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Backdoors for Linear Temporal Logic

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Abstract

In the present paper, we introduce the backdoor set approach into the field of temporal logic for the global fragment of linear temporal logic. We study the parameterized complexity of the satisfiability problem parameterized by the size of the backdoor. We distinguish between backdoor detection and evaluation of backdoors into the fragments of Horn and Krom formulas. Here we classify the operator fragments of globally-operators for past/future/always, and the combination of them. Detection is shown to be fixed-parameter tractable whereas the complexity of evaluation behaves differently. We show that for Krom formulas the problem is paraNP-complete. For Horn formulas, the complexity is shown to be either fixed parameter tractable or paraNP-complete depending on the considered operator fragment.

Keywords Linear temporal logic · Parameterized complexity · Backdoor sets

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1 Introduction

Temporal logic is one of the most important formalism in the area of program verification and validation of specification consistency. Most notable are the seminal contributions of Kripke [21], Pnueli [32], Emerson, Clarke, and Halpern [7,14] to name a few. There exist several different variants of temporal logic from which, best known are the computation tree logic CTL, the linear temporal logic LTL, and the full branching time logic CTL*. In this paper, we will consider the global fragment of LTL for formulas in *separated normal form* (SNF) which has been introduced by Fisher [15]. This normal form is a generalization of the conjunctive normal form from propositional logic to linear temporal logic with future and past modalities interpreted over the flow of time, i.e., the frame of the integers $(\mathbb{Z}, <)$. In SNF the formulas are divided into a past, a present, and a future part. Technically this normal form is not a restriction since one can always translate an arbitrary LTL formula to a satisfiability-equivalent formula in SNF in time linear in the original formula [15]. In fact, the restriction to SNF normal form is crucial for us, because it is known that syntactical restrictions of arbitrary LTL formulas such as Horn or Krom do not lead to tractability [4].

LTL and its two main associated computational problems LTL model checking and LTL satisfiability have been deeply investigated in the past. In this work we focus on the LTL satisfiability problem, i.e., given an LTL formula the question is whether there is a temporal interpretation that satisfies the formula. Sistla and Clarke classified the computational complexity of the satisfiability problem to be PSPACE-complete [36]. Then, later, several restrictions of the unrestricted problem have been considered. These approaches considered operator fragments [29], Horn formulas [4], temporal operator fragments, temporal depth, and number of propositional variables [8], the use of negation [27], an XOR fragment [11], an application of Post's lattice [3], and the SNF fragment [2].

In contrast to LTL satisfiability where the search for fruitful parameterization has so far been rather unsuccessful [26], various important parameterizations have been identified for the satisfiability problem of propositional formulas (SAT) [5,30,37]. One very prominent and well-studied structural parameterization for SAT are so-called backdoor sets. Informally, backdoors are small sets of variables of a SAT instance that represent “clever reasoning shortcuts” through the search space. Backdoor sets have been widely used in the areas of propositional satisfiability [9,10,19,20,33,35,38], and also for material discovery [25], abductive reasoning [31], argumentation [13], planning [22,23], and quantified Boolean formulas [34]. A backdoor set is defined with respect to some fixed *base class* for which the computational problem under consideration is polynomial-time tractable. For instance, in the case of SAT, a backdoor set B for a given CNF formula ϕ into the base class of Horn formulas is a set of variables such that for every assignment of the variables in B it holds that the reduced formula, i.e., the formula obtained after applying the assignment to ϕ , is Horn. Given such a backdoor set one can decide the satisfiability of ϕ in time $O(2^{|B|}p(|\phi|))$ by enumerating the $2^{|B|}$ assignments of the variables in B and for each such assignment solving the remaining formula in time $p(|\phi|)$, where p is a polynomial given by the base class. As a result, once a small backdoor set is identified the satisfiability check is *fixed-parameter tractable* for the parameter backdoor size. Since the backdoor set

Table 1 Results overview

Problem	Operators	HORN	KROM
Detection	Any	FPT (Thm. 5)	FPT (Thm. 5)
Evaluation	\boxtimes	FPT (Thm. 8)	paraNP-c. (Thm. 9)
	\square_F, \square_P	paraNP-c. (Thm. 10)	paraNP-c. (Above)
LTL-SAT	One of \square_F, \square_P	Open	paraNP-c. (Cor. 11)
	$\boxtimes, \square_F, \square_P$	P [2]	NP-c. [2]
	\boxtimes	P [2]	NL [2]

The term “Any” refers to any combination of $\boxtimes, \square_F, \square_P$, whereas “Above” denotes that the lower bound from the cell above applies

is usually not provided with the input, it is crucial that small backdoor sets to a given base class can be found efficiently. When employing the backdoor approach one consequently usually considers two subtasks the so-called *detection* and *evaluation* problem, where the former is the task to identify a small backdoor set and the later concerns the solution of the problem using the backdoor set.

Our Contribution In this paper, we introduce a notion of backdoors for the global fragment of LTL formulas that are given in SNF. Namely, we consider backdoor sets to the base classes that have recently been identified by Artale et al. [2]. These base classes are defined by both restrictions on the allowed temporal operators (i.e., to a subset of $\{\boxtimes, \square_P, \square_F\}$) and restrictions on the clauses to be either HORN or KROM. We show that surprisingly a notion of backdoor sets very similar to the strong backdoor sets employed for SAT [18] can also be successfully applied to LTL formulas. Whereas the detection of these backdoor sets can be achieved via efficient fpt-algorithms for all the considered fragments (using algorithms similar to the algorithms employed in the context of SAT), the evaluation of these backdoor sets turns out to be much more involved. In particular, we obtain tractability of the evaluation problem for HORN formulas using only the always operator. In fact, LTL restricted to only the always operator, is already quite interesting, since it allows one to express “Safety” properties of a system. For almost all of the remaining cases we show that the evaluation problem is paraNP-hard. Moreover, the techniques used to show these results are very different from and more involved than the techniques employed for SAT, i.e., in the context of SAT the backdoor set evaluation problem is trivial. Our results are summarized in Table 1.

2 Preliminaries

Parameterized Complexity A good introduction into the field of parameterized complexity is given by Downey and Fellows [12]. A *parameterized problem* Π is a tuple (Q, κ) such that the following holds. $Q \subseteq \Sigma^*$ is a language over an alphabet Σ , and $\kappa : \Sigma^* \rightarrow \mathbb{N}$ is a computable function; then κ also is called the *parameterization* (of Π).

If there is a deterministic Turing machine M and a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ s.t. for every instance $x \in \Sigma^*$ (i) M decides correctly if $x \in Q$, and (ii) M has a runtime

bounded by $f(\kappa(x)) \cdot |x|^{O(1)}$, then we say that M is an *fpt-algorithm* for Π and that Π is *fixed-parameter tractable* (or in the class FPT). If M is non-deterministic, then Π belongs to the class paraNP. One way to show paraNP-hardness of a parameterized problem (Q, κ) is to show that Q is NP-hard for a specific, fixed value of κ , i.e., there exists a constant $\ell \in \mathbb{N}$ such that $(Q, \kappa)_\ell := \{x \mid x \in Q \text{ and } \kappa(x) = \ell\}$ is NP-hard.

Temporal Logic We assume familiarity with standard notions of propositional logic. Let PROP be a finite set of propositions and \perp/\top abbreviate the constants false/true. The syntax of the global fragment of LTL is defined by the following EBNF:

$$\varphi ::= \perp \mid \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_P \varphi \mid \Box_F \varphi \mid \boxtimes \varphi,$$

where $p \in \text{PROP}$. Here $\Box_P \varphi$ can be read as “ φ holds in every point in the past”, $\Box_F \varphi$ as “ φ holds in every point in the future”, and $\boxtimes \varphi$ as “ φ holds always”. We also will make use of well-known shortcuts such as $\rightarrow, \leftrightarrow$. Now we define the semantics of these formulas. Here, we interpret LTL formulas over the flow of time $(\mathbb{Z}, <)$ (for further information on this approach, see, e.g., Gabbay et al. [17]). Note that all our results can be easily transferred to the case if the formulas are evaluated over the set of natural numbers instead of the set of all integers.

Definition 1 (*Temporal Semantics*) Let PROP be a finite set of propositions. A *temporal interpretation* $\mathfrak{M} = (\mathbb{Z}, <, V)$ is a mapping from propositions to moments of time, i.e., $V : \text{PROP} \rightarrow \mathcal{P}(\mathbb{Z})$. The satisfaction relation \models is then defined as follows where $n \in \mathbb{Z}, \varphi, \psi \in \text{LTL}$

$\mathfrak{M}, n \models \top$	always,
$\mathfrak{M}, n \models \perp$	never,
$\mathfrak{M}, n \models p$	iff $n \in V(p)$,
$\mathfrak{M}, n \models \neg\varphi$	iff $\mathfrak{M}, n \not\models \varphi$,
$\mathfrak{M}, n \models \varphi \vee \psi$	iff $\mathfrak{M}, n \models \varphi$ or $\mathfrak{M}, n \models \psi$,
$\mathfrak{M}, n \models \varphi \wedge \psi$	iff $\mathfrak{M}, n \models \varphi$ and $\mathfrak{M}, n \models \psi$,
$\mathfrak{M}, n \models \Box_F \varphi$	iff for all $k > n$ it holds $\mathfrak{M}, k \models \varphi$,
$\mathfrak{M}, n \models \Box_P \varphi$	iff for all $k < n$ it holds $\mathfrak{M}, k \models \varphi$, and
$\mathfrak{M}, n \models \boxtimes \varphi$	iff for all $k \in \mathbb{Z}$ it holds $\mathfrak{M}, k \models \varphi$.

