Landscape of uncertainty in Hilbert space for one-particle states

Stefan Weigert
Institut für Physik der Universität Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland
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The *functional of uncertainty* $J[\psi]$ assigns to each state $|\psi\rangle$ the product of the variances of the momentum and position operators. Its first and second variations are investigated. Each stationary point is located on one of a countable set of three-dimensional manifolds in Hilbert space. For a harmonic oscillator with given mass and frequency the extremals are identified as displaced squeezed energy eigenstates. The neighborhood of the stationary states is found to have the structure of a saddle, thus completing the picture of the landscape of uncertainty in Hilbert space. This result follows from the diagonalization of the second variation of the uncertainty functional, which is not straightforward since $J[\psi]$ depends nonlinearly on the state $|\psi\rangle$.

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INTRODUCTION

There are various ways to define coherent states for a harmonic oscillator [1]. The annihilation operator a is a linear combination of position and momentum operators, and all its eigenstates are coherent states. Also, they can be obtained by appropriately displacing in phase space the vacuum or ground state $|0\rangle$, defined by the property $a|0\rangle=0$. Finally, the construction of quantum states, which are localized as much as possible about a given point in the classical phase space, i.e., the consideration of states of *minimal* uncertainty, leads to coherent states.

In the following, however, the focus will be on states with stationary, not minimal uncertainty. The appropriate tool to work with [2] is a functional $J[\psi]$, which assigns a value of uncertainty [3] to each state $|\psi\rangle$ in the one-particle Hilbert space $\mathscr{H}$. Such a functional usually is introduced without reference to a Hamiltonian operator: only the operators $\hat{p}$ and $\hat{q}$—constituting the Heisenberg algebra—and the space $\mathscr{H}$ are required for its definition. For the Heisenberg algebra one finds that the states that render the uncertainty stationary are eigenstates of an appropriate harmonic oscillator. The variational approach given in [2] is applicable to any pair of noncommuting operators, not only position and momentum. This feature has been exploited, for example, in [4] to introduce ''intelligent'' spin states that turn the uncertainty relation for spin operators into an equality. Similar relations for number and phase operators have been studied in this spirit [2,5]. In the present work somewhat more ''global'' information is extracted from the uncertainty functional; its behavior near the stationary states will be determined.

In order to establish notation, first the stationary points of the uncertainty functional are derived in a representation-independent way from the requirement that the first variation of the functional $J[\psi]$ vanish. In the present approach the result is naturally expressed in terms of squeezed states, the concept of which was put forward only *after* [2] was published. Then, the quadratic approximation of the functional $J[\psi]$ in Hilbert space is calculated and diagonalized. Based on this result the topography of the *landscape of uncertainty* defined over Hilbert space $\mathscr{H}$ will be discussed.

PRELIMINARIES

The operators of position and momentum, $\hat{q}$ and $\hat{p}$, respectively, obey the commutation relation

$$[\hat{p},\hat{q}]=\frac{\hbar}{i}, \tag{1}$$

and the uncertainty functional $J_\lambda[\psi]$ associated with the algebra (1) is defined as the product of the variances of these operators

$$J_\lambda[\psi]=\Delta_p[\psi]\Delta_q[\psi]-\lambda(\langle\psi|\psi\rangle-1). \tag{2}$$

The variance Δ_r of an operator $\hat{r}$ in a state $|\psi\rangle$ is given by

$$\Delta_r[\psi]=(\langle\psi|\hat{r}^2|\psi\rangle-\langle\psi|\hat{r}|\psi\rangle^2)^{1/2}, \tag{3}$$

and the restriction that the state be normalized,

$$\langle\psi|\psi\rangle=1, \tag{4}$$

is taken into account through the term containing the Lagrangian multiplier λ. Contrary to the quantum-mechanical functional used to derive Schrödinger's equation from a variational principle [6], the one considered here is *nonlinear* in the state $|\psi\rangle$.

