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# A NOVEL PROGRESSIVE GAUSSIAN APPROXIMATE FILTER WITH VARIABLE STEP SIZE BASED ON A VARIATIONAL BAYESIAN APPROACH

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## ABSTRACT

The selection of step sizes in the progressive Gaussian approximate filter (PGAF) is important, and it is difficult to select optimal values in practical applications. Furthermore, in the PGAF, significant integral approximation errors are generated by the repeated approximate calculations of the Gaussian weighted integrals, which results in an inaccurate measurement noise covariance matrix (MNCM). To solve these problems, in this paper, the step sizes and the MNCM are jointly estimated based on the variational Bayesian (VB) approach. By incorporating the adaptive estimates of step sizes and the MNCM into the PGAF framework, a novel PGAF with variable step size is proposed. Simulation results illustrate that the proposed filter has higher estimation accuracy than existing state-of-the-art nonlinear Gaussian approximate filters.

***Index Terms***— Gaussian approximate filter, progressive measurement update, variable step size, variational Bayesian

## 1. INTRODUCTION

Gaussian approximate filters (GAFs) are the most common approach in various nonlinear applications since they can provide a better compromise between the computational cost and the estimation accuracy [1]–[3]. In the framework of GAFs, the posterior PDF is approximated as Gaussian, and a multitude of GAFs have been developed based on various numerical integral techniques [3]–[9]. However, the performance of these standard GAFs may be degraded in applications with large prior uncertainty but high measurement accuracy [10].

One way to handle this problem is the employment of the PGAF, which involves the gradual introduction of measurement information instead of absorbing all the measurements at one time [11], [12]. PGAFs are derived in [11] and [12], but their performances may strongly rely on the number of samples since the continuous PDFs are discretized based on the deterministic Dirac mixture approximation approach. A recursive extended Kalman filter (EKF) is derived in [13], yet it has limited estimation accuracy due to the use of first order linearization. Under the Bayesian estimation framework, in [14], an improved PGAF has been proposed and then developed into a general framework. The associated measurement likelihood is evolved gradually and the intermediate progressive joint PDFs of the state and measurement are approximated as Gaussian, which leads to a higher estimation accuracy [14]. However, due to the lack of powerful theoretical foundations and being highly dependent upon engineering experience when selecting the optimal step sizes, the practicality of the improved PGAF will be normally degraded. Apart from that, integral approximation errors will be generated due to the repeated approximate calculations of the Gaussian weighted integrals, which results in an inaccurate MNCM.

To address the aforementioned problems, a novel PGAF with variable step size and adaptive MNCM is proposed in this paper, where the state together with step sizes and inaccurate MNCM are jointly inferred based on the VB approach. The posterior PDFs of state, step sizes and MNCM are, respectively, updated as Gaussian, truncated Gamma and inverse Wishart distributions. Simulation results of bearing only tracking illustrate that the proposed filter has higher estimation accuracy than existing state-of-the-art nonlinear GAFs.

## 2. A NOVEL PGAF WITH VARIABLE STEP SIZE

Consider the following nonlinear stochastic state-space model

$$\begin{cases} \mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}) + \mathbf{w}_{k-1} \\ \mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k \end{cases} \tag{1}$$

where $k$ is the discrete time index, $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, $\mathbf{z}_k \in \mathbb{R}^m$ is the measurement vector, $\mathbf{w}_k \in \mathbb{R}^n$ and $\mathbf{v}_k \in \mathbb{R}^m$ are independent zero-mean Gaussian white noise vectors satisfying $\mathbf{E}[\mathbf{w}_k\mathbf{w}_l^T] = \mathbf{Q}_k\delta_{kl}$ and $\mathbf{E}[\mathbf{v}_k\mathbf{v}_l^T] = \mathbf{R}_k\delta_{kl}$ respectively, where $\delta_{kl}$ is the Kronecker delta function. The initial state $\mathbf{x}_0$ is a Gaussian random vector with mean $\hat{\mathbf{x}}_{0|0}$ and covariance matrix $\mathbf{P}_{0|0}$, and it is independent from $\mathbf{w}_k$ and $\mathbf{v}_k$.

