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SERVICE DIFFERENTIATION UNDER MIXED TRANSIT OLIGOPOLY: A GRAPHICAL ANALYSIS

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ABSTRACT

In this paper, we consider a mixed transit oligopoly market and analyze strategic interactions between passengers and operators. A number of operators provide vertically differentiated transit service to passengers. Individuals choose transit based on the assumption of willingness to pay (WTP) and minimize their generalized costs. Users are generally heterogeneous in both WTP and value of time (VOT). The efficient transit market is segmented by cut-off VOTs and illustrated in the two-dimensional WTP and VOT space. The market bounds is a concave polygonal chain connected by cut-off points. We formulate the efficient transit market problem for Pareto optimal solutions and develop a randomized market bounds algorithm (RMBA). The algorithm is based on random walk Monte Carlo methods. It is a powerful tool to compute and analyze transit oligopoly. The Pareto frontier can be found by RMBA and shown to converge in probability. A case study on duopoly is conducted.

Keywords: Mixed transit oligopoly; service differentiation; Pareto optimality; randomized market bounds algorithm

1. INTRODUCTION

The market structure of oligopoly is commonly observed in public transport in the real world. Different transit modes such as buses, metros and trams, are operated by different firms and often overlap in routes to share passenger demands. Even for the single transit mode where monopoly prevails, the entry of new operators becomes possible thanks to the policy of transit deregulation (Nash, 1993; Wang and Yang, 2005). The transit market dominated by a few number of operators with different objectives is referred to as a mixed transit oligopoly (De Fraja and Delbono, 1990; Qin *et al.*, 2016).

Bertrand competition, of which firms solely compete in price, is one of the earliest models to describe market behaviors under oligopoly. Vertical product differentiation enables firms to compete in a much richer context (Choudhary *et al.*, 2005). In this paper, we study a mixed transit oligopoly where operators differentiate their services in price and quality. The full price in transit is a passenger's generalized cost per trip, while service quality is measured by transit frequency. Individuals' choices on transit are based on the assumption of WTP given by Zhang *et al.* (2017), and the operator with the lowest generalized cost is selected. Such service differentiation is successful when users' valuation of quality differs, namely they are heterogeneous in VOT. The transit market is "efficient" (Yang *et al.*, 2001) when each operator has a positive demand share. And the corresponding efficient transit market condition is established. Market is segmented by cut-off VOTs and illustrated in the two-dimensional WTP and VOT space.

The Pareto optimal solutions can be found by a constrained mathematical optimization problem under the efficient transit market condition. The major difficulties to solve such formulations are that both an operator's objective and its strategy set depend on operators' strategies. The normalized objectives are generally non-convex and non-quasiconvex. The strategy sets may be non-linear and non-convex. Traditional optimization techniques are thus inapplicable. We transform the mathematical form of market constraints to the graphical determination of market bounds, and reformulate the problem with cut-off points. We develop a randomized market bounds algorithm (RMBA) based on random walk Monte Carlo methods (Dunn and Shultis, 2011). The algorithm is used to find the Pareto frontier and shown to converge in probability.

We organize the remaining parts of the paper as follows. The next section discusses efficient transit market condition, and illustrates market segmentation in the two-dimensional WTP and VOT space. Section 3 investigates formulations of Pareto optimality. Section 4 reformulates the problem with cut-off points, and introduces RMBA. A case study on the market structure of duopoly are analyzed in Section 5. Conclusion is given in Section 6.

2. EFFICIENT TRANSIT MARKET CONDITION AND SEGMENTATION

We consider an oligopoly transit market with n ($n \geq 2, n \in \mathbb{N}$) operators, who are indexed by the set $I = \{i \mid i = 1, 2, \cdots, n\}$. Operators provide services for passengers in a single origin-destination trip and differentiate their services in price and quality. Specifically, an operator $i \in I$ sets its own fare P_i and frequency F_i strategically. Let $\boldsymbol{\chi}_i = (P_i, F_i)$ denote operator i's strategy and $\boldsymbol{\chi}_{-i}$ denote operators' strategies other than operator i. Let the $2n \times 1$ column vector $\boldsymbol{\chi} = (\boldsymbol{\chi}_i, \boldsymbol{\chi}_{-i})^T$ denote the strategies of all operators.

