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Model predictive control of linear systems with preview information: feasibility, stability and inherent robustness

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Abstract—The use of available disturbance predictions within a nominal model predictive control formulation is studied. The main challenge that arises is the loss of recursive feasibility and stability guarantees when a persistent disturbance is present in the model and on the system. We show how standard stabilizing terminal conditions may be modified to account for the use of disturbances in the prediction model. Robust stability and feasibility are established under the assumption that the disturbance change across sampling instances is limited.

Index Terms—Linear systems; Uncertain systems; Predictive control for linear systems; Constrained control

I. INTRODUCTION

The stability and robustness of model predictive controllers has been investigated extensively over the last few decades, and the theoretical foundations are now mature [1]. Ultimately, the purpose of any feedback controller is to reduce the effects of uncertainty on a plant, and accordingly the MPC literature is—by and large—separable into three, depending on whether or how uncertainty is considered: nominal MPC for deterministic (*i.e.* disturbance-free) systems, robust MPC for systems subject to bounded uncertainty and stochastic MPC for systems subject to uncertainties for which a probabilistic description is available [2]. While the fundamental conditions for stabilizing nominal controllers have been known for some time [3], progress has been more recent in robust and stochastic forms of MPC [1]. Even so, nominal MPC remains the most popular choice in practical applications, owing to its greater simplicity with respect to both design and computation. The drawback to nominal MPC is, of course, that the guarantees provided by the controller are indeed only nominal; instances of zero robustness when the MPC controller employs a nominal model but the real system is subject to an additive disturbance are well documented [4]. Consequently, the inherent robustness properties of nominal MPC have been investigated thoroughly, and necessary and sufficient conditions are well established [5], [6].

Despite these successes, a problem that has received relatively little attention is that of regulation of a disturbed system by nominal MPC when a *forecast* or *preview information* of future disturbances is available for use in the prediction model. This problem does not exactly fit the descriptions of robust or stochastic MPC for regulation, but is closely related to topic of tracking, wherein it has been known for some time that a disturbance model must be included in the predictions in order to obtain offset-free tracking [7]; in that context, different classes of disturbance signals have been investigated, including piecewise constant [7]–[9] and periodic [10]–[12].

This *preview control* problem [13] that we study is motivated by applications in, *inter alia*, power systems [14], wind energy [15]–[17], water networks [18] and automotive control [19]. Another motivation is, as we show in Section II, the optimal control problem that arises in *distributed* MPC. The common feature is the availability of (possibly accurate) predictions about an exogenous disturbance acting on the plant, obtained from sensors or otherwise: for example, wind gusts

acting on a turbine may be predicted by LiDAR, while the load disturbance in a power network is partly deterministic (scheduled by contract) and hence predictable. Most approaches to preview MPC focus on formulating a controller in the setting of the application, and then demonstrating the benefits to closed-loop performance of including accurate preview information; stability guarantees are largely lacking. An exception is the schemes based on robust MPC: [18] proposes a controller that uses knowledge of a deterministic demand in a tube-based robust MPC scheme. The approach of [14] uses preview information to bound uncertainties, and a parametrization of the control law in terms of the disturbance is used obtain a robust controller. The uncertain load scheduling problem tackled by [20] employs a robust controller acting on the uncertain error dynamics, thus eliminating the scheduled disturbance in the prediction model.

The aim of this note is to study the direct use of a disturbance prediction within a *nominal* MPC formulation for regulation, and develop conditions for the recursive feasiblity, stability and inherent robustness of such a controller without using the well-known techniques from tracking or resorting to taking a robust or stochastic approach. We consider a linear time-invariant, constrained system subject to an additive disturbance, with the assumption that a sequence of future disturbances (over the horizon of the controller) is known to the controller at the current time, but—given the non-deterministic nature of the disturbance—this sequence may change arbitrarily (within a bounded set) at the next step, thus is *persistent*. It is known that nominal MPC controllers for linear systems *do* possess robustness to bounded additive disturbances [6], but these results are restricted to the case where the prediction model omits the disturbance. Technical challenges arise when the disturbance is included in the prediction model, which mean that feasibility and stability are not assured, even when standard stabilizing terminal conditions are employed.

Our contributions are the following: we modify standard terminal ingredients for nominal MPC [21] to account for the disturbance appearing in the prediction model; we consider a general form of terminal conditions permitting the use of a nonlinear terminal control law. Under the modifications, the ill-posedness of the optimal control problem is removed, and the desirable properties of invariance of the terminal set and monotonic descent of the terminal cost hold for the perturbed model. Then we study the inherent robustness (to changing preview information) of the controlled system; we find that, as in inherently robust nominal MPC, the system states converge to a robust positively invariant set around the origin, the size of which depends on the limit assumed (or imposed) on the step-to-step change of the disturbance predictions. Illustrative examples show the advantage of including the disturbance in the prediction model: the region of attraction is larger than that for same controller omitting the disturbance information.

The paper is organized as follows. Section II defines the problem and presents the basic MPC formulation that will be studied. In Section III, modifications to the optimal control problem, and the standard terminal conditions employed in MPC, are developed to allow for the inclusion of the predicted disturbance. In Section IV, feasibility and stability of the system are studied, under both unchanging and changing disturbances. Illustrative examples are given in Section V,

and conclusions are made in Section VI.

Notation and basic definitions: $\mathbb{R}_+$ is the set of non-negative reals and $\mathbb{N}$ is the set of positive integers. The set of symmetric positive semidefinite $n \times n$ matrices is $\mathbb{S}_+^n$ and the set of positive definite ones is $\mathbb{S}_{++}^n$. We use normal face to denote a real-valued vector, and boldface to denote a sequence of vectors. $|x|$ denotes a generic norm of a vector $x \in \mathbb{R}^n$; $\|x\|_Q^2$ denotes the quadratic form $x^\top Q x$ in $x \in \mathbb{R}^n$, where $Q \in \mathbb{R}^{n \times n}$. For sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, the Minkowski sum is $\mathcal{A} \oplus \mathcal{B} \triangleq \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$. The notation $A^{-1}(X)$ represents the preimage of the set X under the matrix A, *i.e.* $A^{-1}(X) \triangleq \{x : Ax \in X\}$. A set $X \subseteq \mathbb{R}^n$ is positively invariant for a (autonomous) system $x^+ = f(x)$, where $f \colon \mathbb{R}^n \mapsto \mathbb{R}^n$, if $x \in X$ implies $f(x) \in X$. A set $X \subseteq \mathbb{R}^n$ is control invariant for a (non-autonomous) system $x^+ = f(x, u)$ and input set $\mathbb{U}$, where $f \colon \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$, if for all $x \in X$ there exists a $u \in \mathbb{U}$ such that $f(x, u) \in X$. A C-set is a closed and bounded (compact) convex set containing the origin; if the origin is in the interior, then it is a PC-set. An LTI system $x^+ = Ax + Bu$ is reachable (controllable from the origin) if, for any $x_1 \in \mathbb{R}^n$, there exists a sequence of controls that steers the state from $x(0) = 0$ to x_1 in finite time. When it is clear from the context, and in order to keep notation simple, we reserve k as the sampling instant and i as the prediction step; that is, $x(k)$ denotes a state at sampling instant k, while $x(i)$ denotes an i-step ahead prediction of x; where confusion may arise, we write $x(k + i)$ to denote the latter.

II. PROBLEM SETUP AND BASIC FORMULATION

A. System description and control objective

Consider the regulation problem for a discrete-time, linear time-invariant (LTI) system subject to an additive disturbance:

$$x^+ = Ax + Bu + w \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^n$ are, respectively, the state, input, and disturbance at the current time, and x^+ is the successor state. The control objective is to regulate the state to the origin, despite the action of the disturbance, while satisfying any constraints on states and inputs:

$$x \in \mathbb{X} \subset \mathbb{R}^n, \quad u \in \mathbb{U} \subset \mathbb{R}^m$$

Assumption 1 (Basic system assumptions)**.** *The pair (A, B) is reachable, and the matrices (A, B) are known.*

Assumption 2 (Constraint sets)**.** *The sets $\mathbb{X}$ and $\mathbb{U}$ are PC-sets; the set $\mathbb{W}$ is a C-set.*

We make the following assumption about the preview information available to the controller at the time it makes control decisions:

Assumption 3 (Information available to the controller)**.**
  1) *The state $x(k)$ and disturbance $w(k)$ are known exactly at time k; the future disturbances are not known exactly but satisfy $w(k + i) \in \mathbb{W}, i \in \mathbb{N}$.*
  2) *At any time step k, a prediction of future disturbances, over a finite horizon, is available.*

Full state availability is a standard assumption, but disturbance measurement availability is not; this might be restrictive but can arise in problems in, for example, power systems and, as we show in Section II-C, distributed MPC.