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In the PGAF, the measurement likelihood is evolved gradually and the posterior PDF can be formulated as follows [14]

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) \propto p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) \prod_{\gamma=1}^{N} [P(\mathbf{z}_k|\mathbf{x}_k^\gamma)]^{\lambda_\gamma} \tag{2}$$

where $\mathbf{z}_{1:k} = \{\mathbf{z}_j\}_{j=1}^k$, $\sum_{\gamma=1}^N \lambda_\gamma = 1$, $\lambda_\gamma \in (0, 1]$ denotes the progressive step size. $\mathbf{x}_k^\gamma$ denotes the state at the $\gamma$th recursion, and $N$ denotes the number of product terms in the progressive formulation.

Under the framework of progressive measurement update, the intermediate one-step predicted PDF $p(\mathbf{x}_k^\gamma|\mathbf{z}_{1:k}^{\gamma-1})$ and likelihood PDF $p(\mathbf{z}_k|\mathbf{x}_k^\gamma)$ can be written as Gaussian, i.e.

$$p(\mathbf{x}_k^\gamma|\mathbf{z}_{1:k}^{\gamma-1}) = \mathrm{N}(\mathbf{x}_k^\gamma; \hat{\mathbf{x}}_{k|k}^{\gamma-1}, \mathbf{P}_{k|k}^{\gamma-1}) \tag{3}$$

$$p(\mathbf{z}_k|\mathbf{x}_k^\gamma) = \mathrm{N}(\mathbf{z}_k; \mathbf{h}_k(\mathbf{x}_k^\gamma), \mathbf{R}_k^\gamma/\lambda_\gamma) \tag{4}$$

where $\mathrm{N}(\cdot; \mu, \mathbf{\Sigma})$ denotes a Gaussian PDF with mean $\mu$ and covariance matrix $\mathbf{\Sigma}$. $\hat{\mathbf{x}}_{k|k}^\gamma$ denotes the estimate of $\mathbf{x}_k^\gamma$, and $\mathbf{z}_{1:k}^\gamma$ denotes the gradually absorbed measurements sequence. $\mathbf{P}_{k|k}^\gamma$ denotes the estimated error covariance matrix at the $\gamma$th recursion.

The state $\mathbf{x}_k^\gamma$ together with the step size $\lambda_\gamma$ and the inaccurate MNCM $\mathbf{R}_k^\gamma$ are, respectively, assumed to satisfy Gaussian, uniform and inverse Wishart priors, i.e.

$$\begin{aligned} p(\mathbf{x}_k^\gamma, \lambda_\gamma, \mathbf{R}_k^\gamma|\mathbf{z}_{1:k}^{\gamma-1}) \approx & \mathrm{N}(\mathbf{x}_k^\gamma; \hat{\mathbf{x}}_{k|k}^{\gamma-1}, \mathbf{p}_{k|k}^{\gamma-1})\mathrm{U}(\lambda_\gamma; 0, \eta_\gamma) \\ & \times \mathrm{IW}(\mathbf{R}_k^\gamma; u_{k|k-1}^\gamma, \mathbf{U}_{k|k-1}^\gamma) \end{aligned} \tag{5}$$

where $\mathrm{U}(\cdot; m, n)$ represents the uniform PDF over the interval $[m, n]$, and $\mathrm{IW}(\cdot; u, \mathbf{U})$ is the inverse Wishart PDF with degree of freedom (dof) parameter $u$ and inverse scale matrix $\mathbf{U}$. Meanwhile $\eta_\gamma$ satisfies $\eta_\gamma = 1 - \sum_{j=1}^{\gamma-1} \lambda_j$.

