The full phase space of a model in the Calogero-Ruijsenaars family

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Abstract

We complete the recent derivation of a Ruijsenaars type system that arises as a reduction of the natural free system on the Heisenberg double of $\mathrm{SU}(n,n)$. The previous analysis by Marshall focused on a dense open submanifold of the reduced phase space, and here we describe the full phase space wherein Liouville integrability of the system holds by construction.

1 Introduction

The method of Hamiltonian reduction belongs to the set of standard toolkits applicable to study a great variety of problems ranging from geometric mechanics to field theory and harmonic analysis [1, 2]. It is especially useful in the theory of integrable Hamiltonian systems [3], where one of the maxims is that one should view the systems of interests as reductions of obviously solvable 'free' systems [4]. This is often advantageous, for example since the reduction produces global phase spaces on which the reduced free flows are automatically complete, which is an indispensable property of any integrable system.

An interesting application of this method appeared in the recent paper [5] by Marshall, where a deformation of the classical hyperbolic BC_n Sutherland system [4] was derived by reduction of a free system on the Heisenberg double [6] of the Poisson-Lie group $\mathrm{SU}(n,n)$. In a closely related work [7], we investigated the analogous reduction of the Heisenberg double of $\mathrm{SU}(2n)$ and thereby obtained a deformation of the trigonometric variant of the BC_n Sutherland system. In the course of our analysis [7] we noticed that Marshall's considerations were restricted to a proper open submanifold of the reduced phase space. Although this submanifold forms a dense subset, the reduced Hamiltonian flows are complete only on the full phase space. The goal of this paper is to provide a globally valid model of the relevant reduced phase space.

Our motivation basically stems from the fact that the global description of the reduced phase space is a necessary ingredient of the characterization of any Hamiltonian reduction. Besides, we also intend this complement to the papers [5, 7] to serve as a starting point for a future work where the duality aspects of the $\mathrm{SU}(n,n)$ and the $\mathrm{SU}(2n)$ cases should be treated together. Based on experience [8, 9], ideas for finding the pertinent dual systems are readily available, but their technical implementation poses a challenging open problem.

Below, we concentrate on the essential points referring to [5, 7] for more details. In Section 2 the necessary preliminaries are summarized. Then in Section 3 we describe the reduced phase space. In Subsection 3.1, we present the local picture of Marshall using a shortcut that leads to it, and give the global picture in Subsection 3.2. The reader may go directly to Theorem 3.6 to see the result. The reduced Hamiltonians and their integrability is briefly discussed in the last section, and two appendices are included to help readability. Appendix A contains some auxiliary explicit formulas, while Appendix B details a property of the reduced Hamiltonians.

2 Definitions and first steps

All of the material presented in this section is adapted from [5] and [7].

Fix an integer $n > 1$ and introduce the group

$$\mathrm{SU}(n,n) = \{g \in \mathrm{SL}(2n,\mathbb{C}) \mid g^\dagger \boldsymbol{J} g = \boldsymbol{J}\} \tag{2.1}$$

with $\boldsymbol{J} = \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n)$. Then consider the open submanifold $\mathrm{SL}(2n,\mathbb{C})' \subset \mathrm{SL}(2n,\mathbb{C})$ consisting of those elements, K, that admit both Iwasawa-like decompositions of the form

$$K = g_L b_R^{-1} = b_L g_R^{-1}, \qquad g_L, g_R \in \mathrm{SU}(n,n), \quad b_L, b_R \in \mathrm{SB}(2n), \tag{2.2}$$

where $\mathrm{SB}(2n) < \mathrm{SL}(2n,\mathbb{C})$ is the group of upper triangular matrices having positive entries along the diagonal. Both decompositions are unique and the constituent factors depend smoothly on $K \in \mathrm{SL}(2n,\mathbb{C})'$. The manifold $\mathrm{SL}(2n,\mathbb{C})'$ carries the symplectic form [10]

$$\omega = \frac{1}{2}\Im \mathrm{tr}(db_L b_L^{-1} \wedge dg_L g_L^{-1}) + \frac{1}{2}\Im \mathrm{tr}(db_R b_R^{-1} \wedge dg_R g_R^{-1}). \tag{2.3}$$

On this symplectic manifold, which is a symplectic leaf of the Heisenberg double of the Poisson-Lie group $\mathrm{SU}(n,n)$, one has the pairwise Poisson commuting Hamiltonians

$$\mathcal{H}_j(K) = \frac{1}{2j}\mathrm{tr}(K\boldsymbol{J}K^\dagger \boldsymbol{J})^j, \qquad j \in \mathbb{Z}^*. \tag{2.4}$$

They generate complete flows that can be written down explicitly (see Section 4). We are concerned with a reduction of these Hamiltonians based on the symmetry group $G_+ \times G_+$, where

$$G_+ = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n)) < \mathrm{SU}(n,n) \tag{2.5}$$

is the block-diagonal subgroup. Throughout, we refer to the obvious 2×2 block-matrix structure corresponding to $\boldsymbol{J}$. The action of $G_+ \times G_+$ on $\mathrm{SL}(2n,\mathbb{C})'$ is given by the map

$$G_+ \times G_+ \times \mathrm{SL}(2n,\mathbb{C})' \to \mathrm{SL}(2n,\mathbb{C})' \tag{2.6}$$

that works according to

$$(\eta_L, \eta_R, K) \mapsto \eta_L K \eta_R^{-1}. \tag{2.7}$$

One can check that this map is well-defined, i.e. $\eta_L K \eta_R^{-1}$ stays in $\mathrm{SL}(2n,\mathbb{C})'$, and has the Poisson property with respect to the product Poisson structure on the left-hand side [6, 10], where on G_+ the standard Sklyanin bracket is used and the Poisson structure on $\mathrm{SL}(2n,\mathbb{C})'$ is engendered by ω. Moreover, this $G_+ \times G_+$ action is associated with a momentum map in the sense of Lu [11]. The momentum map can be written as $\Phi_+ \colon \mathrm{SL}(2n,\mathbb{C})' \to G_+^* \times G_+^*$, where G_+^* is the subgroup of $\mathrm{SB}(2n)$ containing the elements with vanishing off-diagonal blocks. By utilizing the obvious projection $\pi \colon \mathrm{SB}(2n) \to G_+^*$, which replaces the block off-diagonal components by zeroes, the momentum map obeys the formula

$$\Phi_+(K) = \big(\pi(b_L), \pi(b_R)\big). \tag{2.8}$$

The Hamiltonians $\mathcal{H}_j$ (2.4) are invariant with respect to the symmetry group $G_+ \times G_+$ and Φ_+ is constant along their flows.

The general theory [11] ensures that one can now perform Marsden-Weinstein type reduction. This amounts to imposing the constraint

$$\Phi_+(K) = \mu = (\mu_L, \mu_R) \tag{2.9}$$

with some constant $\mu \in G_+^* \times G_+^*$ and then taking the quotient of $\Phi_+^{-1}(\mu)$ by the corresponding isotropy group, denoted below as G_μ.