The main focus of the note is on the second part of Assumption 3: the availability of disturbance predictions. Our aim is to study how such information can be used within a nominal MPC formulation, and establish conditions for feasibility, stability and inherent robustness of the resulting controller and closed-loop system. Note that we do not assume anything about the *accuracy* of the disturbance predictions, and in fact will permit the predictions to vary over time (in which case it is implicit that earlier predictions were not accurate).

B. A basic MPC formulation with disturbance information

We study an MPC formulation that would arise from the direct (and perhaps naïve) inclusion of disturbance predictions in a nominal MPC problem for regulation, without using techniques from tracking MPC to handle it. As per the preceding assumptions, the controller has knowledge of the current state x and disturbance w but, additionally, a *prediction* of the future disturbances. We define the disturbance prediction at the current time to be a sequence $\mathbf{w} \triangleq \{w(0), w(1), \dots, w(N - 1)\}$, where $w(0) = w$ (the measured disturbance), with that and each future value of the sequence lying within the bounded set $\mathbb{W}$. The optimal control problem for $(x, \mathbf{w})$ is

$$\mathbb{P}(x; \mathbf{w}) \colon \min\{V_N(x, \mathbf{u}; \mathbf{w}) : \mathbf{u} \in \mathcal{U}_N(x; \mathbf{w})\}$$

where $\mathcal{U}_N(x; \mathbf{w})$ is defined by, for $i = 0 \dots N - 1$,

$$\begin{aligned}
x(0) &= x, \\
x(i + 1) &= Ax(i) + Bu(i) + w(i), \\
x(i) &\in \mathbb{X}, \\
u(i) &\in \mathbb{U}, \\
x(N) &\in \mathbb{X}_f.
\end{aligned}$$

In this problem, the cost function comprises, in the usual way, a stage cost plus terminal penalty:

$$V_N(x, \mathbf{u}; \mathbf{w}) \triangleq V_f\big(x(N); \mathbf{w}\big) + \sum_{i=0}^{N-1} \ell\big(x(i), u(i); \mathbf{w}\big),$$

where $\ell \colon \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}_+$, $V_f \colon \mathbb{R}^n \mapsto \mathbb{R}_+$; both depend, in a way that is to be defined, on the disturbance sequence. The domain of this problem is $\mathcal{X}_N(\mathbf{w}) \triangleq \{x \in \mathbb{X} : \mathcal{U}_N(x; \mathbf{w}) \neq \emptyset\}$, and depends on $\mathbf{w}$. The objective of the controller is to steer the system towards its equilibrium despite the disturbances; when the disturbance is *persistent* then regulation is merely to a neighbourhood of the origin. In this case, we aim to achieve regulation to a neighbourhood of the origin using only nominal MPC, *i.e.* avoiding a robust formulations.

The solution of this problem at a state x and with disturbance sequence $\mathbf{w}$ yields an optimal control sequence

$$\mathbf{u}^0(x; \mathbf{w}) \triangleq \{u^0(0; x, \mathbf{w}), u^0(1; x, \mathbf{w}), \dots, u^0(N - 1; x, \mathbf{w})\}.$$

The application of the first control in the sequence defines the control law $u = \kappa_N(x; \mathbf{w}) = u^0(0; x, \mathbf{w})$. At the successor state $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w(0; \mathbf{w})$, the problem is solved again, yielding a new control sequence. Motivated by the limited accuracy of any disturbance prediction, the disturbance sequence used in the problem at x^+ may change arbitrarily to $\mathbf{w}^+$, and is not necessarily equal to the tail of the previous sequence $\mathbf{w}$.

C. A motivating application: distributed MPC

Our problem description can arise naturally in distributed forms of MPC. Consider the two dynamically coupled subsystems

$$\begin{aligned}
x_1^+ &= A_{11}x_1 + B_1 u_1 + A_{12}x_2 \\
x_2^+ &= A_{22}x_2 + B_2 u_2 + A_{21}x_1
\end{aligned}$$

and suppose each subsystem controller $p \in \{1, 2\}$ has an optimal control problem of the form

$$\mathbb{P}_p(x_p; \mathbf{w}_p) \colon \min\{V_{p,N}(x_p, \mathbf{u}_p; \mathbf{w}_p) : \mathbf{u}_p \in \mathcal{U}_{p,N}(x_p; \mathbf{w}_p)\}$$

where $\mathcal{U}_{p,N}(x_p; \mathbf{w}_p)$ is defined by

$$x_p(0) = x_p,$$
$$x_p(i+1) = A_{pp}x_p(i) + B_p u_p(i) + w_p(i),$$
$$x_p(i) \in \mathbb{X}_p,$$
$$u_p(i) \in \mathbb{U}_p,$$
$$x_p(N) \in \mathbb{X}_{p,f}.$$

Suppose that, at some sampling instant, controller 1 solves its problem first (assuming some information about $\mathbf{w}_1$—in the simplest case $\mathbf{w}_1 = \mathbf{0}$) to obtain an optimized future control sequence $\mathbf{u}_1^0(x_1)$ and associated sequence of state predictions, $\mathbf{x}_1^0(x_1)$. If the controller then shares these predictions with controller 2, then controller 2 receives preview information $\mathbf{w}_2 = \mathbf{x}_1^0(x_1)$ available to use in its own problem. In fact, $\mathbf{w}_2$ fits exactly with Assumption 3, for the first element $w_2(0) = x_1$ is known accurately, while the remaining elements are subject to change at the next sampling instant (after controller 1 solves its optimization problem once more).

Distributed MPC schemes essentially differ in how they use this information: it is known that the direct inclusion into the MPC formulation, as we have illustrated here, is problematic, and therefore iterative [22] and robust [23], [24] methods are typically used.

D. Stabilizing conditions for nominal MPC and issues that arise from including preview information

When the prediction model omits the disturbance, an established approach to ensuring desirable properties of the controller and closed-loop system is to design the cost and terminal set to satisfy some standard conditions [21]. In particular, when the model employed is $x^+ = Ax + Bu$ and neither the terminal set nor the cost function depend on $\mathbf{w}$, the latter are constructed to satisfy the following:

Assumption 4 (Terminal set properties). *The set $\mathbb{X}_f \subset \mathbb{X}$ is a PC-set.*

Assumption 5 (Cost function bounds). *The functions $\ell(\cdot, \cdot)$, $V_f(\cdot)$ are continuous, with $\ell(0,0) = 0$, $V_f(0) = 0$ and such that, for some $c_1 > 0$, $c_2 > 0$, $a > 0$,*

$$\ell(x, u) \geq c_1 |x|^a \text{ for all } x \in \mathcal{X}_N, u \in \mathbb{U}$$
$$V_f(x) \geq c_2 |x|^a \text{ for all } x \in \mathbb{X}_f$$

Assumption 6 (Basic stability assumption). *For all $x \in \mathbb{X}_f$,*

$$\min_{u \in \mathbb{U}} \{ V_f(Ax + Bu) + \ell(x, u) : Ax + Bu \in \mathbb{X}_f \} \leq V_f(x).$$

Assumption 7 (Control invariance of $\mathbb{X}_f$). *The set $\mathbb{X}_f$ is control invariant for $x^+ = Ax + Bu$ and the set $\mathbb{U}$.*

In turn, each of the *controllability sets* $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_N$, where

$$\mathcal{X}_{i+1} = \{ x \in \mathbb{X} : \exists u \in \mathbb{U} \text{ such that } Ax + Bu \in \mathcal{X}_i \},$$

with $\mathcal{X}_0 = \mathbb{X}_f$, is control invariant for $x^+ = Ax + Bu$ and $\mathbb{U}$, and these sets are *nested*: $\mathcal{X}_N \supseteq \mathcal{X}_{N-1} \supseteq \cdots \supseteq \mathcal{X}_1 \supseteq \mathcal{X}_0$. Furthermore, the set $\mathcal{X}_N$ (and $\mathcal{X}_{N-1}$) is positively invariant for $x^+ = Ax + Bu$ under the MPC control law $u = \kappa_N(x)$ and admissible with respect to the constraints. Consequently, if $x(0) \in \mathcal{X}_N$ then (i) the optimal control problem is recursively feasible, (ii) constraints are satisfied for all times, and the origin is asymptotically stable for the closed-loop system, with region of attraction $\mathcal{X}_N$ [21].