To estimate the state $\mathbf{x}_k^\gamma$ together with step size $\lambda_\gamma$ and the inaccurate MNCM $\mathbf{R}_k^\gamma$, the joint posterior PDF $p(\mathbf{x}_k^\gamma, \lambda_\gamma, \mathbf{R}_k^\gamma|\mathbf{z}_{1:k})$ needs to be first computed. However, there is not an analytical solution for it, thus the VB approach is utilized to hunt for a free form factored approximate posterior PDF, i.e.

$$p(\mathbf{x}_k^\gamma, \lambda_\gamma, \mathbf{R}_k^\gamma|\mathbf{z}_{1:k}) \approx q(\mathbf{x}_k^\gamma)q(\lambda_\gamma)q(\mathbf{R}_k^\gamma) \tag{6}$$

where $q(\cdot)$ represents the approximate posterior PDF, and $q(\mathbf{x}_k^\gamma), q(\lambda_\gamma), q(\mathbf{R}_k^\gamma)$ are solved by minimizing the Kullback–Leibler divergence (KLD) between the factored approximate posterior PDF $q(\mathbf{x}_k^\gamma)q(\lambda_\gamma)q(\mathbf{R}_k^\gamma)$ and the real joint posterior PDF $p(\mathbf{x}_k^\gamma, \lambda_\gamma, \mathbf{R}_k^\gamma|\mathbf{z}_{1:k})$ [16], [17]. The optimal solution satisfies the following equation

$$\log q(\theta) = \mathrm{E}_{(\mathbf{\Xi}-\theta)}[\log p(\mathbf{\Xi}, \mathbf{z}_{1:k})] + c_\theta \tag{7}$$

where $\mathbf{\Xi}$ is defined as $\mathbf{\Xi} \triangleq \{\mathbf{x}_k^\gamma, \lambda_\gamma, \mathbf{R}_k^\gamma\}$; and $\mathrm{E}[\cdot]$ denotes the expectation operation; $\log(\cdot)$ represents the logarithmic operation; $\theta$ is an arbitrary element of $\mathbf{\Xi}$, and $\mathbf{\Xi} - \theta$ means

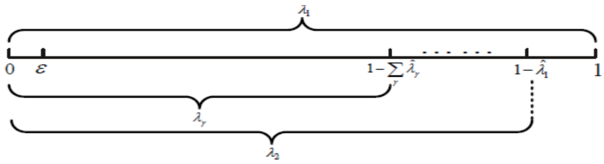


**Fig. 1**: The process of the variable step sizes progressive measurement update

the set of all elements in $\mathbf{\Xi}$ except for $\theta$, and $c_\theta$ is the constant with respect to $\theta$. Since the variational parameters of $q(\mathbf{x}_k^\gamma)$, $q(\lambda_\gamma)$, $q(\mathbf{R}_k^\gamma)$ are coupled, fixed-point iterations are employed to solve (7) [17].

### 2.1. Measurement update

The process of the variable step sizes progressive measurement update is depicted as Fig. 1. The measurement is involved gradually by the step size $\lambda_\gamma$, which will be estimated in the following, before reaching the ultimate threshold $\varepsilon$.

Based on the Bayesian theorem, the joint PDF $p(\mathbf{\Xi}, \mathbf{z}_{1:k})$ can be written as follows

$$\begin{aligned} p(\mathbf{\Xi}, \mathbf{z}_{1:k}) = & c(\gamma)\mathrm{N}(\mathbf{z}_k; \mathbf{h}_k(\mathbf{x}_k^\gamma), \mathbf{R}_k^\gamma/\lambda_\gamma)\mathrm{N}(\mathbf{x}_k^\gamma; \hat{\mathbf{x}}_{k|k}^{\gamma-1}, \mathbf{p}_{k|k}^{\gamma-1}) \\ & \times \mathrm{U}(\lambda_\gamma; 0, \eta_\gamma)\mathrm{IW}(\mathbf{R}_k^\gamma; u_{k|k-1}^\gamma, \mathbf{U}_{k|k-1}^\gamma) \end{aligned} \tag{8}$$

where $c(\gamma)$ is the normalization constant. Exploiting (8) and performing a logarithmic operation on it, we obtain