We pick the following value μ of the momentum map,

$$\mu_L = \begin{bmatrix} e^u \nu(x) & \mathbf{0}_n \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix}, \quad \mu_R = \begin{bmatrix} e^v \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & e^{-v}\mathbf{1}_n \end{bmatrix}, \tag{2.10}$$

where u, v, and x are real constants satisfying

$$u + v \neq 0, \qquad x > 0, \tag{2.11}$$

and $\nu(x)$ is the $n \times n$ upper triangular matrix defined by

$$\nu(x)_{jj} = 1, \quad \nu(x)_{jk} = (1 - e^{-x})e^{\frac{(k-j)x}{2}}, \quad j < k. \tag{2.12}$$

The essential property of $\nu(x)$ is that $\nu(x)\nu(x)^\dagger$ has only two different eigenvalues, one of them with multiplicity 1. (This also holds for $x < 0$ and our assumption $x > 0$ only serves to keep the text shorter.) The corresponding isotropy group G_μ is

$$G_\mu = G_+(\mu_L) \times G_+, \tag{2.13}$$

where the elements $\eta_L \in G_+(\mu_L)$ have the form

$$\eta_L = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \tag{2.14}$$

with $\eta_L(1) \in \mathrm{U}(n)$ satisfying

$$\eta_L(1)\nu(x)\nu(x)^\dagger \eta_L(1)^{-1} = \nu(x)\nu(x)^\dagger \tag{2.15}$$

and $\eta_L(2) \in \mathrm{U}(n)$, coupled by $\det(\eta_L) = 1$. It will turn out that the reduced phase space

$$M = \Phi_+^{-1}(\mu)/G_\mu \tag{2.16}$$

is a smooth manifold. Our task is to characterize this manifold, which carries the reduced symplectic form ω_M defined by the relation

$$\iota_\mu^* \omega = \pi_\mu^* \omega_M, \tag{2.17}$$

where $\iota_\mu \colon \Phi_+^{-1}(\mu) \to \mathrm{SL}(2n, \mathbb{C})'$ is the tautological injection and $\pi_\mu \colon \Phi_+^{-1}(\mu) \to M$ is the natural projection.

Consider the following central subgroup $\mathbb{Z}_{2n}$ of $G_+ \times G_+$,

$$\mathbb{Z}_{2n} = \{(w\mathbf{1}_{2n}, w\mathbf{1}_{2n}) \mid w \in \mathbb{C},\ w^{2n} = 1\}, \tag{2.18}$$

which acts trivially according to (2.7) and is contained in G_μ. Later we shall refer to the factor group

$$\bar{G}_\mu = G_\mu/\mathbb{Z}_{2n} \tag{2.19}$$

as the 'effective gauge group'. Obviously, we have $\Phi_+^{-1}(\mu)/G_\mu = \Phi_+^{-1}(\mu)/\bar{G}_\mu$.

In the end, we shall obtain a model of the quotient space M by explicitly exhibiting a global cross-section of the orbits of G_μ in $\Phi_+^{-1}(\mu)$. The construction uses the generalized Cartan decomposition of $\mathrm{SU}(n, n)$, which says that every $g \in \mathrm{SU}(n, n)$ can be written as

$$g = g_+ \begin{bmatrix} \cosh q & \sinh q \\ \sinh q & \cosh q \end{bmatrix} h_+, \tag{2.20}$$

where $g_+, h_+ \in G_+$ and $q = \mathrm{diag}(q_1, \dots, q_n)$ is a real diagonal matrix verifying

$$q_1 \geq \cdots \geq q_n \geq 0. \tag{2.21}$$

The components q_i are uniquely determined by g, and yield smooth functions on the locus where they are all distinct. In what follows we shall often identify diagonal matrices like q with the corresponding elements of $\mathbb{R}^n$.

As the first step towards describing M, we apply the decomposition (2.20) to g_L in $K = g_L b_R^{-1}$ and impose the right-handed momentum constraint $\pi(b_R) = \mu_R$. It is then easily seen that up to G_μ-transformations every element of $\Phi_+^{-1}(\mu)$ can be represented in the following form:

$$K = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \cosh q & \sinh q \\ \sinh q & \cosh q \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix}. \tag{2.22}$$

Here $\rho \in \mathrm{SU}(n)$ and α is an $n \times n$ complex matrix. Referring to the 2×2 block-matrix notation, we introduce $\Omega = K_{22}$ and record from (2.22) that

$$\Omega = (\sinh q)\alpha + e^v \cosh q. \tag{2.23}$$

It will prove advantageous to seek for Ω in the polar-decomposed form,

$$\Omega = \Lambda T, \tag{2.24}$$

where $T \in \mathrm{U}(n)$ and Λ is a Hermitian, positive semi-definite matrix.

The next step is to implement the left-handed momentum constraint $\pi(b_L) = \mu_L$ by writing $K = b_L g_R^{-1}$ with

$$b_L = \begin{bmatrix} e^u \nu(x) & \chi \\ \mathbf{0}_n & e^{-u}\mathbf{1}_n \end{bmatrix}, \tag{2.25}$$

where χ is an unknown $n \times n$ matrix. Then we inspect the components of the 2×2 block-matrix identity

$$K\boldsymbol{J}K^\dagger = b_L \boldsymbol{J} b_L^\dagger, \tag{2.26}$$

which results by substituting K from (2.22). We find that the (22) component of this identity is equivalent to

$$\Omega\Omega^\dagger = \Lambda^2 = e^{-2u}\mathbf{1}_n + e^{-2v}(\sinh q)^2. \tag{2.27}$$

This uniquely determines Λ in terms of q and also shows that Λ is invertible. An important consequence of the first condition in (2.11) is that we must have

$$q_n > 0, \tag{2.28}$$

and therefore $\sinh q$ is an invertible diagonal matrix. Indeed, if $q_n = 0$, then from (2.23) and (2.27) we would get $(\Omega\Omega^\dagger)_{nn} = e^{2v} = e^{-2u}$, which is excluded by (2.11).

By using the above relations, it is simple algebra to convert the (12) and the (21) components of the identity (2.26) into the equation

$$\chi = \rho(\sinh q)^{-1}[e^{-u}\cosh q - e^{u+v}\Omega^\dagger]. \tag{2.29}$$

Finally, the (11) entry of the identity (2.26) translates into the following crucial equation:

$$\rho(\sinh q)^{-1}T^\dagger(\sinh q)^2 T(\sinh q)^{-1}\rho^\dagger = \nu(x)\nu(x)^\dagger. \tag{2.30}$$

This is to be satisfied by q subject to (2.21), (2.28) and $T \in \mathrm{U}(n)$, $\rho \in \mathrm{SU}(n)$. After finding q, T and ρ, one can reconstruct K (2.22) by applying the formulas derived above.