When disturbances are present in the model, the same ingredients fail to attain the same guarantees, and several issues arise.

- *Ill-posed objective.* For the perturbed dynamics, $(x = 0, u = 0)$ is not, in general, an equilibrium pair for the system. Consequently, employing a cost function that satisfies Assumptions 5

and 6 will *not* ensure that the value function satisfies the conditions of a Lyapunov function.

- *Loss of invariance of $\mathbb{X}_f$.* Control invariance of $\mathbb{X}_f$ for $x^+ = Ax + Bu$ does not imply control invariance for $x^+ = Ax + Bu + w$, even when w is constant. Invariance of $\mathbb{X}_f$ is a *necessary* condition for the controllability sets to be control invariant and nested when state constraints $\mathbb{X}$ are present [25].

- *Loss of nesting of the controllability sets.* The controllability sets are defined by the iteration

$$\mathcal{X}_{i+1} = \mathbb{X} \cap A^{-1} \left(\mathcal{X}_i \oplus -B\mathbb{U} \oplus -\{w(N-i)\} \right), \quad (2)$$

with $\mathcal{X}_0 = \mathbb{X}_f$. The final summand in (2) is problematic, for $\{w(N-i)\}$ is a point, not a set containing the origin, and therefore induces *translation* of the controllability sets between iterations. Moreover, if A is unstable, then with each iteration the controllability set is shifted further away as disturbance propagates through the unstable predictions. The implication is that $\mathcal{X}_N$ is not control invariant for $x^+ = Ax + Bu + w$, and recursive feasibility may not be easy to establish, even if $\mathcal{X}_f$ is control invariant for the terminal dynamics.

III. MODIFICATIONS TO THE COST AND TERMINAL INGREDIENTS

In this section, we develop modifications to the cost $\ell(\cdot, \cdot)$ and terminal ingredients $V_f(\cdot)$, $\mathbb{X}_f$ for the model predictive controller that aim to overcome the first two of the fundamental issues outlined in the previous section. With respect to the third issue, we show in Section IV that recursive feasibility of the control problem can be established without relying on nestedness of the controllability sets; in fact, we show that a different kind of nested property holds here.

Our approach is to modify the cost and terminal set using the available preview information. We assume a constant-disturbance terminal prediction model, take stabilizing ingredients designed for its nominal counterpart, and translate the costs and terminal set to account for the non-zero equilibrium caused by the disturbance; as we shall show, the resulting ingredients achieve the required properties of invariance and monotonicity with respect this new predicted equilibrium instead of the origin. The results in this section are essentially derived from known results on stabilizing MPC [21], but lay the foundations for our main results in Section IV.

A. Constant-disturbance terminal dynamics

In order to synthesize appropriate terminal ingredients, one first needs to consider the terminal (beyond the horizon) dynamics of the prediction model. This motivates the next assumption, regarding the disturbance sequence: by definition, the disturbance predictions are known only up to N steps ahead in the future, and therefore to consider the terminal dynamics we setup the predicted disturbance sequence in the following way:

Assumption 8. *An* admissible *disturbance sequence* $\mathbf{w} = \{w(0), \ldots, w(N-1)\}$ *has* $w(i) \in \mathbb{W}$, $i = 0 \ldots (N-2)$, *and* $w(N-1) \in \mathbb{W}_f \subseteq \mathbb{W}$; *we write* $\mathbf{w} \in \mathcal{W} \triangleq \mathbb{W} \times \cdots \times \mathbb{W} \times \mathbb{W}_f$. *To account for* $i \geq N$, *the sequence is continued indefinitely by concatenation with* $\mathbf{w}_f \triangleq \{w(i)\}_{i \geq N}$, *with* $w(i) = w_f(\mathbf{w}) \triangleq w(N-1)$ *for* $i \geq N$.

That is, the predicted disturbance is, after N steps, constant and equal to $w_f(\mathbf{w}) = w(N-1)$, the final value in the N-length sequence $\mathbf{w}$. This final value lies in a subset $\mathbb{W}_f \subseteq \mathbb{W}$; thus, $w_f : \mathcal{W} \mapsto \mathbb{W}_f$. A constant terminal disturbance assumption is not necessarily restrictive; for example, [11] considers disturbance models that are integrating and reach a constant value.

Assumption 8 implies that, at a time instant k and given the disturbance prediction $\mathbf{w}$, the terminal prediction dynamics are given

by $x^+ = Ax + Bu + w_f(\mathbf{w})$, with $w_f(\mathbf{w})$ constant for all prediction steps $i \geq N$. At the next time instant, however, $w_f(\mathbf{w}^+)$ need not be equal to $w_f(\mathbf{w})$, and we address this issue in Section IV. Until then, for simplicity of notation we refer to $w_f(\mathbf{w})$ as w_f.

B. Equilibria of the terminal dynamics

To proceed, we first assume the availability of terminal ingredients for the nominal system; in the sequel, the overbar notation denotes a function or set pertaining to the nominal dynamics $x^+ = Ax + Bu$.

Assumption 9. *There is known a set $\bar{\mathbb{X}}_f$ and functions $\bar{\ell}(\cdot,\cdot)$, $\bar{V}_f(\cdot)$ satisfying Assumptions 4–7 for the dynamics $x^+ = Ax + Bu$ and constraint sets $(\beta_x \mathbb{X}, \beta_u \mathbb{U})$, where $\beta_x, \beta_u \in [0,1)$.*

The set $\bar{\mathbb{X}}_f$ is a control invariant set (for the nominal dynamics) that resides within $\beta_x \mathbb{X}$, which is the state constraint set scaled to allow for the effect of non-zero disturbances on the terminal dynamics; the selection of suitable scaling factors β_x and β_u is described in the sequel. Assumption 9 implies the existence of a (possibly set-valued) control law, $u_f : \mathbb{R}^n \mapsto 2^{\mathbb{R}^m}$ that induces invariance of the set $\bar{\mathbb{X}}_f$ with respect to the nominal dynamics:

$$u_f(x) = \{ u \in \beta_u \mathbb{U} : Ax + Bu \in \bar{\mathbb{X}}_f,$$
$$\bar{V}_f(Ax + Bu) + \bar{\ell}(x, u) \leq \bar{V}_f(x) \}, \text{ for } x \in \bar{\mathbb{X}}_f. \quad (3)$$

We denote the system $x^+ = Ax + Bu$ under the control law $u = \bar{\kappa}_f(x)$—where, if necessary, $\bar{\kappa}_f(x)$ is an appropriate selection from $u_f(x)$, but is otherwise equal to $u_f(x)$—as $x^+ = f_{\bar{\kappa}_f}(x) = Ax + B\bar{\kappa}_f(x)$ (so that $f_{\bar{\kappa}_f} : \bar{\mathbb{X}}_f \mapsto \bar{\mathbb{X}}_f$). The following result is an immediate consequence of the existence of $\bar{\kappa}_f(\cdot)$ satisfying the conditions of Assumption 4–7.

Lemma 1 (Stabilizing terminal control law). *Suppose Assumptions 1 and 9 hold. The origin is an asymptotically stable equilibrium for the system $x^+ = f_{\bar{\kappa}_f}(x)$, with region of attraction $\bar{\mathbb{X}}_f$.*

The difficulty with translating the terminal ingredients to the new steady state or equilibrium point of the system perturbed by w_f is that this point is not immediately known, since the terminal control law is nonlinear and/or set-valued. The approach we take is to linearize the terminal dynamics and control law around the equilibrium point $x = 0 \in \bar{\mathbb{X}}_f$, for which we make the following assumption.