The demand structure of passengers is based on the assumption of WTP (denoted by ω) given by Zhang *et al.* (2017). It is defined as an individual's reserved maximum amount of the generalized cost for transit service. An individual will use transit service if his or her perceived generalized cost does not exceed the person's WTP. Users are supposed to be heterogeneous in both WTP and VOT. Users' WTP lies in $\omega \in [0, \bar{\omega}]$, while VOT lies in $\beta \in [0, \bar{\beta}]$. $\bar{\omega}$ and $\bar{\beta}$ denote the suprema of WTP and VOT among all potential users, respectively. And we suppose $\bar{\beta} \geq (\bar{\omega} - P_i)/T_i, \forall i \in I$ (the domain can be extended if this is not satisfied).

The cost structure of passengers is assumed as a linear combination of monetary cost and time cost. For a passenger with a VOT β who uses operator i's service, his or her perceived generalized cost C_i is given by

$$C_i = c_i(P_i, F_i) = P_i + \beta T_i, \ \forall i \in I \tag{1}$$

where $T_i = T_i(F_i) > 0$ is the time component of the trip, and $dT_i/dF_i < 0$ for each operator $i \in I$.

As there are several operators, passengers may still face multiple options that satisfy their WTP. We further assume that if a passenger uses transit, the passenger will always choose the operator with the minimum perceived generalized cost. With this assumption, there are actually some *cut-off VOTs* (Wang and Yang, 2005) that partition passengers' choices among operators. If we artificially let $P_0 = P_1,\ T_0 > T_1$ and $P_{n+1} = \bar{\omega},\ T_{n+1} = 0$, then all the cut-off VOTs can be consistently defined as

$$\beta_i = \frac{P_{i+1} - P_i}{T_i - T_{i+1}}, \ i = 0, 1, \cdots, n \tag{2}$$

The corresponding coordinates (β_i, ω_i) are called *cut-off points*, where

$$\omega_i = P_i + \beta_i T_i,\ i = 0, 1, \cdots, n \tag{3}$$

In particular, $(\beta_0, \omega_0) = (0, P_1)$ and $(\beta_n, \omega_n) = \left((\bar{\omega} - P_n)/T_n, \bar{\omega}\right)$.

How these cut-off VOTs and cut-off points work are shown as follows. Suppose that operators are sorted in an ascending order with respect to fare, and their strategies satisfy

$$\begin{cases} 0 \le P_1 < P_2 < \cdots < P_{n-1} < P_n \le P_{\max} < \bar{\omega} \\ T_{\max} \ge T_1(F_1) > T_2(F_2) > \cdots > T_{n-1}(F_{n-1}) > T_n(F_n) \ge T_{\min} \\ 0 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_{n-1} < \beta_n \le \bar{\beta} \end{cases} \tag{4}$$

where $P_{\max}$, $T_{\min}$ and $T_{\max}$ are positive pre-defined limits due to policy or fleet size restrictions. Condition (4) is called the *efficient transit market condition*. The term "efficient" is originated from Yang *et al.* (2001). The efficient transit market condition ensures a positive demand share for each operator. This is given by the following proposition.

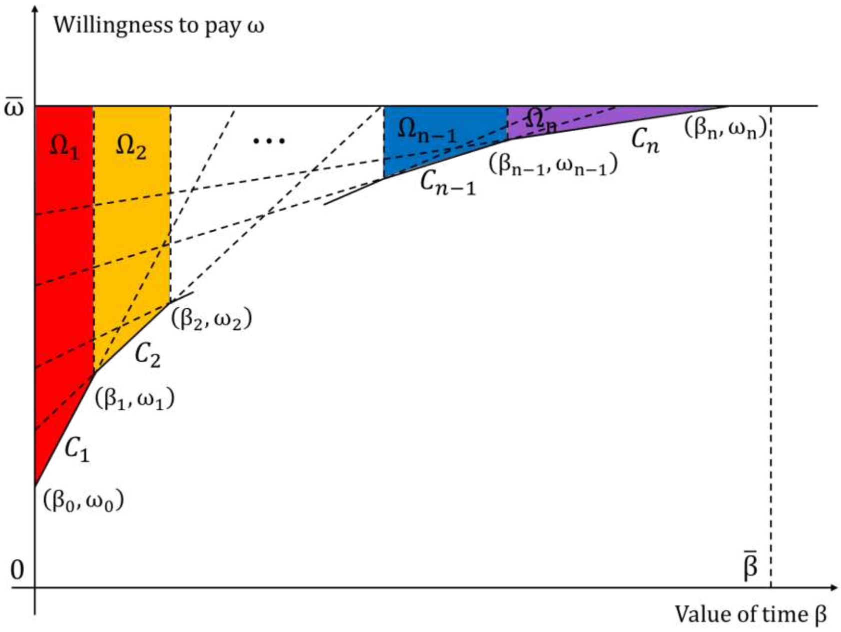

Figure 1. Efficient transit market segmentation under service differentiation.