$$\begin{aligned} \log p(\mathbf{\Xi}, \mathbf{z}_{1:k}) = & 0.5m \log \lambda_\gamma - 0.5(\mathbf{x}_k^\gamma - \hat{\mathbf{x}}_{k|k}^{\gamma-1})^T(\mathbf{P}_{k|k}^{\gamma-1})^{-1} \\ & \times (\mathbf{x}_k^\gamma - \hat{\mathbf{x}}_{k|k}^{\gamma-1}) - 0.5(u_{k|k-1}^\gamma + m + 2)\log|\mathbf{R}_k^\gamma| - 0.5\lambda_\gamma \\ & \times (\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k^\gamma))^T(\mathbf{R}_k^\gamma)^{-1}(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k^\gamma)) \\ & - 0.5\mathrm{tr}(\mathbf{U}_{k|k-1}^\gamma(\mathbf{R}_k^\gamma)^{-1}) + c_{\mathbf{\Xi}} \end{aligned} \tag{9}$$

where $c_{\mathbf{\Xi}}$ is the constant with respect to $\mathbf{\Xi}$.

Let $\theta = \lambda_\gamma$, and utilizing (9) in (7), we obtain

$$\log q^{i+1}(\lambda_\gamma) = 0.5m \log \lambda_\gamma - 0.5\lambda_\gamma \mathrm{tr}(\mathbf{D}_k^{\gamma(i)}\mathrm{E}^i[(\mathbf{R}_k^\gamma)^{-1}]) + c_\lambda \tag{10}$$

where $q^{i+1}(\cdot)$ is the approximate PDF of $q(\cdot)$ at the $i+1$th iteration, and $\mathbf{D}_k^{\gamma(i)}$ can be formulated as

$$\mathbf{D}_k^{\gamma(i)} = \mathrm{E}^i[(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k^\gamma))(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k^\gamma))^T] \tag{11}$$

Thus $q^{i+1}(\lambda_\gamma)$ can be updated as

$$q^{i+1}(\lambda_\gamma) = \mathrm{G}(\lambda_\gamma; \alpha^{i+1}, \beta^{i+1}) \tag{12}$$

where $\mathrm{G}(\cdot; \alpha, \beta)$ represents the Gamma PDF with shape parameter $\alpha$ and rate parameter $\beta$. Note that the Gamma PDF in this paper is truncated and has definitions only over the interval $(0, 1]$, and $\alpha^{i+1}$ and $\beta^{i+1}$ can be given by

$$\alpha^{i+1} = 0.5m + 1 \tag{13}$$

$$\beta^{i+1} = 0.5\text{tr}(\mathbf{D}_k^{\gamma(i)}\text{E}^i[(\mathbf{R}_k^\gamma)^{-1}]) \tag{14}$$

Let $\theta = \mathbf{R}_k^\gamma$, and utilizing (9) in (7), we obtain

$$\log q^{i+1}(\mathbf{R}_k^\gamma) = -0.5(u_{k|k-1}^\gamma + m + 2)\log|\mathbf{R}_k^\gamma|$$
$$- 0.5\text{tr}[(\text{E}^{i+1}[\lambda^\gamma]\mathbf{D}_k^{\gamma(i)} + \mathbf{U}_{k|k-1}^\gamma)(\mathbf{R}_k^\gamma)^{-1}] + c_R \tag{15}$$

Thus $q^{i+1}(\mathbf{R}_k^\gamma)$ can be updated as

$$q^{i+1}(\mathbf{R}_k^\gamma) = \text{IW}(\mathbf{R}_k^\gamma; u_k^{\gamma(i+1)}, \mathbf{U}_k^{\gamma(i+1)}) \tag{16}$$

where the dof parameter $u_k^{\gamma(i+1)}$ and inverse scale matrix $\mathbf{U}_k^{\gamma(i+1)}$ are given by

$$u_k^{\gamma(i+1)} = u_{k|k-1}^\gamma + 1 \tag{17}$$

$$\mathbf{U}_k^{\gamma(i+1)} = \text{E}^{i+1}[\lambda^\gamma]\mathbf{D}_k^{\gamma(i)} + \mathbf{U}_{k|k-1}^\gamma \tag{18}$$