From our viewpoint, a key observation is that (2.30) coincides completely with equation (5.7) in the paper [8], where its general solution was found. The correspondence between the notations used here and in [8] is

$$(\rho, T, \sinh q) \Longleftrightarrow (k_L, k_R^\dagger, e^{\hat{p}}). \tag{2.31}$$

For this reason, we introduce the new variable $\hat{p} \in \mathbb{R}^n$ by the definition

$$\sinh q_k = e^{\hat{p}_k}, \quad k = 1, \dots, n. \tag{2.32}$$

Because of (2.21) and (2.28), the variables $\hat{p}_k$ satisfy

$$\hat{p}_1 \geq \cdots \geq \hat{p}_n. \tag{2.33}$$

We do not see an a priori reason why the very different reduction procedures led to the same equation (2.30) here and in [8]. However, we are going to take full advantage of this situation. We note that essentially every formula written in this section appears in [5] as well (with slightly different notations), but in Marshall's work the previously obtained results about the solutions of (2.30) were not used.

3 The reduced phase space

The statement of Proposition 3.3 characterizes a submanifold of M (2.16), which was erroneously claimed in [5] to be equal to M. After describing this 'local picture', we shall present a globally valid model of M.

3.1 The local picture

By applying results of [8, 12] in the same way as in [7], one can prove the following lemma.

Lemma 3.1. *The constraint surface $\Phi_+^{-1}(\mu)$ contains an element of the form* (2.22) *if and only if $\hat{p}$ defined by* (2.32) *lies in the closed polyhedron*

$$\bar{\mathcal{C}}_x = \{\hat{p} \in \mathbb{R}^n \mid \hat{p}_k - \hat{p}_{k+1} \geq x/2 \ (k = 1, \dots, n-1)\}. \tag{3.1}$$

The polyhedron $\bar{\mathcal{C}}_x$ is the closure of its interior, $\mathcal{C}_x$, defined by strict inequalities. We note that in [5] the elements of the boundary $\bar{\mathcal{C}}_x \setminus \mathcal{C}_x$ were omitted.

For any fixed $\hat{p} \in \bar{\mathcal{C}}_x$, one can write down the solutions of (2.30) for T and ρ explicitly [8]. By inserting those into the formula (2.22), using the relations (2.23), (2.24), (2.27) to determine the matrix α, one arrives at the next lemma. It refers to the $n \times n$ real matrices

$$\theta(x, \hat{p}), \quad \zeta(x, \hat{p}), \quad \kappa(x), \tag{3.2}$$

which belong to the group $\mathrm{SO}(n)$ and are defined by explicit formulas that can be found, for example, at the beginning of Section 3.2 in [7]. For the reader's convenience, we append these formulas at the end of the text.

Proposition 3.2. *For any parameters u, v, x subject to* (2.11)*, and variables $\hat{p} \in \bar{\mathcal{C}}_x$ and $e^{\mathrm{i}\hat{q}}$ from the n-torus $\mathbb{T}_n$, define the matrix*

$$K(\hat{p}, e^{\mathrm{i}\hat{q}}) = \begin{bmatrix} \rho & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{1}_n + e^{2\hat{p}}} & e^{\hat{p}} \\ e^{\hat{p}} & \sqrt{\mathbf{1}_n + e^{2\hat{p}}} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \alpha \\ \mathbf{0}_n & e^{v}\mathbf{1}_n \end{bmatrix} \tag{3.3}$$

by employing

$$\rho = \rho(x, \hat{p}) = \kappa(x)\zeta(x, \hat{p})^{-1} \tag{3.4}$$

and

$$\alpha = \alpha(x, u, v, \hat{p}, e^{\mathrm{i}\hat{q}}) = e^{\mathrm{i}\hat{q}}\sqrt{e^{-2u}e^{-2\hat{p}} + e^{-2v}\mathbf{1}_n}\,\theta(x, \hat{p})^{-1} - e^{v}\sqrt{e^{-2\hat{p}} + \mathbf{1}_n}. \tag{3.5}$$

Then $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ resides in the constraint surface $\Phi_+^{-1}(\mu)$ and the set

$$S = \{K(\hat{p}, e^{\mathrm{i}\hat{q}}) \mid (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n\} \tag{3.6}$$

intersects every orbit of G_μ in $\Phi_+^{-1}(\mu)$.

By arguing verbatim along the lines of [7], and referring to [5] for the calculation of the reduced symplectic form, one can establish the validity of the subsequent proposition.

Proposition 3.3. *The effective gauge group $\bar{G}_\mu$* (2.19) *acts freely on $\Phi_+^{-1}(\mu)$ and thus the quotient space M* (2.16) *is a smooth manifold. The restriction of the natural projection $\pi_\mu \colon \Phi_+^{-1}(\mu) \to M$ to*

$$S^o = \{K(\hat{p}, e^{\mathrm{i}\hat{q}}) \mid (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathcal{C}_x \times \mathbb{T}_n\} \tag{3.7}$$

gives rise to a diffeomorphism between $\mathcal{C}_x \times \mathbb{T}_n$ and the open, dense submanifold of M provided by $\pi_\mu(S^o)$. Taking S^o as model of $\pi_\mu(S^o)$, the corresponding restriction of the reduced symplectic form ω_M becomes the Darboux form

$$\omega_{S^o} = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k. \tag{3.8}$$

Remark 3.4. In the formula (3.3) $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ appears in the decomposed form $K = g_L b_R^{-1}$ and it is not immediately obvious that it belongs to $\mathrm{SL}(2n, \mathbb{C})'$, i.e., that it can be decomposed alternatively as $b_L g_R^{-1}$. However, by defining $b_L(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \mathrm{SB}(2n)$ by the formula (2.25) using χ in (2.29) with the change of variables $\sinh q = e^{\hat{p}}$, the matrix ρ as given above, and $T = e^{\mathrm{i}\hat{q}}\theta(x, \hat{p})^{-1}$ that enters (3.3), we can verify that for these elements $g_R^{-1} = b_L^{-1}K$ satisfies the defining relation of $\mathrm{SU}(n, n)$ (2.1), as required. The reader may perform this verification, which relies only on the constraint equations displayed in Section 2.

3.2 The global picture

The train of thought leading to the construction below can be outlined as follows. Proposition 3.3 tells us, in particular, that any G_μ-orbit passing through S^o intersects S^o in a single point. Direct inspection shows that the analogous statement is false for $S \setminus S^o$, which corresponds to $(\bar{\mathcal{C}}_x \setminus \mathcal{C}_x) \times \mathbb{T}_n$ in a one-to-one manner. Thus a global model of M should result by identifying those points of $S \setminus S^o$ that lie on the same G_μ-orbit. By using the bijective map from $\bar{\mathcal{C}}_x \times \mathbb{T}_n$ onto S given by the formula (3.3), the desired identification will be achieved by constructing such complex variables out of $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n$ that coincide precisely for gauge equivalent elements of S.