Proposition 1. *Under the efficient transit market condition, the transit market is segmented by cut-off VOTs. A passenger who uses transit will choose transit service of operator* $i \in I$ *if the passenger's VOT lies in* $\beta \in (\beta_{i-1}, \beta_i)$.

Proof. Under the efficient transit market condition, we have $C_i = P_i + \beta T_i < P_{i+1} + \beta T_{i+1} = C_{i+1}$ if $\beta < \beta_i$; and $C_i = P_i + \beta T_i < P_{i-1} + \beta T_{i-1} = C_{i-1}$ if $\beta > \beta_{i-1}$ for $\forall i \in I$. Since $\beta_i < \beta_j, \forall i, j \in I, i < j$, we then have

$$\begin{cases} C_i \le C_n = P_n + \beta T_n < P_n + \beta_n T_n = \bar{\omega},\ \forall i \in I \text{ if } \beta \in (\beta_{i-1}, \beta_i) \\ \bar{\omega} = P_n + \beta_n T_n < P_n + \beta T_n = C_n \le C_i,\ \forall i \in I \text{ if } \beta \in (\beta_n, \bar{\beta}] \end{cases} \tag{5}$$

and

$$C_i < C_j,\ \forall i, j \in I, j \ne i \text{ if } \beta \in (\beta_{i-1}, \beta_i) \tag{6}$$

So an individual with WTP $\omega \in [C_i, \bar{\omega}]$ and VOT $\beta \in (\beta_{i-1}, \beta_i)$ will use transit and choose transit service of operator $i \in I$ because it has the minimum perceived generalized cost among all operators. For passengers with VOT $\beta \in (\beta_n, \bar{\beta}]$, they do not use any operator's transit service. ■

By Proposition 1, the transit market segmentation is illustrated in Figure 1. The whole transit market segment is written as $\Omega = \sum_{i=1}^{n} \Omega_i$, where Ω_i is the market segment of operator $i \in I$. Each operator has a cuneiform market segment on the two-dimensional WTP and VOT space. The operators who provide premium services (relatively high fare and frequency) will serve the high end market where passengers have higher WTP and VOT. The whole market bounds is defined by the envelope connected by cut-off points (β_i, ω_i), $i = 0, 1, \cdots, n$. Operator $i \in I$'s market bounds is defined by its cost curve C_i and two cut-off points $(\beta_{i-1}, \omega_{i-1})$ and (β_i, ω_i).

3. FORMULATIONS OF PARETO OPTIMALITY

Suppose that each operator behaves in a non-cooperative manner and chooses its fare and frequency simultaneously. Given other operators' strategies χ_{-i}, operator i's objective is to maximize a weighted combination of its profit π_i and total consumer surplus ψ:

$$\max_{\chi_i = (P_i, F_i)} z_i(\chi) = z_i(\chi_i, \chi_{-i}) = \pi_i + \gamma_i \cdot \psi, \ \forall i \in I \tag{7}$$

where $\gamma_i \in [0,1]$ is the consumer weight that indicates operator i's nature (Zhang *et al.*, 2017). Operators' objectives vary when consumer weights vary. Such transit market is referred to as a mixed transit oligopoly. Note that there should be total consumer surplus ψ in the objective function rather than operator i's consumer surplus ψ_i. Because an originally private operator's consideration of consumer surplus happens only under regulation, and the regulator concerns the overall benefits of passengers. Such objective function is also adopted by papers like Clark *et al.* (2009) and Qin *et al.* (2016). It contrasts the objective function in papers like Ishibashi and Kaneko (2008) and Sanjo (2009) that also take the rivals' interests into consideration.
Operator i's profit is given by

$$\pi_i = P_i \cdot Q_i - K_i, \ \forall i \in I \tag{8}$$

With users' heterogeneity in both WTP and VOT, a bivariate distribution (β, ω) is thus assigned to characterize each user. Let $h(\beta, \omega)$ denote the joint probability density function of (β, ω) and assume it to be differentiable when there is a large group of passengers. According to Figure 1, operator i's realized passenger demand is a function of operators' strategies and written as

