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THE LASCAR GROUPS AND THE FIRST HOMOLOGY GROUPS IN MODEL THEORY

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Abstract. Let p be a strong type of an algebraically closed tuple over $B = \mathrm{acl}^{\mathrm{eq}}(B)$ in any theory T. Depending on a ternary relation $\mathop{\smile\!\!\!\!\!\mid}^{*}$ satisfying some basic axioms (there is at least one such, namely the trivial independence in T), the first homology group $H_1^*(p)$ can be introduced, similarly to [3].

We show that there is a canonical surjective homomorphism from the Lascar group over B to $H_1^*(p)$. We also notice that the map factors naturally via a surjection from the 'relativised' Lascar group of the type (which we define in analogy with the Lascar group of the theory) onto the homology group, and we give an explicit description of its kernel. Due to this characterization, it follows that the first homology group of p is independent from the choice of $\mathop{\smile\!\!\!\!\!\mid}^{*}$, and can be written simply as $H_1(p)$.

As consequences, in any T, we show that $|H_1(p)| \geq 2^{\aleph_0}$ unless $H_1(p)$ is trivial, and we give a criterion for the equality of stp and Lstp of algebraically closed tuples using the notions of the first homology group and a relativised Lascar group.

We also argue how any abelian connected compact group can appear as the first homology group of the type of a model.

In this paper we study the first homology group of a strong type in any theory.

Originally, in [3] and [4], a homology theory only for rosy theories is developed. Namely, given a strong type p in a rosy theory T, the notion of the nth homology group $H_n(p)$ depending on thorn-forking independence relation is introduced. Although the homology groups are defined analogously as in singular homology theory in algebraic topology, the $(n+1)$th homology group for $n > 0$ in the rosy theory context has to do with the nth homology group in algebraic topology. For example as in [3], $H_2(p)$ in stable theories has to do with the fundamental group in topology. This implies that, already in rosy

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theories, $H_1(p)$ is detecting somewhat endemic properties of p existing only in model theory context.

Indeed, in every known rosy example, $H_n(p)$ for $n \geq 2$ is a profinite abelian group. In [5], it is proved to be so when T is stable under a canonical condition, and conversely, every profinite abelian group can arise in this form. On the other hand, we show in this paper that the first homology groups appear to have distinct features as follows.

Let $p = \mathrm{tp}(a/B)$ be a strong type over $B = \mathrm{acl}^{\mathrm{eq}}(B)$ in any theory T. Fix a ternary invariant independence relation $\mathop{\smile\!\!\!\!\!\!\,|}^*$ among small sets satisfying finite character, normality, symmetry, transitivity and extension. (There is at least one such relation, by putting $A \mathop{\smile\!\!\!\!\!\!\,|}_C D$ for any sets A, C, D.) Then we can analogously define $H_1^*(p)$ depending on $\mathop{\smile\!\!\!\!\!\!\,|}^*$, (which of course is the same as $H_1(p)$ when $\mathop{\smile\!\!\!\!\!\!\,|}^*$ is thorn-independence in rosy T). In this note, a canonical epimorphism from the Lascar group over B of T to $H_1^*(p)$ is constructed. Indeed, we also introduce the notion of the relativised Lascar group of a type which is proved to be independent from the choice of the monster model of T, and the homomorphism factors through a surjection from the relativised Lascar group of $\bar{p} = \mathrm{tp}(\mathrm{acl}(aB)/B)$ onto $H_1^*(p)$. Moreover, we can identify its kernel. Roughly, $H_1^*(p)$ has to do with the abelianization of the relativised Lascar group of $\bar{p}$. More precisely, $H_1^*(p) = G/K$, where G is the group of automorphisms of the realization set of $\bar{p}$, and K is the normal subgroup of G fixing each orbit under the action of the derived subgroup of G. Surprisingly, this conclusion is independent from the choice of $\mathop{\smile\!\!\!\!\!\!\,|}^*$ satisfying the axioms.[1] Hence, we can write the first homology group simply as $H_1(p)$, which makes sense in any theory.

Consequently, we show that $|H_1(p)| \geq 2^{\aleph_0}$ unless $H_1(p)$ is trivial, and exhibit a non-profinite example in a rosy theory. In conclusion, we find a criterion for the coincidence of notions of strong types and Lascar types of algebraically closed tuples in any theory, in terms of the triviality of the first homology groups and the abelianness of a relativised Lascar group (Corollary 4.7). It seems reasonable to ask whether this criterion can be applied in verifying or refuting stp$\equiv$Lstp in simple theories.

In Section 1, we introduce/recall basic definitions of the first homology group of a strong type for any theory. In Section 2, as mentioned above, we construct a surjective homomorphism from the Lascar group to the first homology group. In Section 3, we introduce the aforementioned concept of relativized Lascar groups, and in Section 4, we prove

[1]However it is not clear whether the same feature can happen for nth homology groups of types for $n > 1$.

the characterization theorem (Theorem 4.4) of the first homology group and give a criterion for Lstp≡stp. We also argue that the size of the first homology group of a strong type is either 1 or $\geq 2^{\aleph_0}$ (in Theorem 4.8, and a more detailed explanation is given in Section 6). In Section 5, we state that any connected compact abelian group can appear as the first homology group of the type of a model (Theorem 5.2), which follows from a result by Bouscaren, Lascar, Pillay, and Ziegler. We also give a more precise example of a type in a rosy theory having a non-profinite first homology group. We point out here that this paper is a result of merging two notes. The first one, single-authored by Junguk Lee, covered Section 2, Theorem 4.8 in Section 4, and Section 5.2, and the second note, jointly written by the three authors, consisted of Section 1,3, 4 and Section 5.1.

1. Introduction

Throughout this paper, we work in a large saturated model $\mathcal{M}(=\mathcal{M}^{\mathrm{eq}})$ of a complete theory T, and we use standard notations. For example, unless stated otherwise, $a, b, \ldots$, and $A, B, \ldots$ are small but possibly infinite tuples and sets from $\mathcal{M}$, respectively, and $a \equiv_A b$, $a \equiv^s_A b$, $a \equiv^L_A b$ mean $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$, $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$, $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$, respectively. For the general theory of model theory, of the Lascar groups, and of rosy theories, we refer to [6], [11], and [2], respectively. For the homology theory in model theory, see [4, 3]. A particular case of the first homology group with respect to thorn-forking in rosy theories is studied in [7],[8]. The main difference of the first homology groups introduced in this section from those in the references is that the groups are defined with respect to a fixed independence notion in an arbitrary theory as follows, not necessarily thorn-forking/Shelah-forking in rosy/simple theories. However, as the reader will see, all the arguments from the rosy theory context can follow in the general context.

For the rest of this section (and also for Section 4), we fix a ternary automorphism-invariant relation $\mathop{\smile\!\!\!\!\!\!|}^*$ between small sets of $\mathcal{M}$ satisfying

- finite character: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $a \mathop{\smile\!\!\!\!\!\!|}^*_C b$ for any finite tuples $a \in A$ and $b \in B$;
- normality: for any sets A, B and C, if $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}^*_C \mathrm{acl}(BC)$;
- symmetry: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$;
- transitivity: $A \mathop{\smile\!\!\!\!\!\!|}^*_B D$ iff $A \mathop{\smile\!\!\!\!\!\!|}^*_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}^*_C D$, for any sets A and $B \subseteq C \subseteq D$;
- extension: for any sets A and $B \subseteq C$, there is $A' \equiv_B A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^*_B C$.

Throughout this paper we call the above axioms **the basic 5 axioms**. We say that A is $*$-independent from B over C if $A \mathop{\smile\!\!\!\!\!\;\;|}^*_C B$. Notice that there is at least one such relation for any theory, namely, the *trivial independence relation*: For any sets A, B, C, put $A \mathop{\smile\!\!\!\!\!\;\;|}^*_B C$. Of course there is a non-trivial such relation when T is simple or rosy, given by forking or thorn-forking, respectively.

Now, we also fix a strong type p of possibly infinite arity over $B = \mathrm{acl}(B)$. We shall define the first homology group of p with respect to $\mathop{\smile\!\!\!\!\!\;\;|}^*$, analogously to that in the references. Hence we begin by recalling some notations from the references.

Notation 1.1. Let s be an arbitrary finite set of natural numbers. Given any subset $X \subseteq \mathcal{P}(s)$, we may view X as a category where for any $u, v \in X$, $\mathrm{Mor}(u, v)$ consists of a single morphism $\iota_{u,v}$ if $u \subseteq v$, and $\mathrm{Mor}(u, v) = \emptyset$ otherwise. If $f\colon X \to \mathcal{C}$ is any functor into some category $\mathcal{C}$, then for any $u, v \in X$ with $u \subseteq v$, we let f^u_v denote the morphism $f(\iota_{u,v}) \in \mathrm{Mor}_{\mathcal{C}}(f(u), f(v))$. We shall call $X \subseteq \mathcal{P}(s)$ a *primitive category* if X is non-empty and *downward closed*; i.e., for any $u, v \in \mathcal{P}(s)$, if $u \subseteq v$ and $v \in X$ then $u \in X$. (Note that all primitive categories have the empty set $\emptyset \subset \omega$ as an object.)

We use now $\mathcal{C}_B$ to denote the category whose objects are all the small subsets of $\mathcal{M}$ containing B, and whose morphisms are elementary maps over B. For a functor $f : X \to \mathcal{C}_B$ and objects $u \subseteq v$ of X, $f^u_v(u)$ denotes the set $f^u_v(f(u)) (\subseteq f(v))$.

Definition 1.2. By a *$*$-independent functor in p*, we mean a functor f from some primitive category X into $\mathcal{C}_B$ satisfying the following:

(1) If $\{i\} \subset \omega$ is an object in X, then $f(\{i\})$ is of the form $\mathrm{acl}(Cb)$ where $b \models p$, $C = \mathrm{acl}(C) = f^{\emptyset}_{\{i\}}(\emptyset) \supseteq B$, and $b \mathop{\smile\!\!\!\!\!\;\;|}^*_B C$.

(2) Whenever $u (\neq \emptyset) \subset \omega$ is an object in X, we have

$$f(u) = \mathrm{acl}\left(\bigcup_{i \in u} f^{\{i\}}_u(\{i\})\right)$$

and $\{f^{\{i\}}_u(\{i\}) \mid i \in u\}$ is $*$-independent over $f^{\emptyset}_u(\emptyset)$.

We let $\mathcal{A}^*_p$ denote the family of all $*$-independent functors in p.

A $*$-independent functor f is called a *$*$-independent n-simplex* in p if $f(\emptyset) = B$ and $\mathrm{dom}(f) = \mathcal{P}(s)$ with $s \subset \omega$ and $|s| = n + 1$. We call s the *support* of f and denote it by $\mathrm{supp}(f)$.

In the rest we may call a $*$-independent n-simplex in p just an *n-simplex* of p, as far as no confusion arises. We are ready to define the first homology group $H_1^*(p)$ of p depending on our choice of the independence relation $\downarrow^*$.

Definition 1.3. Let $n \geq 0$. We define:

$$S_n(\mathcal{A}_p^*) := \{\, f \in \mathcal{A}_p^* \mid f \text{ is an } n\text{-simplex of } p \,\}$$
$$C_n(\mathcal{A}_p^*) := \text{the free abelian group generated by } S_n(\mathcal{A}_p^*).$$

An element of $C_n(\mathcal{A}_p^*)$ is called an *n-chain* of p. The support of a chain c, denoted by $\operatorname{supp}(c)$, is the union of the supports of all the simplices that appear in c with a non-zero coefficient. Now for $n \geq 1$ and each $i = 0, \ldots, n$, we define a group homomorphism

$$\partial_n^i \colon C_n(\mathcal{A}_p^*) \to C_{n-1}(\mathcal{A}_p^*)$$

by putting, for any n-simplex $f \colon \mathcal{P}(s) \to \mathcal{C}$ in $S_n(\mathcal{A}_p^*)$ where $s = \{s_0 < \cdots < s_n\} \subset \omega$,

$$\partial_n^i(f) := f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and then extending linearly to all n-chains in $C_n(\mathcal{A}_p^*)$. Then we define the *boundary map*

$$\partial_n \colon C_n(\mathcal{A}_p^*) \to C_{n-1}(\mathcal{A}_p^*)$$

by

$$\partial_n(c) := \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

We shall often refer to $\partial_n(c)$ as the *boundary of* c. Next, we define:

$$Z_n(\mathcal{A}_p^*) := \operatorname{Ker} \partial_n$$
$$B_n(\mathcal{A}_p^*) := \operatorname{Im} \partial_{n+1}.$$

The elements of $Z_n(\mathcal{A}_p^*)$ and $B_n(\mathcal{A}_p^*)$ are called *n-cycles* and *n-boundaries* in p, respectively. It is straightforward to check that $\partial_n \circ \partial_{n+1} = 0$. Hence we can now define the group

$$H_n^*(p) := Z_n(\mathcal{A}_p^*)/B_n(\mathcal{A}_p^*)$$

called the nth *$*$-homology group* of p.

Notation 1.4.
(1) For $c \in Z_n(\mathcal{A}_p^*)$, $[c]$ denotes the homology class of c in $H_n^*(p)$.
(2) When n is clear from the context, we shall often omit n in ∂_n^i and in ∂_n, writing simply as ∂^i and ∂.