Turning to the implementation of the above plan, we introduce the space of complex variables

$$\hat{M}_c = \mathbb{C}^{n-1} \times \mathbb{C}^{\times}, \qquad (\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}), \tag{3.9}$$

carrying the symplectic form

$$\hat{\omega}_c = \mathrm{i} \sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j + \frac{\mathrm{i} dz_n \wedge d\bar{z}_n}{2 z_n \bar{z}_n}. \tag{3.10}$$

We also define the surjective map

$$\hat{\mathcal{Z}}_x \colon \bar{\mathcal{C}}_x \times \mathbb{T}_n \to \hat{M}_c, \quad (\hat{p}, e^{\mathrm{i}\hat{q}}) \mapsto z(\hat{p}, e^{\mathrm{i}\hat{q}}) \tag{3.11}$$

by setting

$$\begin{aligned} z_j(\hat{p}, e^{\mathrm{i}\hat{q}}) &= (\hat{p}_j - \hat{p}_{j+1} - x/2)^{\frac{1}{2}} \prod_{k=j+1}^{n} e^{\mathrm{i}\hat{q}_k}, \quad j = 1, \dots, n-1, \\ z_n(\hat{p}, e^{\mathrm{i}\hat{q}}) &= e^{-\hat{p}_1} \prod_{k=1}^{n} e^{\mathrm{i}\hat{q}_k}. \end{aligned} \tag{3.12}$$

The restriction $\mathcal{Z}_x$ of $\hat{\mathcal{Z}}_x$ to $\mathcal{C}_x \times \mathbb{T}_n$ is a diffeomorphism onto the open subset

$$\hat{M}_c^o = \left\{ z \in \hat{M}_c \,\middle|\, \prod_{j=1}^{n-1} z_j \neq 0 \right\}, \tag{3.13}$$

and it verifies the relation

$$\mathcal{Z}_x^* \hat{\omega}_c = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k. \tag{3.14}$$

Thus we manufactured a change of variables $\mathcal{C}_x \times \mathbb{T}_n \longleftrightarrow \hat{M}_c^o$. The inverse $\mathcal{Z}_x^{-1} \colon \hat{M}_c^o \to \mathcal{C}_x \times \mathbb{T}_n$ involves the functions

$$\hat{p}_1(z) = -\log|z_n|, \quad \hat{p}_j(z) = -\log|z_n| - \sum_{k=1}^{j-1} (|z_k|^2 + x/2) \quad (j = 2, \dots, n). \tag{3.15}$$

These extend smoothly to $\hat{M}_c$ wherein $\hat{M}_c^o$ sits as a dense submanifold.

Now we state a lemma, which is a simple adaptation from [8, 13].

Lemma 3.5. *By using the shorthand* $\sigma_j = \prod_{k=j+1}^{n} e^{\mathrm{i}\hat{q}_k}$ *for* $j = 1, \dots, n-1$ *(cf.* (3.12)*), let us define*

$$\sigma_+(e^{\mathrm{i}\hat{q}}) = \mathrm{diag}(\sigma_1, \dots, \sigma_{n-1}, 1) \quad \text{and} \quad \sigma_-(e^{\mathrm{i}\hat{q}}) = \mathrm{diag}(1, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}). \tag{3.16}$$

Then there exist unique smooth functions $\hat{\zeta}(x, z)$*,* $\hat{\theta}(x, z)$ *and* $\hat{\alpha}(x, u, v, z)$ *of* $z \in \hat{M}_c$ *that satisfy the following identities for any* $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n$*:*

$$\hat{\zeta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \sigma_+(e^{\mathrm{i}\hat{q}}) \zeta(x, \hat{p}) \sigma_+(e^{\mathrm{i}\hat{q}})^{-1}, \tag{3.17}$$

$$\hat{\theta}(x, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \sigma_+(e^{\mathrm{i}\hat{q}}) \theta(x, \hat{p}) \sigma_-(e^{\mathrm{i}\hat{q}}), \tag{3.18}$$

$$\hat{\alpha}(x, u, v, z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \sigma_+(e^{\mathrm{i}\hat{q}}) \alpha(x, u, v, \hat{p}, e^{\mathrm{i}\hat{q}}) \sigma_+(e^{\mathrm{i}\hat{q}})^{-1}. \tag{3.19}$$

Here we refer to the functions on $\bar{\mathcal{C}}_x \times \mathbb{T}_n$ *displayed in equations* (3.2) *and* (3.5)*.*

The explicit formulas of the functions on $\hat{M}_c$ that appear in the above identities are easily found by first determining them on $\hat{M}_c^o$ using the change of variables $\mathcal{Z}_x$, and then noticing that they automatically extend to $\hat{M}_c$. The expressions of the functions $\hat{\zeta}$ and $\hat{\theta}$, which depend only on $z_1, \dots, z_{n-1}$, are the same as given in Definition 3.3 in [8]. (For most purposes the above definitions and the formulas of Appendix A suffice.) As for $\hat{\alpha}$, by defining

$$\Delta(z) = \mathrm{diag}(z_n, e^{-\hat{p}_2(z)}, \dots, e^{-\hat{p}_n(z)}) \tag{3.20}$$

we have

$$\hat{\alpha}(x,u,v,z) = \sqrt{e^{-2v}e^{2\hat{p}(z)} + e^{-2u}\mathbf{1}_n}\,\Delta(z)\hat{\theta}(x,z)^{-1} - e^v\sqrt{e^{-2\hat{p}(z)} + \mathbf{1}_n} \tag{3.21}$$

that satisfies relation (3.19) due to the identity

$$\Delta(z(\hat{p}, e^{\mathrm{i}\hat{q}})) = e^{-\hat{p}}e^{\mathrm{i}\hat{q}}\sigma_+(e^{\mathrm{i}\hat{q}})\sigma_-(e^{\mathrm{i}\hat{q}}), \qquad \forall (\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n. \tag{3.22}$$

With these preparations at hand, we can formulate the main result of this paper.