$$Q_i = \tilde{q}_i(\chi_i, \chi_{-i}, \mathbf{X}) = \bar{Q} \cdot \int_{(\beta,\omega)\in\Omega_i} h(\beta, \omega)\, d\omega d\beta, \ \forall i \in I \tag{9}$$

where $\bar{Q} = \bar{q}(\mathbf{X})$ is potential passenger demand that solely depends on exogenous demand variables $\mathbf{X}$. Operator i's cost is generally a function of its frequency and demand, which can also be written as a function of operators' strategies:

$$K_i = k_i(F_i, Q_i) = \tilde{k}_i(\chi_i, \chi_{-i}, \mathbf{X}), \ \forall i \in I \tag{10}$$

Total consumer surplus $\psi = \sum_{i=1}^{n} \psi_i$, where ψ_i is operator i's consumer surplus and is given by

$$\psi_i = \bar{Q} \cdot \int_{(\beta,\omega)\in\Omega_i} \left[\omega - (P_i + \beta T_i)\right] h(\omega, \beta)\, d\omega d\beta, \ \forall i \in I \tag{11}$$

With the detailed form of operator i 's objective function as described above, we continue to consider Pareto optimal solutions under the efficient transit market condition. A strict inequality constraint may lead to an ill-posed optimization problem in the sense that the solution is infeasible but on the boundary of the feasible set. Therefore, we relax the efficient transit market condition (4) to non-strict inequality when considering the problem under constraints. We can do sanity check after a solution is obtained. Let

$$\mathrm{X}(\chi)=\left\{\begin{array}{r} 0\le P_1\le P_2\le\cdots\le P_{n-1}\le P_n\le P_{\max}\le\bar{\omega} \\ \chi=\left(P_1,F_1;\cdots;P_n,F_n\right)^T \mid T_{\max}\ge T_1\left(F_1\right)\ge T_2\left(F_2\right)\ge\cdots\ge T_{n-1}\left(F_{n-1}\right)\ge T_n\left(F_n\right)\ge T_{\min} \\ 0=\beta_0\le\beta_1\le\beta_2\le\cdots\le\beta_{n-1}\le\beta_n\le\bar{\beta} \end{array}\right\} \quad (12)$$

denote the set of all the operators' feasible strategies under the relaxed efficient transit market condition. The Pareto optimal solutions can thus be formulated as

$$\max_{\chi}\ \left(z_1(\chi),z_2(\chi),\cdots,z_n(\chi)\right)\ s.t.\ \chi\in\mathrm{X}(\chi) \quad (13)$$

Various methods can be used to simplify such multi-objective optimization problem, such as linear scalarization or ε-constraint method (Geoffrion, 1967; Yang and Yang, 2011). However, the difficulty is that the feasible strategy set $\mathrm{X}(\chi)$ depends on operators' strategies and may be non-linear or non-convex. What is more, the normalized objective function after scalarization is generally non-convex and non-quasiconvex. So it is still difficult to generate feasible solutions for the simplified single objective optimization problem.

4. RANDOMIZED MARKET BOUNDS ALGORITHM

Here we try another graphical approach to solve the efficient transit market problem. The efficient transit market condition requires a particular shape of market bounds as shown in Figure 1. The market bounds is connected by cut-off points. From Eqs. (2) and (3), operator i 's time component is given by

$$T_i=\frac{\omega_i-\omega_{i-1}}{\beta_i-\beta_{i-1}},\forall i\in I \quad (14)$$

It is also the slope of operator i 's market bounds in Figure 1. Operator i 's fare is given by

$$P_i=\frac{\omega_{i-1}\beta_i-\omega_i\beta_{i-1}}{\beta_i-\beta_{i-1}},\forall i\in I \quad (15)$$

So if every cut-off point is determined, every operator's strategy is also determined. This inspires the idea that the mathematical form of market constraints can be transformed to the graphical determination of market bounds, and the efficient transit market problem can be reformulated with cut-off points. In fact, it is proved (omitted in this paper) that the transit market is efficient if and only if its market bounds is a concave polygonal chain connected by cut-off points with boundary conditions on the two-dimensional WTP and VOT space. As a result, we develop an algorithm for problem (13), which is called *randomized market bounds algorithm*. This algorithm is based on random walk Monte Carlo methods (Dunn and Shultis, 2011). It is relevant to the concept of "random walk" because the generation of whole market bounds is a stochastic process that describes a concave polygonal chain with successive generation of random cut-off points. It is relevant to the concept of "Monte Carlo methods" because we rely on repeated random sampling of market bounds generation to obtain numerical results.