Definition 1.5. A 1-chain $c \in C_1(\mathcal{A}_p^*)$ is called a *1-$*$-shell* (or just a 1-shell) in p if it is of the form

$$c = f_0 - f_1 + f_2$$

where f_i's are 1-simplices of p satisfying

$$\partial^i f_j = \partial^{j-1} f_i \quad \text{whenever } 0 \leq i < j \leq 2.$$

Hence, for $\operatorname{supp}(c) = \{n_0 < n_1 < n_2\}$ and $k \in \{0,1,2\}$, it follows that

$$\operatorname{supp}(f_k) = \operatorname{supp}(c) \smallsetminus \{n_k\}.$$

Notice that the boundary of any 2-simplex is a 1-shell.

Notation/Remark 1.6. Let $p(x) = \operatorname{tp}(a/B)$ be fixed, and let $\bar{p}(\bar{x}) = \operatorname{tp}(\operatorname{acl}(aB)/B)$ (with some enumeration of $\operatorname{acl}(aB)$). Obviously $\bar{p}(\bar{x})$ only depends on p (not on its realizations). By the definitions of the $*$-independent functors and the first homology group, $H_1^*(p)$ and $H_1^*(\bar{p})$ are identical.

If c is a 1-shell, then in $H_1^*(p)$, we shall see in Remark 2.5 that $[-c] = [c']$ where c' is another 1-shell with $\operatorname{supp}(c') = \operatorname{supp}(c)$.

We note now that in [4], the notion of an *amenable* collection of functors into a category is introduced, and due to the 5 axioms that $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies, it is clear that $\mathcal{A}_p^*$ forms such a collection of functors into $\mathcal{C}_B$. Therefore the following corresponding fact holds.

Fact 1.7. *(*[3] *or* [4]*)*

$$H_1^*(p) = \{[c] \mid c \textit{ is a } 1\text{-}*\textit{-shell with } \operatorname{supp}(c) = \{0,1,2\}\}.$$

So if any 1-shell is the boundary of some 2-chain then $H_1^*(p) = 0$.[2]

Remark 1.8. The following Fact 1.9 directly comes from [7, Theorem 2.4] (and above 1.6), since the proof of the theorem only uses the fact that thorn-independence in any rosy theory satisfies the basic 5 axioms. But we point out that corrections should be made in the theorem and other results in [7]. Namely, $p(x)$ there should be changed to $\bar{p}(\bar{x})$ since a vertex of a simplex in p is an algebraically closure over B of a realization of p. In fact it is not clear whether $p(x)$ being a Lascar type implies that $\bar{p}(\bar{x})$ is also a Lascar type (the converse always holds though), unless T is G-compact over B: Let T be G-compact over B, and let p be a Lascar type; i.e., for any $a, b \models p$, we have $a \equiv_B^L b$. We claim that $\bar{p}$ is a Lascar type, too. Since T is G-compact over B, equality of Lstp over B is B-type-definable and a conjunction of those of finite arities. Thus by compactness for $a \models p$ and finite $c \in \operatorname{acl}(aB)$, it suffices to show that $q(x,y) := \operatorname{tp}(ac/B)$ is a Lascar type. Now since p is a Lascar type, for any $a' \models p$ there is $c' \in \operatorname{acl}(a'B)$ such that $ac \equiv_B^L a'c'$. Since there are only finitely many conjugates of c' over $a'B$,

[2]Notice that in Definition 1.2, we take algebraic closures, rather that bounded closures. Hence even when $\mathop{\smile\!\!\!\!\!\mid}^*$ is nonforking in simple T, it is not known whether $H_1^*(p)$ is trivial. We put this as an open question in Question 4.9.

it follows that there are at most finitely many distinct Lascar types in q. This implies that equality of Lstp in q is relatively definable over B. But since q is a strong type over $B = \mathrm{acl}(B)$, there is only one Lascar class in q.

Fact 1.9. *Suppose that $\bar{p}(\bar{x})$ is a Lascar type. Then $H_1^*(p) = 0$.*

For the rest of this paper, for notational simplicity, we suppress B to $\emptyset$ by naming it (and reuse B to mean an arbitrary small set). In particular, $\mathcal{C}$ denotes $\mathcal{C}_B$. **We further suppose (until the end of Section 4) that the fixed strong type p is a type of an algebraically closed set (by assuming $p = \bar{p}$) or that the algebraic closure of its realization is the same as its definable closure.** This process is necessary as pointed out in Remark 1.8, and will not affect in computing $H_1^*(p)$ due to 1.6.

2. Lascar groups and the first homology groups

In this section, we show that there is a canonical epimorphism from the Lascar group $\mathrm{Gal}_L(T)$ onto the first homology group $H_1^*(p)$ of p.

2.1. **Representations of 1-shells.**

Definition 2.1. (1) We introduce some notation which will be used throughout. Let $f\colon \mathcal{P}(s) \to \mathcal{C}$ be an n-simplex in p. For $u \subset s$ with $u = \{i_0 < \ldots < i_k\}$, we shall write $f(u) = [a_0 \ldots a_k]_u$, where each $a_j \models p$ is an algebraically closed tuple as assumed before, if $f(u) = \mathrm{acl}(a_0 \ldots a_k)$, and $\mathrm{acl}(a_j) = f_u^{\{i_j\}}(\{i_j\})$. So, $\{a_0, \ldots, a_k\}$ is $*$-independent. Of course, if we write $f(u) \equiv [b_0 \ldots b_k]_u$, then it means that there is an automorphism sending $a_0 \ldots a_k$ to $b_0 \ldots b_k$.

(2) Let $s = f_{12} - f_{02} + f_{01}$ be a 1-$*$-shell in p such that $\mathrm{supp}(f_{ij}) = \{n_i, n_j\}$ with $n_i < n_j$ for $0 \le i < j \le 2$. Clearly there is a quadruple (a_0, a_1, a_2, a_3) of realizations of p such that $f_{01}(\{n_0, n_1\}) \equiv [a_0 a_1]_{\{n_0,n_1\}}$, $f_{12}(\{n_1, n_2\}) \equiv [a_1 a_2]_{\{n_1,n_2\}}$, and $f_{02}(\{n_0, n_2\}) \equiv [a_3 a_2]_{\{n_0,n_2\}}$. We call this quadruple a *representation of* s. For any such representation of s, call a_0 an *initial point*, a_3 a *terminal point*, and (a_0, a_3) an *endpoint pair* of the representation.

In the next theorem, we will see that the endpoint pairs of representations determine the classes of 1-shells in $H_1^*(p)$, and the group structure of $H_1^*(p)$ can be described by endpoint pairs.

Theorem 2.2. *Let s_0 and s_1 be 1-shells with support $\{0, 1, 2\}$. Suppose they have some representations with the same endpoint pair. Then*

$s_0 - s_1$ is a boundary of a 2-chain, that is, s_0 and s_1 are in the same homology class in $H_1^*(p)$

Proof. Let $s_k := f^k_{12} - f^k_{02} + f^k_{01}$ for $k = 0, 1$, where for each 1-shell f^k_{ij} in p, $\operatorname{supp}(f^k_{ij}) = \{i, j\}$. Suppose s_0 and s_1 have representations (a, b_0, c_0, a') and (a, b_1, c_1, a'), respectively. Take $b \models p$ such that $b \mathop{\smile\!\!\!\!\!\!\;|}^* ab_0b_1c_0c_1a'$. Then, there is a 2-chain $\alpha = (a^0_{01} + a^0_{12} - a^0_{02}) - (a^1_{01} + a^1_{12} - a^1_{02})$, where for each $k = 0, 1$ and $0 \leq i < j \leq 2$, a^k_{ij} is a 2-simplex satisfying the following: For $k = 0, 1$,

(1) $\operatorname{supp}(a^k_{ij}) = \{i, j, 3\}$;
(2) $a^k_{ij} \restriction \mathcal{P}(\{i, j\}) = f^k_{ij}$; and
(3) $a^k_{01}(\{0, 1, 3\}) = [ab_kb]_{\{0,1,3\}}$, $a^k_{12}(\{1, 2, 3\}) = [b_kc_kb]_{\{1,2,3\}}$, $a^k_{02}(\{0, 2, 3\}) = [a'c_kb]_{\{0,2,3\}}$,

and $\partial(a^k_{01} + a^k_{12} - a^k_{02}) = s_k - (f_k - g_k)$, where $f_k = a^k_{01} \restriction \mathcal{P}(\{0, 3\})$ and $g_k = a^k_{02} \restriction \mathcal{P}(\{0, 3\})$. So $\partial\alpha = (s_0 - s_1) - ((f_1 - g_1) - (f_0 - g_0))$. It is enough to show that a 1-chain $(f_1 - g_1) - (f_0 - g_0)$ is a boundary of a 2-chain. (Notice that even if for f_0, f_1 (similarly for g_0, g_1), $f_0(\{0, 3\}) = f_1(\{0, 3\}) = [ab]_{\{0,3\}}$, it need not be that $f_0 = f_1$ since $f_0(\{0\}) = f^0_{01}(\{0\})$ and $f_1(\{0\}) = f^1_{01}(\{0\})$.)

Now by the extension axiom, we can choose $c, c' \models p$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}^* ab$, $c' \mathop{\smile\!\!\!\!\!\!\;|}^* a'b$, and $ca \equiv c'a'$. For $k = 0, 1$, consider 2-simplices $\hat{f}_k$ and $\hat{g}_k$ such that:

(1) $\operatorname{supp}(\hat{f}_k) = \operatorname{supp}(\hat{g}_k) = \{0, 3, 4\}$;
(2) $\hat{f}_k(\{0, 3, 4\}) = [abc]_{\{0,3,4\}}$ and $\hat{g}_k(\{0, 3, 4\}) = [a'bc']_{\{0,3,4\}}$;
(3) $\partial^0 \hat{f}_0 = \partial^0 \hat{f}_1$, $\partial^0 \hat{g}_0 = \partial^0 \hat{g}_1$;
(4) $\partial^1 \hat{f}_k = \partial^1 \hat{g}_k$; and
(5) $\partial^2 \hat{f}_k = f_k$, $\partial^2 \hat{g}_k = g_k$.

Then the 1-chain $(f_1 - g_1) - (f_0 - g_0)$ is the boundary of the 2-chain $(\hat{f}_1 - \hat{g}_1) - (\hat{f}_0 - \hat{g}_0)$. $\square$

Remark 2.3. Notice that in the proof above, that s_0, s_1 have the same support is not at all essential, so Theorem 2.2 still holds even if their supports are distinct. Moreover, if (a, b, c, a) is a representation of some 1-shell s, even if a and bc need not be $*$-independent, $[s] = 0$ in $H_1^*(p)$.

Theorem 2.4. *Let s_0 and s_1 be 1-shells with support $\{0, 1, 2\}$, and let $a, a', a'' \models p$ be such that (a, a') and (a', a'') are the endpoint pairs of representations of s_0 and s_1, respectively. Then there is a 1-shell s with support $\{0, 1, 2\}$ having a representation with the endpoint pair (a, a'') such that $[s] = [s_0] + [s_1]$ in $H_1^*(p)$.*

Proof. Let $s_0 = f^0_{01} + f^0_{12} - f^0_{02}$, and let $s_1 = f^1_{01} + f^1_{12} - f^1_{02}$ with $\mathrm{supp}(f^k_{ij}) = \{i,j\}$. Since (a,a') and (a',a'') are some endpoint pairs of s_0 and s_1 respectively, by Theorem 2.2, we may assume the restrictions of f^0_{01} and f^1_{01} to the domain $\mathcal{P}(\{0\})$ are the same, hence so are the restrictions of f^0_{02} and f^1_{02} to $\mathcal{P}(\{0\})$. Let $b_0, b_1, c_0, c_1 \models p$ be such that the two quadruples (a, b_0, c_0, a') and (a', b_1, c_1, a'') are representations of s_0 and s_1, respectively. Consider two $*$-independent elements $d, e \models p$ with $de \mathop{\smile\!\!\!\!\!\!\;|}^* aa'a''b_0b_1c_0c_1$. Then there is a 2-chain $\alpha = (a^0_{01}+a^0_{12}-a^0_{02}) - b + (a^1_{01} + a^1_{12} - a^1_{02})$, where for $k = 0,1$ and $0 \le i < j \le 2$, a^k_{ij} and b are 2-simplices satisfying the following:

(1) $\mathrm{supp}(a^k_{ij}) = \{i, j, 3+k\}$ and $\mathrm{supp}(b) = \{0, 3, 4\}$;
(2) $a^k_{ij} \restriction \mathcal{P}(\{i,j\}) = f^k_{ij}$;
(3) $a^0_{01}(\{0,1,3\}) = [ab_0d]_{\{0,1,3\}}$, $a^0_{12}(\{1,2,3\}) = [b_0c_0d]_{\{0,2,3\}}$, $a^0_{02}(\{0,2,3\}) = [a'c_0d]_{\{0,2,3\}}$, $a^1_{01}(\{0,1,4\}) = [a'b_1e]_{\{0,1,4\}}$, $a^1_{12}(\{1,2,4\}) = [b_1c_1e]_{\{1,2,4\}}$, $a^1_{02}(\{0,2,4\}) = [a''c_1e]_{\{0,2,4\}}$, and $b(\{0,3,4\}) = [a'de]_{\{0,3,4\}}$;

and $\partial(\alpha) = s_0 + s_1 - s'$, where $s' = a^0_{01} \restriction \mathcal{P}(\{0,3\}) + b \restriction \mathcal{P}(\{3,4\}) - a^1_{02} \restriction \mathcal{P}(\{0,4\})$ is a 1-shell of a support $\{0,3,4\}$ having (a,a'') as its endpoint pair. Now by Remark 2.3, for a 1-shell s of a support $\{0,1,2\}$ obtained from s' by simply changing the support $\{0,3,4\}$ to $\{0,1,2\}$, we have $[s] = [s']$ in $H^*_1(p)$. Thus, there is a 2-chain α' having a 1-chain s_0+s_1-s as its boundary, and so $[s] = [s_0] + [s_1]$. □

Now, we summarize some properties of endpoint pairs of 1-shells which follow from Theorem 2.2 and Theorem 2.4. We define an equivalence relation $\sim$ on the set of pairs of realizations p as follows: For $a, a', b, b' \models p$, $(a,b) \sim (a',b')$ if two pairs (a,b) and (a',b') are endpoint pairs of 1-shells s and s' respectively such that $[s] = [s'] \in H^*_1(p)$. We write $\mathcal{E}^* = p(\mathcal{M}) \times p(\mathcal{M}) / \sim$. We denote the class of $(a,b) \in p(\mathcal{M}) \times p(\mathcal{M})$ by $[a,b]$. By 2.2, if $ab \equiv a'b'$, then $[a,b] = [a',b']$. Now, define a binary operation $+_{\mathcal{E}^*}$ on $\mathcal{E}^*$ as follows: For $[a,b], [b',c'] \in \mathcal{E}^*$, $[a,b] +_{\mathcal{E}^*} [b',c'] = [a,c]$, where $bc \equiv b'c'$. Due to Theorem 2.4, this operation is well-defined.