Let $\theta = \mathbf{x}_k^\gamma$, and utilizing (9) in (7), we obtain

$$\log q^{i+1}(\mathbf{x}_k^\gamma) = -0.5(\mathbf{x}_k^\gamma - \hat{\mathbf{x}}_{k|k}^{\gamma-1})^T(\mathbf{P}_{k|k}^{\gamma-1})^{-1}(\mathbf{x}_k^\gamma - \hat{\mathbf{x}}_{k|k}^{\gamma-1})$$
$$- 0.5(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k^\gamma))^T(\tilde{\mathbf{R}}_k^\gamma)^{-1}(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k^\gamma)) + c_x \tag{19}$$

where

$$\tilde{\mathbf{R}}_k^\gamma = \text{E}^{i+1}[\mathbf{R}_k^\gamma]/\text{E}^{i+1}[\lambda^\gamma] \tag{20}$$

According to a property of the inverse Wishart distribution, $\text{E}^{i+1}[\mathbf{R}_k^\gamma]$ can be computed as [18]

$$\text{E}^{i+1}[\mathbf{R}_k^\gamma] = \mathbf{U}_k^{\gamma(i+1)}/(u_k^{\gamma(i+1)} - m - 1) \tag{21}$$

Since the PDF $q^{i+1}(\lambda^\gamma)$ obeys a truncated Gamma PDF, $\text{E}^{i+1}[\lambda^\gamma]$ is calculated by the definition of the integral. Using (19) - (21), $q^{i+1}(\mathbf{x}_k^\gamma)$ can be formulated as

$$q^{i+1}(\mathbf{x}_k^\gamma) = \frac{\text{N}(\mathbf{z}_k; \mathbf{h}_k(\mathbf{x}_k^\gamma), \tilde{\mathbf{R}}_k^\gamma)\text{N}(\mathbf{x}_k^\gamma; \hat{\mathbf{x}}_{k|k}^{\gamma-1}, \mathbf{P}_{k|k}^{\gamma-1})}{\int \text{N}(\mathbf{z}_k; \mathbf{h}_k(\mathbf{x}_k^\gamma), \tilde{\mathbf{R}}_k^\gamma)\text{N}(\mathbf{x}_k^\gamma; \hat{\mathbf{x}}_{k|k}^{\gamma-1}, \mathbf{P}_{k|k}^{\gamma-1})d\mathbf{x}_k^\gamma} \tag{22}$$

which has the identical form as the posterior PDF of the state in a standard GA filter, consequently, the PDF $q^{i+1}(\mathbf{x}_k^\gamma)$ can be approximated as Gaussian and updated by the modified likelihood PDF $\text{N}(\mathbf{z}_k; \mathbf{h}_k(\mathbf{x}_k^\gamma), \tilde{\mathbf{R}}_k^\gamma)$, i.e.

$$q^{i+1}(\mathbf{x}_k^\gamma) = \text{N}(\mathbf{x}_k^\gamma; \hat{\mathbf{x}}_{k|k}^{\gamma(i+1)}, \mathbf{P}_{k|k}^{\gamma(i+1)}) \tag{23}$$

where $\hat{\mathbf{x}}_{k|k}^{\gamma(i+1)}$ and $\mathbf{P}_{k|k}^{\gamma(i+1)}$ denote the state estimation and estimated error covariance matrix at the $\gamma$th recursion and $i+1$th fixed-point iteration respectively.

When the rest of the progressive increment is less than the pre-set threshold $\varepsilon$, the progressive loop will be terminated and the remaining measurement information will be included at one time. In this paper, the threshold $\varepsilon$ is chosen as $10^{-2}$. After $N$th recursions, we obtain $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k}^N$, $\mathbf{P}_{k|k} = \mathbf{P}_{k|k}^N$.