Assumption 10. *$\bar{\kappa}_f(\cdot)$ is continuous over $\bar{\mathbb{X}}_f$, with $\bar{\kappa}_f(0) = 0$, and continuously differentiable in a neighbourhood of $x = 0$.*

In reality, this is a mild assumption: a ready choice of $\bar{\kappa}_f(\cdot)$ is a linear, stabilizing K. Even if $\bar{\kappa}_f(\cdot)$ is piecewise linear in order to maximize the size of $\bar{\mathbb{X}}_f$ (as in [26]), the control law is typically linear (for a linear system) in a neighbourhood of the origin when $\mathbb{X}$ and $\mathbb{U}$ are PC-sets. Then the linearization of $x^+ = f_{\bar{\kappa}_f}(x)$ yields

$$\Pi \triangleq \left. \frac{\partial \bar{\kappa}_f(x)}{\partial x} \right|_{x=0}; \quad \Phi \triangleq \left. \frac{\partial (Ax + B\bar{\kappa}_f(x))}{\partial x} \right|_{x=0} = A + B\Pi \quad (4)$$

from which the equilibrium point of the linearized system $x^+ = \Phi x + w_f$ is

$$x_f(w_f) = \Psi w_f, \qquad u_f(w_f) = \Pi \Psi w_f$$

where $\Psi \triangleq (I - \Phi)^{-1}$. In view of Lemma 1 and Assumption 10, $\Phi = (A + B\Pi)$ is well defined and strictly stable, hence $\Psi = (I - \Phi)^{-1}$ exists and is unique. This proves the following.

Lemma 2 (Existence of equilibrium). *Suppose Assumptions 1, 9 and 10 hold. The point $x_f(w_f)$ exists and is the unique equilibrium of $x^+ = \Phi x + w_f$, with w_f constant.*

C. Modified terminal conditions: translation and properties

We propose to translate the cost function, terminal sets and control law to this new point (x_f, u_f):

$$\ell(x, u; w_f) \triangleq \bar{\ell}(x - x_f(w_f), u - u_f(w_f))$$
$$V_f(x; w_f) \triangleq \bar{V}_f(x - x_f(w_f))$$
$$\mathbb{X}_f(w_f) \triangleq \bar{\mathbb{X}}_f \oplus \{x_f(w_f)\}$$
$$\kappa_f(x; w_f) \triangleq \bar{\kappa}_f(x - x_f(w_f)) + u_f(w_f)$$

The translated sets and functions are then the ones employed in the optimal control problem with perturbed prediction model. The main result of this section then establishes that the resulting terminal ingredients satisfy the required conditions (counterparts to Assumptions 4–6) for $x^+ = Ax + Bu + w_f$. First, the following assumption is required.

Assumption 11. *There exist scalars $\alpha_x, \alpha_u \in [0,1)$ such that*

$$\Psi \mathbb{W}_f \subseteq \alpha_x \mathbb{X} \qquad \qquad \Pi \Psi \mathbb{W}_f \subseteq \alpha_u \mathbb{U}$$

This assumption essentially limits the size of the terminal disturbance set, $\mathbb{W}_f$, with respect to the state and input constraint sets. Then the following result restores the stability properties.

Proposition 1. *Suppose that Assumptions 1–3, 8–11 hold. For any fixed $w_f \in \mathbb{W}_f$, (i) the set*

$$\mathbb{X}_f(w_f) = \bar{\mathbb{X}}_f \oplus \{x_f(w_f)\} \quad (5)$$

is positively invariant for $x^+ = Ax + B\kappa_f(x; w_f) + w_f$, and (ii) the functions $\ell(x, u; w_f)$ and $V_f(x; w_f)$ satisfy

$$V_f([Ax + B\kappa_f(x; w_f) + w_f]; w_f) + \ell(x, \kappa_f(x; w_f); w_f)$$
$$\leq V_f(x; w_f) \text{ for all } x \in \mathbb{X}_f(w_f).$$

Furthermore, (iii) $\mathbb{X}_f(w_f) \subseteq \mathbb{X}$, and $\kappa_f(\mathbb{X}_f; w_f) \subseteq \mathbb{U}$ if $\alpha_x + \beta_x \leq 1$ and $\alpha_u + \beta_u \leq 1$. Finally, (iv) the point $x_f(w_f)$ is an asymptotically stable equilibrium point for the system $x^+ = Ax + B\kappa_f(x; w_f) + w_f$. The region of attraction is $\mathbb{X}_f(w_f)$.

This result implies that the set $\mathbb{X}_f(w_f)$ is control invariant for the system $x^+ = Ax + Bu + w_f$, w_f constant, and input set $\mathbb{U}$. In particular, the set is positively invariant for $x^+ = Ax + B\kappa_f(x; w_f) + w_f$. Within this set, $V_f(\cdot)$ is a control Lyapunov function. Thus, we have established stability of the predicted terminal dynamics to an equilibrium point under the assumption of a constant disturbance w_f. In the next section, we consider the receding-horizon implementation of the optimal control problem, allowing the terminal disturbance to change across sample times but, at each sampling instant, remain constant beyond the end of the horizon.

Remark 1 (Offset-free regulation and links with tracking and robust MPC). *In view of the regulation objective, the stabilization of the perturbed system under constant disturbances to $x_f(w_f) = \Psi w_f$ may seem unambitious. Ultimately, however, this is an issue of available degrees of freedom: if $x = 0$ is required to be an equilibrium of $x^+ = Ax + Bu + w_f$, then standard results from offset-free tracking inform us this is possible, in general, if and only if $m \geq n$—a strong condition that is typically not met unless the system is over-actuated. A pragmatic solution, then, can be to define some outputs $y = Cx \in \mathbb{R}^p$, with $p \leq m$, and steer the states to values that satisfy $y_f = Cx_f = 0$ (or minimize $\|Cx_f\|^2$). With the regulation goal in mind, such a scheme requires careful (and application-dependent) design of C in order to ensure that some states do not converge to values far from the origin.*

When such an option is not pursued, there is an interesting link with what is known from robust MPC. The smallest neighbourhood of

the origin that the states of a perturbed system $x^+ = \Phi x + w$, under bounded $w \in \mathbb{W}$, can stay within is the minimal robust positively invariant (mRPI) set, given by the Minkowski summation

$$\mathcal{R}_\infty \triangleq \bigoplus_{i=0}^{\infty} \Phi^i \mathbb{W}.$$

There is a strong connection between this set and the equilibrium point $x_f(w_f) = \Psi w_f$ under a constant disturbance w_f: $x_f(w_f) \in \mathcal{R}_\infty$ and, similar to this summation,

$$x_f(w_f) = (I - \Phi)^{-1} w_f = (I + \Phi + \Phi^2 + \dots) w_f = \bigoplus_{i=0}^{\infty} \Phi^i \{w_f\}.$$

IV. FEASIBILITY, STABILITY AND INHERENT ROBUSTNESS UNDER CHANGING DISTURBANCES

In this section, we analyse the properties of the MPC controller with the proposed cost and terminal set modifications, considering the possibility that the disturbance sequence provided to the controller changes over time. Given a disturbance sequence $\mathbf{w}$, the set of states for which the problem $\mathbb{P}(x; \mathbf{w})$ is feasible is $\mathcal{X}_N(\mathbf{w}) = \{x \in \mathbb{X} : \mathcal{U}_N(x; \mathbf{w}) \neq \emptyset\}$. The challenge that arises is a consequence of the receding horizon implementation of MPC: if $\mathbb{P}(x; \mathbf{w})$ is feasible and yields a solution $\mathbf{u}^0(x; \mathbf{w}) \triangleq \{u^0(0; x, \mathbf{w}), u^0(1; x, \mathbf{w}), \dots, u^0(N-1; x, \mathbf{w})\}$, then the first control in this sequence is applied (the implicit control law is $\kappa_N(x; \mathbf{w}) \triangleq u^0(0; x, \mathbf{w})$), the system evolves to $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w$, and the problem to be solved at the subsequent time is $\mathbb{P}(x^+; \mathbf{w}^+)$. The question is, when is this problem feasible, given that the disturbance sequence may have changed (perhaps arbitrarily) from $\mathbf{w}$ to $\mathbf{w}^+$?

A. Unchanging disturbance sequences

Our first results on feasibility and stability consider the artificial situation where the disturbance sequence is shifted in time but otherwise unchanged, *i.e.* the sequence $\mathbf{w}^+$ is the *tail* of the sequence $\mathbf{w}$, and the disturbance acting on the plant is equal to the first element in the sequence at each sampling instant. These results, though immediate consequences of the preceding developments and standard results in MPC, form the basis for the more realistic case of a persistently changing disturbance sequence considered in Sec. IV-B.