Consider the change of the functional J_λ when moving away from the state $|\psi\rangle$ along direction $|e\rangle\equiv\varepsilon|\varepsilon\rangle$, with $\varepsilon\ll 1$ and $\langle\varepsilon|\varepsilon\rangle=1$. It is natural to require $\langle e|\psi\rangle=0$, so that $|\varepsilon\rangle$ becomes an element of the orthocomplement of $|\psi\rangle$ in $\mathscr{H}$, denoted by $\mathscr{H}_\psi$.

Expanding the functional $J_\lambda[\psi+e]=J_\lambda^0+\delta J_\lambda+\delta^2 J_\lambda$ up to terms quadratic in ε, one obtains

$$J_\lambda[\psi+e]=J_\lambda[\psi]+\varepsilon D_\varepsilon J_\lambda[\psi]+\frac{\varepsilon^2}{2}D_\varepsilon^2 J_\lambda[\psi], \tag{5}$$

where the operator D_ε contains functional derivatives

$$D_\varepsilon=\langle\varepsilon|\frac{\delta}{\delta\langle\psi|}+\frac{\delta}{\delta|\psi\rangle}|\varepsilon\rangle. \tag{6}$$

In a first step the extrema (or stationary states) $\{|\psi_0\rangle\}$ of the functional J_λ will be determined by requiring the second

term in Eq. (5) to vanish for arbitrary (unrestricted) variation $|e\rangle$. In a second step, the behavior of the functional in the neighborhood of each stationary state $|\psi_0\rangle$ is investigated, which amounts to diagonalizing a quadratic form associated with the third term in the expansion (5).

FIRST VARIATION

Turning to first order, the equation for the extrema is found to be

$$D_\varepsilon J_\lambda[\psi]=0=\langle\varepsilon|\{'\Delta_p[\psi]\Delta_q[\psi] + '\Delta_q[\psi]\Delta_p[\psi]-\lambda|\psi\rangle\}+\text{ c.c.}, \quad (7)$$

where the "left" derivative of the variance Δ_r is given by

$$'\Delta_r[\psi]\equiv\frac{\delta\Delta_r[\psi]}{\delta\langle\psi|}=\frac{1}{2\Delta_r[\psi]}(\hat{r}^2-2\langle\psi|\hat{r}|\psi\rangle\hat{r})|\psi\rangle. \quad (8)$$

In order that Eq. (7) hold for arbitrary variations $\langle\varepsilon|$, one must have

$$e^{-2\gamma}(\hat{p}^2-2\langle\hat{p}\rangle\hat{p})|\psi\rangle+e^{2\gamma}(\hat{q}^2-2\langle\hat{q}\rangle\hat{q})|\psi\rangle=2\lambda|\psi\rangle, \quad (9)$$

where

$$\gamma=\frac{1}{2}\ln\left(\frac{\Delta_p[\psi]}{\Delta_q[\psi]}\right), \quad (10)$$

with both Δ_q and Δ_p assumed to be different from zero [7]. Here and in the following all expectation values $\langle\cdot\rangle$ are taken in the state $|\psi\rangle$. A second equation results from varying $|\varepsilon\rangle$ instead of $\langle\varepsilon|$, which, however, turns out to be the adjoint of Eq. (9).

It is convenient to rescale the operators $\hat{q}$ and $\hat{p}$,

$$\hat{p}_\gamma=e^{-\gamma}\hat{p}=S_\gamma\,\hat{p}\,S_\gamma^\dagger,$$
$$\hat{q}_\gamma=e^{+\gamma}\hat{q}=S_\gamma\,\hat{q}\,S_\gamma^\dagger, \quad (11)$$

by using the unitary dilation or squeezing operator [8]:

$$S_\gamma=\exp\left(\frac{i}{2\hbar}\gamma(\hat{p}\hat{q}+\hat{q}\hat{p})\right). \quad (12)$$

The value of the multiplier λ in the state $|\psi\rangle$ follows from Eq. (9) and the constraint (4)

$$\lambda=\frac{1}{2}\{\langle\hat{p}_\gamma^2+\hat{q}_\gamma^2\rangle-2(\langle\hat{p}_\gamma\rangle^2+\langle\hat{q}_\gamma\rangle^2)\}. \quad (13)$$