The key part of the randomized market bounds algorithm is randomized market bounds generation. The generation process is a successive iteration process. To ensure that the whole market bounds is a concave polygonal chain on the two-dimensional WTP and VOT space, the consecutive cut-off points have certain graphical relations. The subsequent cut-off point must lie within an angle area whose vertex is the preceding cut-off point. The slopes of angle rays and generation details are described in Table 1.

Table 1. Algorithm: randomized market bounds generation

Algorithm: randomized market bounds generation	
Step 1: *Initialization.*	Set $\beta_0 = 0$. Randomly set $\omega_0 \in [0, P_{\max}]$. Set operator 1's minimum slope $T_1^{\min} = T_{\min}$ and maximum slope $T_1^{\max} = T_{\max}$. Set current operator $i = 1$.
Step 2: *Generation.*	The two rays of minimum and maximum slopes and the suprema of passengers' WTP $\bar{\omega}$ and VOT $\bar{\beta}$ form a shaded area. It can be a quadrilateral or a triangle, which depends on the two intersecting VOTs: $\beta_i^{\min} = \beta_{i-1} + (\bar{\omega} - \omega_{i-1}) / T_i^{\max}$ and $\beta_i^{\max} = \beta_{i-1} + (\bar{\omega} - \omega_{i-1}) / T_i^{\min}$. Randomly set subsequent cut-off point (β_i, ω_i) within the shaded area. Calculate T_i from Eq. (14). Set operator $i+1$'s minimum slope $T_{i+1}^{\min} = \max\left\{T_{\min}, \dfrac{\omega_i}{\beta_i - P_{\max}}\right\}$ and maximum slope $T_{i+1}^{\max} = \min\{T_{\max}, T_i\}$.
Step 3: *Move.*	Set current operator $i = i+1$. If $i < n$, go to Step 2; else if $i = n$, go to Step 4.
Step 4: *Finalization.*	Set $\omega_n = \bar{\omega}$. Calculate two intersecting VOTs: $\beta_n^{\min} = \beta_{n-1} + (\bar{\omega} - \omega_{n-1}) / T_n^{\max}$ and $\beta_n^{\max} = \beta_{n-1} + (\bar{\omega} - \omega_{n-1}) / T_i^{\min}$. Randomly set $\beta_n \in [\beta_n^{\min}, \beta_n^{\max}]$. Note β_n may be larger that $\bar{\beta}$. In this case we extend $\bar{\beta}=\beta_n$. Return the series of cut-off points: $\mathbf{a} = \{(\beta_i, \omega_i),\ i = 0, 1, \cdots, n\}$

With the whole market bounds defined by the series of cut-off points obtained, we are ready to search the Pareto optimal solutions of the efficient transit market problem. The algorithm process is described in Table 2. The algorithm is in linear time complexity of the loop counter. It is also proved (omitted in this paper) that the Pareto frontier candidate obtained from the algorithm converges in probability towards the true Pareto frontier.

Table 2. Algorithm: searching the Pareto optimality

Algorithm: searching the Pareto optimality	
Step 1: *Initialization.*	Set loop counter $N = 0$. Set an empty set of Pareto frontier candidate $s_N = \{\}$.
Step 2: *Calculation.*	Generate a new allocation of cut-off point series $\mathbf{a}$ using the algorithm of randomized market bounds generation. Calculate each operator $i \in I$'s demand Q_i, cost K_i, profit π_i, consumer surplus ψ_i and objective z_i given by Eqs. (9), (10), (8), (11) and (7), respectively.
Step 3: *Comparison.*	Compare the objectives of allocation $\mathbf{a}$ with every allocation in s_N. If allocation $\mathbf{a}$ is not dominated by any allocation in s_N, remove the allocations in s_N that are dominated by $\mathbf{a}$ and put $\mathbf{a}$ into s_N.
Step 4: *Move.*	If N reaches a pre-defined maximum loop counter, stop and return s_N; else let $s_{N+1} = s_N$, set loop counter $N = N+1$ and go to Step 2.

