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A Revisit to Gradient-Descent Bearing-Only Formation Control

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***Abstract*— This paper addresses the problem of bearing-only formation control of multi-agent systems, where each agent can merely obtain the relative bearing measurements of their neighbor neighbors whereas relative distance or position measurements are unavailable. In particular, we revisit a bearing-only formation control law proposed in [1]. Unlike many other existing ones, this control law is gradient-descent, which is favorable from the stability analysis point of view. It has the potential to be extended to handle more complex agent models and moving target formations. Up to now, this control law has not attracted sufficient attention probably because its stability analysis is based on optimization techniques and challenging to generalize. The contribution of this paper is to present a new stability analysis of this formation control law based on Lyapunov approaches. The new stability analysis reveals some new properties of the control law such as exponential convergence rate and lays a foundation for deriving new bearing-only control laws in the future.**

I. Introduction

This paper studies multi-agent formation control that aims to steer a group of agents to form a desired geometric pattern in a distributed manner. We particularly focus on the case where each agent is only able to measure the relative bearings to their nearest neighboring agents while relative distance or position information is unavailable. Compared to the existing formation control approaches that rely on relative position measurements, the bearing-only formation control approach is appealing since it poses minimal requirements on the sensing ability of each agent. In practice, bearing measurements can be obtained by, for example, visual sensing [2] or sensor arrays [3], [4].

Despite the recent advances on bearing-only formation control, many problems in this area are still unsolved. In particular, the existing bearing-only control laws are merely applicable to single-integrator agent models and stationary target formations [1], [5]–[11]. From the practical point of view, it is necessary to study more realistic models and how to track moving target formations. However, it is nontrivial to generalize the existing bearing-only control laws to handle these problems. One reason is that most of the existing bearing-only control laws are not gradient descent. For example, a bearing-only formation control law proposed in [10] is proved to be almost globally stable. This control law is not gradient-descent and the stability is proved by showing that the error between the current formation and the desired target formation converges to zero. A relevant control law proposed in [12] is gradient-descent. This control law can stabilize a target formation that is constrained by desired bearings. However, this control law is not bearing-only because it requires both relative bearing and distance measurements.

In this paper, we revisit a bearing-only formation control law proposed in [1]. Unlike many other existing bearing-only formation control laws, this one is a gradient-descent control law, which is favorable from the stability analysis point of view. It has the potential to generalize to handle more realistic agent models and moving target formations. However, this control law has not attracted sufficient attention up to now probably because its stability analysis is based on optimization techniques and challenging to generalize. The contribution of our work is to present a new stability analysis of this formation control law using standard Lyapunov approaches. Such a new stability analysis is nontrivial since it relies on many new techniques developed based on our recent work of bearing localizability [13]. Our analysis also reveals some new properties of the control law such as exponential convergence rate and nonincreasing formation scale. New control laws could be proposed by generalizing this gradient-descent control law and will be studied in our future work.

II. Preliminaries and Problem Statement

A. Notations of formations

Consider a group of n mobile agents in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$). Let $p_i \in \mathbb{R}^d$ be the position of agent $i \in \{1, \dots, n\}$, and $p = [p_1^\mathrm{T}, \dots, p_n^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^{dn}$ be the configuration of the agents. The interaction among the agents is described by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which consists of a vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edge $(i, j) \in \mathcal{E}$ indicates that agent i can measure the relative bearing of agent j, and hence agent j is a neighbor of i. The set of neighbors of agent i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. This paper only consider undirected graphs where $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. A formation, denoted as $(\mathcal{G}, p)$, is $\mathcal{G}$ with its vertex i mapped to p_i for all $i \in \mathcal{V}$.

Define the *edge vector* and *bearing vector* for edge (i, j), respectively, as

$$e_{ij} := p_j - p_i, \quad g_{ij} := \frac{e_{ij}}{\|e_{ij}\|},$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix. The unit vector g_{ij} represents the relative bearing of p_j with respect to p_i. Note $e_{ij} = -e_{ji}$ and $g_{ij} = -g_{ji}$. Assume that all agents are able to sense a global reference frame. All the bearings in this paper are expressed in this global reference frame. In practice, such

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a global reference frame can be measured by using sensors such as GPS and a compass.