In order to present the results, we need to define some notation. For a disturbance sequence $\mathbf{w} = \{w(0), w(1), \dots, w(N-2), w(N-1)\}$, the associated tail is $\tilde{\mathbf{w}}(\mathbf{w}) \triangleq \{w(1), \dots, w(N-1), w(N-1)\}$ where the first $N-1$ elements of $\mathbf{w}$ are augmented by continuing the final value $w(N-1)$ for one further step. The infinite-length disturbance sequence $\mathbf{w}_N(k)$ is formed from concatenating $\mathbf{w}(k)$ and $\mathbf{w}_f(k)$ where

$$\mathbf{w}_N(k) = \{w(0; \mathbf{w}(k)), \dots, w(N-1; \mathbf{w}(k)), \mathbf{w}_f(k)\}.$$

The latter is the infinite sequence of constant disturbances obtained by holding the final value of $\mathbf{w}(k)$—that is, $\mathbf{w}_f(k) = \{w_f(\mathbf{w}(k))\}$ where $w_f(\mathbf{w}(k)) = w(N-1; \mathbf{w}(k))$. Finally, $\mathbf{w}_i(k)$, for $i = 0 \dots N$, is a version of $\mathbf{w}_N(k)$ of omitting the first $N-i$.

The next result, and those in the remainder of the paper, suppose that Assumptions 1–3, 9–11 hold.

Proposition 2 (Feasibility under unchanging disturbance). *Let* $\mathbf{w} = \{w(0), w(1), \dots, w(N-1)\} \in \mathcal{W}$. *If* $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$, *then (i)* $x \in \mathcal{X}_N(\mathbf{w})$ *implies* $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w(0; \mathbf{w}) \in \mathcal{X}_N(\mathbf{w}^+)$. *(ii) Given* $\mathbf{w}(0)$, *the set*

$$\bigcup_{k=0}^{\infty} \mathcal{X}_N(\mathbf{w}(k)) \text{ with } \mathbf{w}(k+1) = \tilde{\mathbf{w}}(\mathbf{w}(k))$$

is control invariant for $x(k+1) = Ax(k) + Bu(k) + w(k)$ and $\mathbb{U}$, where $w(k) = w(0; \mathbf{w}(k))$. (iii) The unions of controllability sets are nested:

$$\bigcup_{k=0}^{\infty} \mathcal{X}_N(\mathbf{w}_N(k)) \supseteq \bigcup_{k=0}^{\infty} \mathcal{X}_{N-1}(\mathbf{w}_{N-1}(k)) \supseteq \cdots \supseteq \bigcup_{k=0}^{\infty} \mathcal{X}_0(\mathbf{w}_0(k)),$$

where

$$\mathcal{X}_{i+1}(\mathbf{w}_{i+1}) = \mathbb{X} \cap A^{-1}\big(\mathcal{X}_i(\mathbf{w}_i) \oplus -B\mathbb{U} \oplus -\{w(N-i)\}\big),$$

with $\mathcal{X}^0(\mathbf{w}_0) = \mathbb{X}_f(\mathbf{w}_f)$, and $\mathbf{w}_i(k+1) = \tilde{\mathbf{w}}(\mathbf{w}_i(k))$.

Remark 2. *Whereas the individual sets* $\mathcal{X}_N, \mathcal{X}_{N-1}, \dots, \mathcal{X}_0$ *are not necessarily control invariant and nested, their unions are, provided the disturbance sequence updates by taking the tail.*

Remark 3. *Owing to the nilpotency of the dynamics of the disturbance sequence (that is,* $\mathbf{w}_N(k+1) = \tilde{\mathbf{w}}(\mathbf{w}(k))$ *converges from* $\mathbf{w}(0)$ *to* $\mathbf{w}_f$ *in* N *steps), the set unions in Proposition 2 are finitely determined:*

$$\bigcup_{k=0}^{\infty} \mathcal{X}_N(\mathbf{w}(k)) = \bigcup_{k=0}^{N} \mathcal{X}_N(\mathbf{w}(k)).$$

Recursive feasibility leads to the following result.

Proposition 3 (Exponential stability of $x_f(w_f)$). *Let* $\mathbf{w}(0) \in \mathcal{W}$ *and* $x(0) \in \mathcal{X}_N(\mathbf{w}(0))$. *If the disturbance sequence is updated as* $\mathbf{w}(k+1) = \tilde{\mathbf{w}}(\mathbf{w}(k))$, *and the disturbance applied to the system is* $w(k) = w(0; \mathbf{w}(k))$, *then the point* $x_f(\mathbf{w}_f)$ *is an exponentially stable equilibrium for the system* $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w$. *The region of attraction is* $\mathcal{X}_N(\mathbf{w}(0))$.

Proposition 3 highlights the advantage of including the disturbance sequence in the prediction model and adjusting the terminal conditions accordingly: stability of a point is achieved. On the other hand, when the disturbance sequence is *not* included in the prediction model, we have the following well-known result from the inherent robustness of nominal MPC [6], [21], specialized to this setting.

Lemma 3 (Robust stability of nominal MPC). *Let* $\mathbf{w}(0) \in \mathcal{W}$ *and* $x(0) \in \Omega_\beta(\mathbf{w} = \mathbf{0}) \triangleq \{x : V_N^0(x; \mathbf{w} = \mathbf{0}) \leq \beta\} \subset \mathcal{X}_N(\mathbf{w} = \mathbf{0})$. *If* $\mathcal{W}$ *is sufficiently small, then* $\Omega_\beta(\mathbf{0})$ *is a robust positively invariant set for* $x^+ = Ax + B\kappa_N(x; \mathbf{0}) + w$. *Moreover, the system states converge, in finite time, to a robust positively invariant set* $\Omega_\alpha(\mathbf{0}) \subseteq \Omega_\beta(\mathbf{0})$, *the size of which depends on* $\mathbb{W}$.

We do not attempt to qualify what is sufficiently small here; details can be found in, *e.g.*, [21, Chapter 3]. The main point is that the states of the closed-loop system can be guaranteed to converge to a neighbourhood of the origin, rather than an equilibrium point, when the disturbance is omitted from the predictions.

B. Persistently changing disturbance predictions

We next consider the case where the disturbance sequence is permitted to change over sampling instances, but where the rate of change is limited. As a prerequisite to our developments, we consider the composite state $z \triangleq (x, \mathbf{w})$ and define

$$\mathcal{Z}_N \triangleq \{z = (x, \mathbf{w}) : \mathbf{w} \in \mathcal{W}, x \in \mathcal{X}_N(\mathbf{w})\} \subset \mathbb{R}^{n+nN}.$$

In view of the properties of $\mathcal{X}_N$ and $\mathcal{W}$, this is a PC-set. The following lemma is a well established consequence of linearity of the system, continuity of the cost, and compactness of the constraint sets [6].

Lemma 4 ($\mathcal{K}$-continuity of the value function). *The value function* $V_N^0(\cdot)$ *satisfies* $|V_N^0(z_1) - V_N^0(z_2)| \leq \sigma_V(|z_1 - z_2|)$ *over* $\mathcal{Z}_N$, *with* σ_V *a* $\mathcal{K}$-function.

We exploit this continuity in bounding the rate of change of disturbance sequences, as follows, and ultimately in establishing robust stability.

Assumption 12. *The disturbance sequence evolves as* $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w}) + \Delta\mathbf{w}$, *where* $\Delta\mathbf{w} = \mathbf{w}^+ - \tilde{\mathbf{w}}(\mathbf{w}) \in \Delta\mathcal{W} \subseteq \mathcal{W}$. *Moreover,* $\Delta\mathcal{W}$ *is chosen such that*

$$\lambda \triangleq \max\{|\mathbf{w} - \mathbf{w}'| : \mathbf{w} \in \mathcal{W}, \mathbf{w}' \in \mathcal{W}, (\mathbf{w} - \mathbf{w}') \in \Delta\mathcal{W}\}$$

satisfies

$$\lambda \leq \sigma_V^{-1}\big((\rho - \gamma)\alpha\big)$$

for $\rho \in (\gamma, 1)$, *where* $\gamma = (1 - c_1/c_3) \in (0, 1)$ *and* $c_3 \geq c_2 \geq c_1 > 0$, *and* $\alpha > 0$ *such that* $\Omega_\alpha^z \triangleq \{z = (x, \mathbf{w}) : V_N^0(z) \leq \alpha\} \subset \mathcal{Z}_N$.