Theorem 3.6. *Define the smooth map* $\hat{K}\colon \hat{M}_c \to \mathrm{SL}(2n,\mathbb{C})'$ *by the formula*

$$\hat{K}(z) = \begin{bmatrix} \kappa(x)\hat{\zeta}(x,z)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{1}_n + e^{2\hat{p}(z)}} & e^{\hat{p}(z)} \\ e^{\hat{p}(z)} & \sqrt{\mathbf{1}_n + e^{2\hat{p}(z)}} \end{bmatrix} \begin{bmatrix} e^{-v}\mathbf{1}_n & \hat{\alpha}(x,u,v,z) \\ \mathbf{0}_n & e^v\mathbf{1}_n \end{bmatrix}. \tag{3.23}$$

The image of $\hat{K}$ *belongs to the submanifold* $\Phi_+^{-1}(\mu)$ *and the induced mapping* $\pi_\mu \circ \hat{K}$*, obtained by using the natural projection* $\pi_\mu\colon \Phi_+^{-1}(\mu) \to M = \Phi^{-1}(\mu)/G_\mu$*, is a symplectomorphism between* $(\hat{M}_c, \hat{\omega}_c)$*, defined by* (3.9)-(3.10)*, and the reduced phase space* (M, ω_M)*.*

Proof. We start by pointing out that for any $(\hat{p}, e^{\mathrm{i}\hat{q}}) \in \bar{\mathcal{C}}_x \times \mathbb{T}_n$ the identity

$$\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}})) = \begin{bmatrix} \kappa(x)\sigma_+(e^{\mathrm{i}\hat{q}})\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix} K(\hat{p}, e^{\mathrm{i}\hat{q}}) \begin{bmatrix} \sigma_+(e^{\mathrm{i}\hat{q}}) & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix}^{-1} \tag{3.24}$$

is equivalent to the identities listed in Lemma 3.5. We see from this that $\hat{K}(z(\hat{p}, e^{\mathrm{i}\hat{q}}))$ is a G_μ-transform of $K(\hat{p}, e^{\mathrm{i}\hat{q}})$ (3.3), and thus $\hat{K}(z)$ belongs to $\Phi_+^{-1}(\mu)$. Indeed, the right-hand side of (3.24) can be written as $\eta_L K(\hat{p}, e^{\mathrm{i}\hat{q}})\eta_R^{-1}$ with

$$\eta_L = c\begin{bmatrix} \kappa(x)\sigma_+(e^{\mathrm{i}\hat{q}})\kappa(x)^{-1} & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix}, \qquad \eta_R = c\begin{bmatrix} \sigma_+(e^{\mathrm{i}\hat{q}}) & \mathbf{0}_n \\ \mathbf{0}_n & \sigma_+(e^{\mathrm{i}\hat{q}}) \end{bmatrix}, \tag{3.25}$$

where c is a scalar ensuring $\det(\eta_L) = \det(\eta_R) = 1$, and one can check (see the last paragraph of Appendix A) that this (η_L, η_R) lies in the group G_μ (2.13).

To proceed further, we let $\hat{K}_o$ denote the restriction of $\hat{K}$ to the dense open subset $\hat{M}_c^o$ and also let $K_o\colon \mathcal{C}_x \times \mathbb{T}_n \to \mathrm{SL}(2n,\mathbb{C})'$ denote the map defined by the corresponding restriction of the formula (3.3). Notice that, in addition to (2.17), we have the relations

$$\pi_\mu \circ \hat{K}_0 = \pi_\mu \circ K_o \circ \mathcal{Z}_x^{-1} \quad \text{and} \quad (\pi_\mu \circ K_o)^*\omega_M = \sum_{k=1}^n d\hat{q}_k \wedge d\hat{p}_k, \tag{3.26}$$

which follow from (3.24) and the last sentence of Proposition 3.3. By using (3.14) (together with $\hat{K}_o = \iota_\mu \circ \hat{K}_o$ and $K_o = \iota_\mu \circ K_o$) the above relations imply the restriction of the equality

$$(\pi_\mu \circ \hat{K})^*\omega_M = \hat{\omega}_c \tag{3.27}$$

on $\hat{M}_c^o$. This equality is then valid on the full $\hat{M}_c$ since the 2-forms concerned are smooth.

It is a direct consequence of (3.24) and Proposition 3.2 that $\pi_\mu \circ \hat{K}$ is surjective. Since, on account of (3.27), it is a local diffeomorphism, it only remains to demonstrate that the map $\pi_\mu \circ \hat{K}$ is injective. The relation $\pi_\mu(\hat{K}(z)) = \pi_\mu(\hat{K}(z'))$ for $z, z' \in \hat{M}_c$ requires that

$$\hat{K}(z') = \begin{bmatrix} \eta_L(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_L(2) \end{bmatrix} \hat{K}(z) \begin{bmatrix} \eta_R(1) & \mathbf{0}_n \\ \mathbf{0}_n & \eta_R(2) \end{bmatrix}^{-1} \tag{3.28}$$

for some $(\eta_L, \eta_R) \in G_\mu$. Supposing that (3.28) holds, application of the decomposition $\hat{K}(z) = g_L(z) b_R(z)^{-1}$ to the formula (3.23) implies that

$$\hat{\alpha}(z') = \eta_R(1)\hat{\alpha}(z)\eta_R(2)^{-1} \tag{3.29}$$

and

$$g_L(z') = \eta_L g_L(z) \eta_R^{-1}. \tag{3.30}$$

The matrices on the two sides of (3.30) appear in the form (2.20), and standard uniqueness properties of the constituents in this generalized Cartan decomposition now imply that

$$\hat{p}(z') = \hat{p}(z) \tag{3.31}$$

and

$$\eta_R(1) = \eta_R(2) = m \in \mathbb{T}_n. \tag{3.32}$$

We continue by looking at the $(k+1, k)$ components of the equality (3.29) for $k = 1, \dots, n-1$ using that $\hat{\alpha}_{k+1,k}$ depends on z only through $\hat{p}(z)$ and it never vanishes. (This follows from (3.20)-(3.21) by utilizing that $\hat{\theta}(x,z)_{k,k+1} = \theta(x, \hat{p}(z))_{k,k+1}$ by (3.18), which is nonzero for each $\hat{p}(z) \in \bar{\mathcal{C}}_x$ as seen from (A.1).) Putting (3.32) into (3.29), we obtain that $m = C\mathbf{1}_n$ with a scalar C, and therefore

$$\hat{\alpha}(z') = \hat{\alpha}(z). \tag{3.33}$$

The rest is an inspection of this matrix equality. In view of (3.31) and the forms of $\Delta(z)$ (3.20) and $\hat{\alpha}(z)$ (3.21), the last column of the equality (3.33) entails that

$$\hat{\theta}(x,z)_{nk} = \hat{\theta}(x,z')_{nk}, \quad k = 2, \dots, n, \tag{3.34}$$

where we re-instated the dependence on x that was suppressed above. One can check directly from the formulas (3.12), (3.18) and (A.1), (A.2) that

$$\hat{\theta}(x,z)_{nk} = \bar{z}_{k-1} F_k(x, \hat{p}(z)), \quad k = 2, \dots, n, \tag{3.35}$$

where $F_k(x, \hat{p}(z))$ is a smooth, strictly positive function. Hence we obtain that $z_j = z'_j$ for $j = 1, \dots, n-1$. With this in hand, since the variable z_n appears only in $\Delta(z)$, we conclude from (3.33) that $\Delta(z) = \Delta(z')$. This plainly implies that $z_n = z'_n$, whereby the proof is complete.