**Remark 1**: The fixed-point iterations methods employed in the VB approach can be fairly computational costly, thus we suggest that the iterations loops are repeated only when the update is significant, i.e. while

$$\text{Norm}(\boldsymbol{\Xi}^{(i+1)}, \boldsymbol{\Xi}^{(i)}) > \delta \tag{24}$$

the iterations operation is executed, where $\text{Norm}(\cdot,\cdot)$ denotes the 2-norm distance between the items. In this paper, the threshold $\delta$ is selected empirically as $10^{-6}$.

### 2.2. Time update

In the time update, the distribution prior parameters of inaccurate MNCM need to be selected. In this paper, we assumed the prior dof parameter $u_{k|k-1}^\gamma$ and prior scale matrix $\mathbf{U}_{k|k-1}^\gamma$ are as follows

$$\begin{cases} u_{k|k-1}^\gamma = \tau + m + 1 \\ \mathbf{U}_{k|k-1}^\gamma = \tau\mathbf{R}_0 \end{cases} \tag{25}$$

where $\tau$ is the tuning parameter suggested to be selected within $[2, 6]$, and $\mathbf{R}_0$ is the initial inaccurate MNCM. The proposed PGAF with variable step size and adaptive MNCM is shown in Algorithm 1, where $M$ denotes the final fixed-point iteration times.

## 3. SIMULATION

In this simulation, the superior performance of the proposed PGAF as compared with existing methods is illustrated by bearing only tracking. The third-degree spherical radial cubature rule is chosen to implement the proposed PGAF, which leads to a CKF with progressive measurement update. The process and measurement equations are the same as [14]. Where the process noise $\mathbf{w}_k \sim \text{N}(\mathbf{w}_k; 0, 0.001^2\mathbf{I}_{2\times 2})$, and the measurement noise $\mathbf{v}_k \sim \text{N}(\mathbf{v}_k; 0, 2.5 \times 10^{-4})$. The initial true state vector $\mathbf{x}_0$ and the initial estimated error covariance matrix $\mathbf{P}_{0|0}$ are, respectively, set as $\mathbf{x}_0 = [-0.05, 0.001, 0.7, -0.055]^T$ and $\mathbf{P}_{0|0} = 10 * diag([0.1^2 0.005^2 0.1^2 0.01^2])$, the initial state estimate $\hat{\mathbf{x}}_{0|0}$ is chosen randomly from $\text{N}(\hat{\mathbf{x}}_{0|0}; \mathbf{x}_0, \mathbf{P}_{0|0})$. Besides, the tuning parameter of the proposed PGAF is $\tau = 3$. The simulation time is $T = 100s$ and the number of Monte Carlo runs is $M = 1000$. The logarithmic mean square errors (LMSEs) of positions and velocities are chosen as performance metrics, which are formulated similar as in [14].

In this simulation, the standard CKF [3] and the existing EKF with recursive update (RUEKF) [13], the existing Sigma-Point Kalman filter with recursive update (RUSPKF) [15], the existing PGAF [14] with the progressive steps $N = 30$ as well as the proposed novel PGAF are tested. It is clear to see from Fig. 2-3 that the proposed novel PGAF has higher estimation accuracy than existing methods no matter the positions or velocities. Table 1 gives the averaged LMSEs of the proposed method and existing methods over the

**Algorithm 1:** the proposed PGAF with variable step size

**Inputs**: $\hat{\mathbf{x}}_{k-1|k-1}$, $\mathbf{P}_{k-1|k-1}$, $\mathbf{z}_k$, $\mathbf{f}_{k-1}(\cdot)$, $\mathbf{h}_k(\cdot)$, $\mathbf{R}_0$, $\tau$, $N$, $N_{VB}$

**Time update**

1. Compute $\hat{\mathbf{x}}_{k|k-1}$ and $\mathbf{P}_{k|k-1}$ by the time update of the standard GAF [3]
2. Compute $u^{\gamma}_{k|k-1}$ and $\mathbf{U}^{\gamma}_{k|k-1}$ by (25)