For g_{ij}, define $P_{g_{ij}} := I_d - g_{ij}g_{ij}^{\mathrm{T}}$, where $I_d \in \mathbb{R}^{d\times d}$ is the identity matrix. Note that $P_{g_{ij}}$ is an orthogonal projection matrix that geometrically projects any vector onto the orthogonal compliment of g_{ij}. It can be verified that $P_{g_{ij}}$ is positive semi-definite and $\mathrm{Null}(P_{g_{ij}}) = \mathrm{span}\{g_{ij}\}$. This orthogonal projection matrix is widely used in bearing-based control and estimation problems because it is able to describe parallel bearing vectors in arbitrary dimensions [10], [13].

When there are leaders, without loss of generality, suppose the first n_ℓ agents are leaders and the rest $n_f = n - n_\ell$ agents are followers. Let $\mathcal{V}_\ell = \{1,\dots,n_\ell\}$ and $\mathcal{V}_f = \mathcal{V}\setminus\mathcal{V}_\ell$ be the sets of leaders and followers, respectively. The positions of the leaders and followers are denoted as $p_\ell = [p_1^{\mathrm{T}},\dots,p_{n_\ell}^{\mathrm{T}}]^{\mathrm{T}}$ and $p_f = [p_{n_\ell+1}^{\mathrm{T}},\dots,p_n^{\mathrm{T}}]^{\mathrm{T}}$, respectively. Then $p = [p_\ell^{\mathrm{T}}, p_f^{\mathrm{T}}]^{\mathrm{T}}$.

Oriented graphs are widely used in this paper. An *orientation* of an undirected graph is the assignment of a direction to each edge. An *oriented graph* is an undirected graph together with an orientation. Consider an arbitrary oriented graph of $\mathcal{G}$. Let m be the number of undirected edges in $\mathcal{G}$. Hence, the oriented graph has m directed edges. Suppose edge (i,j) in $\mathcal{G}$ corresponds to the kth directed edge in the oriented graph where $k \in \{1,\dots,m\}$. The edge and bearing vectors for the kth directed edge can be expressed as

$$e_k := e_{ij} = p_j - p_i, \quad g_k := \frac{e_k}{\|e_k\|}.$$

Denote $e = [e_1^{\mathrm{T}},\dots,e_m^{\mathrm{T}}]^{\mathrm{T}}$ and $g = [g_1^{\mathrm{T}},\dots,g_m^{\mathrm{T}}]^{\mathrm{T}}$. The *incidence matrix* $H \in \mathbb{R}^{m\times n}$ of the oriented graph is the $\{0,\pm 1\}$-matrix with rows indexed by edges and columns by vertices. Specifically, all the entries in the kth row of H are zero except $[H]_{ki}$ and $[H]_{kj}$. We have $[H]_{ki} = -1$ since vertex i is the tail of edge k, and $[H]_{kj} = 1$ since vertex j is the head of edge k. For a connected graph, it holds that $H\mathbf{1}_n = 0$ and $\mathrm{rank}(H) = n-1$, where $\mathbf{1}_n = [1,\dots,1]^{\mathrm{T}} \in \mathbb{R}^n$ [14]. Note that $e = (H\otimes I_d)p := \bar{H}p$, where $\otimes$ denotes the Kronecker product.

B. Bearing-Only Formation Control Law

Suppose each agent can be modeled as a single integrator: $\dot{p}_i = u_i$, where u_i is the control input to be designed. The problem of bearing-only formation control is formally stated as below.

Problem 1 (**Bearing-Only Formation Tracking Control**). *Design u_i for agent $i \in \mathcal{V}$ based merely on the bearing measurements $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ such that $g_{ij} \to g_{ij}^*$ for all $(i,j)\in\mathcal{E}$ as $t\to\infty$.*

In this problem, the target formation is specified by bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$. It has two key properties: existence and uniqueness. We only consider feasible bearing constraints so that the target formation defined above exists. The uniqueness property is important because if the target formation is not unique, the formation may converge to undesired formation shapes even if the bearing constraints are achieved. In this paper, we consider two cases: leaderless and leader-follower. In the leaderless case, since there are no leaders, the scale of the target formation is not specified. With bearing constraints only, the geometric pattern of the target formation can be uniquely determined if it is bearing rigid [10]. In the leader-follower case, there are some stationary leaders. The target formation can be uniquely determined if it is bearing localizable [13]. Preliminaries to bearing localizability will be introduced later.