We note in passing that by continuing the above line of arguments the free action of G_μ is easily confirmed. Indeed, for $z' = z$ (3.30) also implies, besides (3.32), the equalities $\eta_L(2) = m$ and $\eta_L(1)\kappa(x)\hat{\zeta}(x,z)^{-1} = \kappa(x)\hat{\zeta}(x,z)^{-1} m$. Since $m = C\mathbf{1}_n$, as was already established, we must have $(\eta_L, \eta_R) = C(\mathbf{1}_{2n}, \mathbf{1}_{2n}) \in \mathbb{Z}_{2n}$ (2.18). By using that the image of $\hat{K}$ intersects every G_μ-orbit, we can conclude that $\bar{G}_\mu$ (2.19) acts freely on $\Phi_+^{-1}(\mu)$. $\square$

Remark 3.7. Observe from Theorem 3.6 that $\hat{S} = \{\hat{K}(z) \mid z \in \hat{M}_c\}$ is a global cross-section for the action of G_μ on $\Phi_+^{-1}(\mu)$. Hence $\hat{S}$ carrying the pull-back of ω as well as $(\hat{M}_c, \hat{\omega}_c)$ yield globally valid models of the reduced phase space (M, ω_M). The submanifold of $\hat{S}$ corresponding to $\hat{M}_c^o$ (3.13) is gauge equivalent to S^o (3.7) that features in Proposition 3.3.

4 Discussion

In this paper we clarified the global structure of the reduced phase space M (2.16), and thus completed the previous analysis [5] that dealt with the submanifold parametrized by $\mathcal{C}_x \times \mathbb{T}_n$. In terms of the model $\hat{M}_c$ (3.9) of M, the complement of the submanifold in question is simply the zero set of the product of the complex variables. The phase space $\hat{M}_c$ and the embedding of $\mathcal{C}_x \times \mathbb{T}_n$ into it coincides with what occurs for the so-called $\widetilde{\text{III}}$-system of Ruijsenaars [13, 8], which is the action-angle dual of the standard trigonometric Ruijsenaars-Schneider system. This circumstance is not surprising in light of the fact [5] that the reduced 'main Hamiltonian' arising from $\mathcal{H}_1$ (2.4) is a $\widetilde{\text{III}}$-type Hamiltonian coupled to external fields. We display this Hamiltonian below after exhibiting the corresponding Lax matrices.

The unreduced free Hamiltonians $\mathcal{H}_j$, for any $j \in \mathbb{Z}^*$, mentioned in Section 2, can be written alternatively as

$$\mathcal{H}_j(K) = \frac{1}{2j}\mathrm{tr}(K\boldsymbol{J}K^\dagger\boldsymbol{J})^j = \frac{1}{2j}\mathrm{tr}(K^\dagger\boldsymbol{J}K\boldsymbol{J})^j. \tag{4.1}$$

One can verify (for example by using the standard r-matrix formula of the Poisson bracket on the Heisenberg double [6]) that the Hamiltonian flow generated by $\mathcal{H}_j$ reads

$$K(t_j) = \exp\left[\mathrm{i}t_j\left((K(0)\boldsymbol{J}K(0)^\dagger\boldsymbol{J})^j - \frac{1}{2n}\mathrm{tr}(K(0)\boldsymbol{J}K(0)^\dagger\boldsymbol{J})^j\mathbf{1}_{2n}\right)\right]K(0) \tag{4.2}$$

$$= K(0)\exp\left[\mathrm{i}t_j\left((\boldsymbol{J}K(0)^\dagger\boldsymbol{J}K(0))^j - \frac{1}{2n}\mathrm{tr}(\boldsymbol{J}K(0)^\dagger\boldsymbol{J}K(0))^j\mathbf{1}_{2n}\right)\right]. \tag{4.3}$$

Since the exponentiated elements reside in the Lie algebra $\mathfrak{su}(n,n)$, these alternative formulas show that the flow stays in $\mathrm{SL}(2n,\mathbb{C})'$, as it must, and imply that the building blocks g_L and g_R of $K = b_L g_R^{-1} = g_L b_R^{-1}$ follow geodesics on $\mathrm{SU}(n,n)$, while b_L and b_R provide constants of motion. Equivalently, the last statement means that

$$K\boldsymbol{J}K^\dagger\boldsymbol{J} = b_L\boldsymbol{J}b_L^\dagger\boldsymbol{J} \quad \text{and} \quad K^\dagger\boldsymbol{J}K\boldsymbol{J} = (b_R^{-1})^\dagger\boldsymbol{J}b_R^{-1}\boldsymbol{J} \tag{4.4}$$

stay constant along the unreduced free flows.

To elaborate the reduced Hamiltonians, note that for an element K of the form (2.22) we have

$$(b_R^{-1})^\dagger\boldsymbol{J}b_R^{-1}\boldsymbol{J} = \begin{bmatrix} e^{-2v}\mathbf{1}_n & -e^{-v}\alpha \\ e^{-v}\alpha^\dagger & e^{2v}\mathbf{1}_n - \alpha^\dagger\alpha \end{bmatrix}. \tag{4.5}$$

By using this, as explained in Appendix B, one can prove that on $\Phi_+^{-1}(\mu)$ the Hamiltonians $\mathcal{H}_j$ can be written (for all j), up to additive constants, as linear combinations of the expressions

$$h_k = \mathrm{tr}(\alpha^\dagger\alpha)^k, \qquad k = 1, \dots, n. \tag{4.6}$$

Since in this way the Hermitian matrix $L = \alpha^\dagger\alpha$ generates the commuting reduced Hamiltonians, it provides a Lax matrix for the reduced system. By inserting α from (3.5), we obtain the explicit formula

$$\begin{aligned} L(\hat{p}, e^{\mathrm{i}\hat{q}}) =& (e^{2v} + e^{-2u})e^{-2\hat{p}} + (e^{2v} + e^{-2v})\mathbf{1}_n \\ &- \sqrt{e^{-2u}e^{-2\hat{p}} + e^{-2v}\mathbf{1}_n}\, e^{\mathrm{i}\hat{q}}\theta(x,\hat{p})^{-1}e^{v}\sqrt{e^{-2\hat{p}} + \mathbf{1}_n} \\ &- e^{v}\sqrt{e^{-2\hat{p}} + \mathbf{1}_n}\,\theta(x,\hat{p})e^{-\mathrm{i}\hat{q}}\sqrt{e^{-2u}e^{-2\hat{p}} + e^{-2v}\mathbf{1}_n}. \end{aligned} \tag{4.7}$$

On the other hand, the Lax matrix of Ruijsenaars's $\widetilde{\mathrm{III}}$-system can be taken to be [13, 8]

$$\tilde{L}(\hat{p}, e^{\mathrm{i}\hat{q}}) = e^{\mathrm{i}\hat{q}}\theta(x,\hat{p})^{-1} + \theta(x,\hat{p})e^{-\mathrm{i}\hat{q}}. \tag{4.8}$$

The similarity of the structures of these Lax matrices as well as the presence of the external field couplings in (4.7) is clear upon comparison. The extension of the Lax matrix $\alpha^\dagger\alpha$ (4.7) to the full phase space $M \simeq \hat{M}_c$ is of course given by $\hat{\alpha}^\dagger\hat{\alpha}$ by means of (3.21).