We say that φ is *satisfiable* if there is a temporal interpretation \mathfrak{M} such that $\mathfrak{M}, 0 \models \varphi$. Then \mathfrak{M} is also referred to as a (*temporal*) *model* (of φ). Sometimes we also directly write $\mathfrak{M}(p)$ instead of $V(p)$.

Table 2 exemplifies the semantics with some basic formulas. As shown by Fisher et al. every LTL formula considered over the frame $(\mathbb{Z}, <)$ has a satisfiability-equivalent formula in the separated normal form SNF [16], which can be constructed in linear time [15]). We follow the notation of SNF formulas by Artale et al. [2] and directly restrict them to the relevant global fragment of this study:

$$\lambda ::= \perp \mid p \mid \Box_F \lambda \mid \Box_P \lambda \mid \boxtimes \lambda, \tag{1}$$

$$\varphi ::= \lambda \mid \neg\lambda \mid \varphi \wedge \varphi \mid \boxtimes(\neg\lambda \vee \dots \vee \neg\lambda \vee \lambda \vee \dots \vee \lambda), \tag{2}$$

where λ is called a *temporal literal* and φ is said to be in *clausal normal form*.

Table 2 Temporal semantics

	< 0	0	1	2	3	4	5	> 5
p	1	1	0	1	0	1	1	0
q	0	0	1	1	0	0	1	1
r	0	0	0	0	1	0	0	0
$p \wedge q$	0	0	0	1	0	0	1	0
$\boxtimes p$	0	0	0	0	0	0	0	0
$\square_F q$	0	0	0	0	0	1	1	1
$\square_F \square_F q$	0	0	0	0	1	1	1	1
$\square_P p$	1	1	1	0	0	0	0	0
$\boxtimes(\square_P p)$	0	0	0	0	0	0	0	0
$\boxtimes(p \vee \square_F q \vee \square_P p \vee r)$	1	1	1	1	1	1	1	1

Table 3 Considered normal forms

Class	Description	Restrictions on n, m
CNF	No restrictions on (2)	–
HORN	At most one positive temporal literal	$m \leq 1$
KROM	Binary clauses	$n + m \leq 2$

Restrictions refer to Eq. (2)

Note that the operator name G instead of \square_F often occurs in literature. Yet, in contrast to $G\varphi$, for $\square_F \varphi$ it is not required that φ holds in the present world. We distinguish fragments of LTL by adding superscripts and subscripts as follows. If $O \subseteq \{\square_F, \square_P, \boxtimes\}$ is an operator subset then LTL^O is the fragment of LTL consisting of formulas that are allowed to only use temporal operators from O for temporal literals, i.e., it is a constraint on the allowed operators in equation (1) from above. We also consider restrictions of the clausal normal form in (2): $\boxtimes(\neg\lambda_1 \vee \dots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \dots \vee \lambda_{n+m})$. Table 3 lists the relevant cases for this study. If $\alpha \in \{\text{CNF}, \text{HORN}, \text{KROM}\}$ then LTL_α is the set of formulas where the subformulas of the type $\boxtimes(\neg\lambda_1 \vee \dots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \dots \vee \lambda_{n+m})$ (3), obey the normal form α .

The following lemma shows a log-space constructible normal form which prohibits deep nesting of temporal operators of the investigated formulas.

Proposition 2 ([2, Lemma 2]) *Let $\mathcal{L} \in \{LTL_{\alpha}^{\square_F, \square_P}, LTL_{\alpha}^{\square_F}, LTL_{\alpha}^{\square_P}, LTL_{\alpha}^{\boxtimes}\}$ be a formula class for $\alpha \in \{\text{CNF}, \text{HORN}, \text{KROM}\}$. For any formula $\varphi \in \mathcal{L}$, one can construct, in log-space, a satisfiability-equivalent \mathcal{L} -formula $\Psi \wedge \boxtimes \Phi$, where Ψ is a conjunction of propositional variables from Φ , and Φ is a conjunction of clauses of the form (3) containing only \square_F, \square_P for $LTL_{\alpha}^{\square_F, \square_P}$, \square_F for $LTL_{\alpha}^{\square_F}$, \square_P for $LTL_{\alpha}^{\square_P}$, and only \boxtimes for LTL_{α}^{\boxtimes} , in which the temporal operators are not nested.*

In the following sections we consider only formulas given in this normal form $\Psi \wedge \boxtimes \Phi$.

3 Introduction of Backdoors for the Global Fragment of LTL

In the following, we will introduce a notion of backdoors for formulas in the global fragment of linear temporal logic. The definition of these backdoors turns out to be very similar to the definition of the so-called strong backdoor sets for propositional formulas [18]. The main difference is that whenever a propositional variable is in the backdoor set then also all of its temporal literals are required to be in the backdoor set as well. A consequence of this is that in contrast to propositional formulas, where a backdoor set needs to consider all assignments of the backdoor set variables, we only need to consider assignments that are consistent between propositional variables and their temporal literals.

Let \mathcal{O} be a set of operators. An assignment $\theta: \text{Vars}(\phi) \cup \{Ox \mid x \in \text{Vars}(\phi) \wedge O \in \mathcal{O}\} \rightarrow \{0, 1\}$ is *consistent* if for every $x \in \text{Vars}(\phi)$ it holds that if $\theta(\boxtimes x) = 1$, then also $\theta(\square_P x) = 1$, $\theta(\square_F x) = 1$, and $\theta(x) = 1$.

Definition 3 (Backdoors) Let \mathcal{C} be a class of CNF-formulas, \mathcal{O} be a set of operators, and ϕ be an $\text{LTL}_{\text{CNF}}^{\mathcal{O}}$ formula. A set $X \subseteq \text{Vars}(\phi)$ is a (*strong*) $(\mathcal{C}, \mathcal{O})$ -backdoor if for every consistent assignment $\theta: X \cup \{Ox \mid x \in X, O \in \mathcal{O}\} \rightarrow \{0, 1\}$ it holds that $\phi[\theta]$ is in \mathcal{C} .

The *reduct* $\phi[\theta]$ is defined similarly to that for standard CNF-formulas, i.e., all clauses that contain a satisfied literal are deleted, and all falsified literals are deleted from their clauses. Here empty clauses are substituted by false, and the empty formula by true. Sometimes if the context of \mathcal{O} is clear, we omit to state it and just mention the backdoor class \mathcal{C} .

Example 4 Let $\varphi = p_1 \wedge p_2 \wedge \boxtimes(\neg \square_P p_4 \vee \square_P p_2 \vee \square_P p_3)$ be the considered formula. Then $B = \{p_3\}$ is a strong $(\text{KROM}, \{\square_P, \boxtimes\})$ -backdoor as the following assignments have to be examined:

p_3	$\square_P p_3$	$\boxtimes p_3$	$\varphi[\theta]$	
0	0	0	$p_1 \wedge p_2 \wedge \boxtimes(\neg \square_P p_4 \vee \square_P p_2)$	★
0	0	1	Irrelevant as inconsistent	
0	1	0	$p_1 \wedge p_2$	♡
0	1	1	Irrelevant as inconsistent	
1	0	0	$p_1 \wedge p_2 \wedge \boxtimes(\neg \square_P p_4 \vee \square_P p_2)$	★
1	0	1	Irrelevant as inconsistent	
1	1	0	$p_1 \wedge p_2$	♡
1	1	1	$p_1 \wedge p_2$	

First, observe that all relevant rows lead to a KROM-formula. Note that for the rows marked with ★ the reduct just removed the temporal literal $\square_P p_3$. All other rows are either inconsistent (and hence irrelevant) or delete the clause $(\neg \square_P p_4 \vee \square_P p_2 \vee \square_P p_3)$ completely, because $\square_P p_3$ is set to true. At first glance, our definition of backdoor sets for LTL is almost purely syntactical, and thereby is an accordance to

strong backdoor sets for the propositional satisfiability problem. For instance consider the assignments marked with the \heartsuit . In these cases we delete the clause $(\neg \Box_P p_4 \vee \Box_P p_2 \vee \Box_P p_3)$ completely because $\Box_P p_3$ is set to true. However, we also know that, because $\boxtimes p_3$ is set to false, the clause will not be satisfied solely by $\Box_P p_3$ in all possible worlds of a satisfying model. This indicates that solving the formula using the backdoor will not be as simple as it was for the propositional satisfiability problem, where it was sufficient to enumerate all assignments of the backdoor set and solve the reduced formula. Nevertheless, as we will show in Sect. 5.1 our backdoor sets can still be used for the efficient evaluation of LTL formulas.

To exploit backdoor sets to obtain efficient fpt-algorithms for LTL one needs to accomplish two tasks: first, one needs to find a small backdoor set, and then one needs to show how the backdoor set can be exploited to efficiently evaluate the formula. This leads to the following problem definitions for every class \mathcal{C} of formulas and set of operators \mathcal{O} .

Problem: $\text{Eval}^{\mathcal{O}}(\mathcal{C})$ — Backdoor evaluation to $\text{LTL}_{\mathcal{C}}^{\mathcal{O}}$.
Input: $\text{LTL}_{\text{CNF}}^{\mathcal{O}}$ formula ϕ , strong $(\mathcal{C}, \mathcal{O})$ -backdoor X .
Parameter: $|X|$.
Question: Is ϕ satisfiable?