The bearing-only control considered in this paper is

$$\dot{p}_i(t) = \sum_{j\in\mathcal{N}_i}\left(g_{ij}(t) - g_{ij}^*\right), \quad i \in \mathcal{V}. \tag{1}$$

This control law was originally proposed in [1, Equation (13)]. One key property of (1) is that it is a gradient-descent control law. Specifically, consider the Lyapunov function

$$\begin{aligned} V &= \frac{1}{4}\sum_{(i,j)\in\mathcal{E}} \|e_{ij}\|\|g_{ij} - g_{ij}^*\|^2 \\ &= \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} \|e_{ij}\|(1 - g_{ij}^{\mathrm{T}}g_{ij}^*). \end{aligned} \tag{2}$$

It can be verified that (1) is a gradient-descent control for V when there are no leaders. Another property of (1), which is also a common property for many bearing-only control laws, is that the control input is always bounded. That is because $\|\dot{p}_i\| \le \sum_{j\in\mathcal{N}_i}\|g_{ij}(t) - g_{ij}^*\| \le 2|\mathcal{N}_i|$.

Control law (1) can successfully solve Problem 1. Its asymptotic stability has been analyzed in [1]. The novelty of this paper is to present a new stability analysis based on Lyapunov approaches. This stability analysis reveals some new properties of control law (1) and lay a foundation to analyze new bearing-only control laws in the future.

In order to analyze the formation stability, following [1], we make the following assumption.

Assumption 1 (**Collision Avoidance**). *Assume no neighboring agents collide with each other during the formation evolvement. Specifically, $\|e_{ij}\|$ is bounded from below by a positive constant for all t and all $(i,j)\in\mathcal{E}$.*

Assumption 1 ensures that the bearing vector between any pair of neighboring agents is always well defined. This assumption may be dropped by considering discontinuous systems where the bearing can be properly defined even when two agents collocate. This assumption may also be fulfilled by designing collision avoidance control algorithms. These problems are nontrivial to solve and will be addressed in our future work.

III. Leaderless Formation Control

This section presents a Lyapunov-based stability analysis of control law (1) in the leaderless case (i.e., there are no leaders).

Consider an arbitrary oriented graph of $\mathcal{G}$. Then, control law (1) can be written in a matrix-vector form as

$$\dot{p} = -\bar{H}^{\mathrm{T}}(g - g^*), \tag{3}$$

where H, $g(t)$, and g^* are the incidence matrix, current bearing vectors, and target bearing vectors, respectively. The Lyapunov function in (2) becomes

$$V=\frac{1}{2}\sum_{k=1}^{m}\|e_k\|\|g_k-g_k^*\|^2=\sum_{k=1}^{m}\|e_k\|(1-g_k^{\mathrm{T}}g_k^*)\geq 0. \tag{4}$$

The matrix-vector form of V is

$$\begin{aligned}V&=\sum_{k=1}^{m}(e_k^{\mathrm{T}}g_k-e_k^{\mathrm{T}}g_k^*)\\&=e^{\mathrm{T}}(g-g^*)\\&=p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\geq 0.\end{aligned} \tag{5}$$

It can be seen from (4) and (5) that $V=0 \Leftrightarrow g=g^*$ since $\|e_k\|\neq 0$ for all k as assumed. As a result, the steady state of (3) is characterized as below.

Lemma 1 (**Steady State Value**). *Under Assumption 1, $\dot{p}=-\bar{H}^{\mathrm{T}}(g-g^*)=0$ if and only if $g=g^*$.*

Proof. The sufficiency is obvious. To prove the necessity, note that $\bar{H}^{\mathrm{T}}(g-g^*)=0 \Rightarrow p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)=0$, which implies $g=g^*$ by (5). □

Define the centroid and scale of the formation, respectively, as

$$\bar{p}=\frac{1}{n}\sum_{i=1}^{n}p_i=\frac{1}{n}(\mathbf{1}_n\otimes I_d)^{\mathrm{T}}p,$$

$$s=\sum_{i=1}^{n}\|p_i-\bar{p}\|^2=\|p-\mathbf{1}_n\otimes\bar{p}\|^2.$$

The scale s characterizes how far the agents are from the formation centroid. The centroid and scale satisfy the following properties.

