\mathbb{H}}\,\mathrm{d}t,$$
$$S_{-\infty}(m) := \frac{1}{4}\int_{-\infty}^0 \|\Phi(m(t))\|^2_{\mathbb{H}}\,\mathrm{d}t, \quad S_{-T_1,-T_2}(m) := \frac{1}{4}\int_{-T_1}^{-T_2} \|\Phi(m(t))\|^2_{\mathbb{H}}\,\mathrm{d}t.$$

Proposition Appendix C.8. *For $a \in \mathrm{V} \cap \mathcal{M}$, one has*

$$V(a) = \inf\Bigg\{ S_T(m) : T > 0, m \in \mathcal{X}_T(a) \Bigg\},$$

Proof. For simplicity, we will only consider the case $\lambda_1 = \lambda_2 = 1$, it is not difficult to generalize to the case $\lambda_1 \in \mathbb{R}$ and $\lambda_2 > 0$.

By (3.51), if m is a weak solution of equation (C.4), then

$$m \in L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H}) \cap C([0,T];\mathcal{M}),$$

and

$$\Phi(m) = m \times h - m \times (m \times h) = m \times h + \Pi_m h.$$

So $m \in \mathcal{X}_T(a)$ and $|\Phi(m)|^2 = 2|\Pi_m h|^2$. By Proposition Appendix C.6, m is also a solution of equation (C.4) with $\Pi_m h$ instead of h. Therefore

$$U(a) \geq \inf \left\{ S_T(m) : T > 0, m \in \mathcal{X}_T(a) \right\}.$$

On the other hand, if $m \in \mathcal{X}_T(a)$, one can construct h as in the proof of Proposition Appendix C.5, such that m is the solution of equation (C.4) and $|\Phi(m)|^2 = 2|h|^2$. Hence

$$U(a) \leq \inf \left\{ S_T(m) : T > 0, m \in \mathcal{X}_T(a) \right\}.$$

The proof is complete. □

By Proposition Appendix C.3, we also have the following result:

Proposition Appendix C.9. *For $a \in \mathrm{V} \cap \mathcal{M}$, one has*

$$V(a) = \inf \left\{ S_{-T}(m) : T > 0, m \in \mathcal{X}_{-T}(a) \right\},$$

Finally, we can show that

Proposition Appendix C.10. *For $a \in \mathrm{V} \cap \mathcal{M}^+$, we have*

$$U(a) = V(a).$$

We need following two Lemmata to prove Proposition Appendix C.10.

Lemma Appendix C.11. *For all $T > 0$, $\varepsilon > 0$, there exists $\eta > 0$ such that for all $a \in \mathrm{V}$ with $\|a - \zeta_+\|_{\mathrm{V}} < \eta$, there exists $m \in \mathcal{X}_T(a)$ and such that $S_T(m) < \varepsilon$.*

Proof. It can be seen that the map $S_T : L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H}) \longrightarrow \mathbb{R}$ is continuous and $\Phi(\zeta_+) = 0$ (by (C.8)). Moreover, as constructed in the proof of Proposition Appendix C.5, there exists a continuous map

$$\begin{aligned} R : \mathrm{V} &\longrightarrow L^2(0,T;D(A)) \cap H^1(0,T;\mathbb{H}) \\ a &\longmapsto m \end{aligned}$$

such that $m(T) = a$, $m(0) = \zeta_+$ and $R(\zeta_+) = \zeta_+$. Hence the map $S_T \circ R : \mathrm{V} \longrightarrow \mathbb{R}$ is also continuous and $(S_T \circ R)(\zeta_+) = 0$. Therefore the proof is complete. □