Problem: $\text{Detect}^{\mathcal{O}}(\mathcal{C})$ — Backdoor detection to $\text{LTL}_{\mathcal{C}}^{\mathcal{O}}$.
Input: $\text{LTL}_{\text{CNF}}^{\mathcal{O}}$ formula ϕ , integer $k \in \mathbb{N}$.
Parameter: k .
Task: Find a strong $(\mathcal{C}, \mathcal{O})$ -backdoor of size $\leq k$ if one exists.

Of course, this approach is only meaningful if one considers target classes that have polynomial time solvable satisfiability problems. Artale et al. have shown [2] that satisfiability for $\text{LTL}_{\text{HORN}}^{\boxtimes}$ and $\text{LTL}_{\text{KROM}}^{\boxtimes}$ are solvable in P. Adding \Box_F, \Box_P to the set of allowed operators makes the KROM fragment NP-complete whereas for HORN formulas the problem stays in P. Accordingly, we will consider in the following only KROM and HORN formulas. Moreover, note that when considering arbitrary CNF formulas instead of HORN or KROM formulas, then $\text{LTL}_{\text{CNF}}^{\mathcal{O}}$ is known to be NP-complete for any (even empty) subset $\mathcal{O} \subseteq \{\Box_F, \Box_P, \boxtimes\}$ [2].

4 Backdoor Set Detection

In this section, we show that finding strong \mathcal{C} -backdoor sets (under the parameter size of the set) is fixed-parameter tractable if \mathcal{C} is either HORN or KROM. The algorithms that we will present are very similar to the algorithms that are known for the detection of strong backdoors for propositional CNF formulas [18].

We first show how to deal with the fact that we only need to consider consistent assignments. The following observation is easily witnessed by the fact that if one of $\Box_P x, \Box_F x, x$ does not hold then $\neg \boxtimes x$ is true.

Observation 1 *Let $\phi := \Psi \wedge \boxtimes \Phi$ be an $\text{LTL}^{\Box_P, \Box_F, \boxtimes}$ formula. Then any clause C of Φ containing $\neg \boxtimes x$ and (at least) one of $\Box_P x, \Box_F x$ or x for some variable $x \in \text{Vars}(\phi)$ is tautological and can be removed from Φ (without changing the satisfiability of ϕ).*

Observe that the tautological clauses above are exactly the clauses that are satisfied by every consistent assignment. It follows that once these clauses are removed from the formula, it holds that for every clause C of ϕ there is a consistent assignment θ such that C is not satisfied by θ .

Theorem 5 *For every $\mathcal{O} \subseteq \{\ast, \square_P, \square_F\}$ and $\mathcal{C} \in \{\text{HORN}, \text{KROM}\}$ the problem $\text{Detect}^{\mathcal{O}}(\mathcal{C})$ is in FPT.*

Proof Let $\mathcal{O} \subseteq \{\ast, \square_P, \square_F\}$. We will reduce $\text{Detect}^{\mathcal{O}}(\text{HORN})$ to the problem VertexCover which is well-known to be fixed-parameter tractable (parameterized by the solution size) and which can actually be solved very efficiently in time $O(1.2738^k + kn)$ [6], where k is the size of the vertex cover and n the number of vertices in the input graph. Recall that given an undirected graph G and an integer k , VertexCover asks whether there is a subset $C \subseteq V(G)$ of size at most k (which is called a vertex cover of G) such that $C \cap e \neq \emptyset$ for every $e \in E(G)$. Given an $\text{LTL}^{\mathcal{O}}$ formula $\phi := \Psi \wedge \ast \Phi$, we will construct an undirected graph G such that ϕ has a strong HORN-backdoor of size at most k if and only if G has a vertex cover of size at most k . The graph G has vertex set $\text{Vars}(\phi)$ and there is an edge between two vertices x and y in G if and only if there is a clause that contains at least two literals from $\{x, y\} \cup \{Ox, Oy \mid O \in \mathcal{O}\}$. Note that if $x = y$, the graph G contains a self-loop. We claim that a set $X \subseteq \text{Vars}(\phi)$ is a strong HORN-backdoor if and only if X is a vertex cover of G .

Towards showing the forward direction, let $X \subseteq \text{Vars}(\phi)$ be a strong HORN-backdoor set of ϕ . We claim that X is also a vertex cover of G . Suppose for a contradiction that X is not a vertex cover of G , i.e., there is an edge $\{x, y\} \in E(G)$ such that $X \cap \{x, y\} = \emptyset$. Because $\{x, y\} \in E(G)$, we obtain that there is a clause C in Φ that contains at least two literals from $\{x, y\} \cup \{Ox, Oy \mid O \in \mathcal{O}\}$. Moreover, because of Observation 1 there is a consistent assignment $\theta: X \cup \{Ox \mid x \in X \wedge O \in \mathcal{O}\} \rightarrow \{0, 1\}$ that falsifies all literals of C over variables in X . Consequently, $\phi[\theta]$ contains a sub-clause of C that still contains at least two literals from $\{x, y\} \cup \{Ox, Oy \mid O \in \mathcal{O}\}$. As a reason for this, $\phi[\theta] \notin \text{HORN}$, contradicting our assumption that X is a strong HORN-backdoor set of ϕ .

Towards showing the reverse direction, let $X \subseteq V(G)$ be a vertex cover of G . We claim that X is also a strong HORN-backdoor of ϕ . Suppose for a contradiction that this is not the case, then there is an (consistent) assignment $\theta: X \cup \{Ox \mid x \in X \wedge O \in \mathcal{O}\} \rightarrow \{0, 1\}$ and a clause C in $\phi[\theta]$ containing two positive literals say over variables x and y . We obtain that C contains at least two positive literals from $\{x, y\} \cup \{Ox, Oy \mid O \in \mathcal{O}\}$ and consequently G contains the edge $\{x, y\}$, contradicting our assumption that X is a vertex cover of G .

Now we will reduce $\text{Detect}^{\mathcal{O}}(\text{KROM})$ to the 3-HittingSet problem, which is well-known to be fixed-parameter tractable (parameterized by the solution size) [1]. Recall that given a universe U , a family \mathcal{F} of subsets of U of size at most three, and an integer k , 3-HittingSet asks whether there is a subset $S \subseteq U$ of size at most k (which is called a hitting set of \mathcal{F}) such that $S \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Given an $\text{LTL}^{\mathcal{O}}$ formula $\phi := \Psi \wedge \ast \Phi$, we will construct a family \mathcal{F} of subsets (of size at most three) of a universe U such that ϕ has a strong KROM-backdoor of size at most k if and only if \mathcal{F} has a hitting set of size at most k . The universe U is equal to $\text{Vars}(\phi)$ and \mathcal{F} contains

the set $\text{Vars}(C)$ for every set C of exactly three literals contained in some clause of Φ . We claim that a set $X \subseteq \text{Vars}(\phi)$ is a strong KROM-backdoor if and only if X is a hitting set of \mathcal{F} .

Towards showing the forward direction, let $X \subseteq \text{Vars}(\phi)$ be a strong KROM-backdoor set of ϕ and suppose for a contradiction that there is a set $F \in \mathcal{F}$ such that $X \cap F = \emptyset$. It follows from the construction of \mathcal{F} that Φ contains a clause C containing at least three literals over the variables in F . Moreover, because of Observation 1 there is a consistent assignment $\theta: X \cup \{Ox \mid x \in X \wedge O \in \mathcal{O}\} \rightarrow \{0, 1\}$ that falsifies all literals of C over variables in X . Consequently, $\phi[\theta]$ contains a sub-clause of C that still contains at least three literals over the variables in F . As a result, $\phi[\theta] \notin \text{KROM}$, contradicting our assumption that X is a strong KROM-backdoor set of ϕ .

Towards showing the reverse direction, let $X \subseteq U$ be a hitting set of \mathcal{F} and suppose for contradiction that there is an (consistent) assignment $\theta: X \cup \{Ox \mid x \in X \wedge O \in \mathcal{O}\} \rightarrow \{0, 1\}$ and a clause C in $\phi[\theta]$ containing at least three literals. Let C' be a set of at exactly three literals from C . It follows from the construction of \mathcal{F} , that \mathcal{F} contains the set $\text{Vars}(C')$, however, $\text{Vars}(C') \cap X = \emptyset$ contradicting our assumption that X is a hitting set of G . \square

Having shown that the detection problem is fixed-parameter tractable, we now proceed to the backdoor set evaluation problem. We begin by investigating this problem for the class HORN and show that the problem lies in FPT.

5 Backdoor Set Evaluation

5.1 Formulas Using only the Always Operator

We showed in the previous section that strong backdoors can be found to the classes HORN and KROM in FPT time. In fact, this result holds independently of the considered temporal operators. In this section, we will consider the question of efficiently *using* a backdoor set to decide the satisfiability of a formula in the case of formulas restricted to the \boxtimes operator. We will show that this problem is in FPT for the class of HORN formulas but not for KROM formulas. Our fixed-parameter tractability result for HORN formulas largely depends on the special semantics of formulas restricted to the \boxtimes operators. Consequently, we will start by stating some properties of these formulas necessary to obtain our tractability result.