Lemma 2 (**Centroid and Scale of Formation**). *Under the action of control law* (3)*, the centroid $\bar{p}$ is invariant. The scale s monotonically decreases if and only if $g\neq g^*$. As a consequence, $\|p\|$ and $\|e\|$ are bounded from above for all t.*

Proof. First, since $(\mathbf{1}_n\otimes I_d)^{\mathrm{T}}\bar{H}^{\mathrm{T}}=0$, we have $(\mathbf{1}_n\otimes I_d)^{\mathrm{T}}\dot{p}=0$ and hence $\dot{\bar{p}}=0$. Second, $\dot{s}=2(p-\mathbf{1}_n\otimes\bar{p})^{\mathrm{T}}\dot{p}=-2(p-\mathbf{1}_n\otimes\bar{p})^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)=-2p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\leq 0$. According to (5), $\dot{s}=0$ if and only if $g=g^*$.

We next analyze the boundedness. Since $\dot{s}\leq 0$, it follows that $s(0)\geq s(t)$ and hence $\sqrt{s(0)}\geq\|p-\mathbf{1}_n\otimes\bar{p}\|\geq\|p\|-\|\mathbf{1}_n\otimes\bar{p}\|$. As a result, $\|p\|\leq\sqrt{s(0)}+\|\mathbf{1}_n\otimes\bar{p}\|$ for all t. Since $e=\bar{H}p$, we have $\|e\|\leq\|\bar{H}\|\|p\|\leq\|\bar{H}\|(\sqrt{s(0)}+\|\mathbf{1}_n\otimes\bar{p}\|)$. □

The property of the formation scale is important because it shows the boundedness of $\|e\|$, which will be critical for the Lyapunov-based stability analysis shown later. The reason that the formation scale is nonincreasing is that the Lyapunov function contains the distance term $\|e_k\|$. While control law (3) is the gradient-descent control aiming at minimizing the Lyapunov function, it reduces either the bearing errors to zero or the inter-neighbor distances to zero. Numerical simulation shows that under certain initial conditions the formation scale may decrease to zero, which means all the agents converge to the same point. This extreme case is not of particular interest and it is excluded by Assumption 1.

The global stability of (3) is proved below.

Theorem 1 (**Single-Integrator Leaderless Control**). *Under Assumption 1, $g(t)$ converges to g^* globally asymptotically under the action of control law* (3)*.*

Proof. Define the bearing error as $\delta_g=g-g^*$. Since $\|e_k\|$ is bounded from below as assumed in Assumption 1 and bounded from above according to Lemma 2, suppose $0<\alpha\leq\|e_k\|\leq\beta$ for all k and all t. Then, V in (4) satisfies

$$\frac{\alpha}{2}\|\delta_g\|^2\leq V\leq\frac{\beta}{2}\|\delta_g\|^2.$$

Since $\dot{g_k}=P_{g_k}\dot{e}_k/\|e_k\|$ and $P_{g_k}e_k=0$, it follows that $e^{\mathrm{T}}\dot{g}=0$. As a result, the time derivative of V in (5) is

$$\begin{aligned}\dot{V}&=e^{\mathrm{T}}\dot{g}+(g-g^*)^{\mathrm{T}}\dot{e}\\&=0+(g-g^*)^{\mathrm{T}}\bar{H}\dot{p}\\&=-(g-g^*)^{\mathrm{T}}\bar{H}\bar{H}^{\mathrm{T}}(g-g^*)\\&=-\delta_g^{\mathrm{T}}\bar{H}\bar{H}^{\mathrm{T}}\delta_g\leq 0.\end{aligned}$$

Since $\bar{H}^{\mathrm{T}}\delta_g=0\Leftrightarrow\delta_g=0$ by Lemma 1, we have $\dot{V}=0$ if and only if $\delta_g=0$. As a result, $\dot{V}$ is negative definite with respect to δ_g. According to [15, Theorem 4.2], $\delta_g=0$ is globally asymptotically stable. □

As shown in Theorem 1, the convergence of the bearing errors does not require any conditions of the bearings. However, in order to get a unique formation shape, g^* should be designed such that the target formation is infinitesimally bearing rigid [10]. In this case, when g converges to g^*, the formation also converges to a desired geometric shape. Moreover, in the leaderless case, the scale of the final formation is determined by the initial configuration. In order to have a desired final formation scale, leaders must be introduced.