Remark 2.5. The pair $(\mathcal{E}^*, +_{\mathcal{E}^*})$ forms a commutative group which is isomorphic to $H^*_1(p)$. More specifically, for $a, b, c \models p$ and $\sigma \in \mathrm{Aut}(\mathcal{M})$, we have:

- $[a,b] + [b,c] = [a,c]$;
- $[a,a]$ is the identity element;
- $-[a,b] = [b,a]$;
- $\sigma([a,b]) := [\sigma(a), \sigma(b)] = [a,b]$; and
- $f : \mathcal{E}^* \to H^*_1(p)$ sending $[a,b] \mapsto [s]$, where (a,b) is an endpoint pair of s, is a group isomorphism.

From now on, we identify $\mathcal{E}^*$ and $H_1^*(p)$. Notice that, indeed, the group structure of $\mathcal{E}^*$ depends only on the types of (a,b)'s with $[a,b]\in\mathcal{E}^*$. Hence one may similarly define an equivalence relation on

$$\{q(x,y):\ p(x)\cup p(y)\subseteq q(x,y) \text{ a complete type over } \emptyset\}$$

to form $\mathcal{E}^*_{\mathrm{tp}}$, and give a corresponding group operation to conclude that $\mathcal{E}^*$ and $\mathcal{E}^*_{\mathrm{tp}}$ are isomorphic.

Due to the same proof in [7], we can restate Fact 1.9 as follows using the endpoint notion:

Fact 2.6. *Let $a,b\models p$ be such that $a\equiv^L b$. Then any 1-shell having (a,b) as its endpoint pair is the boundary of a 2-chain; i.e., $[a,b]=0$ in $H_1^*(p)$.*

2.2. **The Lascar group and the first homology groups.** Here, using the notion of an ordered bracket, for each $a\models p$ we define a map φ_a^* from the automorphism group over $B(=\emptyset)$ into the first homology group of p as follows: For $\sigma\in\operatorname{Aut}(\mathcal{M})$, we let $\varphi_a^*(\sigma)=[a,\sigma(a)]$. This map will be proven to be a surjective homomorphism not depending on the choice of $a\models p$. Thus, we get a canonical epimorphism from $\operatorname{Aut}(\mathcal{M})$ onto $H_1^*(p)$ and we study its kernel.

Theorem 2.7. *(1) Each φ_a^* is an epimorphism.*
(2) For $a,b\models p$, $\varphi_a^*=\varphi_b^*$. So we get a canonical map φ_p^* from $\operatorname{Aut}(\mathcal{M})$ into $H_1^*(p)$.
(3) There is a canonical epimorphism Φ_p^* from $\operatorname{Gal}_L(\mathcal{M})$ onto $H_1^*(p)$.

Proof. (1) Fix $a\models p$. At first, surjectivity of φ_a^* comes from the fact that for $b\models p$, there is a $\sigma\in\operatorname{Aut}(\mathcal{M})$ such that $\sigma(a)=b$. It is enough to show that φ_a is a homomorphism. For $\sigma,\tau\in\operatorname{Aut}(\mathcal{M})$,

$$\begin{aligned}
\varphi_a^*(\sigma\tau) &= [a,\sigma\tau(a)]\\
&= [a,\sigma(a)]+[\sigma(a),\sigma\tau(a)]\\
&= [a,\sigma(a)]+\sigma[a,\tau(a)]\\
&= [a,\sigma(a)]+[a,\tau(a)]\\
&= \varphi_a^*(\sigma)+\varphi_a^*(\tau).
\end{aligned}$$

So φ_a^* is a homomorphism.

(2) Choose $a,b\models p$. Then there exists $\tau\in\operatorname{Aut}(\mathcal{M})$ such that $b=\tau(a)$. For $\sigma\in\operatorname{Aut}(\mathcal{M})$,

$$\begin{aligned}
\varphi_b^*(\sigma) &= \varphi_a^*(\tau^{-1}\sigma\tau)\\
&= \varphi_a^*(\tau^{-1})+\varphi_a^*(\sigma)+\varphi_a^*(\tau)\\
&= \varphi_a^*(\sigma).
\end{aligned}$$

Thus $\varphi_a^* = \varphi_b^*$, and we get a canonical epimorphism

$$\varphi_p^* \ : \ \mathrm{Aut}(\mathcal{M}) \to H_1^*(p),$$

defined by: $\varphi_p^* := \varphi_a^*$ for some $a \models p$.

(3) By Fact 2.6, the kernel of φ_p^* contains $\mathrm{Autf}(\mathcal{M})$ and φ_p^* induces a canonical epimorphism Φ_p^* from $\mathrm{Gal}_L(\mathcal{M})$ onto $H_1^*(p)$. □

3. The relativised Lascar groups

In this section, we introduce some candidates for the notion of Lascar group of a strong type p, which are intended to be the Lascar group relativised to p. We begin by presenting several automorphism groups. Let $\Sigma(\bar{x})$ be a partial type over $\emptyset$ (with $\bar{x}$ of possibly infinite length, and realizations of Σ need not be algebraically closed.) Recall that $\mathrm{Autf}_B(\mathcal{M})$ is the subgroup of $\mathrm{Aut}_B(\mathcal{M})$ generated by

$$\{f \in \mathrm{Aut}_B(\mathcal{M}) : \ f \in \mathrm{Aut}_M(\mathcal{M}) \text{ for some model } (B \subseteq) M \prec \mathcal{M}\},$$

and

$$\mathrm{Gal_L}(T, B) := \mathrm{Aut}_B(\mathcal{M})/\,\mathrm{Autf}_B(\mathcal{M})$$

(which does not depend on the choice of the monster model $\mathcal{M}$).

Definition 3.1. (1) $\mathrm{Aut}(\Sigma(\mathcal{M})) := \{\sigma \restriction \Sigma(\mathcal{M}) : \ \sigma \in \mathrm{Aut}(\mathcal{M})\}$;
(2) $\mathrm{Autf_{res}}(\Sigma(\mathcal{M})) := \{\sigma \restriction \Sigma(\mathcal{M}) : \ \sigma \in \mathrm{Autf}(\mathcal{M})\}$;
(3) for a cardinal $\lambda > 0$, $\mathrm{Autf}^{\lambda}_{\mathrm{fix}}(\Sigma(\mathcal{M})) :=$

$$\{\sigma \restriction \Sigma(\mathcal{M}) : \ \sigma \in \mathrm{Aut}(\mathcal{M}) \text{ such that for any } a_i \models \Sigma \text{ and } \bar{a} = (a_i)_{i<\lambda},\ \bar{a} \equiv^L \sigma(\bar{a})\};$$

and
(4) $\mathrm{Autf_{fix}}(\Sigma(\mathcal{M})) :=$

$$\{\sigma \restriction \Sigma(\mathcal{M}) : \ \sigma \in \mathrm{Aut}(\mathcal{M}) \text{ such that } \bar{a} \equiv^L \sigma(\bar{a}) \text{ where } \bar{a} \text{ is some enumeration of } \Sigma(\mathcal{M})\}.$$

It is straightforward to see that each of the groups $\mathrm{Autf_{res}}(\Sigma(\mathcal{M})) \leq \mathrm{Autf_{fix}}(\Sigma(\mathcal{M})) \leq \mathrm{Autf}^{\lambda}_{\mathrm{fix}}(\Sigma(\mathcal{M}))$ is a normal subgroup of $\mathrm{Aut}(\Sigma(\mathcal{M}))$.

Definition 3.2. (1) $\mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}(\Sigma(\mathcal{M})) := \mathrm{Aut}(\Sigma(\mathcal{M}))/\,\mathrm{Autf_{res}}(\Sigma(\mathcal{M}))$;
(2) $\mathrm{Gal}^{\mathrm{fix},\lambda}_{\mathrm{L}}(\Sigma(\mathcal{M})) := \mathrm{Aut}(\Sigma(\mathcal{M}))/\,\mathrm{Autf}^{\lambda}_{\mathrm{fix}}(\Sigma(\mathcal{M}))$; and
(3) $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(\Sigma(\mathcal{M})) := \mathrm{Aut}(\Sigma(\mathcal{M}))/\,\mathrm{Autf_{fix}}(\Sigma(\mathcal{M}))$.

Remark 3.3. We have $\mathrm{Autf_{fix}}(\Sigma(\mathcal{M})) = \mathrm{Autf}^{\omega}_{\mathrm{fix}}(\Sigma(\mathcal{M}))$. So $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(\Sigma(\mathcal{M})) = \mathrm{Gal}^{\mathrm{fix},\omega}_{\mathrm{L}}(\Sigma(\mathcal{M}))$.

Proof. We will show by induction on $\lambda \geq \omega$ that tuples $(a_j)_{j<\lambda}, (b_j)_{j<\lambda}$ with $a_j, b_j \models \Sigma$ are Lascar-equivalent iff all their corresponding countable subtuples are. The base case is clear. Suppose the statement is true for all cardinal numbers smaller than λ, and assume that corresponding countable subtuples of $(a_j)_{j<\lambda}$ and $(b_j)_{j<\lambda}$ are Lascar-equivalent. By the inductive hypothesis, for every $i < \lambda$ there is $n_i < \omega$ such that the Lascar distance of $a_{<i}(:= (a_j)_{j<i})$ and $b_{<i}$ is equal to n_i. If there is $n < \omega$ such that $\{i \in \lambda : n = n_i\}$ is cofinal in λ, then the Lascar distance of $(a_i)_{i<\lambda}$ and $(b_i)_{i<\lambda}$ is n and we are done. So let us assume it is not the case, and hence there are $(i_k < \lambda)_{k<\omega}$ such that $n_{i_k} \geq k$. Then by compactness, for each $k < \omega$, there is a finite subset I_k of i_k such that the Lascar distance of $a_{I_k} := (a_j)_{j \in I_k}$ and b_{I_k} is at least k. Considering the countable set $I := \bigcup_{k<\omega} I_k$, we get that a_I and b_I are not Lascar equivalent, a contradiction. $\square$

Remark 3.4. (1) In [11] or [6], how to endow $\mathrm{Gal_L}(T)$ with a canonical topology to make it a topological group is explained as follows. For fixed small submodels M and N of $\mathcal{M}$, it easily follows that if $f(M) \equiv_N g(M)$ for $f, g \in \mathrm{Aut}(\mathcal{M})$, then $fg^{-1} \in \mathrm{Autf}(\mathcal{M})$. Hence there are the canonical maps $\mu : \mathrm{Aut}(\mathcal{M}) \to S_M(N)$ (where $S_M(N)$ denotes the Stone space of types over N of all conjugates of M) mapping f to $\mathrm{tp}(f(M)/N)$, and $\nu : S_M(N) \to \mathrm{Gal_L}(T)$ such that $\nu\mu : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal_L}(T)$ is the quotient map sending f to $f\,\mathrm{Autf}(\mathcal{M})$. The quotient topology under the map ν is given to $\mathrm{Gal_L}(T)$.

(2) Analogously, we consider $\nu' : S_M(N) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ such that $\nu'\mu : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ is the quotient map sending f to $(f \restriction \Sigma(\mathcal{M}))\,\mathrm{Autf_{res}}(\Sigma(\mathcal{M}))$. Again, we put on $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ the quotient topology with respect to ν'. Notice that $\nu' = \xi\nu$, where $\xi : \mathrm{Gal_L}(T) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ is given by $\xi(h\,\mathrm{Autf}(\mathcal{M})) = (h \restriction \Sigma(\mathcal{M}))\,\mathrm{Autf_{res}}(\Sigma(\mathcal{M}))$ (it is easy to see that ξ is well-defined).