The main reduced Hamiltonian found in [5] reads as follows:

$$\begin{aligned}\mathcal{H}_1(K(\hat{p}, e^{\mathrm{i}\hat{q}})) = &-\frac{e^{-2u}+e^{2v}}{2}\sum_{j=1}^n e^{-2\hat{p}_j} + \\ &+\sum_{j=1}^n \cos(\hat{q}_j)\big[1+(1+e^{2(v-u)})e^{-2\hat{p}_j}+e^{2(v-u)}e^{-4\hat{p}_j}\big]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^n\left[1-\frac{\sinh^2\left(\frac{x}{2}\right)}{\sinh^2(\hat{p}_j-\hat{p}_k)}\right]^{\frac{1}{2}}.\end{aligned} \tag{4.9}$$

Liouville integrability holds since the functional independence of the involutive family obtained by reducing $\mathcal{H}_1,\dots,\mathcal{H}_n$ (4.1) is readily established and the projections of the free flows (4.3) to M are automatically complete. Similarly to its analogue in [7], the Hamiltonian (4.9) can be identified as an Inozemtsev type limit of a specialization of van Diejen's 5-coupling deformation of the hyperbolic BC_n Sutherland Hamiltonian [14]. This fact suggests that it should be possible to extract the local form of dual Hamiltonians from [15] and references therein, which contain interesting results about closely related quantum mechanical systems and their bispectral properties. Indeed, in several examples, classical Hamiltonians enjoying action-angle duality correspond to bispectral pairs of Hamiltonian operators after quantization. In a future work, we wish to explore the action-angle dual of the Hamiltonian (4.9) in the reduction framework and employ the duality together with the traditional projection method for studying the associated dynamics.

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A Explicit formulas for the matrices θ, ζ, and κ

In this appendix we collect the explicit expressions of the matrices θ, ζ, and κ for the reader's convenience. More detailed information about these matrices can be found in the paper [8]. At an arbitrary point $\hat{p} \in \bar{\mathcal{C}}_x$ (3.1) the components of $\theta(x,\hat{p})$ are defined as follows

$$\theta(x,\hat{p})_{jk} = \frac{\sinh\left(\frac{x}{2}\right)}{\sinh(\hat{p}_k-\hat{p}_j)}\prod_{\substack{m=1\\(m\neq j,k)}}^n\left[\frac{\sinh(\hat{p}_j-\hat{p}_m-\frac{x}{2})\sinh(\hat{p}_k-\hat{p}_m+\frac{x}{2})}{\sinh(\hat{p}_j-\hat{p}_m)\sinh(\hat{p}_k-\hat{p}_m)}\right]^{\frac{1}{2}}, \quad j\neq k, \tag{A.1}$$

and

$$\theta(x,\hat{p})_{jj} = \prod_{\substack{m=1\\(m\neq j)}}^n\left[\frac{\sinh(\hat{p}_j-\hat{p}_m-\frac{x}{2})\sinh(\hat{p}_j-\hat{p}_m+\frac{x}{2})}{\sinh^2(\hat{p}_j-\hat{p}_m)}\right]^{\frac{1}{2}}. \tag{A.2}$$

Note that $\theta(x,\hat{p})$ is an orthogonal matrix of determinant 1. Next, with the help of the vector $r(x,\hat{p}) \in \mathbb{R}^n$ defined by

$$r(x,\hat{p})_j = \left[\frac{1-e^{-x}}{1-e^{-nx}}\right]^{\frac{1}{2}} \prod_{\substack{k=1 \\ (k\neq j)}}^{n} \left[\frac{1-e^{2\hat{p}_j-2\hat{p}_k-x}}{1-e^{2\hat{p}_j-2\hat{p}_k}}\right]^{\frac{1}{2}}, \quad j=1,\dots,n, \tag{A.3}$$

the entries of the real $n \times n$ matrix $\zeta(x,\hat{p})$ can be written as

$$\begin{aligned} &\zeta(x,\hat{p})_{nn} = r(x,\hat{p})_n, \quad \zeta(x,\hat{p})_{ij} = \delta_{ij} - \frac{r(x,\hat{p})_i r(x,\hat{p})_j}{1+r(x,\hat{p})_n}, \\ &\zeta(x,\hat{p})_{in} = -\zeta(x,\hat{p})_{ni} = r(x,\hat{p})_i, \quad i,j \neq n. \end{aligned} \tag{A.4}$$

Finally, by introducing the vector $v = v(x)$:

$$v(x)_j = \left[\frac{n(e^x-1)}{1-e^{-nx}}\right]^{\frac{1}{2}} e^{-\frac{jx}{2}}, \quad j=1,\dots,n, \tag{A.5}$$

the elements of the $n \times n$ matrix $\kappa(x)$ read

$$\begin{aligned} \kappa(x)_{nn} = \frac{v(x)_n}{\sqrt{n}}, \quad \kappa(x)_{ij} = \delta_{ij} - \frac{v(x)_i v(x)_j}{n+\sqrt{n}v(x)_n}, \\ \kappa(x)_{in} = -\kappa(x)_{ni} = \frac{v(x)_i}{\sqrt{n}}, \quad i,j \neq n. \end{aligned} \tag{A.6}$$

It can be shown that both $\kappa(x)$ and $\zeta(x,\hat{p})$ are orthogonal matrices of determinant 1. The main feature of $\kappa(x)$ is that (with $\nu(x)$ in (2.12)) the matrix $\kappa(x)^{-1}\nu(x)\nu(x)^\dagger\kappa(x)$ is diagonal. This implies that $\eta_L(1) = \kappa(x)\tau\kappa(x)^{-1} \in \mathrm{U}(n)$ satisfies (2.15) for any $\tau \in \mathbb{T}_n$, which we used in the main text (see (3.25)). In the above we assumed that $x > 0$, otherwise the definition of the matrices ζ and κ would need different formulas.