IV. Leader-Follower Formation Control

This section presents the stability analysis of control law (1) in the leader-follower case. In particular, suppose the leaders are stationary and satisfy $p_i(t)=p_i^*$ for all t and $i\in\mathcal{V}_\ell$. The target formation in the leader-follower case can be defined as below.

Definition 1 (**Target Formation**). *In the target formation $(\mathcal{G},p^*)$, the inter-neighbor bearings $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ are constant, and the positions of the leaders $\{p_i^*\}_{i\in\mathcal{V}_\ell}$ are stationary.*

In order to prove the formation stability, we only need to show that the followers converge to their desired positions in the target formation, i.e., $p_i(t)\to p_i^*$ for $i\in\mathcal{V}_f$.

The target formation is jointly determined by the bearings and the positions of the leaders. Its uniqueness is described by *bearing localizability* as shown in the following subsection.

A. Preliminaries to Bearing Localizability

Bearing localizability characterizes whether the target formation in Definition 1 is unique. The definition of bearing localizability is given below.

Definition 2 (Bearing Localizability). *The target formation $(\mathcal{G}, p^*)$ is called bearing localizable if the value of p^* can be uniquely determined by the inter-neighbor bearings $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ and the positions of the leaders $\{p_i^*\}_{i\in\mathcal{V}_\ell}$.*

By definition, the formation in Figure 1(a) is not bearing localizable. That is because multiple formations that have different geometric shapes may have the same bearings and leader positions, and consequently the bearings and leader positions are not able to determine a unique formation.

In order to characterize the necessary and sufficient condition of bearing localizability, we introduce a matrix termed *bearing Laplacian* [13]. Specifically, for the target formation, define a matrix $\mathcal{B}\in\mathbb{R}^{dn\times dn}$ with the ijth block of submatrix as

$$[\mathcal{B}]_{ij} = \begin{cases} \mathbf{0}_{d\times d}, & i\neq j,\ (i,j)\notin\mathcal{E}, \\ -P_{g_{ij}^*}, & i\neq j,\ (i,j)\in\mathcal{E}, \\ \sum_{k\in\mathcal{N}_i} P_{g_{ik}^*}, & i=j,\ i\in\mathcal{V}. \end{cases}$$

The matrix $\mathcal{B}$ is a matrix-weighted graph Laplacian matrix. It is called the *bearing Laplacian* since it characterizes both the underlying graph and the bearings of the target formation. The bearing Laplacian matrix plays important roles in bearing-based control and estimation problems [13], [16]. According to the partition of leader and follower agents, partition $\mathcal{B}$ as

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{\ell\ell} & \mathcal{B}_{\ell f} \\ \mathcal{B}_{f\ell} & \mathcal{B}_{ff} \end{bmatrix},$$

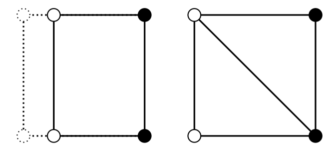
where $\mathcal{B}_{ff}\in\mathbb{R}^{dn_f\times dn_f}$. A necessary and sufficient condition for bearing localizability of the target formation is given below.

Lemma 3 (Condition for Bearing Localizability [13]). *The target formation $(\mathcal{G}, p^*)$ is bearing localizable, i.e., p^* can be uniquely determined by $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ and $\{p_i^*\}_{i\in\mathcal{V}_\ell}$, if and only if $\mathcal{B}_{ff}$ is nonsingular.*

In the leader-follower case, we only consider bearing localizable target formations.

Assumption 2 (Bearing Localizability). *Assume that the target formation $(\mathcal{G}, p^*)$ is bearing localizable, i.e., $\mathcal{B}_{ff}$ of the target formation is positive definite.*

An example of bearing localizable formations is given in Figure 1. More examples and other conditions for bearing localizability can be found in [13, Section 4]. In order to ensure bearing localizability, there must exist sufficient and appropriate leader agents. Details of the leader selection problem can be found in [13] and are omitted here. It is worth noting that at least two leaders are required to ensure bearing localizability.



(a) (b)

Fig. 1: The target formation in (a) is not bearing localizable. The one in (b) is bearing localizable. Solid dots represent leaders and hollow dots represent followers.

B. Exponential Stability Analysis

Since the leaders are stationary, control law (1) can be written as

$$\dot{p} = -\begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}}(g-g^*). \tag{6}$$

The initial value is $p(0)=[(p_\ell^*)^{\mathrm{T}}, p_f^{\mathrm{T}}(0)]^{\mathrm{T}}$, where $p_f(0)$ can be arbitrarily chosen. To analyze the formation stability, we first introduce two useful results.