Plugging this expression back into Eq. (9), one obtains

$$\{(\hat{p}_\gamma-\langle\hat{p}_\gamma\rangle)^2+(\hat{q}_\gamma-\langle\hat{q}_\gamma\rangle)^2\}|\psi\rangle = \{\langle(\hat{p}_\gamma-\langle\hat{p}_\gamma\rangle)^2\rangle+\langle(\hat{q}_\gamma-\langle\hat{q}_\gamma\rangle)^2\rangle\}|\psi\rangle, \quad (14)$$

suggesting use of the shifted operators

$$\hat{P}=\hat{p}_\gamma-\langle\hat{p}_\gamma\rangle=\tilde{T}_\alpha\,\hat{p}_\gamma\tilde{T}_\alpha^\dagger,$$
$$\hat{Q}=\hat{q}_\gamma-\langle\hat{q}_\gamma\rangle=\tilde{T}_\alpha\,\hat{q}_\gamma\tilde{T}_\alpha^\dagger, \quad (15)$$

where the operator for phase-space translations is given by

$$\tilde{T}_\alpha=\exp\left(-\frac{i}{\hbar}(\xi_\gamma\hat{p}_\gamma-\eta_\gamma\hat{q}_\gamma)\right), \quad (16)$$

and the complex number α denotes a point in phase space:

$$\alpha=\frac{1}{\sqrt{2\hbar}}(\langle\hat{q}_\gamma\rangle+i\langle\hat{p}_\gamma\rangle)\equiv\frac{1}{\sqrt{2\hbar}}(\xi_\gamma+i\,\eta_\gamma). \quad (17)$$

Note that $\tilde{T}_\alpha$ is defined in terms of the rescaled operators $\hat{p}_\gamma$ and $\hat{q}_\gamma$; the expression in terms of the original operators follows from the relation $\tilde{T}_\alpha=S_\gamma T_\alpha S_\gamma^\dagger$.

Rewriting Eq. (14) leads to

$$\frac{1}{2}(\hat{P}^2+\hat{Q}^2)|\psi\rangle=\frac{1}{2}\langle(\hat{P}^2+\hat{Q}^2)\rangle|\psi\rangle. \quad (18)$$

Consequently, a state $|\psi\rangle$ is a stationary point of the functional $J_\lambda[\psi]$ if it is an eigenstate of the operator $\hat{H}_0=(\hat{P}^2+\hat{Q}^2)/2$, which is formally identical to the Hamiltonian of a harmonic oscillator. The operator $\hat{h}_0=(\hat{p}^2+\hat{q}^2)/2$ is unitarily equivalent to $\hat{H}_0$ according to (11) and (15). If the normalized eigenstates of $\hat{h}_0$ are denoted by $|n\rangle$, $n=0,1,2,\ldots$, one can express the solutions of Eq. (18) in the form

$$|n;\gamma,\alpha\rangle=T_\alpha S_\gamma|n\rangle, \quad n=0,1,2,\ldots\ . \quad (19)$$

These states have also been introduced in Ref. [9] in a different context.

The value of the uncertainty in the states $|n;\gamma,\alpha\rangle$ increases linearly with n, just as does the energy of a harmonic oscillator $\hat{H}_0$:

$$J_\lambda[n]=\hbar(n+\tfrac{1}{2}), \quad (20)$$

as follows from

$$\Delta_p^2[n]=\Delta_q^2[n]=\hbar(n+\tfrac{1}{2})=\Delta^2(n). \quad (21)$$

Consequently, the set of all states with stationary uncertainty can be labeled by one discrete index $n=0,1,2,\ldots$, and three continuous (real) parameters: the real number $\gamma\in\mathbb{R}$, corresponding to a scaling of the position and momentum axis, and the complex number α, fixing one point in the complex plane.