(3) The topology on $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ defined above is the same as the quotient topology induced from $\mathrm{Gal_L}(T)$ by ξ. In particular, the topology on $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ does not depend on the choice of models M and N above, and ξ is a continuous map: For a subset W of $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$, we have that W is open iff $\nu'^{-1}[W] = \nu^{-1}[\xi^{-1}[W]]$ is open in $S_M(N)$ iff $\xi^{-1}[W]$ is open in $\mathrm{Gal_L}(T)$.

(4) In a similar manner, by considering the map $\nu'' : S_M(N) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$ such that $\nu''\mu : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$ is the quotient map, we equip the group $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$ with the topology induced by ν'', which coincides with the topology induced

on $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$ by the quotient map $\mathrm{Gal}_{\mathrm{L}}(T) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$. Since the quotient of a topological group by a normal subgroup is always a topological group with respect to the quotient topology, using the fact that $\mathrm{Gal}_{\mathrm{L}}(T)$ is a topological group ([11, Theorem 16]) we obtain the following corollary.

Corollary 3.5. *With the topologies defined above,* $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ *and* $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$ *are topological groups.*

Proposition 3.6. *The group* $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},\lambda}(\Sigma(\mathcal{M}))$ *(for* $\lambda \leq \omega$*, so in particular* $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$*) does not depend on the choice of the monster model* $\mathcal{M}$*.*

Proof. Consider two monster models $\mathcal{M} \prec \mathcal{M}'$, such that $\mathcal{M}'$ is $|\mathcal{M}|^+$-saturated and $|\mathcal{M}|$-strongly homogeneous. We define a map

$$\eta : \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},\lambda}(\Sigma(\mathcal{M})) \to \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},\lambda}(\Sigma(\mathcal{M}'))$$

by

$$\eta([f \restriction \Sigma(\mathcal{M})]) = [f' \restriction \Sigma(\mathcal{M}')],$$

where $f' \in \mathrm{Aut}(\mathcal{M}')$ is any extension of $f \in \mathrm{Aut}(\mathcal{M})$. Let us check that η is well-defined. Suppose that two automorphisms $f_1, f_2 \in \mathrm{Aut}(\mathcal{M})$ determine the same element in $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},\lambda}(\Sigma(\mathcal{M}))$; i.e. $g := (f_1 f_2^{-1}) \restriction \Sigma(\mathcal{M})$ belongs to $\mathrm{Autf}_{\mathrm{fix}}^{\lambda}(\Sigma(\mathcal{M}))$. Take any $f_1', f_2' \in \mathrm{Aut}(\mathcal{M}')$ extending f_1 and f_2 respectively. To see that $g' := (f_1' f_2'^{-1}) \restriction \Sigma(\mathcal{M}')$ is in $\mathrm{Autf}_{\mathrm{fix}}^{\lambda}(\Sigma(\mathcal{M}'))$, take any λ-tuple a' of elements of $\Sigma(\mathcal{M}')$. Pick $a \in \mathcal{M}$ which is Lascar equivalent to a'. Then $g'(a') \equiv^L g'(a) = g(a) \equiv^L a \equiv^L a'$, which shows that $g' \in \mathrm{Autf}_{\mathrm{fix}}^{\lambda}(\mathcal{M}')$ (by Remark 3.3), and so η is well-defined.

Now it is clear that η is an injective homomorphism. To see that it is onto, consider any element $[g \restriction \Sigma(\mathcal{M}')]$ of $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},\lambda}(\Sigma(\mathcal{M}'))$, where $g \in \mathrm{Aut}(\mathcal{M}')$. By the argument in Remark 3.4(1), we can find $g' \in \mathrm{Aut}(\mathcal{M}')$ such that $gg'^{-1} \in \mathrm{Autf}(\mathcal{M}')$ and $g'[\mathcal{M}] = \mathcal{M}$. Then $\eta([g' \restriction \Sigma(\mathcal{M})]) = [g' \restriction \Sigma(\mathcal{M}')] = [g \restriction \Sigma(\mathcal{M}')]$. This shows that η is an isomorphism. □

Notation 3.7. Due to above Proposition 3.6, we write $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},n}(\Sigma), \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma)$ for the groups $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},n}(\Sigma(\mathcal{M})), \mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(\Sigma(\mathcal{M}))$, respectively.

Question 3.8. Is the group $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{res}}(\Sigma(\mathcal{M}))$ independent from the choice of the monster model $\mathcal{M}$? What is an example in which $\mathrm{Autf}_{\mathrm{res}}(\Sigma(\mathcal{M}))$ differs from $\mathrm{Autf}_{\mathrm{fix}}(\Sigma(\mathcal{M}))$?

Remark 3.9. If there are realizations $a_i \in \Sigma(\mathcal{M})$ and a small submodel $(B \subseteq) M$ of $\mathcal{M}$ such that $M \subseteq \mathrm{dcl}(Ba_i |\ i < \lambda)$, then $\mathrm{Autf}_{\mathrm{res}}(\Sigma(\mathcal{M})) =$

$\mathrm{Autf}_{\mathrm{fix}}(\Sigma(\mathcal{M}))$. In particular, if $M \subseteq \mathrm{dcl}(Ba_0)$, then $\mathrm{Autf}_{\mathrm{res}}(\Sigma(\mathcal{M})) = \mathrm{Autf}_{\mathrm{fix}}(\Sigma(\mathcal{M})) = \mathrm{Autf}^1_{\mathrm{fix}}(\Sigma(\mathcal{M}))$.

Fact 3.10. *Recall from Section 2.2 that we have a canonical epimorphism*
$$\psi^*_p : \mathrm{Aut}(p(\mathcal{M})) \to H^*_1(p)$$
sending each $\sigma \in \mathrm{Aut}(p(\mathcal{M}))$ *to* $[a, \sigma(a)]$ *for some/any realization* a *of* p. *Due to Fact 2.6,* $\mathrm{Ker}(\psi^*_p)$ *contains* $\mathrm{Autf}_{\mathrm{fix}}(p(\mathcal{M}))$. *Hence, this induces a canonical epimorphism* $\Psi^*_p :\ \mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p) \to H^*_1(p)$ *as well.*

Remark 3.11. Note that $\mathrm{Aut}(p(\mathcal{M}))/\,\mathrm{Ker}(\psi^*_p)$ is isomorphic to $H^*_1(p)$, which is independent from the choice of the monster model. Since $H^*_1(p)$ is abelian, $\mathrm{Ker}(\psi^*_p)$ contains the derived subgroup of $\mathrm{Aut}(p(\mathcal{M}))$. We shall figure out what $\mathrm{Ker}(\psi^*_p)$ is, and it will turn out that even the kernel (so $H^*_1(p)$ too) is independent from the choice of $\mathop{\smile}\limits^{*}$.

4. Characterization of the first homology groups

The goal of this section is to identify what $\mathrm{Ker}(\psi^*_p)$ is. In [7],[8], the 2-chains in p (in the sense of thorn-independence) with 1-shell boundaries are classified when T is rosy. However, again, the only properties of thorn-forking used there are *the basic 5 axioms*: finite character, normality, symmetry, transitivity, and extension. Therefore, the same conclusion can be obtained in our context of $*$-independence in any T.

In particular we obtain the following from [7, 3.14]:

Remark 4.1. Let $s = f_{01} + f_{12} - f_{02}$ be a 1-$*$-shell with $\mathrm{supp}(f_{ij}) = \{i, j\}$. Then s is the boundary of some 2-$*$-chain in p iff s is the boundary of some 2-$*$-chain
$$\alpha = \sum_{i=0}^{2n} (-1)^i a_i$$
with 2-$*$-simplicies a_i, which is a *chain-walk* from f_{01} to f_{12}. We call the 2-$*$-chain α a *chain-walk* from f_{01} to f_{12} if,

(1) there are non-zero numbers $k_0, \ldots, k_{2n+1}$ (not necessarily distinct) such that $k_0 = k_{2n} = 1$, $k_{2n+1} = 2$, and for $i \leq 2n$, $\mathrm{supp}(a_i) = \{0, k_i, k_{i+1}\}$;
(2) $\partial^2 a_0 = f_{01}$, $\partial^2 a_{2n} = f_{12}$; and
(3) for $0 \leq i < n$,
$$\partial^0 a_{2i} = \partial^0 a_{2i+1},\ \partial^2 a_{2i+1} = \partial^2 a_{2i+2}.$$

Note that actually in [7, 3.14], it is given as a chain-walk from f_{01} to $-f_{02}$ but the same proof gives a chain-walk from f_{01} to f_{12}.

Now due to the fact that $\partial(\alpha) = s$ and α is a chain-walk, we can directly obtain the following fact.

Theorem 4.2. *A 1-$*$-shell s in p is the boundary of a 2-chain if and only if there is a representation (a, b, c, a') of s such that for some $n \geq 0$ there is a finite sequence $(d_i)_{0 \leq i \leq 2n+2}$ of realizations of p satisfying the following conditions:*

(1) $d_0 = a$, $d_{2n+1} = c$ and $d_{2n+2} = a'$;
(2) $\{d_j, d_{j+1}, b\}$ is $*$-independent for each $0 \leq j \leq 2n+1$; and
(3) there is a bijection

$$\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \leq n$.

Proof. Let s be a 1-shell with $\mathrm{supp}(s) = \{0, 1, 2\}$ in p, which is the boundary of a 2-chain. Then, by Remark 4.1, we have a 2-$*$-chain-walk $\sum_{i=0}^{2n}(-1)^i a_i$ from f_{01} to f_{12} with the boundary s. Then there are $d_0, d_1, \ldots, d_{2n+1}, b \models p$ such that

(1) $\{d_j, d_{j+1}, b\}$ is $*$-independent for each $0 \leq j \leq 2n$;
(2) $a_{2i}(\{0, 1, 2\}) \equiv [d_{2i}bd_{2i+1}]_{\{0,1,2\}}$ and $a_{2i+1}(\{0, 1, 2, \}) \equiv [d_{2i+2}bd_{2i+1}]_{\{0,1,2\}}$ for each $0 \leq i \leq n-1$; and
(3) $f_{02}(\{0, 2\}) \equiv [d_{2i_0}d_{2i_0+1}]_{\{0,2\}}$ for some $0 \leq i_0 \leq n-1$.

Take a tuple $(d_0, b, d_{2n+1}, d_{2i_0})$, which is a representation of s. Since α is a 2-$*$-chain-walk, there is a bijection

$$\sigma : \{0, 1, \cdots, n\} \smallsetminus \{i_0\} \to \{0, 1, \cdots, n-1\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \neq i_0 \leq n$. Set $\sigma' := \sigma \cup \{(i_0, n)\}$ and $d_{2n+2} := d_{2i_0}$. We get a desired result from the bijection σ' and the sequence $(d_0, d_1, \ldots, d_{2n+1}, d_{2n+2})$.

Conversely, we assume that there are a representation (a, b, c, a'), a finite sequence $(d_i)_{0 \leq i \leq 2n+2}$ of realizations of p, and a bijection σ on $\{0, 1, \ldots, n\}$ for some $n \geq 0$ such that

(1) $d_0 = a$, $d_{2n+1} = c$ and $d_{2n+2} = a'$;
(2) $\{d_j, d_{j+1}, b\}$ is $*$-independent for each $0 \leq j \leq 2n+1$; and
(3) $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \leq n$.

Put $i_0 = \sigma^{-1}(n)$. From the subsequence $(d_i)_{0 \leq i \leq 2n+1}$, we get a 2-$*$-chain-walk $\alpha = \sum_{i=0}^{2n}(-1)^i a_i$ from f_{01} to f_{12} with the boundary $f_{01} - a_{2i_0} \restriction \{0, 2\} + f_{12}$. Since $d_{2i_0}d_{2i_0+1} \equiv d_{2n+2}d_{2n+1} = a'c$, we can make $a_{2i_0} \restriction \{0, 2\} = f_{02}$ so that $\partial\alpha = s$.

$\square$

Corollary 4.3. *For $a, a' \models p$, $[a, a'] = 0$ in $H_1^*(p)$ if and only if for some $n \geq 0$ there is a finite sequence $(d_i)_{0\leq i\leq 2n+2}$ of realizations of p satisfying the following conditions:*

(1) $d_0 = a$, and $d_{2n+2} = a'$;
(2) $\{d_j, d_{j+1}\}$ is $*$-independent for each $0 \leq j \leq 2n+1$; and
(3) there is a bijection

$$\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \leq n$.

Proof. Fix $a, a' \models p$. The left-to-right direction is clear from Theorem 4.2. For the right-to-left direction, we assume that there is a finite sequence $(d_i)_{0\leq i\leq 2n+2}$ of realizations of p satisfying the following conditions:

(1) $d_0 = a$, and $d_{2n+2} = a'$;
(2) $\{d_j, d_{j+1}\}$ is $*$-independent for each $0 \leq j \leq 2n+1$; and
(3) there is a bijection

$$\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \leq n$.

Take $b \models p$ such that $b \mathop{\smile\!\!\!\!\!\!\mid}^* d_0d_1 \ldots d_{2n+2}$. Then the tuple (a, b, d_{2n+1}, a') and the sequence $(d_i)_{0\leq i\leq 2n+2}$ gives a 1-$*$-shell which is a boundary of a 2-$*$-chain by Theorem 4.2. $\square$

By now, as promised, we can identify $\mathrm{Ker}(\psi_p^*)$.