Lemma 4. *For any p satisfying $p_i\neq p_j$ for all $(i,j)\in\mathcal{E}$, it holds that*

$$(p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\leq 0, \tag{7}$$

$$(p-p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\geq 0, \tag{8}$$

where the equalities hold if and only if $g=g^*$.

Proof. Inequality (7) holds because $(p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*) = (e^*)^{\mathrm{T}}(g-g^*) = \sum_{k=1}^m \|e_k^*\|((g_k^*)^{\mathrm{T}}g_k - 1)\leq 0$. Since $\|e_k^*\|\neq 0$, the equality holds when $g_k=g_k^*$ for all k. Inequality (8) can be obtained by combining (7) and (5). □

Lemma 5. *For any p satisfying $p_i\neq p_j$ for all $(i,j)\in\mathcal{E}$, it holds that*

$$p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\geq \frac{1}{2\max_k\|e_k\|}p^{\mathrm{T}}\mathcal{B}p, \tag{9}$$

where $\mathcal{B}$ is the bearing Laplacian of the target formation $(\mathcal{G}, p^*)$. When $g-g^*$ is sufficiently small so that $g_k^{\mathrm{T}}g_k^*\geq 0$ for all k, it holds that

$$p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\leq \frac{1}{\min_k\|e_k\|}p^{\mathrm{T}}\mathcal{B}p. \tag{10}$$

Proof. Note that $\mathcal{B}$ can be expressed as $\mathcal{B}=\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}$ where $\mathrm{diag}(P_{g_k^*})=\mathrm{blkdiag}(P_{g_1^*},\dots,P_{g_m^*})$ [13, Lemma 2]. It follows that

$$\begin{aligned} p^{\mathrm{T}}\mathcal{B}p &= p^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}p = e^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})e \\ &= \sum_{k=1}^m e_k^{\mathrm{T}}(I_d - g_k^*(g_k^*)^{\mathrm{T}})e_k = \sum_{k=1}^m \|e_k\|^2(1-(g_k^{\mathrm{T}}g_k^*)^2) \\ &= \sum_{k=1}^m \|e_k\|^2(1-g_k^{\mathrm{T}}g_k^*)(1+g_k^{\mathrm{T}}g_k^*). \end{aligned} \tag{11}$$

Since $1+g_k^{\mathrm{T}}g_k^*\leq 2$, it is implied by (11) that

$$\begin{aligned} p^{\mathrm{T}}\mathcal{B}p &\leq 2\max_k\|e_k\|\sum_{k=1}^m\|e_k\|(1-g_k^{\mathrm{T}}g_k^*) \\ &= 2\max_k\|e_k\|p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*). \end{aligned}$$

Inequality (9) follows immediately.

Suppose that $g-g^*$ is sufficiently small so that $g_k^{\mathrm{T}} g_k^* \geq 0$ for all k (i.e., the angle between g_k and g_k^* is less than $\pi/2$). Since $1+g_k^{\mathrm{T}} g_k^* \geq 1$, it is implied by (11) that

$$\begin{aligned} p^{\mathrm{T}} \mathcal{B} p &\geq \min_k \|e_k\| \sum_{k=1}^m \|e_k\|(1-g_k^{\mathrm{T}} g_k^*) \\ &= \min_k \|e_k\| p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*). \end{aligned}$$

Inequality (10) follows immediately. □

Lemma 5 establishes the equivalence between $p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*)$ and $p^{\mathrm{T}} \mathcal{B} p$. Since the bearing Laplacian is the key to characterize bearing localizability, Lemma 5 bridges the quantity $p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*)$ with bearing localizability. This result especially (9) is widely used in this paper.

The global exponential stability of (6) is analyzed as below.

Theorem 2 (Single-Integrator Leader-Follower Control). *Under Assumptions 1 and 2, $p(t)$ converges to p^* globally and exponentially fast under the action of control law* (6).