Lemma Appendix C.12. *Assume that $m \in \mathcal{X}_{-\infty}(a)$ for some $a \in \mathrm{V} \cap \mathcal{M}$. Then for all $\varepsilon > 0$, there exists $T_\varepsilon > 0$ and $m_\varepsilon \in \mathcal{X}_{-T_\varepsilon}(a)$, such that*

$$S_{-T_\varepsilon}(m_\varepsilon) \leq S_{-\infty}(m) + \varepsilon.$$

Proof. Let us fix $\varepsilon > 0$ and assume that $S_{-\infty}(m) < \infty$. Since $m \in \mathcal{X}_{-\infty}(a)$, $m \in C((-\infty, 0]; \mathrm{V})$ and $m(-\infty) = \zeta_+$. So

$$\|m(t) - \zeta_+\|_{\mathrm{V}} \longrightarrow 0, \qquad t \to -\infty.$$

Hence for $\eta > 0$, there exists $T_\eta > 0$ such that

$$\|m(t) - \zeta_+\|_{\mathrm{V}} < \eta, \qquad t \leq -T_\eta.$$

Therefore by Lemma Appendix C.11 (with $T = 1$, $a = m(-T_\eta)$), there exists $w \in \mathcal{X}_{-T_\eta - 1, -T_\eta}(m(-T_\eta))$ such that

$$S_{-T_\eta-1,-T_\eta}(w) < \varepsilon.$$

Let

$$m_\varepsilon(t) = \begin{cases} w(t), & t \in [-T_\eta - 1, -T_\eta]; \\ m(t), & t \in [-T_\eta, 0]. \end{cases}$$

Then $m_\varepsilon \in \mathcal{X}_{-T_\eta - 1}$ and

$$\begin{aligned} S_{-T_\eta-1}(m_\varepsilon) &= S_{-T_\eta-1,-T_\eta}(w) + S_{-T_\eta}(m) \\ = \ & S_{-T_\eta-1,-T_\eta}(w) + \left(S_{-\infty}(m) - \frac{1}{4}\int_{-\infty}^{-T_\eta} \|\Phi(m(t))\|_{\mathbb{H}}^2 \, \mathrm{d}t\right) \leq S_{-\infty}(m) + \varepsilon. \end{aligned}$$

Let $T_\varepsilon = T_\eta + 1$, then the proof is complete. □

Proof of Proposition Appendix C.10. We first prove that

$$V(a) = \inf \left\{ S_{-\infty}(m) : m \in \mathcal{X}_{-\infty}(a) \right\}. \tag{C.10}$$

For $T > 0$ and $m \in \mathcal{X}_{-T}(a)$, we define

$$\bar{m}(t) = \begin{cases} m(t), & t \in [-T, 0]; \\ \zeta_+, & t \leq -T. \end{cases}$$

Then $\bar{m} \in \mathcal{X}_{-\infty}(a)$. $\Delta\zeta_+ = \phi'(\zeta_+) = \frac{\mathrm{d}\zeta_+}{\mathrm{d}t} = 0$, so $\Phi(m) = \Phi(\bar{m})$. Therefore $S_{-\infty}(\bar{m}) = S_{-T}(m)$. Hence by Proposition Appendix C.9, we have

$$V(a) \geq \inf \{S_{-\infty}(m) : m \in \mathcal{X}_{-\infty}(a)\}.$$

On the other hand, we can assume $\inf \{S_{-\infty}(m) : m \in \mathcal{X}_{-\infty}(a)\} < \infty$, so there exists $m \in \mathcal{X}_{-\infty}(a)$ such that $S_{-\infty}(m) < \infty$. By Lemma Appendix C.12, one has

$$\inf \{S_{-T}(m) : m \in \mathcal{X}_{-T}(a)\} \leq S_{-\infty}(m) + \varepsilon, \qquad \varepsilon > 0.$$

Since ε is arbitrary, by Proposition Appendix C.9 again,

$$V(a) \leq \inf \{S_{-\infty}(m) : m \in \mathcal{X}_{-\infty}(a)\}.$$

Hence (C.10) has been proved.

Then the rest part of the proof of Proposition Appendix C.10, i.e. prove that

$$U(a) = \inf \{S_{-\infty}(m) : m \in \mathcal{X}_{-\infty}(a)\}$$

can be done by the same way as in the proof of Proposition Appendix C.8. $\square$

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