B On the reduced Hamiltonians

In this appendix we prove the claim, made in Section 4, that on the momentum surface $\Phi_+^{-1}(\mu)$ the Hamiltonians $\mathcal{H}_j$, $j \in \mathbb{Z}^*$ (4.1) are linear combinations of h_k, $k=1,\dots,n$ (4.6). This will be achieved by establishing the form of the integer powers of the matrix displayed in (4.5), which we denote here by $\mathcal{L}$, i.e.

$$\mathcal{L} = \begin{bmatrix} e^{-2v}\mathbf{1}_n & -e^{-v}\alpha \\ e^{-v}\alpha^\dagger & e^{2v}\mathbf{1}_n - \alpha^\dagger\alpha \end{bmatrix}. \tag{B.1}$$

Lemma B.1. *For any positive integer j, the j-th power of the $2n \times 2n$ matrix $\mathcal{L}$ (B.1) reads*

$$\mathcal{L}^j = \begin{bmatrix} \mathcal{L}_{11}^j & \mathcal{L}_{12}^j \\ \mathcal{L}_{21}^j & \mathcal{L}_{22}^j \end{bmatrix}, \tag{B.2}$$

where $\mathcal{L}^j_{11}, \mathcal{L}^j_{12}, \mathcal{L}^j_{21}, \mathcal{L}^j_{22}$ are $n \times n$ blocks of the form

$$\begin{aligned} \mathcal{L}^j_{11} &= \sum_{m=1}^{j} a_m^{(j)} (\alpha\alpha^\dagger)^{j-m}, \quad \mathcal{L}^j_{12} = \alpha \sum_{m=1}^{j} b_m^{(j)} (\alpha^\dagger\alpha)^{j-m}, \\ \mathcal{L}^j_{21} &= \alpha^\dagger \sum_{m=1}^{j} c_m^{(j)} (\alpha\alpha^\dagger)^{j-m}, \quad \mathcal{L}^j_{22} = (-1)^j (\alpha^\dagger\alpha)^j + \sum_{m=1}^{j} d_m^{(j)} (\alpha^\dagger\alpha)^{j-m}, \end{aligned} \tag{B.3}$$

with the $4j$ coefficients $a_m^{(j)}, b_m^{(j)}, c_m^{(j)}, d_m^{(j)}$, $m = 1, \dots, j$ depending only on the parameter v.

Proof. We proceed by induction on j. For $j = 1$ the statement clearly holds, and supposing that (B.2)-(B.3) is valid for some fixed integer $j > 0$ we simply calculate the $(j+1)$-th power $\mathcal{L}^{j+1} = \mathcal{L}\mathcal{L}^j$. This proves the statement. ☐

Our claim of linear expressibility follows at once, that is for any positive integer j we have

$$\mathcal{H}_j = (-1)^j h_j + \sum_{k=1}^{j-1} \frac{k}{j} (a_{j-k}^{(j)} + d_{j-k}^{(j)}) h_k + \frac{n}{2j} (a_j^{(j)} + d_j^{(j)}). \tag{B.4}$$

Incidentally, one also obtains a recursion for the coefficients $a_m^{(j)}, b_m^{(j)}, c_m^{(j)}, d_m^{(j)}$ from the proof of Lemma B.1. If they are required, this should enable one to establish the values of the constants that occur in (B.4).

As for the negative powers of $\mathcal{L}$, one readily checks that the inverse of $\mathcal{L}$ is

$$\mathcal{L}^{-1} = \begin{bmatrix} e^{2v}\mathbf{1}_n - \alpha\alpha^\dagger & e^{-v}\alpha \\ -e^{-v}\alpha^\dagger & e^{-2v}\mathbf{1}_n \end{bmatrix}, \tag{B.5}$$

which has essentially the same form as $\mathcal{L}$ does, thus the blocks of $\mathcal{L}^{-j}$ ($j > 0$) can be expressed similarly as in Lemma B.1. In fact, conjugating $\mathcal{L}^{-1}$ with the $2n \times 2n$ involutory block-matrix

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix}, \tag{B.6}$$

leads to the following formula

$$\boldsymbol{C}\mathcal{L}^{-1}\boldsymbol{C} = \begin{bmatrix} e^{-2v}\mathbf{1}_n & -e^{-v}\alpha^\dagger \\ e^{-v}\alpha & e^{2v}\mathbf{1}_n - \alpha\alpha^\dagger \end{bmatrix}, \tag{B.7}$$

which implies that the blocks of $\mathcal{L}^{-j}$ are obtained from those of $\mathcal{L}^j$ by reversing their order and interchanging the role of α and $\alpha^\dagger$. Furthermore, since $\mathrm{tr}((\alpha\alpha^\dagger)^k) = \mathrm{tr}((\alpha^\dagger\alpha)^k)$ we get

$$\mathcal{H}_{-j} = -\mathcal{H}_j \qquad \forall j \in \mathbb{Z}^*. \tag{B.8}$$

References

[1] J.E. Marsden and T.S. Ratiu, Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems, Springer, 2nd edition, 2002

[2] P. Etingof, Calogero-Moser Systems and Representation Theory, European Mathematical Society, 2007

[3] O. Babelon, D. Bernard and M. Talon, Introduction to Classical Integrable Systems, Cambridge University Press, 2007

[4] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rep. **71**, 313-400 (1981)

[5] I. Marshall, *A new model in the Calogero-Ruijsenaars family*, Commun. Math. Phys. **338**, 563-587 (2015); `arXiv:1311.4641 [math-ph]`

[6] M.A. Semenov-Tian-Shansky, *Dressing transformations and Poisson groups actions*, Publ. RIMS **21**, 1237-1260 (1985)

[7] L. Fehér and T.F. Görbe, *On a Poisson-Lie deformation of the trigonometric BC_n Sutherland system*, Nucl. Phys. B **901**, 85-114 (2015); `arXiv:1508.04991 [math-ph]`

[8] L. Fehér and C. Klimčík, *Poisson-Lie interpretation of trigonometric Ruijsenaars duality*, Commun. Math. Phys. **301**, 55-104 (2011); `arXiv:0906.4198 [math-ph]`

[9] B.G. Pusztai, *The hyperbolic BC(n) Sutherland and the rational BC(n) Ruijsenaars-Schneider-van Diejen models: Lax matrices and duality*, Nucl. Phys. B **856**, 528-551 (2012); `arXiv:1109.0446 [math-ph]`

[10] A. Alekseev and A. Malkin, *Symplectic structures associated to Lie-Poisson groups*, Commun. Math. Phys. **162**, 147-174 (1994); `arXiv:hep-th/9303038`

[11] J-H. Lu, *Momentum mappings and reduction of Poisson actions*, pp. 209-226 in: Symplectic Geometry, Groupoids, and Integrable Systems, Springer, 1991

[12] L. Fehér and C. Klimčík, *Self-duality of the compactified Ruijsenaars-Schneider system from quasi-Hamiltonian reduction*, Nucl. Phys. B **860**, 464-515 (2012); `arXiv:1101.1759 [math-ph]`

[13] S.N.M. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems III. Sutherland type systems and their duals*, Publ. RIMS **31**, 247-353 (1995)

[14] J.F. van Diejen, *Deformations of Calogero-Moser systems*, Theor. Math. Phys. **99**, 549-554 (1994); `arXiv:solv-int/9310001`

[15] J.F. van Diejen and E. Emsiz, *Spectrum and eigenfunctions of the lattice hyperbolic Ruijsenaars-Schneider system with exponential Morse term*, to appear in Ann. Henri Poincaré (2015); `arXiv:1508.03829 [math-ph]`