Proof. Define the position error as $\delta_p = p - p^*$. Note that $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$ since $p_\ell = p_\ell^*$. As a result,

$$\delta_p^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{n_f} \end{bmatrix} = \delta_p^{\mathrm{T}}.$$

Consider the Lyapunov function $V = \|\delta_p\|^2/2$. The time derivative of V is

$$\begin{aligned} \dot{V} = \delta_p^{\mathrm{T}} \dot{\delta}_p = \delta_p^{\mathrm{T}} \dot{p} &= -\delta_p^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{n_f} \end{bmatrix} \bar{H}^{\mathrm{T}}(g-g^*) \\ &= -\delta_p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*). \end{aligned}$$

According to Lemma 4, $\dot{V} \leq 0$ and $\dot{V}=0$ if and only if $g=g^*$. Since the target formation is bearing localizable as assumed, $g=g^*$ implies $p=p^*$. As a result, $\dot{V}=0 \Leftrightarrow \delta_p = 0$ and hence $\dot{V}$ is negative definite in δ_p. It follows that $\delta_p=0$ is globally asymptotically stable.

In order to prove exponential stability, note that

$$\begin{aligned} \dot{V} &= -\delta_p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*) \\ &= -(p-p^*)^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*) \\ &= -p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*) + (p^*)^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*) \\ &\leq -p^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g-g^*). \end{aligned} \tag{12}$$

Substituting (9) into (12) gives

$$\dot{V} \leq -\frac{1}{2\max_k \|e_k\|} p^{\mathrm{T}} \mathcal{B} p. \tag{13}$$

Since $\mathcal{B} p^* = 0$, we have $p^{\mathrm{T}} \mathcal{B} p = (p-p^*)^{\mathrm{T}} \mathcal{B} (p-p^*) = \delta_p^{\mathrm{T}} \mathcal{B} \delta_p$. Furthermore, since $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$, we have $\delta_p^{\mathrm{T}} \mathcal{B} \delta_p = \delta_{p_f}^{\mathrm{T}} \mathcal{B}_{ff} \delta_{p_f} \geq \lambda_{\min}(\mathcal{B}_{ff}) \|\delta_{p_f}\|^2 = \lambda_{\min}(\mathcal{B}_{ff}) \|\delta_p\|^2$. Substituting into (13) gives

$$\dot{V} \leq -\frac{\lambda_{\min}(\mathcal{B}_{ff})}{2\max_k \|e_k\|} \|\delta_p\|^2. \tag{14}$$

Note that

$$\begin{aligned} \max_k \|e_k\| \leq \|e\| = \|\bar{H} p\| &= \|\bar{H}(p-p^*+p^*)\| \\ &\leq \|\bar{H}\delta_p\| + \|\bar{H} p^*\| \\ &\leq \|\bar{H}\|(\|\delta_p\| + \|p^*\|). \end{aligned} \tag{15}$$

Since $\dot{V} \leq 0$, we have $\|\delta_p(t)\| \leq \|\delta_p(0)\|$. Substituting (15) into (14) yields

$$\dot{V} \leq -\underbrace{\frac{\lambda_{\min}(\mathcal{B}_{ff})}{\|\bar{H}\|(\|\delta_p(0)\| + \|p^*\|)}}_{\gamma} \frac{\|\delta_p\|^2}{2} = -\gamma V,$$

which indicates exponential convergence rate. □

V. Simulation

Figure 2 shows simulation results in the leaderless case. The target formation is a square with four agents and five edges as shown in Fig. 2(b). As can be seen, the bearing error converges to zero. The formation scale also decreases, which is consistent with Lemma 2.

Figure 3 shows a simulation example which demonstrates that the formation scale may decrease to zero under certain initial conditions. In this example, the initial configuration has exactly the opposite values as the desired bearings. In order to avoid such extreme case, leaders should be introduced to specify the final formation scale.

Figure 4 shows a simulation example in the leader-follower case. As can be seen, the bearing error converges to zero. The formation scale is determined by the two leaders.

VI. Conclusions

This paper presented a new stability analysis of the bearing-only formation control law proposed in [1]. The new stability analysis is based on standard Lyapunov approaches and reveals some new properties of the control law. The results presented in this paper lay a foundation for studying new bearing-only formation control laws that can handle more complex agent models and moving target formations in the future.