Let $\mathfrak{M} = (\mathbb{Z}, <, V)$ be a temporal interpretation. We denote by $\text{Vars}(\mathfrak{M})$ the set of propositions (in the following referred to as variables) for which V is defined. For a set of variables $X \subseteq \text{Vars}(\mathfrak{M})$, we denote by $\mathfrak{M}|_X$ the *projection* of \mathfrak{M} onto X , i.e., the temporal interpretation $\mathfrak{M}|_X = (\mathbb{Z}, <, V|_X)$, where $V|_X$ is only defined for the variables in X and $V|_X(x) = V(x)$ for every $x \in X$. For an integer z , we denote by $\mathbf{A}(\mathfrak{M}, z)$ the assignment $\theta: \text{Vars}(\mathfrak{M}) \rightarrow \{0, 1\}$ holding at world z in \mathfrak{M} , i.e., $\theta(v) = 1$ if and only if $z \in \mathfrak{M}(v)$ for every $v \in \text{Vars}(\mathfrak{M})$. Moreover, for a set of worlds $Z \subseteq \mathbb{Z}$ we denote by $\mathbf{A}(\mathfrak{M}, Z)$ the set of all assignments occurring in some world in Z of \mathfrak{M} , i.e., $\mathbf{A}(\mathfrak{M}, Z) := \{\mathbf{A}(\mathfrak{M}, z) \mid z \in Z\}$. We also set $\mathbf{A}(\mathfrak{M})$ to be $\mathbf{A}(\mathfrak{M}, \mathbb{Z})$. For an

Table 4 An example for the notions $\mathbf{G}(\mathbb{A}, V)$ and $\mathbf{G}(\mathbb{A}, V, \theta)$

	\mathbb{A}				θ	$\mathbf{G}(\mathbb{A}, V)(\boxed{*}v_i)$	$\mathbf{G}(\mathbb{A}, V, \theta)(v_i)$
	α_1	α_2	α_3	α_4			
v_1	0	1	0	1	1	0	1
v_2	1	1	1	1	0	1	0
v_3	1	1	0	0	1	0	1
v_4	1	1	1	1	0	1	0

assignment $\theta: X \rightarrow \{0, 1\}$, we denote by $\mathbf{W}(\mathfrak{M}, \theta)$ the set of all worlds $z \in \mathbb{Z}$ of \mathfrak{M} such that $\mathbf{A}(\mathfrak{M}, z)$ is equal to θ on all variables in X .

Let $\varphi := \Psi \wedge \boxed{*}\Phi \in \text{LTL}_{\text{CNF}}^{\boxed{*}}$. We denote by $\text{CNF}(\Phi)$ the propositional CNF formula obtained from Φ after replacing each occurrence of $\boxed{*}x$ in Φ with the same fresh propositional variable (with the same name). For instance, $\boxed{*}a \wedge \boxed{*}a$ is replaced by $\boxed{*}a \wedge \boxed{*}a$, where $\boxed{*}a$ is a fresh propositional variable. For a set of variables V and a set of assignments \mathbb{A} of the variables in V , we denote by $\mathbf{G}(\mathbb{A}, V): \{\boxed{*}v \mid v \in V\} \rightarrow \{0, 1\}$ the assignment defined by setting $\mathbf{G}(\mathbb{A}, V)(\boxed{*}v) = 1$ if and only if $\alpha(v) = 1$ for every $\alpha \in \mathbb{A}$. Moreover, if $\theta: V \rightarrow \{0, 1\}$ is an assignment of the variables in V , we denote by $\mathbf{G}(\mathbb{A}, V, \theta)$ the assignment defined by setting $\mathbf{G}(\mathbb{A}, V, \theta)(v) = \theta(v)$ and $\mathbf{G}(\mathbb{A}, V, \theta)(\boxed{*}v) = \mathbf{G}(\mathbb{A}, V)(\boxed{*}v)$ for every $v \in V$. An example for these notions is given in Table 4. For a set \mathbb{A} of assignments over V and an assignment $\theta: V' \rightarrow \{0, 1\}$ with $V' \subseteq V$, we denote by $\mathbb{A}(\theta)$ the set of all assignments $\alpha \in \mathbb{A}$ such that $\alpha(v) = \theta(v)$ for every $v \in V'$.

For a set \mathbb{A} of assignments over some variables V and a subset $V' \subseteq V$, we denote by $\mathbb{A}|_{V'}$ the *projection* of \mathbb{A} onto V' , i.e., the set of assignments $\alpha \in \mathbb{A}$ restricted to the variables in V' .

Intuitively the next lemma describes the translation of a temporal model into separate satisfiability checks for propositional formulas.

Lemma 6 *Let $\varphi := \Psi \wedge \boxed{*}\Phi \in \text{LTL}^{\boxed{*}}$. Then, φ is satisfiable if and only if there is a set \mathbb{A} of assignments of the variables in φ and an assignment $\alpha_0 \in \mathbb{A}$ such that: α_0 satisfies Ψ and for every assignment $\alpha \in \mathbb{A}$ it holds that $\mathbf{G}(\mathbb{A}, \text{Vars}(\varphi), \alpha)$ satisfies the propositional formula $\text{CNF}(\Phi)$.*

Proof Towards showing the forward direction assume that $\varphi := \Psi \wedge \boxed{*}\Phi$ is satisfiable and let \mathfrak{M} be a temporal interpretation witnessing this. It is easy to check from the definition that the set of assignments $\mathbb{A} := \mathbf{A}(\mathfrak{M})$ together with the assignment $\alpha_0 := \mathbf{A}(\mathfrak{M}, 0)$ satisfy the conditions of the lemma.

Towards showing the reverse direction assume that $\mathbb{A} := \{\alpha_0, \dots, \alpha_{|\mathbb{A}|-1}\}$ is as given in the statement of the lemma. We claim that the temporal interpretation \mathfrak{M} defined below satisfies the formula φ . Let $\mathbb{Z}_{<0}$ be the set of all integers smaller than 0 and let $\mathbb{Z}_{>|\mathbb{A}|}$ be the set of all integers greater than $|\mathbb{A}|$. Then for every variable $v \in \text{Vars}(\varphi)$, the set $\mathfrak{M}(v)$ contains the set $\{z \mid \alpha_z(v) = 1 \wedge 0 \leq z \leq |\mathbb{A}|\}$. Moreover, if $\alpha_0(v) = 1$, $\mathfrak{M}(v)$ also contains the set $\mathbb{Z}_{<0}$ and if $\alpha_{|\mathbb{A}|}(v) = 1$, $\mathfrak{M}(v)$ additionally contains the set $\mathbb{Z}_{>|\mathbb{A}|}$. It is easy to verify that $\mathfrak{M}, 0 \models \varphi$. □

Informally, the following lemma shows that for deciding the satisfiability of an LTL^{\boxtimes} formula, we only need to consider sets of assignments \mathbb{A} , whose size is linear (instead of exponential) in the number of variables.

Lemma 7 *Let $\varphi := \Psi \wedge \boxtimes \Phi \in \text{LTL}^{\boxtimes}$ and $X \subseteq \text{Vars}(\varphi)$. Then φ is satisfiable if and only if there is a set Θ of assignments of the variables in X , an assignment $\theta_0 \in \Theta$, a set \mathbb{A} of assignments of the variables in $\text{Vars}(\varphi)$, and an assignment $\alpha_0 \in \mathbb{A}$ such that:*

- (C1) *the set Θ is equal to $\mathbb{A}|_X$,*
- (C2) *the assignment θ_0 is equal to $\alpha_0|_X$,*
- (C3) *\mathbb{A} and α_0 satisfy the conditions stated in Lemma 6, and*
- (C4) *$|\mathbb{A}(\theta)| \leq |\text{Vars}(\varphi) \setminus X| + 1$ for every $\theta \in \Theta$.*

Proof Note that the reverse direction follows immediately from Lemma 6, because the existence of the set of assignments \mathbb{A} and the assignment α_0 satisfying condition (C3) imply the satisfiability of φ .

Towards showing the forward direction assume that φ is satisfiable. Because of Lemma 6 there is a set \mathbb{A} of assignments of the variables in φ and an assignment $\alpha_0 \in \mathbb{A}$ that satisfy the conditions of Lemma 6. Let Θ be equal to $\mathbb{A}|_X$ and θ_0 be equal to $\alpha_0|_X$. Observe that setting Θ and θ_0 in this way already satisfies (C1) to (C3). We will show that there is a subset of \mathbb{A} that still satisfies (C1)–(C3) and additionally (C4). Towards showing this consider any subset \mathbb{A}' of \mathbb{A} that satisfies the following three conditions: (1) $\alpha_0 \in \mathbb{A}'$, (2) for every $\theta \in \Theta$ it holds that $\mathbb{A}'(\theta) \neq \emptyset$, and (3) for every variable v of φ and every $b \in \{0, 1\}$ it holds that there is an assignment $\alpha \in \mathbb{A}$ with $\alpha(v) = b$ if and only if there is an assignment $\alpha' \in \mathbb{A}'$ with $\alpha'(v) = b$. Note that conditions (1) and (2) ensure that \mathbb{A}' satisfies (C1) and (C2) and condition (3) ensures (C3). Accordingly, any subset \mathbb{A}' satisfying conditions (1)–(3) still satisfies (C1)–(C3). It remains to show how to obtain such a subset \mathbb{A}' that additionally satisfies (C4). We define \mathbb{A}' as follows. Let \mathbb{A}'_0 be a subset of \mathbb{A} containing α_0 as well as one arbitrary assignment $\alpha \in \mathbb{A}(\theta)$ for every $\theta \in \Theta$. Note that \mathbb{A}'_0 already satisfies conditions (1) and (2) as well as condition (3) for every variable $v \in X$. Observe furthermore that if there is a variable v of φ such that condition (3) is violated by \mathbb{A}'_0 then it is sufficient to add at most one additional assignment to \mathbb{A}'_0 in order to satisfy condition (3) for v . Let \mathbb{A}' be obtained from \mathbb{A}'_0 by adding (at most $|\text{Vars}(\varphi) \setminus X|$) assignments in order to ensure condition (3) for every variable $v \in \text{Vars}(\varphi) \setminus X$. Then \mathbb{A}' satisfies the conditions of the lemma. \square

We are now ready to show the tractability of the evaluation of strong HORN-backdoor sets.