5. DUOPOLY CASE STUDY

In this section we consider a numerical case study for the mixed duopoly market. We suppose a bivariate normal distribution for individuals' characteristics on the two-dimensional WTP and VOT space, which is given by

$$(\beta,\omega) \sim \mathcal{N}(\mu,\Sigma), \mu = (60,12)^T, \Sigma = \begin{pmatrix} 400 & 40 \\ 40 & 16 \end{pmatrix} \tag{16}$$

The mean and standard deviation of VOT are 60HK\$/hr and 20HK\$/hr. The mean and standard deviation of WTP are 12HK\$ and 4HK\$. Such assigned distribution guarantees that more than 99.7% (68-95-99.7 rule in statistics) of the VOT values lie in $[0,120]$ (HK\$/hr) and 99.7% of the WTP values lie in $[0,24]$ (HK\$). Therefore, we initially denote $\bar{\beta}=120$ HK\$/hr and $\bar{\omega}=24$ HK\$. Note the covariance is 40 and the correlation is $0.5\neq0$, which means that WTP and VOT are not statistically independent. The probability density function is drawn in Figure 2.

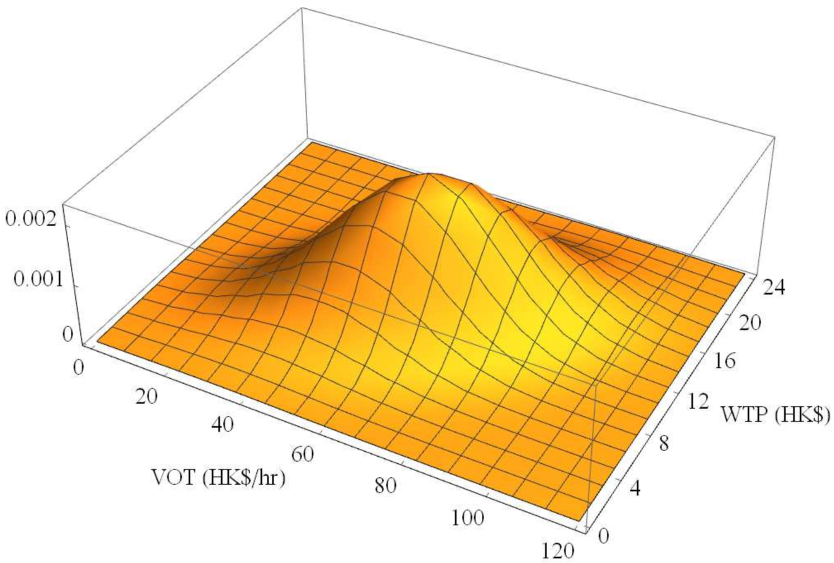

Figure 2. Bivariate normal distribution of individuals' characteristics.

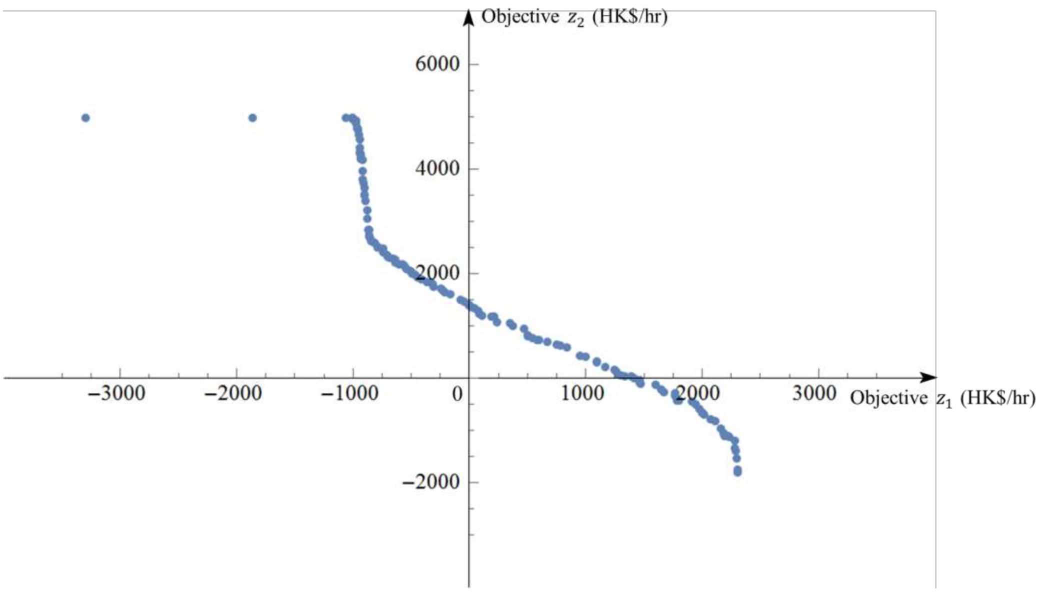

Figure 3. Pareto frontier of under mixed transit duopoly.