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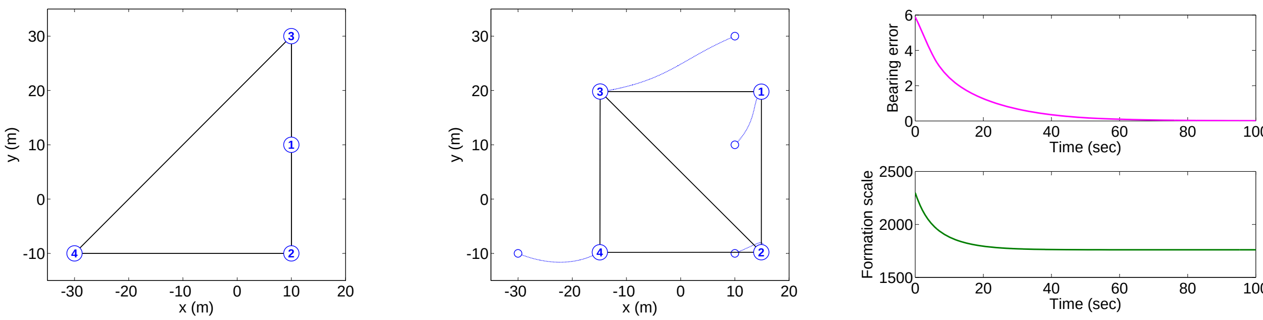
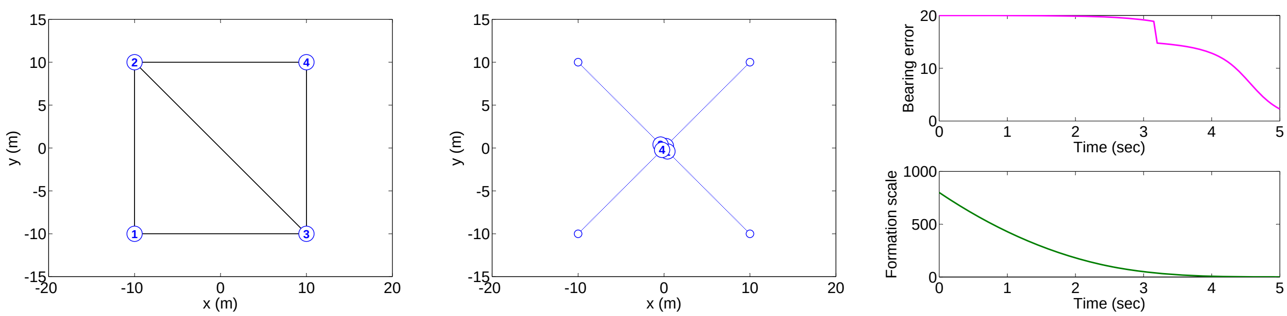


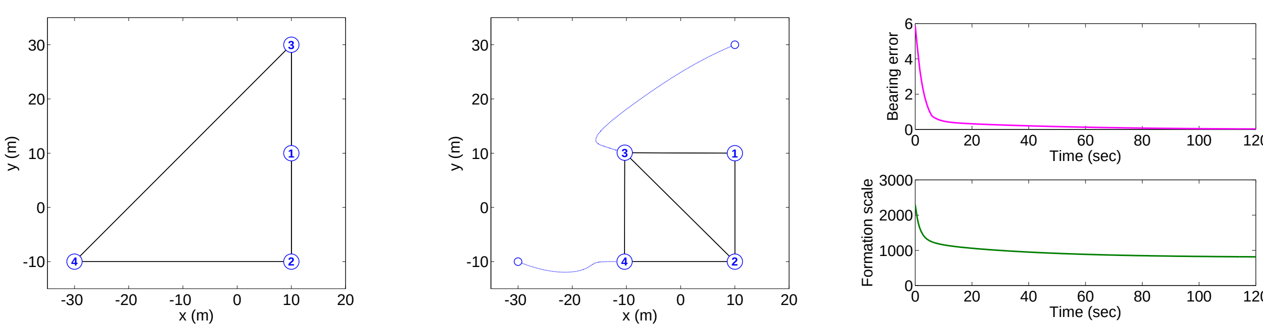
(a) Initial configuration (b) Final configuration (c) Bearing error and formation scale

Fig. 2: Simulation results for a leaderless case.



(a) Initial configuration (b) Final configuration (c) Bearing error and formation scale

Fig. 3: Simulation results for a leaderless case where the formation scale decreases to zero.



(a) Initial configuration (b) Final configuration (c) Bearing error and formation scale

Fig. 4: Simulation results for the leader-follower case where agent 1 and agent 2 are stationary leaders.

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