**Measurement update**

3. Initialize the progressive loop: $\hat{\mathbf{x}}^{0}_{k|k} \leftarrow \hat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}^{0}_{k|k} \leftarrow \mathbf{P}_{k|k-1}$, $\hat{\mathbf{R}}^{0}_{k} \leftarrow \mathbf{R}_0$

**For** $\gamma = 1 : N$

4. Initialize the fix-point iterations loop: $\hat{\mathbf{x}}^{\gamma-1(0)}_{k|k} \leftarrow \hat{\mathbf{x}}^{\gamma-1}_{k|k}$, $\mathbf{P}^{\gamma-1(0)}_{k|k} \leftarrow \mathbf{P}^{\gamma-1}_{k|k}$, $\mathrm{E}^{0}[\mathbf{R}^{\gamma}_{k}] \leftarrow \hat{\mathbf{R}}^{\gamma-1}_{k}$

**While** $\mathrm{Norm}(\mathbf{\Xi}^{(\mathrm{i})}, \mathbf{\Xi}^{(\mathrm{i}-1)}) > \delta$ **AND** i$\leq N_{VB}$

5. Compute $\mathbf{D}^{\gamma(i)}_{k}$ by (11) with $\hat{\mathbf{x}}^{\gamma-1(i-1)}_{k|k}$ and $\mathbf{P}^{\gamma-1(i-1)}_{k|k}$
6. Compute $\alpha^{i+1}$ and $\beta^{i+1}$ by (13) and (14)
7. Compute $\mathrm{E}^{i+1}[\lambda^{\gamma}]$ by the definition of the integral
8. Compute $u^{\gamma(i+1)}_{k}$ and $\mathbf{U}^{\gamma(i+1)}_{k}$ by (17) and (18)
9. Compute $\mathrm{E}^{i+1}[\mathbf{R}^{\gamma}_{k}]$ and $\hat{\mathbf{R}}^{\gamma}_{k}$ by (21) and (20)
10. compute $\hat{\mathbf{z}}^{\gamma}_{k|k}$,$\mathbf{P}^{zz,\gamma(i+1)}_{k|k}$ and $\mathbf{P}^{xz,\gamma}_{k|k}$ by Gaussian weight integration rule
11. $\mathbf{K}^{\gamma(i+1)}_{k} = \mathbf{P}^{xz,\gamma}_{k|k}(\mathbf{P}^{zz,\gamma(i+1)}_{k|k})^{-1}$
12. $\hat{\mathbf{x}}^{\gamma(i+1)}_{k|k} = \hat{\mathbf{x}}^{\gamma-1}_{k|k} + \mathbf{K}^{\gamma(i+1)}_{k}(\mathbf{z}_k - \hat{\mathbf{z}}^{\gamma}_{k|k})$
13. $\mathbf{P}^{\gamma(i+1)}_{k|k} = \mathbf{P}^{\gamma-1(i+1)}_{k|k} - \mathbf{K}^{\gamma(i+1)}_{k}\mathbf{P}^{zz,\gamma(i+1)}_{k|k}(\mathbf{K}^{\gamma(i+1)}_{k})^{T}$

**End While**

14. $\hat{\mathbf{x}}^{\gamma}_{k|k} \leftarrow \hat{\mathbf{x}}^{\gamma(M+1)}_{k|k}$, $\mathbf{P}^{\gamma}_{k|k} \leftarrow \mathbf{P}^{\gamma(M+1)}_{k|k}$, $\hat{\mathbf{R}}^{\gamma}_{k} \leftarrow \mathrm{E}^{M+1}[\mathbf{R}^{\gamma}_{k}]$
15. Compute the remaining step size
16. **If** the remaining step size $< \varepsilon$, **break** the progressive loop