For each $n\in\mathbb{N}_0$ there is a three-dimensional manifold $\mathscr{M}_n$ of states in Hilbert space $\mathscr{H}$ with constant (stationary) uncertainty. Two manifolds $\mathscr{M}_n$ and $\mathscr{M}_{n'}$ do not have any point in common if $n\neq n'$. In particular, for $n=0$ one obtains the set $\mathscr{M}_0$ of standard coherent states [10] that *minimize* the uncertainty. The manifolds $\mathscr{M}_n$ do *not* provide a foliation of Hilbert space into disjoint regions since their dimension is negligible compared to the dimension of $\mathscr{H}$. Qualitatively, this situation can be visualized in a low-dimensional example. Imagine three orthonormal unit vectors in $\mathbb{R}^3$ and attach to each of their tips a straight line. If these lines do not intersect they correspond to the manifolds $\mathscr{M}_n$.

SECOND VARIATION

Having found the stationary points of the functional, their neighborhoods will be investigated now by studying the quadratic approximation of $J_\lambda[\psi]$ at $|n;\gamma,\alpha\rangle$. Consider the third term in the expansion (5), evaluated at the state $|\psi_0\rangle=|n;\gamma,\alpha\rangle$:

$$\frac{1}{2}D_\varepsilon^2 J_\lambda[\psi_0]=\frac{1}{2}\left\{2\langle\varepsilon|\frac{\delta^2 J_\lambda[\psi]}{\delta\langle\psi|\,\delta|\psi\rangle}|\varepsilon\rangle+\frac{\delta^2 J_\lambda[\psi]}{\delta|\psi\rangle^2}|\varepsilon\rangle|\varepsilon\rangle+\langle\varepsilon|\langle\varepsilon|\frac{\delta^2 J_\lambda[\psi]}{\delta\langle\psi|^2}\right\}_{\psi=\psi_0}. \tag{22}$$

Due to the nonlinearity of $J_\lambda[\psi_0]$, its diagonalization is not straightforward; therefore a general discussion of the second variation seems appropriate before calculating it explicitly.

General structure

The second variation $\delta^2 J_\lambda$ of the functional in Eq. (22) is defined on *rays* of Hilbert space $\mathscr{H}$, as it must be: it is invariant under the transformation

$$|\psi_0+e\rangle\to e^{i\beta}|\psi_0+e\rangle, \tag{23}$$

that is, under the simultaneous transformations

$$|\psi_0\rangle\to e^{i\beta}|\psi_0\rangle,\quad |\varepsilon\rangle\to e^{i\beta}|\varepsilon\rangle. \tag{24}$$

However, it is not defined on the rays of the space $\mathscr{H}_{\psi_0}$ but on its *vectors*: replacing only $|\varepsilon\rangle\to e^{i\beta}|\varepsilon\rangle$ modifies the last two terms of Eq. (22) and, *a fortiori*, the value of uncertainty. The space $\mathscr{H}_{\psi_0}$ can be parametrized by the expansion coefficients of its elements $|\varepsilon\rangle$ in an orthonormal basis of $\mathscr{H}_{\psi_0}$, $\{|\chi_n\rangle, n=1,2,\dots\}$, say, all of which fulfill $\langle\psi_0|\chi_n\rangle=0$; one obtains

$$|\varepsilon\rangle=\sum_n \varepsilon_n|\chi_n\rangle,\quad \varepsilon_n\in\mathbb{C}. \tag{25}$$

For a fixed phase convention of the χ basis each ray $|\psi_0+e\rangle$ determines uniquely one set of values $\{\varepsilon_n\}$ and *vice versa*. The second variation $\delta^2 J_\lambda$ becomes a function of the parameters $\{\varepsilon_n,\varepsilon_n^*\}$.

Plugging the expansion of $|\varepsilon\rangle$ into Eq. (22), one finds from decomposing the coefficients into real and imaginary parts, $\varepsilon_n=\varepsilon_n'+i\varepsilon_n''$, that one can write

$$\delta^2 J_\lambda[\psi_0+e]=\frac{\varepsilon^2}{2}\,\vec{\varepsilon}\cdot\mathbf{J}\cdot\vec{\varepsilon}, \tag{26}$$

where $\mathbf{J}$ is a quadratic, real symmetric matrix acting on the elements of the space T_{ψ_0}, spanned by the vectors $\vec{\varepsilon}=(\varepsilon_1',\varepsilon_1'',\varepsilon_2',\dots)$. The scalar product of two elements of $\mathscr{H}_{\psi_0}$ can be expressed as

$$\langle\mu|\nu\rangle=\vec{\mu}\cdot(\mathbf{E}+i\mathbf{S})\cdot\vec{\nu}, \tag{27}$$

where $\mathbf{E}$ is the unit matrix in T_{ψ_0}, and the symplectic matrix $\mathbf{S}$ consists of (2×2) blocks along the diagonal, each of which is a standard symplectic matrix.