Theorem 8 $\text{Eval}^{\boxtimes}(\text{HORN})$ *is in FPT.*

Proof Let $\varphi := \Psi \wedge \boxtimes \Phi \in \text{LTL}^{\boxtimes}$ and let $X \subseteq \text{Vars}(\varphi)$ be a strong HORN-backdoor of φ . The main idea of the algorithm is as follows: For every set Θ of assignments of the variables in X and every $\theta_0 \in \Theta$, we will construct a propositional HORN-formula F_{Θ, θ_0} , which is satisfiable if and only if there is a set \mathbb{A} of assignments of the variables in $\text{Vars}(\varphi)$ and an assignment $\alpha_0 \in \mathbb{A}$ satisfying the conditions of Lemma 7. It then

follows from Lemma 7 that φ is satisfiable if and only if there is such a set Θ of assignments and an assignment $\theta_0 \in \Theta$ for which F_{Θ, θ_0} is satisfiable. Because there are at most $2^{2^{|X|}}$ such sets Θ and at most $2^{|X|}$ such assignments θ_0 and for each of these sets the formula F_{Θ, θ_0} is a HORN-formula, it follows that checking whether there are Θ and θ_0 such that the formula F_{Θ, θ_0} is satisfied (and as a result decide the satisfiability of φ) can be done in time $O(2^{2^{|X|}} \cdot 2^{|X|} \cdot |F_{\Theta, \theta_0}|)$. Since we will show below that the length of the formula F_{Θ, θ_0} can be bounded by an (exponential) function of $|X|$ times a polynomial in the input size, i.e., the length of the formula φ , this implies that $\text{Eval}^{\boxtimes}(\text{HORN})$ is in FPT.

The remainder of the proof is devoted to the construction of the formula F_{Θ, θ_0} for a fixed set of assignments Θ and a fixed assignment $\theta_0 \in \Theta$ (and to show that it enforces the conditions of Lemma 7).

Let $R := \text{Vars}(\varphi) \setminus X$ and $r := |R| + 1$. For a propositional formula F , a subset $V \subseteq \text{Vars}(F)$, an integer i and a label s , we denote by $\text{copy}(F, V, i, s)$ the propositional formula obtained from F after replacing each occurrence of a variable $v \in V$ with a novel variable v_s^i . We need the following auxiliary formulas. For every $\theta \in \Theta \setminus \{\theta_0\}$, let $F_{\Theta, \theta_0}^\theta$ be the formula (where the notation $\Phi[\mathbf{G}(\Theta, X, \theta)]$ refers to the formula that is obtained after applying the assignment $\mathbf{G}(\Theta, X, \theta)$ in the usual sense, that is, removing satisfied clauses and deleting falsified literals):

$$\bigwedge_{1 \leq i \leq r} \text{copy}(\text{CNF}(\Phi[\mathbf{G}(\Theta, X, \theta)]), R, i, \theta).$$

Moreover, let $F_{\Theta, \theta_0}^{\theta_0}$ be the formula:

$$\text{copy}(\Psi[\theta_0] \wedge \text{CNF}(\Phi[\mathbf{G}(\Theta, X, \theta_0)]), R, 1, \theta_0) \wedge \bigwedge_{2 \leq i \leq r} \text{copy}(\text{CNF}(\Phi[\mathbf{G}(\Theta, X, \theta_0)]), R, i, \theta_0).$$

Observe that because X is a strong HORN-backdoor set (and the formula Ψ only consists of unit clauses), it holds that the formula $F_{\Theta, \theta_0}^\theta$ is HORN for every $\theta \in \Theta$. We also need the propositional formula F_{cons} that enforces the consistency between the propositional variables $\boxtimes x$ and the variables in $\{x_\theta^i \mid \theta \in \Theta \wedge 1 \leq i \leq r\}$ for every $x \in \text{Vars}(\varphi) \setminus X$. The formula F_{cons} consists of the following clauses: for every $\theta \in \Theta, i$ with $1 \leq i \leq r$, and $v \in R$, the clause $\boxtimes v \rightarrow v_\theta^i = \neg \boxtimes v \vee v_\theta^i$ and for every $v \in R$ the clause

$$\neg \boxtimes v \rightarrow \bigvee_{\theta \in \Theta \wedge 1 \leq i \leq r} \neg v_\theta^i = \boxtimes v \vee \bigvee_{\theta \in \Theta \wedge 1 \leq i \leq r} \neg v_\theta^i.$$

Observe that F_{cons} is a HORN formula.

Finally the formula F_{Θ, θ_0} is defined as: $\bigwedge_{\theta \in \Theta} F_{\Theta, \theta_0}^\theta \wedge F_{\text{cons}}$.

Note that F_{Θ, θ_0} is HORN and the length of F_{Θ, θ_0} is at most

$$\begin{aligned}
 |F_{\Theta, \theta_0}| &\leq \sum_{\theta \in \Theta} |F_{\Theta, \theta}^\theta| + |F_{\text{cons}}| \\
 &\leq 2^{|X|} (|\text{Vars}(\varphi) \setminus X| + 1)(|\Phi| + |\Psi|) + 2 \cdot 2^{|X|} \cdot (|\text{Vars}(\varphi) \setminus X| + 1)^2
 \end{aligned}$$

and consequently bounded by a function of $|X|$ times a polynomial in the input size. It is now relatively straightforward to verify that $F_{\Theta, \theta}$ is satisfiable if and only if there is a set \mathbb{A} of assignments of the variables in $\text{Vars}(\varphi)$ and an assignment $\alpha_0 \in \mathbb{A}$ satisfying the conditions of Lemma 7. Informally, for every $\theta \in \Theta$, each of the r copies of the formula $\text{CNF}(\Phi[\mathbf{G}(\Theta, X, \theta)])$ represent one of the at most r assignments in $\mathbb{A}(\theta)$, the formula $F_{\Theta, \theta_0}^{\theta_0}$ ensures (among other things) that the assignment chosen for α_0 satisfies Ψ and the formula F_{cons} ensures that the “global assignments” represented by the propositional variables $\boxtimes x$ are consistent with the set of local assignments in \mathbb{A} represented by the variables in $\{x_\theta^i \mid \theta \in \Theta \wedge 1 \leq i \leq r\}$ for every $x \in \text{Vars}(\varphi) \setminus X$. □

Surprisingly, the next result will show that KROM formulas turn out to be quite challenging. Backdoor set evaluation of this class of formulas is proved to be paraNP-complete which witnesses an intractability degree in the parameterized sense.

Theorem 9 $\text{Eval}^{\boxtimes}(\text{KROM})$ is paraNP-complete (the NP-completeness already holds for backdoor sets of size two).

Proof The membership in paraNP follows because the satisfiability of $\text{LTL}_{\text{CNF}}^{\boxtimes}$ can be decided in NP [2, Table 1].

We show paraNP-hardness of $\text{Eval}^{\boxtimes}(\text{KROM})$ by giving a polynomial time reduction from the NP-hard problem 3COL to $\text{Eval}^{\boxtimes}(\text{KROM})$ for backdoors of size two. In 3COL one asks whether a given input graph $G = (V, E)$ has a coloring $f: V(G) \rightarrow \{1, 2, 3\}$ of its vertices with at most three colors such that $f(v) \neq f(u)$ for every edge $\{u, v\}$ of G . Given such a graph $G = (V, E)$, we will construct an $\text{LTL}_{\text{CNF}}^{\boxtimes}$ formula $\phi := \Psi \wedge \boxtimes \Phi$, which has a strong KROM-backdoor B of size two, such that the graph G has a 3-coloring if and only if ϕ is satisfiable.

For the remainder we will assume that there exists an arbitrary but fixed ordering of the vertices $V(G) = \{v_1, \dots, v_n\}$. Further for the construction we assume w.l.o.g. that any undirected edge $e = \{v_i, v_j\} \in E$ follows this ordering, i.e., $i < j$. The formula ϕ contains the following variables:

- (V1) The variables b_1 and b_2 . These variables make up the backdoor set B , i.e., $B := \{b_1, b_2\}$.
- (V2) For every i with $1 \leq i \leq n$, the variable v_i .
- (V3) For every $e = \{v_i, v_j\} \in E(G)$ with $1 \leq i, j \leq n$ the variables $e_{i,j}^{b_1 b_2}, e_{i,j}^{\bar{b}_1 b_2}$, and $e_{i,j}^{b_1 \bar{b}_2}$.

We set Ψ to be the empty formula and the formula Φ contains the following clauses:

- (C1) For every i with $1 \leq i \leq n$, the clause $\neg \boxtimes v_i$. Informally, this clause ensures that v_i has to be false at least at one world, which will later be used to assign a color to the vertex v_i of G . Observe that the clause is KROM.