Theorem 4.4. *For each $h \in K := \mathrm{Ker}(\psi_p^*)$ and $a \models p$, there is an automorphism h' in the derived subgroup G' of $G := \mathrm{Aut}(p(\mathcal{M}))$ such that $h(a) = h'(a)$. Thus, $K(\geq G')$ is the normal subgroup of G consisting of all automorphisms fixing all orbits of elements of $p(\mathcal{M})$ under the action of G', and $H_1^*(p) = G/K$.*

Proof. If the first statement is true then the second statement clearly follows since $G' \leq K = \mathrm{Ker}(\psi_p^*)$ (as $G/K \cong H_1^*(p)$ is abelian). So let us prove the first statement.

Let $h \in K$ and $a \models p$. Thus $[a, h(a)] = 0$ (in $\mathcal{E}^* = H_1^*(p)$). By Corollary 4.3, there are an integer $n \geq 0$ and a finite sequence $(d_i)_{0\leq i\leq 2n+2}$ of realizations of p such that

(1) $d_0 = a$, and $d_{2n+2} = h(a)$;
(2) $\{d_j, d_{j+1}\}$ is $*$-independent for each $0 \leq j \leq 2n+1$; and
(3) there is a bijection

$$\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \leq n$.

By (3), we have the following automorphisms in G:

- $\eta_i : d_{2i} \mapsto d_{2i+1}$ for $0 \le i \le n$;
- $\eta'_j : d_{2j+1} \mapsto d_{2j+2}$ for $0 \le j \le n$; and
- $f_i : d_{2i}d_{2i+1} \mapsto d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \le i \le n$.

Thus for $0 \le i \le n$, $\eta_i\eta'_{i-1}\eta_{i-1}\cdots\eta'_0\eta_0(d_0) = d_{2i+1}$, $\eta'_i\eta_i\cdots\eta'_0\eta_0(d_0) = d_{2i+2}$, and

$$\eta'_n\eta_n\eta'_{n-1}\eta_{n-1}\cdots\eta'_0\eta_0(d_0) = d_{2n+2} = h(a) \quad (\dagger).$$

Moreover, $\eta'_{\sigma(i)}(d_{2\sigma(i)+1}) = f_i\eta_i^{-1}f_i^{-1}(d_{2\sigma(i)+1})$ for $0 \le i \le n$, so $\eta'_i(d_{2i+1}) = f_{\sigma^{-1}(i)}\eta^{-1}_{\sigma^{-1}(i)}f^{-1}_{\sigma^{-1}(i)}(d_{2i+1}) \quad (\ddagger)$. From ($\dagger$) and ($\ddagger$), we get that $g_0(a) = h(a)$, where

$$g_0 := (f_{\sigma^{-1}(n)}\eta^{-1}_{\sigma^{-1}(n)}f^{-1}_{\sigma^{-1}(n)})\eta_n\cdots(f_{\sigma^{-1}(0)}\eta^{-1}_{\sigma^{-1}(0)}f^{-1}_{\sigma^{-1}(0)})\eta_0.$$

We claim that $g_0 \in G'$: Since G/G' is abelian (so $gkG' = kgG'$), we get that $g_0G' = g_1G'$, where

$$g_1 := \eta^{-1}_{\sigma^{-1}(n)}\eta_n\cdots\eta^{-1}_{\sigma^{-1}(0)}\eta_0.$$

Moreover, since σ is a permutation of $\{0,1,\ldots,n\}$, it follows (using again that G/G' is abelian) that $g_1G' = 1_GG' = G'$, so $g_1, g_0 \in G'$. □

Remark 4.5. Due to above Theorem 4.4, $H_1^*(p)$, which of course does not depend on the choice of a monster model, is always the same regardless of our choice of independence $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfying the 5 basic axioms. Hence, we can denote it simply by $H_1(p)$. Moreover, $H_1(p)$ can also be considered as a quotient group of $\mathrm{Gal_L}(T)$ or $\mathrm{Gal_L^{fix}}(p)$, which equivalently endow $H_1(p)$ with a topological group structure.

Remark 4.6. If p is a strong type of a model, then for $G := \mathrm{Gal_L}(T)$, $G \cong \mathrm{Gal_L^{fix,1}}(p) = \mathrm{Gal_L^{fix}}(p)$ and $H_1(p) \cong G/G'$ as topological groups.

Proof. Suppose p is a strong type of a small submodel M of $\mathcal{M}$. By Remark 3.9, $\mathrm{Autf}^1_{\mathrm{fix}}(p(\mathcal{M})) = \mathrm{Autf}_{\mathrm{fix}}(p(\mathcal{M})) = \mathrm{Autf}_{\mathrm{res}}(p(\mathcal{M}))$. It remains to show that $\mathrm{Gal_L^{fix,1}}(p(\mathcal{M})) \cong \mathrm{Gal_L}(T)$. Consider the projection $\pi_{\mathrm{fix},1} : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal_L^{fix,1}}(p(\mathcal{M}))$ sending σ to $(\sigma \restriction p(\mathcal{M}))\,\mathrm{Autf}^1_{\mathrm{fix}}(p(\mathcal{M}))$. Suppose σ is in the kernel of $\pi_{\mathrm{fix},1}$. Then $\sigma(M) \equiv^L M$, and there is $\tau \in \mathrm{Autf}(\mathcal{M})$ such that $\sigma \restriction M = \tau \restriction M$. Thus $\tau^{-1} \circ \sigma \restriction M = \mathrm{id}_M$. So $\tau^{-1} \circ \sigma \in \mathrm{Autf}(\mathcal{M})$ and $\sigma \in \mathrm{Autf}(\mathcal{M})$. Therefore, the kernel of $\pi_{\mathrm{fix},1}$ is a subgroup of $\mathrm{Autf}(\mathcal{M})$. Also, it is easy to check that $\mathrm{Autf}(\mathcal{M})$ is a subgroup of the kernel of $\pi_{\mathrm{fix},1}$. Thus we have that $\mathrm{Gal_L}(T) \cong \mathrm{Gal_L^{fix,1}}(p(\mathcal{M}))$, witnessed by the isomorphism induced from $\pi_{\mathrm{fix},1}$ (actually this is an isomorphism of topological groups).

Now let $\varphi_p : \mathrm{Aut}(\mathcal{M}) \to H_1(p)$ be the epimorphism sending $\sigma \in \mathrm{Aut}(\mathcal{M})$ to $[M, \sigma(M)]$, as defined in 2.7. It is enough to show that the kernel of φ_p is generated by the automorphisms in $\mathrm{Aut}(\mathcal{M})' \cup \mathrm{Autf}(\mathcal{M})$, so $\mathrm{Ker}(\varphi_p) = \mathrm{Aut}(\mathcal{M})' \, \mathrm{Autf}(\mathcal{M})$. It is clear from Theorem 4.4 that $\mathrm{Aut}(\mathcal{M})' \, \mathrm{Autf}(\mathcal{M}) \subseteq \mathrm{Ker}(\varphi_p)$. Conversely, suppose $\sigma \in \mathrm{Ker}(\varphi_p)$. By Theorem 4.4, there is $\tau \in \mathrm{Aut}(\mathcal{M})'$ such that $\sigma \restriction M = \tau \restriction M$. So $(\tau^{-1} \circ \sigma) \restriction M = \mathrm{id}_M$ and $\tau^{-1} \circ \sigma \in \mathrm{Autf}(\mathcal{M})$. Thus, $\sigma \in \tau \, \mathrm{Autf}(\mathcal{M}) \subseteq \mathrm{Aut}(\mathcal{M})' \, \mathrm{Autf}(\mathcal{M})$. Therefore $H_1(p) \cong G/G'$, and furthermore they are homeomorphic as topological groups. $\square$

As a corollary to Theorem 4.4, we get the following characterization of the equality of strong types and Lascar strong types, *for any theory*:

Corollary 4.7. *The following conditions are equivalent :*

(1) p, a strong type of an algebraically closed tuple, is a Lascar strong type;
(2) $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},1}(p)$ is abelian and $H_1(p) = 0$; and
(3) Both $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},1}(p)$ and $H_1(p)$ are trivial.

In particular, if $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(p)$ is abelian, then p is a Lascar strong type if and only if $H_1(p) = 0$. Moreover, if p is a strong type of a model, then p is a Lascar strong type if and only if $\mathrm{Gal}_{\mathrm{L}}(T)$ is abelian and $H_1(p) = 0$.

Proof. The implication from (3) to (2) is trivial. The implications from (1) to (2) and (3) are easy. Suppose p is a Lascar strong type. Then, by the definition, $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},1}(p) = 0$ and $H_1(p) = 0$. It is enough to show (2) $\Rightarrow$ (1). Suppose $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},1}(p)$ is abelian and $H_1(p) = 0$. It is enough to show that for any $a \in p(\mathcal{M})$ and any $\sigma \in \mathrm{Aut}(p(\mathcal{M}))$, $a \equiv^L \sigma(a)$. Choose $a \in p(\mathcal{M})$ and $\sigma \in \mathrm{Aut}(p(\mathcal{M}))$ arbitrarily. Since $H_1(p) = 0$, there is $\tau \in \mathrm{Aut}(p(\mathcal{M}))'$ such that $\sigma(a) = \tau(a)$ by Theorem 4.4. The derived group of $\mathrm{Aut}(p(\mathcal{M}))$ is a subgroup of $\mathrm{Autf}_{\mathrm{fix}}^1(p(\mathcal{M}))$, because $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix},1}(p)$ is abelian. Therefore, we have $a \equiv^L \sigma(a) (= \tau(a))$.

The rest comes from Remark 4.6. $\square$

Next, we consider the orbit equivalence relation $\equiv^{H_1}$ on $p(\mathcal{M})$ under the action of K (equivalently G') in Theorem 4.4 (i.e., for $a, b \models p$, $a \equiv^{H_1} b$ iff there is $f \in K$ (or $\in G'$) such that $b = f(a)$ iff $[a, b] = 0 \in H_1(p)$). We show now that this equivalence relation is an F_σ-relation (in any theory); i.e., there are countably many B-type-definable reflexive, symmetric relations $R_i(x, y)$ such that

$$p(x) \wedge p(y) \models x \equiv^{H_1} y \leftrightarrow \bigvee_{i<\omega} R_i(x, y) :$$

Consider an invariant, symmetric, reflexive relation R such that for $\bar{a},\bar{b}\in\mathcal{M}$, $R(\bar{a},\bar{b})$ if and only if there are $\sigma,\tau\in\mathrm{Aut}(\mathcal{M})$ such that $\bar{b}=[\sigma,\tau](\bar{a})$, where $[\sigma,\tau]:=\sigma^{-1}\tau^{-1}\sigma\tau$, if and only if there are c_1, c_2, and c_3 such that $ac_3\equiv c_1c_2$ and $c_2c_3\equiv c_1b$. Define

$$R_i(\bar{x},\bar{y})\equiv\begin{cases}\bar{x}=\bar{y} & \text{if } i=0\\ \overbrace{R\circ\cdots\circ R}^{i}(\bar{x},\bar{y}) & \text{if } i\geq 1.\end{cases}$$

Then by Theorem 4.4,

$$p(x)\wedge p(y)\models x\equiv^{H_1} y\leftrightarrow\bigvee_{i<\omega}R_i(x,y).$$

Next, define the H_1-*distance on* p as follows: For $a,b\models p$,

$$d_{H_1}(a,b):=\begin{cases}\min\{n|R_n(a,b)\} & \text{if } a\equiv^{H_1} b\\ \infty & \text{otherwise,}\end{cases}$$

and the H_1-diameter on p by:

$$d_{H_1}(p):=\max\{d_R(a,b)|\ a,b\models p\}.$$

Applying the Newelski's result from [9] on the possible cardinality of the set of classes of bounded invariant equivalence relations on a type, we know that the cardinality of $H_1^*(p)$ is at least $2^{\aleph_0}$ if the equivalence relation $\equiv^{H_1}$ on p is not type-definable, and in the other case, we also have that the possible cardinality of H_1 is one or at least $2^{\aleph_0}$ by Appendix A.

Theorem 4.8. *For any theory T:*

(1) The equivalence relation $\equiv^{H_1}$ on p is type-definable if and only if $d_{H_1}(p)$ is finite.
(2) The cardinality of $H_1(p)$ is one or $\geq 2^{\aleph_0}$.

We finish this section by posing the following question for simple theories.

Question 4.9. In a simple theory, is the first homology group of a strong type always trivial?

5. Examples

5.1. **Topological groups and the first homology groups of types.** In this subsection, we argue that all connected abelian compact groups can occur as the first homology groups of strong types (Here, compact topological spaces are Hausdorff by definition). At first, note that in [11] M. Ziegler showed (using a result of E. Bouscaren, D. Lascar, and

A. Pillay) that any compact group occurs as the Lascar Galois group of a complete theory.

Fact 5.1. [11] *Let G be a compact group. Then there is a complete theory T_G whose Lascar Galois group is isomorphic to G.*

From Remark 4.6, we also know that the first homology group of a strong type of a model is isomorphic to the abelianization of the connected component of Lascar Galois group. So, if we take G in Fact 5.1 as an abelian and connected group, we conclude that the first homology group of a strong type of model in T_G is isomorphic to the Lascar Galois group G itself.