This assumption aims to bound the persistent change in the disturbance predictions across sampling times, in order that the unanticipated change is not so large that the stability of the controller is lost. The constants c_1 and c_2 in this assumption are the same ones from Assumption 5, which bound the stage and terminal costs. The constant c_3 is from the upper bound on the MPC cost, derived in the proof of Proposition 3; γ is the rate of the value function descent, $V_N^0(x^+, \tilde{\mathbf{w}}(\mathbf{w})) \leq \gamma V_N^0(x; \mathbf{w})$, in this result.

We then have the following, main result.

Theorem 1. *Suppose Assumptions 1–3, 9–12 hold, and let* $z(0) = \big(x(0), \mathbf{w}(0)\big) \in \Omega_\beta^z = \{(x, \mathbf{w}) : V_N^0(z) \leq \beta\} \subset \mathcal{Z}_N$, *for some* $\beta \geq \alpha$. *The set* Ω_β^z *is positively invariant for the composite system*

$$x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w,$$
$$\mathbf{w}^+ \in \tilde{\mathbf{w}}(\mathbf{w}) \oplus \Delta\mathcal{W}$$

and the states of the system enter Ω_α^z *in finite time and remain therein.*

Corollary 1. *If* $x(0) \in \Omega_\beta\big(\mathbf{w}(0)\big) = \{x : \big(x, \mathbf{w}(0)\big) \in \Omega_\beta^z\} \subset \mathcal{X}_N\big(\mathbf{w}(0)\big)$, *then the system state* $x(k)$ *remains within* $\Omega_\beta\big(\mathbf{w}(k)\big) = \{x : \big(x, \mathbf{w}(k)\big) \in \Omega_\beta^z\} \subset \mathcal{X}_N\big(\mathbf{w}(k)\big)$ *provided that the disturbance sequence update rate is limited as specified. Morever, all optimization problems remain feasible, and for all* k *after some finite time* k', *the state enters and remains in a set* $\Omega_\alpha\big(\mathbf{w}(k')\big) = \{x : \big(x, \mathbf{w}(k')\big) \in \Omega_\alpha^z\}$ *where* $\alpha \leq \beta$.

Corollary 2. *The region of attraction for the controlled system is*

$$\bigcup_{k=0}^{\infty} \Omega_\beta\big(\mathbf{w}(k)\big) \text{ with } \mathbf{w}(k+1) \in \tilde{\mathbf{w}}\big(\mathbf{w}(k)\big) \oplus \Delta\mathcal{W}.$$

Given $\mathbf{w}(0)$, the set $\Omega_\beta\big(\mathbf{w}(0)\big)$ can be chosen to be the largest sublevel set of the value function within $\mathcal{X}_N\big(\mathbf{w}(0)\big)$; our result says then that the system converges to the smallest sublevel set that satisfies Assumption 12. Comparing this result with the result of Lemma 3, the potential advantage of the disturbance sequence inclusion becomes clear: the regions of attraction with and without the disturbance predictions are—using the same terminal sets in each optimal control problem—$\bigcup_{k=0} \Omega_\beta\big(\mathbf{w}(k)\big)$ and $\Omega_\beta(\mathbf{0})$ respectively. Care must be taken, however, before conclusions are made, because a conventional nominal formulation (omitting the disturbance) permits the use of a larger terminal set ($\mathbb{X}_f \subseteq \mathbb{X}$ compared with $\bar{\mathbb{X}}_f \subseteq \beta_x\mathbb{X}$). The example in the next section shows that, despite this, the proposed scheme can lead to a larger robust region of attraction.

Remark 4 (Comparison with robust MPC). *We have studied a nominal formulation including disturbance predictions and analysed its inherent robustness to persistently changing predictions. Theorem 1, based on Assumption 12, offers interesting theoretical insight but is not constructive and, thus, is of limited practical use: several constants and sublevel sets of the value function would need to be determined, which has to be done numerically. Robust tracking MPC techniques, which could also be applied to this problem, offer an attractive alternative, with robust-by-design guarantees and a region of feasibility that is equal to the region of attraction and possible to characterize explicitly. However, robust techniques rely on constraint restrictions and robust invariant sets [1], neither of which a nominal formulation requires, which introduces a degree of complexity to the design.*

V. ILLUSTRATIVE EXAMPLES

Consider the neutrally stable dynamics

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w \tag{6}$$

with state and input constraints $\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 10\}$, $\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 3\}$. The disturbance takes values in the C-set $\mathbb{W} = \{w = (w_1, w_2) \in \mathbb{R}^2 : |w_1| \leq 2, \ w_1 = w_2\}$. The nominal stage and terminal costs are quadratic:

$$\bar{\ell}(x, u) = x^\top Q x + u^\top R u \qquad \bar{V}_f(x) = x^\top P x,$$

where $Q = I$, $R = 1$, and P is solution of the Lyapunov equation $(A + BK_f)^\top P(A + BK_f) - P = -(Q + K_f^\top R K_f)$ for a stabilizing gain $K_f \in \mathbb{R}^{m \times n}$ characterizing the nominal terminal control law $\bar{\kappa}_f(x) = K_f x$. The matrices Φ and Π are determined according to (4), respectively, using the nominal terminal control law, $\Phi = A_{K_f} = A + BK_f$ and $\Psi = (I - A_{K_f})^{-1}$. The nominal terminal set, $\bar{\mathbb{X}}_f$, and scaling constants β_x, β_u, α_x and α_u are designed in line with the requirements of Assumptions 9 and 11: $\bar{\mathbb{X}}_f$ is the maximal constraint admissible invariant set for $x^+ = A_{K_f} x$ within the state set $\beta_x\mathbb{X}$ and input set $\beta_u\mathbb{U}$, while $\Psi\mathbb{W}_f$ and $K_f\Psi\mathbb{W}_f$ must fit within $\alpha_x\mathbb{X}$ and $\alpha_u\mathbb{U}$ respectively. With the assumption that $\mathbb{W}_f = \mathbb{W}$, we obtain $\beta_x = 0.672$, $\beta_u = 0.7$, $\alpha_x = 0.328$ and $\alpha_u = 0.3$.

We consider controlling the system from different initial states using a horizon of $N = 3$ and the following disturbance predictions at each sampling instant k:

$$\mathbf{w}(k) = \begin{cases} \mathbf{w}_0 = \{+0.9, -0.9, -0.9\} & \mod(k, 5) = 0 \\ \mathbf{w}_1 = \{+0.9, +0.9, +0.9\} & \mod(k, 5) = 1 \\ \mathbf{w}_2 = \{-0.9, -0.9, -0.9\} & \mod(k, 5) = 2 \\ \mathbf{w}_3 = \{-0.9, +0.9, -0.9\} & \mod(k, 5) = 3 \\ \mathbf{w}_4 = \{0, 0, 0\} & \mod(k, 5) = 4 \end{cases}$$

The disturbance prediction cycles persistently between five different sequences, while the disturbance applied to the plant at time k is always the first in the sequence $\mathbf{w}(k)$: *i.e.* $w(k) = w\big(0; \mathbf{w}(k)\big)$, so that $w(0) = +0.9$, $w(1) = +0.9$, $w(2) = -0.9$, *etc.* Following Assumption 12, the value of the maximum disturbance variation is $\lambda = 3.17$.

Figure 1 shows the trajectories of the closed-loop system from five different initial states, together with the level sets of the value function, $\Omega_\beta(\mathbf{w}_i) = \{x : V_N^0(x; \mathbf{w}_i) \leq \beta\}$ for $i = 0 \ldots 4$ and $\beta = 100$. Each of the trajectories begins in $\Omega_\beta(\mathbf{w}_0)$—the level set corresponding to the disturbance sequence at $k = 0$—and each subsequent state lies within the appropriate $\Omega_\beta(\mathbf{w}(k))$; however, $x(k) \in \Omega_\beta(\mathbf{w}(k))$ does not imply $x(k+1) \in \Omega_\beta(\mathbf{w}(k))$, as can be seen with the trajectory initialized at $x(0) = [1.9\ 2.5]^\top$, which begins in $\Omega_\beta(\mathbf{w}_0)$ and moves outside to $\Omega_\beta(\mathbf{w}_1)$. Since the disturbances and their predictions are switching between non-decaying values, the states do not converge to zero, but rather to a neighbourhood of the origin.