Now one can determine the orthonormal eigenvectors $\vec{\varepsilon}_k$ and eigenvalues J_k, $k=1,2,\dots,$ of the matrix $\mathbf{J}$. Translating the vectors $\vec{\varepsilon}_k$ of T_{ψ_0} into elements $|\varepsilon_k\rangle$ of $\mathscr{H}_{\psi_0}$, one is led to a set of directions in Hilbert space $\mathscr{H}$,

$$|\psi_k\rangle\sim|\psi_0\rangle+\varepsilon|\varepsilon_k\rangle \tag{28}$$

with associated eigenvalues J_k. The eigendirections $\{|\psi_k\rangle\}$ do not have to (and, in general, will not) be orthogonal since they have been derived from a *nonlinear* functional, $J_\lambda[\psi]$.

It remains to show that different vectors $\vec{\varepsilon}_k$ and $\vec{\varepsilon}_l$ do not define states $|\psi_k\rangle$ and $|\psi_l\rangle$ that belong to the *same* ray in Hilbert space $\mathscr{H}$. Suppose that the normalized states $|\psi_k\rangle$ and $|\psi_l\rangle$ belonged to the same ray. Then, the modulus of their scalar product

$$|\langle\psi_k|\psi_l\rangle|^2=\frac{1}{N^2}|1+\varepsilon^2\langle\varepsilon_k|\varepsilon_l\rangle|^2=1, \tag{29}$$

with $N=1+\varepsilon^2$, would necessarily be equal to 1. However, using $\vec{\varepsilon}_k\cdot\vec{\varepsilon}_l=0$ for $k\neq l$ in Eq. (27) one finds that

$$\langle\varepsilon_k|\varepsilon_l\rangle=i\vec{\varepsilon}_k\cdot\mathbf{S}\cdot\vec{\varepsilon}_l; \tag{30}$$

as a result, Eq. (29) only holds if

$$|\langle\varepsilon_k|\varepsilon_l\rangle|^2=1+\frac{2}{\varepsilon^2},\quad \varepsilon\neq0. \tag{31}$$

This is a contradiction since the product of two normalized elements of $\mathscr{H}_{\psi_0}$ cannot exceed 1: $|\langle\varepsilon_k|\varepsilon_l\rangle|\leqslant1$. Consequently, the directions $|\psi_k\rangle$ associated with different $\vec{\varepsilon}_k$ indeed define different rays in Hilbert space.

Explicit calculation

The explicit evaluation of $\delta^2 J_\lambda$ as given in Eq. (22) simplifies considerably if the neighborhood of the state $|n_0\rangle=|n_0;0,0\rangle$ is investigated, since the expectation values of position and momentum vanish; quantitatively the results will be the same for arbitrary states $|n_0;\gamma,\alpha\rangle$. From a straightforward, lengthy calculation one obtains

$$\frac{1}{2}D_\varepsilon^2 I_\lambda[n_0]=\hbar\langle\varepsilon|\frac{1}{2}(\hat{p}^2+\hat{q}^2)|\varepsilon\rangle-\Delta^2(n_0)\langle\varepsilon|\varepsilon\rangle-\frac{1}{2}\{(\langle\varepsilon|\hat{p}|n_0\rangle+\text{ c.c.})^2+(\langle\varepsilon|\hat{q}|n_0\rangle+\text{ c.c.})^2\}-\frac{1}{8\Delta^2(n_0)}(\langle\varepsilon|(\hat{p}^2-\hat{q}^2)|n_0\rangle+\text{ c.c.})^2; \tag{32}$$

here, the multiplier has been given the value it takes at the stationary state under consideration, as is familiar from isoperimetric problems [11]: $\lambda|_{|n_0\rangle}=\Delta^2(n_0)\equiv\hbar(n_0+1/2)$.