Theorem 5.2. *For each abelian connected compact group G, there is a strong type of a model of a complete theory whose first homology group is isomorphic to G.*

Remark 5.3. There is a strong type p in a theory with trivial first homology group, which is not a Lascar strong type. In other words, in Corollary 4.7 (2), we cannot omit the condition of abelianness of $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{fix}}(p)$ to conclude that a given strong type p is a Lascar strong type. If G is a non-trivial connected compact group whose commutator subgroup is itself, then the first homology group of a strong type of a model of T_G is trivial. In this case, the strong type is not a Lascar strong type because the Lascar Galois group is not trivial. For example, we can take $G := SU(3)$ as such a group.

5.2. **Some computation of the first homology group of a type.** Here we give a more concrete example of a strong type in a rosy theory with a non-trivial first homology group. In [7], S. Kim, and the second and third authors considered the structures $\mathcal{M}_n = (M; S; g_{1/n})$ (which were earlier studied in [1]) for each $n \in \mathbb{N} \setminus \{0\}$, where

(1) M is a saturated circle;
(2) $g_{1/n}$ is the clockwise rotation by $2\pi/n$ radians; and
(3) S is a ternary relation such that $S(a,b,c)$ holds if a,b,c are distinct and b comes before c going around the circle clockwise starting at a,

and it was shown that the unique strong 1-type p_n in $S_1(\emptyset)$ has the trivial first homology group for every n, and is actually a Lascar strong type. Now, we consider a structure $\mathcal{M} = (M; S; g_{1/n} : n \in \mathbb{N} \setminus \{0\})$ expanding the structures $\mathcal{M}_n$ by adding all rotation functions by $2\pi/n$-radians for each $n \in \mathbb{N} \setminus \{0\}$ at the same time (when we write g_r for $r = m/n$ in $\mathbb{Q} \cap [0,1)$, it means $g_{1/n}^m$). We show that $\mathrm{Th}(\mathcal{M})$ is a rosy theory. In [2], C. Ealy and A. Onshuus gave a sufficient condition for a theory to be rosy.

Fact 5.4. *Any theory T which weakly eliminates imaginaries and for which the algebraic closure defines a pregeometry is rosy of thorn U-rank* 1*.*

At first, we show that $\mathrm{Th}(\mathcal{M})$ has weak elimination of imaginaries. In [10], B. Poizat defined a theory T to have *weak elimination of imaginaries* if every definable set has a smallest algebraically closed set over which it is definable. By repeating the argument from [7], we obtain the following sufficient condition for weak elimination of imaginaries in an $\aleph_0$-categorical theory:

Theorem 5.5. *Let T be $\aleph_0$-categorical and let $\mathcal{M} = (M, \dots)$ be a saturated model of T. Suppose that if $X \subset M^1$ is definable over each of two algebraically closed sets A_0 and A_1, then X is definable over $B := A_0 \cap A_1$.*

Then, for any subset Y of M^n, if Y is both A_0-definable and A_1-definable, then it is B-definable. Furthermore, in this case, T has weak elimination of imaginaries.

Proof. Let $A_0 = \mathrm{acl}(A_0)$, $A_1 = \mathrm{acl}(A_1)$, and $B = A_0 \cap A_1$. We use induction on n. If $n = 1$, the conclusion holds by assumption. Let us show that the conclusion holds for $n + 1$ assuming it holds for n. Put $A_0 = \mathrm{acl}(A_0)$, $A_1 = \mathrm{acl}(A_1)$, and $B = A_0 \cap A_1$. Since, by $\aleph_0$-categoricity, the algebraic closure of a finite set is finite, we may assume that A_0 and A_1 are finite, and so is B. Let $Y \subset M^{n+1}$ be A_i-definable by a formula $\phi_i(x_0, \dots, x_n; \bar{a}_i)$ with $\bar{a}_i \subset A_i$ for $i = 0, 1$. Then, for each $c \in M$, the fiber of Y over c, $Y_c := \{\bar{x} \in M^n |\ \phi_i(\bar{x}, c; \bar{a}_i)\}$ is cB-definable by induction. By $\aleph_0$-categoricity, there are only finitely many formulas over $\emptyset$ modulo T, and it easily follows that for each $c \in M^1$, $\phi_i(x_0, \dots, x_{n-1}, c, \bar{a}_i)$ is B-definable. Thus, Y is B-definable.

Since (again by $\aleph_0$-categoricity) there is no infinite descending chain of algebraically closed sets generated by finitely many elements, we conclude that any definable set has a smallest algebraically closed set over which it is definable. Thus, T weakly eliminates imaginaries. □

As a corollary to Theorem 5.5, it was shown in [7] that for each $n \geq 2$, $\mathrm{Th}(\mathcal{M}_n)$ has weak elimination of imaginaries.

Fact 5.6 ([7])**.** *For each $n \geq 2$, $\mathrm{Th}(\mathcal{M}_n)$ weakly eliminates imaginaries.*

Next, we will see that the theory of $\mathcal{M}$ has quantifier-elimination.

Definition 5.7. Let M be the non-standard circle which is the universe of $\mathcal{M}$. For $A \subset M$, let $\mathrm{cl}(A) := \{g_r(a) | \ a \in A, \ r \in \mathbb{Q} \cap [0, 1)\}$. Later, we will see that $\mathrm{cl}(A) = \mathrm{dcl}(A) = \mathrm{acl}(A)$ in the home sort of $\mathcal{M}$. It is also easy to see that $\mathrm{cl}(A)$ is a substructure of $\mathcal{M}$.

Theorem 5.8. *The theory of $\mathcal{M}$ has quantifier-elimination.*

Proof. Take two small subsets $A, B \subset M$ such that $A = \mathrm{cl}(A)$ and $B = \mathrm{cl}(B)$ in M, and a partial isomorphism $f : A \to B$. Take $a \in M \setminus A$. We will find $b \in M \setminus B$ such that the map $f \cup \{(a, b)\}$ can be extended to an embedding from $\mathrm{cl}(Aa)$ to $\mathrm{cl}(Bb)$ in $\mathcal{M}$. Then, the quantifier-elimination in $\mathrm{Th}(\mathcal{M})$ comes from a standard argument. We divide A into two parts: $A_0 := \{x \in A | \ S(a, x, g_{1/2}(a))\}$ and $A_1 := \{x \in A | \ S(g_{1/2}(a), x, a)\}$. Then B is also divided into two parts: $B_0 = f(A_0)$ and $B_1 = f(A_1)$. Take arbitrary $b \in M$ such that for all $y_0 \in B_0, \ y_1 \in B_1$, we have that $S(y_1, b, y_0)$. Then b is a desired element. □

Theorem 5.9. *The theory of $\mathcal{M}$ weakly eliminates imaginaries, and is rosy of thorn U-rank 1.*

Proof. By quantifier elimination, in the structure $\mathcal{M}$ there is no infinite descending chain of algebraic closures of finite sets. It is enough to show that if $X \subset M^n$ is $A_0(= \mathrm{acl}(A_0))$-definable and $A_1(= \mathrm{acl}(A_1))$-definable, then X is $A_0 \cap A_1(= B)$-definable. (Then X has a smallest algebraically closed set over which it is definable, and $\mathrm{Th}(\mathcal{M})$ has weak elimination of imaginaries.)

Let $A_i = \mathrm{acl}(A_i) = \mathrm{cl}(A_i)$ for $i = 0, 1$ and put $B = A_0 \cap A_1$. Let $X \subset M^m$ be A_i-definable in $\mathcal{M}$. Then X is definable over A_i for $i = 0, 1$ in some reduct $\mathcal{M}_n$ of $\mathcal{M}$. Since $\mathcal{M}_n$ weakly eliminates imaginaries, X is definable over B in $\mathcal{M}_n$ by a formula $\psi(\bar{x}, \bar{b})$. Then X is B-definable in $\mathcal{M}$ by the same formula $\psi(\bar{x}, \bar{b})$.

By quantifier elimination, it is easily verified that the algebraic closure in $\mathcal{M}$ gives a trivial pregeometry (i.e. $\mathrm{acl}(A) = \cup_{a \in A} \mathrm{acl}(\{a\})$). Thus, by Fact 5.4, $\mathrm{Th}(\mathcal{M})$ is a rosy theory of thorn U-rank 1. □

There is only one 1-strong type over the empty set in $\mathcal{M}$: $p_0(x) \equiv \{x = x\}$.

In $\mathcal{M}$, for a fixed $a \in M$, we observe that the types in $S_1(a)$ correspond to elements of the unit circle, where the points with rational spherical coordinates are tripled. Using this observation, we compute the first homology group of p_0 in $\mathcal{M}$:

Theorem 5.10. *In $\mathcal{M}$, the first homology group of p_0 is isomorphic to $\mathbb{R}/\mathbb{Z}$.*

We start with defining a distance-like notion between two points on M. We fix an infinitesimal ϵ. For a subset $Y \subset \mathbb{Q}$, we define $Y^* := Y \cup \{y \pm \epsilon | y \in Y\}$. We write $X_{\mathbb{Q}}$ for $X \cap \mathbb{Q}$ for a subset X in $\mathbb{R}$.

Definition 5.11. Let $a, b \in M$ be two elements. We define *the S-distance from a to b, denoted by* $\mathrm{Sd}(a, b)$. *The S-distance has values in* $[0, 1) \cup [0, 1)^*_{\mathbb{Q}} \cup \{1 - \epsilon\}$. *Let $r \in (0, 1)_{\mathbb{Q}}$ and $r' \in (0, 1) \setminus \mathbb{Q}$.*

(1) $\mathrm{Sd}(a, b) = 0$ *if $b = a$;*
(2) $\mathrm{Sd}(a, b) = \epsilon$ *if for all $s \in (0, 1)_{\mathbb{Q}}$, $\mathcal{M} \models S(a, b, g_s(a))$;*
(3) $\mathrm{Sd}(a, b) = 1 - \epsilon$ *if for all $s \in (0, 1)_{\mathbb{Q}}$, $\mathcal{M} \models S(g_s(a), b, a)$;*
(4) $\mathrm{Sd}(a, b) = r$ *if $b = g_r(a)$;*
(5) $\mathrm{Sd}(a, b) = r-\epsilon$ *if for $s \in (0, 1)_{\mathbb{Q}}$ with $s < r$, $\mathcal{M} \models S(g_s(a), b, g_r(a))$;*
(6) $\mathrm{Sd}(a, b) = r+\epsilon$ *if for $t \in (0, 1)_{\mathbb{Q}}$ with $r < t$, $\mathcal{M} \models S(g_r(a), b, g_t(a))$;*
(7) $\mathrm{Sd}(a, b) = r'$ *if for $s < t \in [0, 1)_{\mathbb{Q}}$ such that $s < r' < t$, $\mathcal{M} \models S(g_s(a), b, g_t(a))$.*

In Appendix B, using Dedekind cuts, we develop multivalued operations $+^*$ and $-^*$ to make $\mathbb{R} \cup \mathbb{Q}^*$ a group-like structure. Now, we extend the values of S-distance to $\mathbb{R} \cup \mathbb{Q}^*$. Since $g_k = \mathrm{id}$ for all $k \in \mathbb{Z}$, we write $\mathrm{Sd}(a, b) = r$ for $r \in \mathbb{R} \cup \mathbb{Q}^*$ if $\mathrm{Sd}(a, b) = r'$, where r' is the unique number in $[0, 1) \cup [0, 1)^*_{\mathbb{Q}}$ such that $r \in r' +^* n$ for some $n \in \mathbb{Z}$. Then this values depend only on the type of (a, b), that is, for $a_0, a_1, b_0, b_1 \in M$, if $a_0b_0 \equiv a_1b_1$, then $\mathrm{Sd}(a_0, b_0) = \mathrm{Sd}(a_1, b_1)$ (taking values in $[0, 1) \cup [0, 1)^*_{\mathbb{Q}}$). Then the following fact is easily verified:

Fact 5.12. *Let $a, b, c \in M$.*

(1) $\mathrm{Sd}(b, a) = 1 -^* \mathrm{Sd}(a, b)$.
(2) $\mathrm{Sd}(a, c) = \mathrm{Sd}(a, b) +^* \mathrm{Sd}(b, c)$ *modulo $\mathbb{Z}^*$, that is,* $\mathrm{Sd}(a, b) +^* \mathrm{Sd}(b, c) -^* \mathrm{Sd}(a, c) \subset \mathbb{Z}^*$.

By (1), Sd *is not symmetric, that is, for some $a, b \in M$,* $\mathrm{Sd}(a, b) \neq \mathrm{Sd}(b, a)$ *and so it is called a directed distance.*

Now, we assign to each 1-simplex f a value n_f in $\mathbb{R} \cup \mathbb{Q}^*$ as follows. There are $a, b \in M$ such that $[a, b] = f$; we define n_f as $\mathrm{Sd}(a, b)$. Then n_f is well-defined, that is, it does not depend on the choice of a and b, because if $a_i, b_i \in M$ satisfy $[a_0, b_0] = [a_1, b_1] = f$, then $a_0b_0 \equiv a_1b_1$ and $\mathrm{Sd}(a_0, b_0) = \mathrm{Sd}(a_1, b_1)$. We also assign to each 1-shell $s = f_{01} + f_{12} - f_{02}$ a multivalue n_s in $\mathbb{R} \cup \mathbb{Q}^*$ as follows: $n_s = n_{f_{01}} +^* n_{f_{12}} -^* n_{f_{02}}$. This value is also related to the distance of endpoints. Let (a, a') be an endpoint pair of s. Then $\mathrm{Sd}(a, a') = n_s$ modulo $\mathbb{Z}^*$. Using this assignments, we give a necessary and sufficient condition for a 1-shell to be the boundary of a 2-chain:

Theorem 5.13. *A 1-shell $s = f_{12} - f_{02} + f_{01}$ is the boundary of a 2-chain in p if and only if*

$$n_s = n_{01} +^* n_{12} +^* n_{20} \subset \mathbb{Z}^*,$$

where $n_{01} = n_{f_{01}},\ n_{12} = n_{f_{12}},\ n_{20} = -^* n_{f_{02}}$. Moreover, it is equivalent to the condition that the two endpoints of s are Lascar equivalent over $\emptyset$.