Figure 2 compares the union of $\beta = 100$ sublevel sets over $\mathbf{w} \in \mathcal{W}$ with the corresponding level set for a conventional MPC controller, omitting the disturbance. As we pointed out in the previous section, the region of attraction for the conventional controller is at least as
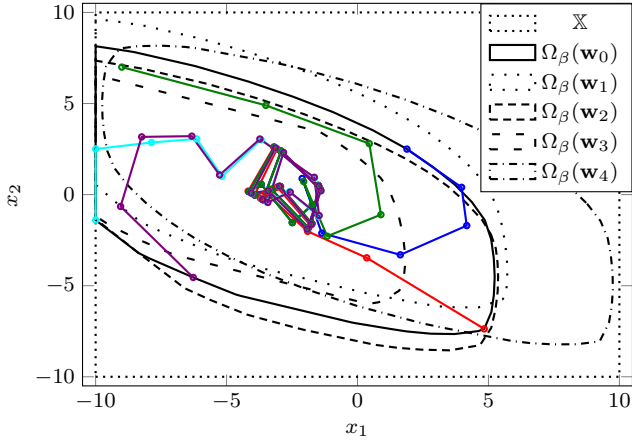
Fig. 1. Closed-loop trajectories of the system under switching disturbances, from different initial states, and the level sets $\Omega_\beta(\mathbf{w}_i)$, $i = 0 \ldots 4$, for $\beta = 100$, corresponding to the disturbance sequences $\mathbf{w}_i$, $i = 0 \ldots 4$.
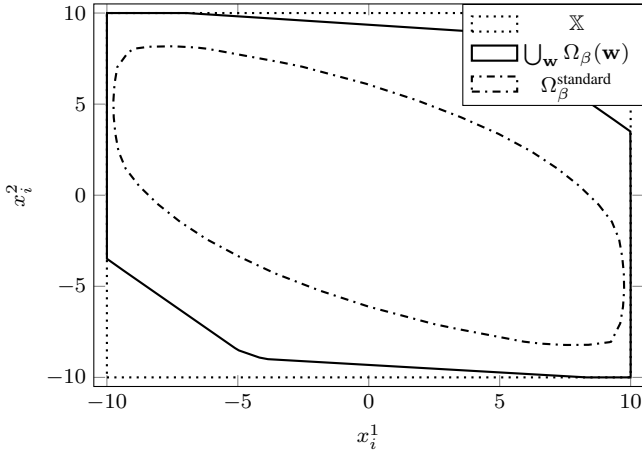

Fig. 2. The union of level sets $\Omega_\beta(\mathbf{w})$ over $\mathbf{w} \in \mathcal{W}$ for $\beta = 100$, and the corresponding level set $\Omega_\beta^{\text{standard}}$ for a conventional MPC formulation.

large as $\Omega_\beta(\mathbf{w} = \mathbf{0})$ in the proposed approach, because the latter requires the terminal set to fit within $\beta_x \mathbb{X}$, rather than merely $\mathbb{X}$ as the conventional controller requires. Despite this, the *overall* region of attraction for the proposed approach—as the union of $\Omega_\beta(\mathbf{w})$, following Corollary 1—is larger.

VI. CONCLUSIONS

We have studied the use of available disturbance predictions within a conventional nonlinear MPC formulation for regulation. Modifications to standard terminal conditions were presented. For unchanging disturbance predictions, recursive feasibility is guaranteed and exponential stability of the closed-loop system is established around an equilibrium point close to the origin. For arbitrarily changing disturbance sequences, stability of a robust positively invariant set is guaranteed, the size of which is related to the permitted step-to-step change of the disturbance sequence.

APPENDIX

A. Proof of Proposition 1

Consider some $w_f \in \mathbb{W}_f$. The point $x_f = x_f(w_f) = \Psi w_f = (I - \Phi)^{-1} w_f$ exists and is unique by Lemma 2. Let $\mathbb{X}_f(w_f) = \bar{\mathbb{X}}_f \oplus \{x_f\}$, and consider some $x \in \mathbb{X}_f(w_f)$ and a corresponding

$z = x - x_f \in \bar{\mathbb{X}}_f$. The successor states are $z^+ = Az + B\bar{\kappa}_f(z)$, which is in $\bar{\mathbb{X}}_f$ by construction, and $x^+ = Ax + B\kappa_f(x; w_f) + w_f$, which, using the control law definition $\kappa_f(x; w_f) = \bar{\kappa}_f(x - x_f) + \Pi x_f$, may be rewritten as

$$
\begin{aligned}
x^+ &= Ax + B\bar{\kappa}_f(x - x_f) + B\Pi x_f + w_f \\
&= A(z + x_f) + B\bar{\kappa}_f(z) + B\Pi x_f + w_f \\
&= Az + B\bar{\kappa}_f(z) + (A + B\Pi)x_f + w_f \\
&= z^+ + \Phi x_f + w_f \\
&= z^+ + x_f
\end{aligned}
$$

where the last line follows from $\Phi x_f + w_f = \Phi \Psi w_f + w_f = (\Phi\Psi + I)w_f$, and since $\Psi = (I - \Phi)^{-1} = (I - \Phi)^{-1}\Phi + I$, then $(\Phi\Psi + I) = \Psi$ and $\Psi w_f = x_f$. Then $x^+ \in \mathbb{X}_f(w_f)$ because $z^+ \in \bar{\mathbb{X}}_f$. This establishes positive invariance of $\mathbb{X}_f(w_f)$, for any fixed $w_f \in \mathbb{W}_f$, under $u = \kappa_f(x; w_f)$.

To prove constraint admissibility, by Assumption 11, we have that $\Psi \mathbb{W}_f \subseteq \alpha_x \mathbb{X}$ and $\Pi \Psi \mathbb{W}_f \subseteq \alpha_u \mathbb{U}$, with $\alpha_x, \alpha_u \in [0, 1)$. On the other hand, by Assumption 9, $\bar{\mathbb{X}}_f \subseteq \beta_x \mathbb{X}$ and $\bar{\kappa}_f(\bar{\mathbb{X}}_f) \subseteq \beta_u \mathbb{U}$, with $\beta_x, \beta_u \in [0, 1)$. Then $\mathbb{X}_f(\mathbb{W}_f) = \bar{\mathbb{X}}_f \oplus \Psi \mathbb{W}_f \subseteq \beta_x \mathbb{X} \oplus \alpha_x \mathbb{X} \subseteq \mathbb{X}$, if $\alpha_x + \beta_x \leq 1$. Similarly, $\kappa_f(\mathbb{X}_f; \mathbb{W}_f) = \bar{\kappa}_f(\bar{\mathbb{X}}_f) \oplus \Pi \Psi \mathbb{W}_f \subseteq \beta_u \mathbb{U} \oplus \alpha_u \mathbb{U} \subseteq \mathbb{U}$ if $\alpha_u + \beta_u \leq 1$.