We consider a duopoly transit market where two operators provide differentiated transit services in both fare and frequency. We set $n=2$. Potential demand $\bar{Q}=1500$ /hr. Fare and time component extrema are $P_{\max}=20$ HK\$, $T_{\min}=1/60$ hr and $T_{\max}=1/4$ hr. Consumer weight $\{\gamma_1,\gamma_2\}=\{0,0.5\}$.

Time component function $T_i = 1/(2F_i)$, $\forall i \in I$. Operator's cost function $K_i = \phi_i F_i + \lambda_i Q_i$, $\forall i \in I$, where $\{\phi_1, \phi_2\} = \{500, 400\}$ HK$ and $\{\lambda_1, \lambda_2\} = \{0.4, 0.6\}$ HK$. Market variables can be calculated using RMBA when a random market bounds is generated. The obtained Pareto frontier with a pre-defined 1.0×10^6 loops are drawn in Figure 3.

6. CONCLUSION

This paper studies service differentiation under a mixed transit oligopoly, where users are generally heterogeneous in WTP and VOT. The efficient transit market condition is established. The transit market segmentation is illustrated in the two-dimensional WTP and VOT space. The Pareto optimality is formulated as a constrained multi-objective optimization problem when the efficient transit market condition is considered.

The intrinsic nature of this efficient transit market problem is complex mainly because both the objective function and feasible strategy set depend on operators' strategies. The general forms of individuals' characteristics and cost structures may yield non-convexity and even complicate the mathematical formulations. The market bounds in the efficient transit market is a concave polygonal chain connected by cut-off points. Therefore, we reformulate the problem with cut-off points and develop RMBA to solve the efficient transit market problem. The convergence in probability of this algorithm is proved. RMBA provides a powerful tool to compute and analyse transit oligopoly. A case study on transit duopoly is conducted.

7. REFERENCES

Choudhary, V., Ghose, A., Mukhopadhyay, T. and Rajan, U. (2005) Personalized pricing and quality differentiation. *Management Science*, 51(7), pp. 1120-1130.

Clark, D. J., Jørgensen, F. and Pedersen, P. A. (2009) Strategic interactions between transport operators with several goals. *Journal of Transport Economics and Policy (JTEP)*, 43(3), pp. 385-403.

De Fraja, G. and Delbono, F. (1990) Game theoretic models of mixed oligopoly. *Journal of Economic Surveys*, 4(1), pp. 1-17.

Dunn, W. L. and Shultis, J. K. (2011) *Exploring Monte Carlo Methods*. Elsevier.

Geoffrion, A. M. (1967) Solving bicriterion mathematical programs. *Operations Research*, 15(1), pp. 39-54.

Ishibashi, K. and Kaneko, T. (2008) Partial privatization in mixed duopoly with price and quality competition. *Journal of Economics*, 95(3), pp. 213-231.

Nash, C. A. (1993) British bus deregulation. *The Economic Journal*, 103(419), pp. 1042-1049.

Qin, F., Zhang, X. and Plant, E. (2017) The welfare effects of nationalization in a mixed duopoly public transport market. *Operational Research*, 17(2), pp. 593-618.

Sanjo, Y. (2009) Bertrand competition in a mixed duopoly market. *The Manchester School*, 77(3), pp. 373-397.

Wang, J. Y. and Yang, H. (2005) A game-theoretic analysis of competition in a deregulated bus market. *Transportation Research Part E: Logistics and Transportation Review*, 41(4), pp. 329-355.

Yang, H., Kong, H. Y. and Meng, Q. (2001) Value-of-time distributions and competitive bus services. *Transportation Research Part E: Logistics and Transportation Review*, 37(6), pp. 411-424.

Yang, H. and Yang, T. (2011) Equilibrium properties of taxi markets with search frictions. *Transportation Research Part B: Methodological*, 45(4), pp. 696-713.

Zhang, J., Yang, H. and Lindsey R. (2017) Public transit service frequency and fares with heterogeneous users under monopoly and alternative regulatory policies. *Working paper*.