Equation (32) suggests using the eigenstates $\{|n\rangle, n_0\neq n\in\mathbb{N}_0\}$ of the operator $\hat{h}_0=\frac{1}{2}(\hat{p}^2+\hat{q}^2)$ as χ basis for the space $\mathscr{H}_{n_0}$:

$$|\varepsilon\rangle=\sum_{n=0}^{\infty}\varepsilon_n|n\rangle \quad (n\neq n_0). \tag{33}$$

The matrix **J** will have nonzero off-diagonal elements only if $n=n_0\pm1,n_0\pm2$. Consequently, the eigenvalues J_n of the matrix **J** for all the other values of n can be read off from the combination of the first two terms of Eq. (32):

$$J_n=\hbar(n-n_0), \quad |n-n_0|>2, \tag{34}$$

each of the values J_n occurring twice. This result is intuitively plausible: if one starts from the state $|n_0\rangle$ and rotates it slightly towards another state $|n'<(n_0-2)\rangle$, the uncertainty decreases, whereas adding a component of a state $|n'>(n_0+2)\rangle$, having itself a larger uncertainty than $|n_0\rangle$, entails an increase of the uncertainty.

There remains to investigate a (8×8) matrix $\tilde{\mathbf{J}}$ with off-diagonal elements resulting from the third and fourth contributions in (32), with entries for $n_0-2\leq n\leq n_0+2$; the cases $n_0=0,1$ have to be considered separately. This matrix decomposes into two (4×4) blocks since the third term has matrix elements only for states with quantum numbers differing by 1 from n_0 (case I), and the last term requires a difference of 2 between the initial and final quantum number (case II).

Case I: Expanding $|\varepsilon\rangle=\varepsilon_{n_0+1}|n_0+1\rangle+\varepsilon_{n_0-1}|n_0-1\rangle$, the first two terms are found to contribute

$$\hbar(\varepsilon'^2_{n_0+1}+\varepsilon''^2_{n_0+1})-(\varepsilon'^2_{n_0-1}+\varepsilon''^2_{n_0-1}). \tag{35}$$

Using $\langle n_0|\hat{q}|\varepsilon\rangle=\sqrt{\hbar/2}(\varepsilon_{n_0+1}\sqrt{n_0+1}+\varepsilon_{n_0+1}\sqrt{n_0})$ and a similar expression for $\langle n_0|\hat{p}|\varepsilon\rangle$, one obtains a contribution that can be written as

$$-\hbar\vec{\varepsilon}\cdot\tilde{\mathbf{M}}\cdot\vec{\varepsilon}, \tag{36}$$

and the matrix $\tilde{\mathbf{M}}$ decomposes into two (2×2) blocks $\tilde{\mathbf{M}}'$ and $\tilde{\mathbf{M}}''$, coupling the real and imaginary components among themselves only, respectively. Explicitly, one has

$$\tilde{\mathbf{M}}'=\begin{bmatrix} n_0 & \sqrt{n_0(n_0+1)} \\ \sqrt{n_0(n_0+1)} & n_0+1 \end{bmatrix}, \tag{37}$$

and $\tilde{\mathbf{M}}''$ is identical to $\tilde{\mathbf{M}}'$ except for the sign of the off-diagonal elements. The determinant of both matrices, $\tilde{\mathbf{M}}'$ and $\tilde{\mathbf{M}}''$, vanishes: the associated two zero eigenvalues correspond to shifts in position and momentum leaving the uncertainty invariant:

$$|\varepsilon_p(n_0)\rangle\sim i\hat{p}|n_0\rangle, \quad |\varepsilon_q(n_0)\rangle\sim i\hat{q}|n_0\rangle. \tag{38}$$

The remaining two nonzero eigenvalues are both given by $-2\hbar(n_0+1/2)$ but no immediate physical interpretation of the associated eigendirections has been found.