Proof. ($\Rightarrow$) Let α be a 2-chain with boundary s. By Remark 4.1, we may assume that $\alpha = \sum_{i=0}^{2n}(-1)^i a_i$ is a chain-walk from f_{01} to f_{12} with $\mathrm{supp}(\alpha) = \{0,1,2\}$. Let $[3] = \{0,1,2\}$. By Theorem 4.2 and the extension axiom, there are independent elements b and $d_0, d_1, \cdots, d_{2n+2}$ such that

- $a_i([3]) \equiv [bd_id_{i+1}]_{[3]}$ if i is even, and $a_i([3]) \equiv [bd_{i+1}d_i]_{[3]}$ if i is odd;
- For some even number $0 \le i_0 \le 2n$, $[d_{i_0}d_{i_0+1}]_{\{1,2\}} \equiv f_{12}(\{1,2\})$; and
- For each even number $0 \le j_0 \ne i_0 \le 2n$, there is an odd number $0 \le j_1 \le 2n$ such that $[d_{j_0}d_{j_0+1}]_{\{1,2\}} \equiv [d_{j_1+1}d_{j_1}]_{\{1,2\}}$.

Then $\mathrm{Sd}(d_1,d_0) +^* \mathrm{Sd}(d_0,d_{2n+2}) = -^* n_{01} -^* n_{20}$ and by Fact 5.12 (1), $\mathrm{Sd}(d_1,d_2) +^* \mathrm{Sd}(d_2,d_3) +^* \cdots +^* \mathrm{Sd}(d_{2n+1},d_{2n+2}) = n +^* n_{12}$. By Fact 5.12 (2), $\mathrm{Sd}(d_1,d_{dn+2}) \in (-^* n_{01} -^* n_{20}) \cap (n +^* n_{12})$. Since $\{0\}^* = (-^* n_{01} -^* n_{20}) +^* (n_{01} +^* n_{20})$ and $(n +^* n_{12}) +^* (n_{01} +^* n_{20}) = n +^* (n_{01} +^* n_{12} +^* n_{20})$, these two equations imply that $n +^* (n_{01} +^* n_{12} +^* n_{20}) \subset \{0\}^*$. Therefore, $n_{01} +^* n_{12} +^* n_{20} \subset \{0\}^* -^* \{n\}^* = \{-n\}^*$ for $n \in \mathbb{N} \subset \mathbb{Z}$.

($\Leftarrow$) Suppose $n_{01} +^* n_{12} +^* n_{20} \subset \{n\}^*$ for some $n \in \mathbb{Z}$. There are independent elements a, b, c, a' such that

$$[ab]_{\{0,1\}} = f_{01}(\{0,1\}),\ [bc]_{\{1,2\}} = f_{12}(\{1,2\}),\ [a'c]_{\{0,2\}} = f_{02}(\{0,2\}).$$

So, $\mathrm{Sd}(a,b) = n_{01}$, $\mathrm{Sd}(b,c) = n_{12}$, $\mathrm{Sd}(c,a) = n_{20}$, and $\mathrm{Sd}(a,a') \in n_{01} +^* n_{12} +^* n_{20}$. Thus $\mathrm{Sd}(a,a') \in \{n\}^*$ and $\mathrm{Sd}(a,a') \in \{0\}^*$. Since $\{a,a'\}$ is independent, $\mathrm{Sd}(a,a') \in \{0\}^* \setminus \{0\}$, that is, $\mathrm{Sd}(a,a') = \epsilon$ or $\mathrm{Sd}(a,a') = 1-\epsilon$.

We will find $d \in M$ such that $a \equiv_d a'$ and $d \mathop{\smile\!\!\!\!\!\mid}\ abca'$, where $\mathop{\smile\!\!\!\!\!\mid}$ is the thorn-forking independence. Consider a partial type $\Sigma(x) = \{s < \mathrm{Sd}(x,a) < t \leftrightarrow s < \mathrm{Sd}(x,a') < t\}_{s<t\in[0,1]_{\mathbb{Q}}}$. Consider finitely many pairs (s_i,t_i) with $s_i < t_i$ and a formula

$$\bigwedge(s_i < \mathrm{Sd}(x,a) < t_i \leftrightarrow s_i < \mathrm{Sd}(x,a') < t_i).$$

We may assume $s_i \leq s_0 < t_0 \leq t_i$. It is enough to show that the formula

$$s_0 < \mathrm{Sd}(x,a) < t_0 \leftrightarrow s_0 < \mathrm{Sd}(x,a') < t_0$$

is satisfiable. Suppose the formula $s_0 < \mathrm{Sd}(x,a) < t_0$ is satisfiable. Then, there is a pair (s,t) such that $s_0 < s < t < t_0$ and $s < \mathrm{Sd}(x,a) < t$ is satisfiable. Let $e \in M$ be an element independent from a such that $s < \mathrm{Sd}(e,a) < t$ holds. Since $\mathrm{Sd}(a,a') \in \{0\}^* \setminus \{0\}$, there is a pair (s',t') such that $s' < t'$, $s_0 < s+s' < t+t' < t_0$, and $s' < \mathrm{Sd}(a,a') < t'$. Then, $s < \mathrm{Sd}(e,a) < t$ and $s' < \mathrm{Sd}(a,a') < t'$ imply $s+s' < \mathrm{Sd}(e,a') < t+t'$. Since $s_0 < s+s' < t+t' < t_0$, $s_0 < \mathrm{Sd}(e,a') < t_0$ and $s_0 < \mathrm{Sd}(x,a') < t_0$ is satisfiable. By the same argument, $s_0 < \mathrm{Sd}(x,a') < t_0 \rightarrow s_0 < \mathrm{Sd}(x,a) < t_0$.

Therefore, there is $d \in M$ such that $\Sigma(d)$ and $\mathrm{Sd}(d,a) = \mathrm{Sd}(d,a')$. Moreover, we may assume that $\{a,b,c,a',d\}$ is independent by taking $d \mathop{\smile\!\!\!\!\!\!\;\,|}_{aa'} bc$. Consider the 2-chain $\alpha = a_0 + a_1 - a_2$, where

- $\mathrm{supp}(a_0) = \{0,1,3\}$, $\mathrm{supp}(a_1) = \{1,2,3\}$, and $\mathrm{supp}(a_2) = \{0,2,3\}$;
- $a_0(\{0,1,3\}) = [abd]_{\{0,1,3\}}$, $a_1(\{1,2,3\}) = [bcd]_{\{1,2,3\}}$, and $a_2(\{0,2,3\}) = [a'cd]_{\{0,2,3\}}$;
- $a_0 \restriction \mathcal{P}(\{0,1\}) = f_{01}$, $a_1 \restriction \mathcal{P}(\{1,2\}) = f_{12}$, and $a_2 \restriction \mathcal{P}(\{0,2\}) = f_{02}$; and
- $a_0 \restriction \mathcal{P}(\{0,3\}) = a_2 \restriction \mathcal{P}(\{0,3\})$, $a_0 \restriction \mathcal{P}(\{1,3\}) = a_1 \restriction \mathcal{P}(\{1,3\})$, and $a_1 \restriction \mathcal{P}(\{2,3\}) = a_2 \restriction \mathcal{P}(\{2,3\})$.

Then $\partial\alpha = f_{01} + f_{12} - f_{02} + (a_2 \restriction \mathcal{P}(\{0,3\}) - a_0 \restriction \mathcal{P}(\{0,3\})) = f_{01} + f_{12} - f_{02}$.

We show the 'moreover' part. Let a, a' be endpoints of s. If $a \equiv^L a'$, then s is a boundary of a 2-chain, and $n_s \subset \{n\}^*$ for some $n \in \mathbb{Z}$. Conversely, we assume that $n_s \subset \{n\}^*$ for some $n \in \mathbb{Z}$. In the proof of the right-to-left implication, we found $d \in M$ such that $a \equiv_d a'$. Consider the substructure $\mathrm{cl}(d) = \mathrm{dcl}(d) = \mathrm{acl}(d)$ generated by d. Then $a \equiv_{cl(d)} a'$, and $a \equiv^L a'$. □

Now we are ready to prove Theorem 5.10. Define a map $\Phi : H_1(p_0) \to (\mathbb{R} \cup \mathbb{Q}^*)/\mathbb{Z}^*$ by sending $[s]$ to $n_s +^* \mathbb{Z}^*$ (note that $(\mathbb{R} \cup \mathbb{Q}^*)/\mathbb{Z}^* \cong \mathbb{R}/\mathbb{Z}$, as shown in Appendix B). It is easy to see that this map is surjective. Since for an endpoint pair (a,b) of s, $n_s +^* \mathbb{Z}^* = \mathrm{Sd}(a,b) +^* \mathbb{Z}^*$, the map Φ depends only on the endpoint pairs of 1-shells. By Theorem 2.4, given 1-shells s_0 and s_1, and endpoint pairs (a,b) and (b,c) of s_0 and s_1, there is a 1-shell s such that $[s] = [s_0] + [s_1]$ and (a,c) is an endpoint pair of s, so the map Φ is a group homomorphism. Moreover, by Theorem 5.13 it is injective, and therefore it is an isomorphism. This completes the proof of Theorem 5.10.

6. Appendix

6.1. Appendix A. We show that the possible number of equivalence classes of a bounded type-definable equivalence relation on a strong type is 1 or at least $2^{\aleph_0}$. Let $T(=T^{eq})$ be any theory in a language $\mathcal{L}$ and let $\mathcal{M}$ be a monster model of T. Fix a small subset $A=\operatorname{acl}(A)$, and choose a strong type $p(x)$ over A with x of possibly infinite length.

Theorem 6.1. *Let $E(x,y)$ be a bounded A-type-definable equivalence relation on $p(x)$, and denote the set of E-classes on p by p/E. Then,*

$$|p/E|=1 \text{ or } |p/E|\geq 2^{\aleph_0}.$$

Proof. For convenience, we assume that $A=\emptyset$. We consider two cases:

Case 1. p/E is finite: Let $a_0,\cdots,a_n\models p$ be representatives of all distinct classes in p/E, and put $\bar a=(a_0,a_1,\ldots,a_n)$. At first, we show that E is relatively definable on p. Consider two types $E(x,a_0)$ and $\bigvee_{i>0}E(x,a_i)$ partitioning p. By compactness, $p(x)\models E(x,a_0)\leftrightarrow\phi(x,a_0)$ for some formula $\phi(x,z)$ such that $E(x,a_0)\models\phi(x;a_0)$. Since $a_0\equiv a_i$, $p(x)\models E(x,a_i)\leftrightarrow\phi(x,a_i)$ for all $i\leq n$. Thus, $p(x)\wedge p(y)\models E(x,y)\leftrightarrow\psi(x,y;\bar a)$, where $\psi(x,y;\bar z)=\bigvee_i[\phi(x,z_i)\wedge\phi(y,z_i)]$. Since E is invariant, $p(x)\wedge p(y)\wedge\psi(x,y;\bar z)\wedge\operatorname{tp}(\bar a)(\bar z)\models\psi(x,y;\bar a)(\leftrightarrow E(x,y))$. By compactness, there is a formula $\psi'(\bar z)$ in $\operatorname{tp}(\bar a)(\bar z)$ such that $p(x)\wedge p(y)\wedge\psi(x,y;\bar z)\wedge\psi'(\bar z)\models\psi(x,y;\bar a)$. Take $\theta(x,y)\equiv\exists\bar z(\psi'(\bar z)\wedge\psi(x,y;\bar z))$. Then $p(x)\wedge p(y)\models\theta(x,y)\leftrightarrow\psi(x,y;\bar a)$. Therefore, E is relatively definable on p by the formula θ. Moreover, we may assume $\theta(x,y)$ is a reflexive and symmetric relation by replacing it with $x=y\vee(\theta(x,y)\wedge\theta(y,x))$.

Next, we find a finite $\emptyset$-definable equivalence relation E' such that $p(x)\wedge p(y)\models E(x,y)\leftrightarrow E'(x,y)$. Since E is an equivalence relation,

$$\begin{aligned}
p(x)\wedge p(y)\wedge p(z) \quad\models\quad & \bigvee_i\theta(x,a_i)\wedge\bigvee_i\theta(y,a_i)\wedge\bigvee_i\theta(z,a_i)\\
\wedge\quad & \bigwedge_i(\theta(x,a_i)\to\bigwedge_{i\neq j}\neg\theta(x,a_j))\\
\wedge\quad & \bigwedge_i(\theta(y,a_i)\to\bigwedge_{i\neq j}\neg\theta(y,a_j))\\
\wedge\quad & \bigwedge_i(\theta(z,a_i)\to\bigwedge_{i\neq j}\neg\theta(z,a_j))\\
\wedge\quad & (\theta(x,y)\wedge\theta(y,z)\to\theta(x,z)).\ (*)
\end{aligned}$$

Again by compactness, there is $\delta(x)\in p(x)$ such that

$$\delta(x)\wedge\delta(y)\wedge\delta(z)\models(*).$$

Define a definable equivalence relation $E'(x,y) \equiv [\neg\delta(x) \wedge \neg\delta(y)] \vee [\delta(x) \wedge \delta(y) \wedge \forall z(\delta(z) \rightarrow (\theta(z,x) \leftrightarrow \theta(z,y)))]$.