Table 5 Given a graph $G = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\})$ together with a 3-coloring $f(v_i) = i$ for $1 \leq i \leq 3$, leads to the depicted temporal interpretation \mathfrak{M} satisfying $\mathfrak{M} \models \phi$ given as a table

	b_1	b_2	v_1	v_2	v_3	$e_{1,2}^{b_1b_2}$	$e_{1,2}^{\bar{b}_1\bar{b}_2}$	$e_{1,2}^{b_1\bar{b}_2}$	$e_{1,3}^{b_1b_2}$	$e_{1,3}^{\bar{b}_1\bar{b}_2}$	$e_{1,3}^{b_1\bar{b}_2}$	$e_{2,3}^{b_1b_2}$	$e_{2,3}^{\bar{b}_1\bar{b}_2}$	$e_{2,3}^{b_1\bar{b}_2}$
1	0	0	0	1	1	1	0	*	1	*	0	*	1	0
2	1	0	1	0	1	1	0	*	1	*	0	*	1	0
3	0	1	1	1	0	1	0	*	1	*	0	*	1	0

Each row of the table corresponds to a world as indicated by the first column of the table. Each column represents the assignments of a variable as indicated in the first row. A “*” indicates that the assignment is not fixed, i.e., the assignment does not influence whether $\mathfrak{M} \models \phi$

- (C2) For every $e = \{v_i, v_j\} \in E(G)$ with $1 \leq i, j \leq n$, the clauses $v_i \vee \boxed{*} e_{i,j}^{b_1b_2} \vee b_1 \vee b_2, v_i \vee \boxed{*} e_{i,j}^{\bar{b}_1\bar{b}_2} \vee \neg b_1 \vee \neg b_2$, and $v_i \vee \boxed{*} e_{i,j}^{b_1\bar{b}_2} \vee b_1 \vee \neg b_2$ as well as the clauses $v_j \vee \neg \boxed{*} e_{i,j}^{b_1b_2} \vee b_1 \vee b_2, v_j \vee \neg \boxed{*} e_{i,j}^{\bar{b}_1\bar{b}_2} \vee \neg b_1 \vee \neg b_2$, and $v_j \vee \neg \boxed{*} e_{i,j}^{b_1\bar{b}_2} \vee b_1 \vee \neg b_2$. Observe that all of these clauses are KROM after deleting the variables in B .
- (C3) The clause $\neg b_1 \vee \neg b_2$. Informally, this clause excludes the color represented by setting b_1 and b_2 to true. Observe that the clause is KROM.

It follows from the definition of ϕ that $\phi[\theta] \in \text{LTL}_{\text{KROM}}^{\boxed{*}}$ for every assignment θ of the variables in B . As a consequence, B is a strong KROM-backdoor of size two of ϕ as required. Moreover, since ϕ can be constructed in polynomial time, it only remains to show that G has a 3-coloring if and only if ϕ is satisfiable.

Towards showing the forward direction assume that G has a 3-coloring and let $f: V(G) \rightarrow \{1, 2, 3\}$ be such a 3-coloring for G . We will show that ϕ is satisfiable by constructing a temporal interpretation \mathfrak{M} such that $\mathfrak{M} \models \phi$. The interpretation \mathfrak{M} is defined as follows:

- For every i with $1 \leq i \leq n$, we set $\mathfrak{M}(v_i) = \mathbb{Z} \setminus \{f(v_i)\}$.
- We set $\mathfrak{M}(b_1) = \{2\}$ and $\mathfrak{M}(b_2) = \{3\}$.
- For every $e = \{v_i, v_j\} \in E(G)$:
 - if $f(v_i) = 1$ set $\mathfrak{M}(e_{i,j}^{b_1b_2}) = \mathbb{Z}$, else set $\mathfrak{M}(e_{i,j}^{b_1b_2}) = \emptyset$.
 - if $f(v_i) = 2$ set $\mathfrak{M}(e_{i,j}^{\bar{b}_1\bar{b}_2}) = \mathbb{Z}$, else set $\mathfrak{M}(e_{i,j}^{\bar{b}_1\bar{b}_2}) = \emptyset$.
 - if $f(v_i) = 3$ set $\mathfrak{M}(e_{i,j}^{b_1\bar{b}_2}) = \mathbb{Z}$, else set $\mathfrak{M}(e_{i,j}^{b_1\bar{b}_2}) = \emptyset$.

An example for such a temporal interpretation resulting for a simple graph is illustrated in Table 5. Towards showing that $\mathfrak{M} \models \phi$, we consider the different types of clauses given in (C1)–(C3).

- The clauses in (C1) hold because $\mathfrak{M}, f(v_i) \not\models v_i$ for every i with $1 \leq i \leq n$.
- For every $e = \{v_i, v_j\} \in E(G)$, we have to show that the clauses given in (C2) are satisfied for every world. Because f is a 3-coloring of G , we obtain that $f(v_i) \neq f(v_j)$. W.l.o.g. we assume in the following that $f(v_i) = 1$ and $f(v_j) = 2$. We first consider the clauses given in (C2) containing v_i . Because $\mathfrak{M}(v_i) = \mathbb{Z} \setminus \{1\}$, it only remains to consider the world 1. In this world b_1 and b_2 are false. It follows that all clauses containing either $\neg b_1$ or $\neg b_2$ are satisfied in this world. As a reason

for this, it only remains to consider clauses of the form $v_i \vee \boxtimes e_{i,j}^{b_1 b_2} \vee b_1 \vee b_2$. But these are satisfied because $f(v_i) = 1$ implies that $\mathfrak{M}(e_{i,j}^{b_1 b_2}) = \mathbb{Z}$.

Consider now the clauses given in (C2) that contain v_j . Using the same argumentation as used above for v_i , we obtain that we only need to consider world 2 and moreover we only need to consider clauses of the form $v_j \vee \neg \boxtimes e_{i,j}^{b_1 b_2} \vee \neg b_1 \vee b_2$.

Because $f(v_i) = 1$, we obtain that $\mathfrak{M}(e_{i,j}^{b_1 b_2}) = \emptyset$, which implies that these clauses are also satisfied.

- The clause $\neg b_1 \vee \neg b_2$ is trivially satisfied, because there is no world in which b_1 and b_2 hold simultaneously.

Towards showing the reverse direction assume that ϕ is satisfiable and let \mathfrak{M} be a temporal interpretation witnessing this. First note that because of the clauses added by C1, it holds that $\mathfrak{M}(v_i) \neq \mathbb{Z}$ for every i with $1 \leq i \leq n$. Let $w: V(G) \rightarrow \mathbb{Z}$ be defined such that for every i with $1 \leq i \leq n$, $w(v_i)$ is an arbitrary world in $\mathbb{Z} \setminus \mathfrak{M}(v_i)$. We define $f: V(G) \rightarrow \{1, 2, 3\}$ by setting:

- $f(v_i) = 1$ if $\mathfrak{M}, w(v_i) \not\models b_1 \vee b_2$,
- $f(v_i) = 2$ if $\mathfrak{M}, w(v_i) \not\models \neg b_1 \vee b_2$, and
- $f(v_i) = 3$ if $\mathfrak{M}, w(v_i) \not\models b_1 \vee \neg b_2$.

Note that because of the clause added by (C3), f assigns exactly one color to every vertex v_i of G . We claim that f is a 3-coloring of G . To show this it suffices to show that for every $e = \{v_i, v_j\} \in E(G)$, it holds that $f(v_i) \neq f(v_j)$. Assume for a contradiction that this is not the case, i.e., there is an edge $e = \{v_i, v_j\} \in E(G)$ such that $f(v_i) = f(v_j)$. W.l.o.g. assume furthermore that $f(v_i) = f(v_j) = 1$. Consider the clause $v_i \vee \boxtimes e_{i,j}^{b_1 b_2} \vee b_1 \vee b_2$ (which was added by C2). Then, because of the definition of w and f , we obtain that $\mathfrak{M}, w(v_i) \not\models v_i \vee b_1 \vee b_2$. It follows that $\mathfrak{M}, w(v_i) \models \boxtimes e_{i,j}^{b_1 b_2}$. Consider now the clause $v_j \vee \neg \boxtimes e_{i,j}^{b_1 b_2} \vee b_1 \vee b_2$ (which was added by C2). Then, again because of the choice of w and f , we obtain that $\mathfrak{M}, w(v_j) \not\models v_j \vee b_1 \vee b_2$. As a consequence, $\mathfrak{M}, w(v_j) \models \neg \boxtimes e_{i,j}^{b_1 b_2}$ contradicting $\mathfrak{M}, w(v_i) \models \boxtimes e_{i,j}^{b_1 b_2}$. This completes the proof of the theorem. □

5.2 Globally in the Past and Globally in the Future

Now we turn to a more flexible fragment where we can talk about the past as well as about the future and show it is possible to encode NP-complete problems into the HORN-fragment yielding a paraNP lower bound.

Theorem 10 $\text{Eval}^{\square_F, \square_P}(\text{HORN})$ is paraNP-complete (the NP-completeness already holds for backdoor sets of size four).

Proof The membership in paraNP follows as the satisfiability of $\text{LTL}_{\text{CNF}}^{\square_F, \square_P}$ can be decided in NP [2, Table 1].