**End For**

17. Absorbing the remaining measurement information at one time and update $\hat{\mathbf{x}}^{\gamma}_{k|k}$ and $\mathbf{P}^{\gamma}_{k|k}$
18. $\hat{\mathbf{x}}_{k|k} \leftarrow \hat{\mathbf{x}}^{\gamma}_{k|k}$, $\mathbf{P}_{k|k} \leftarrow \mathbf{P}^{\gamma}_{k|k}$

**Outputs**: $\hat{\mathbf{x}}_{k|k}$, $\mathbf{P}_{k|k}$

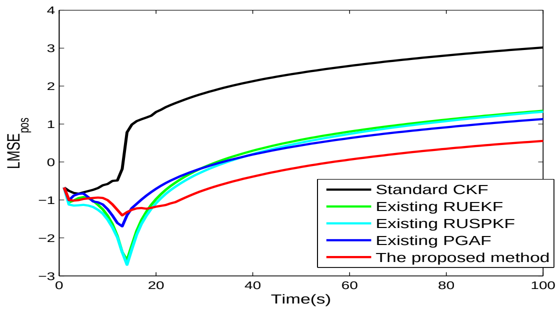


**Fig. 2**: $\mathrm{LMSE_{pos}}$ of the proposed method and existing methods when $N = 30$

last 20s, which shows the significantly improvement on estimation accuracy of the proposed method, and the improved rates are calculated with respect to the exiting PGAF. Fig. 4 demonstrates the variation of the optimal step sizes along with the progressive steps at three different instants in the proposed

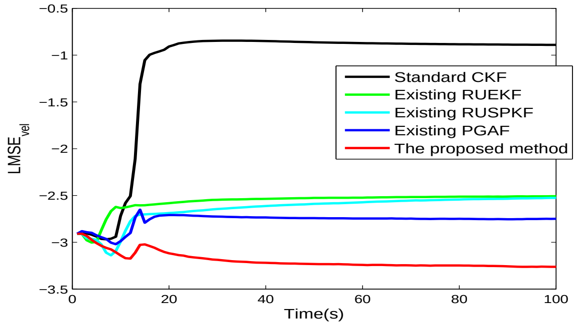


**Fig. 3**: $\mathrm{LMSE_{vel}}$ of the proposed method and existing methods when $N = 30$

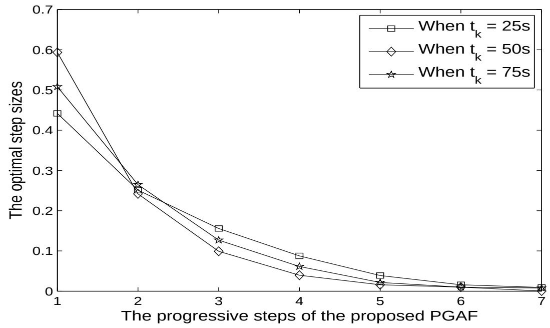


**Fig. 4**: The variation of the optimal step sizes at different instants in the proposed PGAF

**Table 1**: Averaged LMSEs of the proposed method and existing methods over the last 20s

| Estimators | Positions | Velocities | Improved rates |
|---|---|---|---|
| Standard CKF | 2.921 | -0.887 | -4159% |
| Existing RUEKF | 1.245 | -2.510 | -57.26% |
| Existing RUSPKF | 1.212 | -2.533 | -48.73% |
| Existing PGAF | 1.033 | -2.751 | 0 |
| The proposed filter | 0.462 | -3.256 | 65.89% |

PGAF. Meanwhile the simulation also revealed that the proposed novel PGAF has significantly fewer progressive steps in spite of higher estimation accuracy than existing methods.

## 4. CONCLUSION

In this paper, a novel PGAF with variable step size and adaptive MNCM was proposed based on the VB approach to select the optimal step sizes and restrain the integral approximation errors. The performance of the proposed filter was tested in the simulation of bearing only tracking. Simulation results showed that the proposed PGAF had higher estimation accuracy and fewer progressive steps than existing methods due to the modified MNCM and optimal selections of step sizes.

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