Next, we prove the claimed properties of the functions $\ell(\cdot, \cdot)$ and $V_f(\cdot)$. For some $w_f \in \mathbb{W}_f$, $x \in \mathbb{X}_f(w_f)$ and corresponding $z = x - x_f \in \bar{\mathbb{X}}_f$, we have $\bar{V}_f(z^+) - \bar{V}_f(z) \leq -\bar{\ell}(z, \bar{\kappa}_f(z))$ by construction (Assumption 9). Then, for $x^+ = Ax + B\kappa_f(x; w_f) + w_f$,

$$
\begin{aligned}
V_f(x^+) - V_f(x) &= \bar{V}_f(x^+ - x_f) - \bar{V}_f(x - x_f) \\
&= \bar{V}_f(z^+ + x_f - x_f) - \bar{V}_f(z + x_f - x_f) \\
&= \bar{V}_f(z^+) - \bar{V}_f(z) \\
&\leq -\bar{\ell}(z, \bar{\kappa}_f(z)) \\
&= -\bar{\ell}(x - x_f, \bar{\kappa}_f(x - x_f)) \\
&= -\bar{\ell}(x - x_f, \kappa_f(x; w_f) - \Pi x_f) \\
&= -\ell(x, u; w_f) \text{ where } u = \kappa_f(x, w_f)
\end{aligned}
$$

as required. Finally, the asymptotic stability of x_f for the system $x^+ = Ax + B\kappa_f(x; w_f) + w_f$ follows from the bounds on $V_f(\cdot)$: for all $x \in \mathbb{X}_f(w_f)$

$$
\begin{aligned}
c_1 |x - x_f|^a &\leq V_f(x; w_f) \leq c_2 |x - x_f|^a \\
V_f(x^+; w_f) &- V_f(x; w_f) \leq -c_1 |x - x_f|^a.
\end{aligned}
$$

Thus, $x \to x_f$ in the limit. ∎

B. Proof of Proposition 2

For (i), given $x \in \mathcal{X}_N(\mathbf{w})$ there exists a $\mathbf{u}(x; \mathbf{w}) \in \mathcal{U}_N(x; \mathbf{w})$ with associated state predictions $\mathbf{x}(x; \mathbf{w}) = \{x^0(0; x, \mathbf{w}), x^0(1; x, \mathbf{w}), \ldots, x^0(N; x, \mathbf{w})\}$ with $x^0(0; x, \mathbf{w}) = x$. The successor state $x^+ = Ax + Bu^0(0; x, \mathbf{w}) + w(0; \mathbf{w}) = x^0(1; x, \mathbf{w})$, and so, by Proposition 1, the sequences

$$
\begin{aligned}
\tilde{\mathbf{x}}(x^+; \mathbf{w}) = \{&x^0(1; x, \mathbf{w}), \ldots, x^0(N; x, \mathbf{w}), \\
&Ax^0(N; x, \mathbf{w}) + B\kappa_f(x^0(N; x, \mathbf{w}); w_f) + w_f\} \\
\tilde{\mathbf{u}}(x^+; \mathbf{w}) = \{&u^0(1; x, \mathbf{w}), \ldots, u^0(N; x, \mathbf{w}), \\
&\kappa_f(x^0(N; x, \mathbf{w}); w_f)\} \\
\tilde{\mathbf{w}}(\mathbf{w}) = \{&w(1), w(2), \ldots, w(N-2), w(N-1)\}
\end{aligned}
$$

are feasible for all constraints that define $\mathcal{U}_N(x^+, \tilde{\mathbf{w}}(\mathbf{w}))$; in fact, the same solution omitting the terminal steps is also feasible for all constraints that define $\mathcal{U}_{N-1}(x^+, \tilde{\mathbf{w}}(\mathbf{w}))$. Thus, $x^+ \in \mathcal{X}_{N-1}(\mathbf{w}^+) \subseteq \mathcal{X}_N(\mathbf{w}^+)$ if $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$.

Part (ii) follows directly from the above and recursion. If $x \in \mathcal{X}_N(\mathbf{w})$ and $x^+ \in \mathcal{X}_{N-1}(\mathbf{w}^+) \subseteq \mathcal{X}_N(\mathbf{w}^+)$ under $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$ and

$u = \kappa_N(x; \mathbf{w})$, then, given $x(0) \in \mathcal{X}_N\big(\mathbf{w}(0)\big)$ and the disturbance update rule $\mathbf{w}(k+1) = \tilde{\mathbf{w}}\big(\mathbf{w}(k)\big)$, the state trajectory $\{x(k)\}_k$ remains within the union of $\mathcal{X}_N\big(\mathbf{w}(k)\big)$. Hence, the latter set is positively invariant for $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w$, and control invariant for $x^+ = Ax + Bu + w$ and $\mathbb{U}$.

The nested property of the set union follows from the key observation that, under the tail-updating law, $\mathbf{w}_N(k+1) = \mathbf{w}_{N-1}(k)$. Thus, since $x(k) \in \mathcal{X}_N\big(\mathbf{w}_N(k)\big)$ implies $x(k+1) \in \mathcal{X}_{N-1}\big(\mathbf{w}_{N-1}(k)\big) \subseteq \bigcup_k \mathcal{X}_{N-1}\big(\mathbf{w}_{N-1}(k)\big)$, it also implies $x(k+1) \in \mathcal{X}_N\big(\mathbf{w}_N(k+1)\big) \subseteq \bigcup_k \mathcal{X}_N\big(\mathbf{w}_N(k)\big)$. Therefore, $\bigcup_k \mathcal{X}_N\big(\mathbf{w}_N(k)\big) \supseteq \bigcup_k \mathcal{X}_{N-1}\big(\mathbf{w}_{N-1}(k)\big)$. ∎

C. Proof of Proposition 3

From the definitions of V_f, ℓ and V_N, and the bounds in Assumption 5, we have, for all $x \in \mathcal{X}_N(\mathbf{w})$

$$V_N^0(x; \mathbf{w}) \geq \ell\big(x, \kappa_f(x; w_f); w_f\big) \geq c_1 |x - x_f|^a,$$

while the fact that the costs are continuous and the sets $\mathbb{X}_f$, $\mathbb{X}$, $\mathbb{U}$ are PC-sets means there exists a $c_3 \geq c_2 > 0$ such that

$$V_N^0(x; \mathbf{w}) \leq V_f(x; w_f) \leq c_2 |x - x_f|^a \text{ for all } x \in \mathbb{X}_f(w_f)$$
$$\implies V_N^0(x; \mathbf{w}) \leq c_3 |x - x_f|^a \text{ for all } x \in \mathbb{X}_N(\mathbf{w})$$

Recursive feasibility and the descent property of V_f yields, for all $x \in \mathbb{X}_N(\mathbf{w})$,

$$V_N^0(x^+; \tilde{\mathbf{w}}) \leq V_N^0(x; \mathbf{w}) - \ell\big(x, \kappa_f(x; w_f); w_f\big) \leq \gamma V_N^0(x; \mathbf{w})$$

where $\gamma \triangleq (1 - c_1/c_3) \in (0, 1)$ (since $c_3 > c_1 > 0$). Thus, $V_N^0\big(x(k); \mathbf{w}(k)\big) \leq \gamma^k V_N^0\big(x(0); \mathbf{w}(0)\big)$, and, because $c_1 |x - x_f|^a \leq V_N^0(x; \mathbf{w}) \leq c_3 |x - x_f|^a$,

$$c_1 |x(k) - x_f|^a \leq \gamma^k c_3 |x(0) - x_f|^a \implies$$
$$|x(k) - x_f| \leq c\delta^k |x(0) - x_f|$$

where $c \triangleq (c_3/c_1)^{1/a} > 0$ and $\delta \triangleq \gamma^{1/a} \in (0, 1)$. ∎

D. Proof of Theorem 1

Consider some $z = (x, \mathbf{w}) \in \Omega_N^z$. We have $V_N^0\big(x^+, \tilde{\mathbf{w}}(\mathbf{w})\big) \leq \gamma V_N^0(x, \mathbf{w}) \leq \gamma\beta$. By $\mathcal{K}$-continuity,

$$V_N^0(x^+, \mathbf{w}^+) \leq V_N^0\big(x^+, \tilde{\mathbf{w}}(\mathbf{w})\big) + \sigma_V\big(|\mathbf{w}^+ - \tilde{\mathbf{w}}(\mathbf{w})|\big).$$

Therefore, since $\sigma_V\big(|\mathbf{w}^+ - \tilde{\mathbf{w}}(\mathbf{w})|\big) \leq \sigma_V(\lambda) \leq (\rho - \gamma)\alpha$,

$$V_N^0(x^+, \mathbf{w}^+) \leq \gamma\beta + (\rho - \gamma)\alpha \leq \rho\beta < \beta.$$

Moreover, if $\big(x(0), \mathbf{w}(0)\big) \in \Omega_\beta^z \setminus \Omega_\alpha^z$, then

$$V_N^0\big(x(1), \mathbf{w}(1)\big) \leq \gamma V_N^0\big(x(0), \mathbf{w}(0)\big) + (\rho - \gamma) V_N^0\big(x(0), \mathbf{w}(0)\big)$$
$$\leq \rho V_N^0\big(x(0), \mathbf{w}(0)\big).$$

Consequently, $V_N^0\big(x(k), \mathbf{w}(k)\big) \leq \rho^k \beta$, from which it follows that $V_N^0\big(x(k'), \mathbf{w}(k')\big) \leq \alpha$ after some finite k'. ∎

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