We show paraNP-hardness of $\text{Eval}^{\square_F, \square_P}(\text{HORN})$ by describing a polynomial time reduction again from 3COL to $\text{Eval}^{\square_F, \square_P}(\text{HORN})$ for backdoors of size four. Recall that in 3COL one asks whether a given input graph $G = (V, E)$ has a coloring

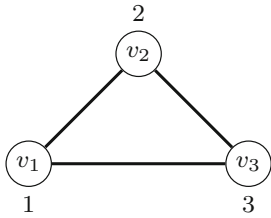
$f : V(G) \rightarrow \{1, 2, 3\}$ of its vertices with at most three colors such that $f(v) \neq f(u)$ for every edge $\{u, v\}$ of G . Given such a graph $G = (V, E)$, we will construct an $\text{LTL}_{\text{CNF}}^{\square_F, \square_P}$ formula $\phi := \Psi \wedge \boxtimes \Phi$, which has a strong HORN-backdoor B of size four, such that the graph G has a 3-coloring if and only if ϕ is satisfiable.

For the remainder we will assume that $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{e_1, \dots, e_m\}$. The formula ϕ contains the following variables:

- (V1) The variables c_1, c_2, c_3, p'_n . These variables make up the backdoor set B , i.e., $B := \{c_1, c_2, c_3, p'_n\}$.
- (V2) The variable s , which indicates the starting world.
- (V3) For every i with $1 \leq i \leq n$, three variables v_i^1, v_i^2, v_i^3 .
- (V4) For every i with $1 \leq i \leq n$ the variable p_i .

We set Ψ to be the formula s and the formula Φ contains the following clauses:

- (C1) The clauses $c_1 \vee c_2 \vee c_3, \neg c_1 \vee \neg c_2 \vee \neg c_3, c_1 \vee \neg c_2 \vee \neg c_3, \neg c_1 \vee \neg c_2 \vee c_3$, and $\neg c_1 \vee c_2 \vee \neg c_3$. Informally, these clauses ensure that in every world it holds that exactly one of the variables c_1, c_2, c_3 is true. Note that $c_1 \vee c_2 \vee c_3$ is not HORN, however, all of its variables are contained in the backdoor set B .
- (C2) For every i and c with $1 \leq i \leq n$ and $1 \leq c \leq 3$, the clauses $v_i^c \rightarrow \square_F v_i^c$ and $v_i^c \rightarrow \square_P v_i^c$; note that $v_i^c \rightarrow \square_F v_i^c$ corresponds to the clause $\neg v_i^c \vee \square_F v_i^c$. Informally, these clauses ensure that the variable v_i^c either holds in every world or in no world for every i and c as above. Observe that both of these clauses are HORN.
- (C3) Informally, the following set of clauses ensures together that for every i with $1 \leq i \leq n$, it holds that p_i is true in every world apart from the i -th world (where p_i is false). Here, the first world is assumed to be the starting world.
 - (C3-1) The clauses $s \rightarrow \neg p_1, s \rightarrow \square_F p_1$, and $s \rightarrow \square_P p_1$. Informally, these ensure that p_1 is only false in the starting world (and otherwise true).
 - (C3-2) The clause $p_i \wedge \square_F p_i \rightarrow \square_F p_{i+1}$ for every i with $1 \leq i < n$. Informally, these clauses (together with the clauses from C3-1) ensure that for every i with $2 \leq i \leq n$, it holds that p_i is true in every world after the i -th world.
 - (C3-3) The clause $\neg p_i \rightarrow \neg \square_F p_{i+1}$ for every i with $1 \leq i < n$. Informally, these clauses (together with the clauses from C3-1 and C3-2) ensure that for every i with $2 \leq i \leq n$, it holds that p_i is false at the i -th world. Observe that the clauses from C3-1 to C3-3 already ensure that $\neg p_i \wedge \square_F p_i$ holds if and only if we are at the i -th world of the model for every i with $1 \leq i \leq n$.
 - (C3-4) The clauses $\neg p_n \wedge \square_F p_n \rightarrow p'_n$ and $\neg p_n \wedge \square_F p_n \leftarrow p'_n = \neg p_n \wedge \square_F p_n \vee \neg p'_n = (\neg p_n \vee \neg p'_n) \wedge (\square_F p_n \vee \neg p'_n)$. Informally, these clauses (together with the clauses from C3-1 to C3-3) ensure that p'_n only holds in the n -th world of the model. Observe that all these clauses are HORN after removing the backdoor set variable p'_n .
 - (C3-5) The clause $p'_n \rightarrow \square_P p_n$. Informally, this clause (together with the clauses from C3-1 to C3-4) ensures that p_n is only false in the n -th world of the model.
 - (C3-6) The clause $p_i \wedge \square_P p_i \rightarrow \square_P p_{i-1}$ for every i with $2 \leq i \leq n$. Informally, these clauses (together with the clauses from C3-1 to C3-5) ensure that p_i is true before the i -th world for every i with $2 \leq i < n$.



	s	c_1	c_2	c_3	p'_n	v_1^1	v_1^2	v_1^3	v_2^1	v_2^2	v_2^3	v_3^1	v_3^2	v_3^3	p_1	p_2	p_3
< 1	0	*	*	*	0	1	0	0	0	1	0	0	0	1	1	1	1
1	1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	1	1
2	0	0	1	0	0	1	0	0	0	1	0	0	0	1	1	0	1
3	0	0	0	1	1	1	0	0	0	1	0	0	0	1	1	1	0
> 3	0	*	*	*	0	1	0	0	0	1	0	0	0	1	1	1	1

Fig. 1 Left: A graph G with vertices $v_1, v_2,$ and v_3 together with a 3-coloring given by the numbers above and below respectively of every vertex. Right: A temporal interpretation \mathfrak{M} that corresponds to the given 3-coloring of G and satisfies $\mathfrak{M} \models \phi$ given as a table. Each row of the table corresponds to a world (or a set of worlds) as indicated by the first column of the table. Each column represents the assignments of a variable as indicated in the first row. A “*” indicates that the assignment is not fixed, i.e., the assignment does not influence whether $\mathfrak{M} \models \phi$

Observe that all of the above clauses are HORN or become HORN after removing all variables from B . Note furthermore that all the above clauses ensure that $\Box_P p_i \wedge \Box_F p_i$ holds if and only if we are at the i -th world of the model for every i with $1 \leq i \leq n$.

- (C4) For every i and j with $1 \leq i \leq n$ and $1 \leq j \leq 3$ the clauses $\Box_F p_i \wedge \Box_P p_i \wedge v_i^j \rightarrow c_j$ and $\Box_F p_i \wedge \Box_P p_i \wedge c_j \rightarrow v_i^j$. Informally, these clauses ensure that in the i -th world for every $1 \leq i \leq n$, the variables c_1, c_2, c_3 are a copy of the variables v_i^1, v_i^2, v_i^3 . Observe that all of these clauses are HORN.
- (C5) For every edge $e = \{v_i, v_j\} \in E(G)$ and every c with $1 \leq c \leq 3$, the clause $\neg v_i^c \vee \neg v_j^c$. Informally, these clauses ensure that the 3-partition (of the vertices of G) given by the (global) values of the variables $v_1^1, v_1^2, v_1^3, \dots, v_n^1, v_n^2, v_n^3$ is a valid 3-coloring for G . Observe that all of these clauses are HORN.

It follows from the definition of ϕ that $\phi[\theta] \in \text{LTL}_{\text{HORN}}^{\Box_F, \Box_P}$ for every assignment θ of the variables in B . Consequently, B is a strong HORN-backdoor of size four of ϕ as required. Moreover, since ϕ can be constructed in polynomial time, it only remains to show that G has a 3-coloring if and only if ϕ is satisfiable.

Towards showing the forward direction assume that G has a 3-coloring and let $f: V(G) \rightarrow \{1, 2, 3\}$ be such a 3-coloring for G . We will show that ϕ is satisfiable by constructing a temporal interpretation \mathfrak{M} such that $\mathfrak{M} \models \phi$. \mathfrak{M} is defined as follows:

- For every j with $1 \leq j \leq 3$, we set $\mathfrak{M}(c_j) = \{i \mid f(v_i) = j\}$.
- We set $\mathfrak{M}(p'_n) = \{n\}$.
- For every i and c with $1 \leq i \leq n$ and $1 \leq c \leq 3$, we set $\mathfrak{M}(v_i^c) = \mathbb{Z}$ if $c = f(v_i)$ and otherwise we set $\mathfrak{M}(v_i^c) = \emptyset$.
- For every i with $1 \leq i \leq n$, we set $\mathfrak{M}(p_i) = \mathbb{Z} \setminus \{i\}$.

An example for such a temporal interpretation resulting for a simple graph is illustrated in Figure 1. It is straightforward (but a little tedious) to verify that $\mathfrak{M} \models \phi$ by considering all the clauses of ϕ .

Towards showing the reverse direction assume that ϕ is satisfiable and let \mathfrak{M} be a temporal interpretation witnessing this. We will start by showing the following series of claims for \mathfrak{M} .