Claim 6.2. *The equivalence relation E' is finite.*

Proof. First, $\neg\delta(x)$ defines an E'-class. We show that on δ, the E'-classes are of the form of $\theta(x,a_i) \wedge \delta(x)$. By the choice of δ, it is partitioned by $\{\theta(x,a_i) \wedge \delta(x)\}_{i\le n}$.

1) We show that $\models \theta(x,a_i) \wedge \delta(x) \rightarrow E'(x,a_i)$: Choose $b \models \theta(x,a_i) \wedge \delta(x)$. Take $c \models \delta(x) \wedge \theta(x,a_i)$. Since θ is transitive on δ and $\theta(b,a_i)$ holds, $\theta(c,b)$ holds. Conversely, if $d \models \delta(x) \wedge \theta(x,b)$, then by transitivity of θ on δ, $\theta(d,a_i)$ holds. Therefore, $E'(b,a_i)$ holds.

2) For $i \neq j$, $\neg E'(a_i,a_j)$: Suppose that for some $i \neq j$, $E'(a_i,a_j)$ holds. Then $\theta(a_i,a_j)$ holds, but it is impossible, since $a_i, a_j \models p$ and θ coincides with E on $p \times p$.

By 1) and 2), the E'-classes are of the form $\theta(x,a_i) \wedge \delta(x)$ or $\neg\delta(x)$, so E' is a finite equivalence relation. $\square$

By the proof of Claim 6.2, E' and E give the same equivalence relation on $p \times p$. Since E' is finite and p is a strong type, $p/E = p/E'$ and there is only one E-class in p.

Case 2. p/E is infinite. Let $\kappa = |p/E|$. If E is definable, then by compactness, $|p/E| \geq \kappa'$ for any small κ' and E is not bounded. So E is not definable but type-definable; write $E(x,y) \equiv \bigwedge_{i<\lambda} \phi_i(x,y)$, where each $\phi_i(x,y)$ is a formula and λ is an infinite cardinal. Furthermore we assume that for each $i,j < \lambda$ there is $k < \lambda$ such that $\phi_k(x,y) \equiv \phi_i(x,y) \wedge \phi_j(x,y)$. We may assume $\phi_i(x,y)$ is reflexive and symmetric (by replacing it with $x = y \vee (\phi_i(x,y) \wedge \phi_i(y,x))$) for each $i < \lambda$. Let $\{a_k \models p\}_{k<\kappa}$ be a set of representatives of all E-classes.

Claim 6.3. *For each $i < \lambda$ and $k < \kappa$, $\phi_i(x,a_k)(\mathcal{M})$ contains infinitely many E-classes.*

Proof. Fix $i < \lambda$. By compactness, there are finitely many $k_0 < k_1 < \cdots < k_n$ such that $p \models \bigvee_j \phi_i(x,a_{k_j})$. By the Pigeonhole Principle, some $\phi_i(x,a_{k_l})$ contains infinitely many a_k's so that $\phi_i(x,a_{k_l})$ contains infinitely many E-classes. Since $a_n \equiv a_m$ for all $n,m < \kappa$ and E is invariant, each $\phi_i(x,a_k)$ contains infinitely many E-classes. $\square$

Claim 6.4. *For each $i < \lambda$ and $k < \kappa$, there are $j < \lambda$ and $k_0, k_1 < \kappa$ such that*

$$\models \forall x[(\phi_j(x, a_{k_0}) \vee \phi_j(x, a_{k_1})) \to \phi_i(x, a_k)] \wedge [\neg \exists x(\phi_j(x, a_{k_0}) \wedge \phi_j(x, a_{k_1}))].$$

Proof. Fix $i < \lambda$ and $k < \kappa$. By Claim 6.3, $\phi_i(x, a_k)$ contains infinitely many E-classes. Choose two different E-classes in $\phi_i(x, a_k)$ and let a_{k_0} and a_{k_1} be representatives of two classes respectively. Since $E(x, a_{k_0})(\mathcal{M})$ and $E(x, a_{k_1})(\mathcal{M})$ are disjoint, by compactness, for some $j_0, j_1 < \lambda$, $\phi_{j_0}(x, a_{k_0})(\mathcal{M})$ and $\phi_{j_1}(x, a_{k_1})(\mathcal{M})$ are disjoint subsets of $\phi_i(x, a_k)(\mathcal{M})$. Take $j < \lambda$ such that $\phi_j(x, y) \equiv \phi_i(x, y) \wedge \phi_{j_0}(x, y) \wedge \phi_{j_1}(x, y)$ and we are done. □

By Claim 6.3, 6.4 and the fact that the cofinality of λ is at least $\aleph_0$, we get a binary tree $\mathcal{B} : 2^{<\omega} \to \omega \times \kappa$ such that for each $b \in 2^{<\omega}$, $\mathcal{B}(b \frown 0) = (j, k_0)$ and $\mathcal{B}(b \frown 1) = (j, k_1)$, where $j < \omega$ and $k_0, k_1 < \kappa$ are given by the Claim 6.4 for $(i, k) := \mathcal{B}(b)$. Then, for each $\tau \in 2^\omega$, we obtain a set of formulas $\{\phi_{i(\tau \restriction n)}(x, a_{k(\tau \restriction n)})\}$, where $\mathcal{B}(\tau \restriction n) = (i(\tau \restriction n), k(\tau \restriction n))$ for each $n \in \omega$. By the choice of $\mathcal{B}$, for $\tau_0 \neq \tau_1 \in 2^\omega$, $\bigcap_n \phi_{i(\tau_0 \restriction n)}(x, a_{k(\tau_0 \restriction n)})(\mathcal{M})$ and $\bigcap_n \phi_{i(\tau_1 \restriction n)}(x, a_{k(\tau_1 \restriction n)})(\mathcal{M})$ are disjoint, and each of them contains at least one E-class. Thus, p/E has at least $2^{\aleph_0}$ many elements. □

6.2. **Appendix B.** We shall see how to recover the ordered group $(\mathbb{R}, +)$ of real numbers from a dense linear order extending $(\mathbb{Q}, <)$ using Dedekind cuts. Consider the language $\mathcal{L}_{od,\mathbb{Q}} = \{<\} \cup \{r\}_{r \in \mathbb{Q}}$ and an $\mathcal{L}_{od,\mathbb{Q}}$-structure $\mathcal{U} = (U, <, r : r \in \mathbb{Q})$ which is a saturated dense linear order extending $(\mathbb{Q}, <)$. Then $\mathrm{Th}(\mathcal{U})$ has quantifier elimination.

Consider $S_1(\emptyset)$, the space of 1-types over the empty set (which we will denote just by S_1). By quantifier elimination, any 1-type p is equivalent to a type of one of the following forms (where $r \in \mathbb{Q}$ and $r' \in \mathbb{R} \setminus \mathbb{Q}$):

(1) $\{x = r\}$;
(2) $\{l < x < r \mid l < r\}$;
(3) $\{r < x < u \mid r < u\}$; and
(4) $\{l < x < u \mid l < r' < u\}$.

For a subset $Y \subset \mathbb{Q}$, we write $Y^* := Y \cup \{y \pm \epsilon \mid y \in Y\}$, where ϵ is an infinitesimal. So we can identify S_1 with the set $\mathbb{R} \cup \mathbb{Q}^*$ in the following way : For $r \in \mathbb{Q}$ and $r' \in \mathbb{R} \setminus \mathbb{Q}$,

(1) $\{x = r\} \leftrightarrow r$;
(2) $\{l < x < r \mid l < r\} \leftrightarrow (r - \epsilon)$;
(3) $\{r < x < u \mid r < u\} \leftrightarrow (r + \epsilon)$; and
(4) $\{l < x < u \mid l < r' < u\} \leftrightarrow r'$.

Next, we define a group-like structure on S_1. Define a plus-like operation $+^* : S_1 \times S_1 \to \mathcal{P}(S_1)$ as follows :

$$p_1 +^* p_2 := \{p|\ p \models (l_1 + l_2 < x < u_1 + u_2),\ p_i \models l_i < x < u_i\},$$

and define a minus-like operation $-^* : S_1 \to S_1$ as follows:

$$(-^* p) := \{-u < x < -l|\ p \models l < x < u\}.$$

We extend $+^*$ and $-^*$ to operations defined on $\mathcal{P}(S_1)$: For $A, B \subset S_1$,

$$A +^* B := \bigcup_{a \in A, b \in B} a +^* b, \text{ and } (-^* A) := \bigcup_{a \in A} (-^* a).$$

We identify each element $a \in S_1$ with its singleton $\{a\} \in \mathcal{P}(S_1)$. Then $+^*$ and $-^*$ are commutative, associative and distributive. For any $p_1, \cdots, p_k \in S_1$ and $k \geq 1$, we have

$$|p_1 +^* \cdots +^* p_k| \leq 3.$$

We write $p_1 -^* p_2$ for $p_1 +^* (-^* p_2)$. These two notions are naturally assigned to $\mathbb{R} \cup \mathbb{Q}^*$ and they are defined as follows:

(1) (a) If both r_1 and r_2 are in $\mathbb{R}$ and $r = r_1 + r_2$, then

$$r_1 +^* r_2 := \begin{cases} \{r\} & \text{if } r \in \mathbb{R} \setminus \mathbb{Q} \\ \{r\} & \text{if } r \in \mathbb{Q} \text{ and } r_1, r_2 \in \mathbb{Q} \\ \{r - \epsilon,\ r, r + \epsilon\} & \text{if } r \in \mathbb{Q} \text{ and } r_1, r_2 \notin \mathbb{Q} \end{cases}$$

(b) If $r_1 \in \mathbb{R} \setminus \mathbb{Q}$ and $r_2 = q \pm \epsilon \in \mathbb{Q}^*$, then $r_1 +^* r_2 := \{r_1 + q\}$;
(c) If $r_1 \in \mathbb{Q}$ and $r_2 = q \pm \epsilon \in \mathbb{Q}^*$, then $r_1 +^* r_2 := \{(r_1 + q) \pm \epsilon\}$;
(d) If $r_1 = p \pm \epsilon$ and $r_2 = q \pm \epsilon \in \mathbb{Q}^*$, then $r_1 +^* r_2 := \{(p + q) \pm \epsilon\}$;
(e) If $r_1 = p \pm \epsilon$ and $r_2 = q \mp \epsilon \in \mathbb{Q}^*$, then $r_1 +^* r_2 := \{(p + q) - \epsilon, (p + q), (p + q) + \epsilon\}$.

(2) (a) If $r_1 \in \mathbb{R}$, then $-^* r_1 := -r_1$;
(b) If $r_1 = p \pm \epsilon \in \mathbb{Q}^*$, then $-^* r_1 := -p \mp \epsilon$.

Now, we induce a group structure from $(S_1, +^*, -^*)$. Define an equivalence relation $\equiv_0$ on S_1 by

$$p_1 \equiv_0 p_2 \text{ iff } p_1 -^* p_2 \subset \{0 - \epsilon,\ 0,\ 0 + \epsilon\},$$

and denote by $[p]_0$ the equivalence class of an element $p \in S_1$ with respect to that relation. Since $\{0 - \epsilon,\ 0,\ 0 + \epsilon\}$ is closed under $+^*$ and $-^*$, $+^*$ and $-^*$ can be extended on $S_1/\equiv_0$. Then $(S_1/\equiv_0, +^*, -^*, [\mathrm{tp}(0)]_0)$ is a group. Actually, it is isomorphic to $(\mathbb{R}, +, -, 0)$.

Theorem 6.5. $(S_1/\equiv_0, +^*, -^*, [\mathrm{tp}(0)]_0) \cong (\mathbb{R}, +, -, 0)$.

Define an equivalence relation $\equiv_{\mathbb{Z}}$ on S_1 by

$$p_1 \equiv_{\mathbb{Z}} p_2 \text{ iff } p_1 -^* p_2 \subset \mathbb{Z}^*,$$

and denote by $[p]_{\mathbb{Z}}$ the corresponding equivalence class. As above, since $\mathbb{Z}^*$ is closed under $+^*$ and $-^*$, we can extend $+^*$ and $-^*$ to on $S_1/\equiv_{\mathbb{Z}}$. Then $(S_1/\equiv_{\mathbb{Z}}, +^*, -^*, [\mathrm{tp}(0)]_{\mathbb{Z}})$ is isomorphic to $(\mathbb{R}/\mathbb{Z}, +, -, 0)$ as a group.

Theorem 6.6. $(S_1/\equiv_{\mathbb{Z}}, +^*, -^*, [0]_{\mathbb{Z}}) \cong (\mathbb{R}/\mathbb{Z}, +, -, 0)$.

The equivalences $\equiv_0$ and $\equiv_{\mathbb{Z}}$ are defined on $\mathbb{R} \cup \mathbb{Q}^*$,

$$(\mathbb{R} \cup \mathbb{Q}^*)/\equiv_0 \cong \mathbb{R} \text{ and } (\mathbb{R} \cup \mathbb{Q}^*)/\equiv_{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}.$$

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