Case II: From analogous reasoning one obtains four more eigenvalues stemming from the last term in Eq. (32), given by $\pm2\hbar,-\hbar(n_0^2+n_0+1)/(2n_0+1)$, and one zero eigenvalue associated with the squeezing transformation:

$$|\varepsilon_s(n_0)\rangle\sim i(\hat{p}\hat{q}+\hat{q}\hat{p})|n_0\rangle. \tag{39}$$

For convenience the eight nontrivial eigenvalues of the matrix $\tilde{\mathbf{J}}$ are exhibited in units of $\hbar$,

$$0_3,-2,+2,-(2n_0+1)_2,-\frac{n_0^2+n_0+1}{2n_0+1}, \tag{40}$$

four of which are negative (the multiplicities are indicated by indices).

In fact, the plane tangent to the surface of constant uncertainty at the point $|\psi\rangle$, the three-dimensional manifold $\mathscr{M}_{n_0}$, is spanned by a linear combination of the Hilbert space directions given in Eqs. (38) and (39), i.e.,

$$|\psi(\mathscr{M}_{n_0})\rangle\sim|n_0\rangle+\xi|\varepsilon_p(n_0)\rangle+\eta|\varepsilon_q(n_0)\rangle+\gamma|\varepsilon_s(n_0)\rangle, \tag{41}$$

with *real* coefficients ξ,η,γ. One can directly check that the right-hand side of Eq. (32) vanishes for these states.

Finally, for $n_0=0,1$, these results are modified slightly. In particular, for $|n_0\rangle=|0\rangle$, the matrix **J** has *no* negative eigenvalue in agreement with the fact that the coherent states have minimal uncertainty.

CONCLUSIONS

In summary, first- and second-order variations of the functional of uncertainty $J[\psi]$ have been investigated in detail. It is expected that the technique developed here to study the second variation will be of interest in quantum optics where uncertainty relations for number and phase are discussed [12,13].

The eigenstates of any given harmonic oscillator are guaranteed to be states with stationary uncertainty since it is exactly this set of states that is obtained if one requires the first-order variation of $J[\psi]$ to vanish. More precisely, there is a countable set of three-dimensional manifolds $\mathscr{M}_n$ of states such that the value of the uncertainty remains unchanged under arbitrary variations. Each point on these manifolds $\mathscr{M}_n$ corresponds to an eigenstate $|n_0\rangle$, say, of a harmonic oscillator with a definitely chosen scale of position and momentum and prescribed expectation values of these quantities.

Physically speaking, "kinematic" considerations thus single out a "dynamical" object, namely the number operator (and all its displaced rescaled versions) of the algebra spanned by $\hat{p}$ and $\hat{q}$. This observation provides another justification of the fact that the Hamiltonian of the harmonic oscillator plays a particular role with respect to the Heisenberg algebra.

According to [10] it is possible to base completeness relations on any (nonzero) state $|\psi\rangle\in\mathscr{H}$, since the family of operators T_α, $\alpha\in\mathbb{C}$, acts irreducibly in Hilbert space $\mathscr{H}$. Thus, one arrives at a countable number of resolutions of unity based on the states given in Eq. (19):

$$\frac{1}{\pi}\int_{\mathbb{C}}d\alpha|n;\gamma,\alpha\rangle\langle n;\gamma,\alpha|=1, \quad n=0,1,2,\ldots, \tag{42}$$

usually written for $n=0$ only. These resolutions of unity for $n=1,2,\ldots,$ occupy a specific position among all possible ones since they involve states of stationary uncertainty only.

When collecting the results stemming from diagonalizing the second variation of $J[\psi]$ about its stationary points, the following picture of the landscape of uncertainty in Hilbert space $\mathscr{H}$ associated with the Heisenberg algebra emerges. The neighborhood of a state such as $|n_0\rangle$ is found to be an infinite-dimensional saddle. There are $[2(n_0-1)+4]=2(n_0+1)$ directions in the space $\mathscr{H}$ along which the uncertainty decreases (if $n_0 \geqslant 2$). With respect to the remaining variations the point $|n_0\rangle$ represents a minimum of the uncertainty—except for the three directions defined by the manifold $\mathscr{M}_{n_0}$ along which the uncertainty remains constant.

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