- (M1) For every $a \in \mathbb{Z}$ exactly one of $\mathfrak{M}, a \models c_1, \mathfrak{M}, a \models c_2,$ and $\mathfrak{M}, a \models c_3$ holds.
- (M2) For every $i, c, a,$ and a' with $1 \leq i \leq n, 1 \leq c \leq 3,$ and $a, a' \in \mathbb{Z},$ it holds that $\mathfrak{M}, a \models v_i^c$ if and only if $\mathfrak{M}, a' \models v_i^c.$
- (M3) For every i with $1 \leq i \leq n$ and every $a \in \mathbb{Z},$ it holds that $\mathfrak{M}, a \models p_i$ if and only if $a \neq i.$
- (M4) For every i and j with $1 \leq i \leq n$ and $1 \leq j \leq 3,$ it holds that $\mathfrak{M}, i \models c_j$ if and only if $\mathfrak{M}, i \models v_i^j.$

(M1) holds because of the clauses added by (C1). Towards showing (M2) consider the clauses added by (C2) and assume for a contradiction that there are $i, c, a,$ and a' as in the statement of (M2) such that w.l.o.g. $\mathfrak{M}, a \models v_i^c$ but $\mathfrak{M}, a' \not\models v_i^c.$ Then, $a \neq a'.$ If $a < a',$ then we obtain a contradiction because of the clause $v_i^c \rightarrow \Box_F v_i^c$ and if on the other hand $a' < a,$ we obtain a contradiction to the clause $v_i^c \rightarrow \Box_P v_i^c.$ This completes the proof of (M2). Considering the explanations for the clauses the proof of (M3) is now reasonably straightforward, however, for completeness we now provide a detailed proof. We will show (M3) with the help of the following series of claims.

- (M3-1) For every $a \in \mathbb{Z}$ it holds that $\mathfrak{M}, a \models p_1$ if and only if $a \neq 1$ (here we assume that 1 is the starting world).
- (M3-2) For every i and a with $1 \leq i \leq n, a \in \mathbb{Z},$ and $a > i,$ it holds that $\mathfrak{M}, a \models p_i.$
- (M3-3) For every i with $1 \leq i \leq n,$ it holds that $\mathfrak{M}, i \not\models p_i.$
- (M3-4) For every $a \in \mathbb{Z},$ it holds that $\mathfrak{M}, a \models p'_n$ if and only if $a = n.$
- (M3-5) For every $a \in \mathbb{Z},$ it holds that $\mathfrak{M}, a \not\models p_n$ if and only if $a = n.$

Because of the clause $s \rightarrow \neg p_1$ (added by C3-1) and the fact that $s \in \Psi,$ we obtain that $\mathfrak{M}, 1 \not\models p_1.$ Moreover, because of the clauses $s \rightarrow \Box_F p_1$ and $s \rightarrow \Box_P p_1,$ we obtain that $\mathfrak{M}, a \models p_1$ for every $a \neq 1.$ This completes the proof for (M3-1).

We show (M3-2) via induction on $i.$ The claim clearly holds for $i = 1$ because of (M3-1). Now assume that the claim holds for p_{i-1} and we want to show it for $p_i.$ Because of the induction hypothesis, we obtain that $\mathfrak{M}, i \models p_{i-1} \wedge \Box_F p_{i-1}.$ Moreover, because ϕ contains the clause $p_{i-1} \wedge \Box_F p_{i-1} \rightarrow \Box_F p_i$ (which was added by (C3-2)), we obtain that $\mathfrak{M}, i \models \Box_F p_i.$ This completes the proof of (M3-2).

We show (M3-3) via induction on $i.$ The claim clearly holds for $i = 1$ because of (M3-1). Now assume that the claim holds for p_{i-1} and we want to show it for $p_i.$ Because of the induction hypothesis, we obtain that $\mathfrak{M}, (i - 1) \not\models p_{i-1}.$ Furthermore, because of (M3-2), we know that $\mathfrak{M}, i \models \Box_F p_i.$ Since ϕ contains the clause $\neg p_{i-1} \rightarrow \neg \Box_F p_i$ (which was added by (C3-3)), we obtain $\mathfrak{M}, (i - 1) \models \neg \Box_F p_i,$ which because $\mathfrak{M}, i \models \Box_F p_i$ can only hold if $\mathfrak{M}, i \not\models p_i.$ This completes the proof of (M3-3).

Towards showing (M3-4), first note that because of (M3-2) and (M3-3), we have that $\mathfrak{M}, a \models \neg p_n \wedge \Box_F p_n$ if and only if $a = n.$ Then, because of the clauses (added by C3-4) ensuring that $\neg p_n \wedge \Box_F p_n \leftrightarrow p'_n,$ the same applies to p'_n (instead of $\neg p_n \wedge \Box_F p_n).$ This completes the proof of (M3-4).

It follows from (M3-2) and (M3-3) that (M3-5) holds for every $a \in \mathbb{Z}$ with $a \geq n.$ Moreover, because of (M3-4), we have that $\mathfrak{M}, n \models p'_i.$ Because of the clause $p'_n \rightarrow \Box_P p_n$ (which was added by (C3-5)), we obtain $\mathfrak{M}, a \models p_n$ for every $a < n.$ This completes the proof of (M3-5).

We are now ready to prove (M3). It follows from (M3-2) and (M3-3) that (M3) holds for every i and a with $a \geq i$. Furthermore, we obtain from (M3-5) that (M3) already holds if $i = n$. We complete the proof of (M3) via an induction on i starting from $i = n$. Because of the induction hypothesis, we obtain that $\mathfrak{M}, i + 1 \models p_{i+1} \wedge \Box_P p_{i+1}$. Accordingly, because of the clause $p_{i+1} \wedge \Box_P p_{i+1} \rightarrow \Box_P p_i$ (added by (C3-6)), we obtain that $\mathfrak{M}, i + 1 \models \Box_P p_i$, which completes the proof of (M3).

Towards showing (M4) first note that it follows from (M3) that $\mathfrak{M}, i \models \Box_F p_i \wedge \Box_P p_i$. Now suppose that there are i and j such that either $\mathfrak{M}, i \models c_j$ but $\mathfrak{M}, i \not\models v_i^j$ or $\mathfrak{M}, i \not\models c_j$ but $\mathfrak{M}, i \models v_i^j$. In the former case, consider the clause $\Box_F p_i \wedge \Box_P p_i \wedge c_j \rightarrow v_i^j$ (which was added by (C4)). Since $\mathfrak{M}, i \models \Box_F p_i \wedge \Box_P p_i$, we obtain that $\mathfrak{M}, i \models v_i^j$; a contradiction. In the later case, consider the clause $\Box_F p_i \wedge \Box_P p_i \wedge v_i^j \rightarrow c_j$ (which was added by (C4)). Since $\mathfrak{M}, i \models \Box_F p_i \wedge \Box_P p_i$, we obtain that $\mathfrak{M}, i \models c_j$; again a contradiction. This completes the proof of the claims (M1)–(M4).

It follows from (M1) and (M4) that for every i and a with $1 \leq i \leq n$ and $a \in \mathbb{Z}$ there is exactly one c with $1 \leq c \leq 3$, such that $\mathfrak{M}, a \models v_i^c$. Moreover, because of (M2) the choice of c is independent of a . Accordingly, the coloring f that assigns the unique color c to every vertex v_i such that $\mathfrak{M}, a \models v_i^c$ forms a partition of the vertex set of G . Also f is a valid 3-coloring because for every $\{v_i, v_j\} \in E(G)$ it holds that $\mathfrak{M}, a \not\models \neg v_i^c \vee \neg v_j^c$ for every $a \in \mathbb{Z}$ (using the clause added by C5) and hence v_i and v_j must be assigned distinct colors by f . □

Corollary 11 *Let $O \in \{\Box_F, \Box_P\}$ then $\text{Eval}^O(\text{KROM})$ is paraNP-complete (the NP-completeness already holds for backdoor sets of size zero).*

Proof Satisfiability of $\text{LTL}_{\text{KROM}}^O$ is NP-hard [2, Theorem 5]. □

6 Conclusion and Discussion

We lift the well-known concept of backdoor sets from propositional logic up to the clausal fragment of linear temporal logic LTL. From the investigated cases we obtain a comprehensive picture of the parameterized complexity for the problem of backdoor set evaluation. The evaluation parameterized by the size of the backdoor into KROM formulas becomes in all cases paraNP-complete and as a result is unlikely to be solvable in FPT whereas the case of backdoor evaluation into the fragment HORN behaves differently. While allowing only \boxtimes makes the problem fixed-parameter tractable, allowing both, \Box_F and \Box_P , makes it paraNP-complete. The last open case, i.e., the restriction to either \Box_F or \Box_P is open for further research and might yield an FPT result. We want to note here that all of our results still hold if LTL is evaluated over the natural numbers instead of the integers.

Satisfiability of $\text{LTL}_{\text{CNF}}^{\boxtimes}$ is NP-complete, for HORN/KROM it is in P/NL [2]. With the help of our backdoor notion, we achieved for a HORN-backdoor an FPT membership. However, for KROM this surprisingly was not possible (paraNP-c., Theorem 9). For the “full global” fragment only for HORN satisfiability is in P and for KROM it is NP-complete [2]. Here in both cases, our notion of backdoors was not fruitful. This is, however, natural since applying the backdoor approach to a novel problem is never a

simple nor straightforward task. We see our work as a first attempt to come up with such a notion for LTL, and, given the notorious difficulty of the LTL-satisfiability problem, we believe our tractability result for LTL formulas restricted to the always operator that are almost HORN is an encouraging result that justifies further investigation of this approach. As mentioned earlier, LTL restricted to the always operator, is already pretty interesting, since it allows one to express “Safety” properties of a system (e.g., $\Box(\neg x)$, where x encodes something bad to happen). Also, see the work of Kupferman and Vardi on this topic [24]. Moreover, our intractability results for the remaining fragments of LTL indicate that a different notion of “closeness” is required to obtain tractability results for these fragments.

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