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Chaos and quantum-nondemolition measurements

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The problem of chaotic behavior in quantum mechanics is investigated against the background of the theory of quantum-nondemolition (QND) measurements. The analysis is based on two relevant features: The outcomes of a sequence of QND measurements are unambiguously predictable, and these measurements actually can be performed on one single system without perturbating its time evolution. Consequently, QND measurements represent an appropriate framework to analyze the conditions for the occurrence of "deterministic randomness" in quantum systems. The general arguments are illustrated by a discussion of a quantum system with a time evolution that possesses nonvanishing algorithmic complexity.

I. INTRODUCTION

In this paper it is demonstrated that the theory of quantum-nondemolition (QND) measurements is an appropriate framework for the analysis of deterministic randomness in quantum systems. Within the theory of classical mechanics the occurrence of deterministic randomness, or, in other words, the existence of exponentially instable motion, has far-reaching consequences. To predict the time evolution of a chaotic system over long times with fixed accuracy involves exponentially increasing resources of computing time and/or storage, hence quickly becomes practically impossible. In quantum mechanics the situation is not clear cut because it is difficult to separate in a physically relevant manner the deterministic time evolution given by Schrödinger's equation from the probabilistic element which enters quantum-mechanical predictions via measurements. One way out of this difficulty is to follow directly the time evolution of wave functions in computer experiments, a point of view put forward, e.g., by Chirikov.[1] Yet, the very consequences of possible deterministic randomness still have to be investigated on the level of actual measurements. It is important to note that the existence of a *dynamic* source of unpredictability in quantum mechanics may turn out to be even more fundamental than in classical mechanics which, considered as a clever approximation to quantum mechanics,[2] is only valid for finite—but possibly long times.

The consideration of QND measurements in this context is promising for the following reasons. The fact that Heisenberg operators at different times may commute with themselves leads to the possibility of predicting exactly the outcomes of later *measurements*—not only of the evolution of the wave function. Furthermore an even closer analogy to classical mechanics arises from the possibility of actually performing these measurements on one single system. This allows one to apply the concept of algorithmic complexity to the time evolution of an appropriately chosen set of basis vectors.

This paper is organized as follows. First, some general aspects of classical and quantum mechanics are reviewed. The subsequent discussion of classically chaotic systems emphasizes the intrinsic difficulty of long-time predictions in such systems. In Sec. III the concept of quantum-nondemolition measurements is developed as far as it is necessary for the subsequent arguments. In Sec. IV it is explained that by performing quantum-nondemolition measurements in quantum mechanics one possibly faces problems which typically occur in classically chaotic systems. These features are briefly illustrated by the configurational quantum cat map.[3] It is associated with the motion of a charged particle in a bounded two-dimensional region of space which is subject to external time-periodic electromagnetic fields.

II. CLASSICAL CHAOS

A. Structure of classical and quantum mechanics

From a general point of view dynamical theories in physics consist of three parts. First of all a particular set of "states" is defined corresponding to all possible states of the physical system to be modeled. Secondly, a group of deterministic automorphisms of the state space is postulated—also called a flow in state space— which aims at mimicking the actual time evolution of the underlying physical system. (For simplicity, only autonomous systems are considered here.) Thirdly, a relationship of the statements derived from the theory to experimental observations has to be established, be it of deterministic or probabilistic type. Phenomenological theories inherently containing stochastic elements are not taken into account.

Within this general framework one has to answer the basic question of (nonstationary) physics: Given some initial state, what does the future behavior of the system look like?

The spaces of states and, consequently, the explicit form of the groups of automorphisms called "time evolution" are completely different in classical and quantum mechanics. In the former, Hamilton's equations map a sympletic phase space to itself, and in the latter, Schrödinger's equation determines the time evolution of rays in Hilbert space.[4] Yet, in both theories one can follow exactly the path in state space which is traced out

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deterministically in the course of time starting from an arbitrary initial state. Hence, on this level the structures of classical and quantum mechanics are similar. With respect to the relationship of states to possible outcomes of experiments, however, these theories are fundamentally different.

Predictions stemming from classical theories are unambiguous if the initial state has been defined exactly. Smooth phase-space densities introduce a statistical element and are not taken into account here. When measuring final positions and momenta of the constituents of the physical system (i.e., determining the phase-space point) one principally expects the result to be in perfect agreement with the theoretical forecast. The assumption that the interaction between the observer and the physical system during a measurement can be made arbitrarily small even allows one to follow the unperturbed time evolution of one single system by consecutive observations.

Although the time evolution of a particular initial state in Hilbert space is deterministic, as a rule there is no possibility to predict the outcome of one single observation from quantum theory. A large number of equivalent measurements on an ensemble of equally prepared systems is necessary in order to compare theoretical predictions with the behavior of real systems. Clearly the probabilistic nature of quantum-mechanical predictions strongly contrasts the deterministic character of classical predictions. Furthermore, it is generally impossible to keep track of the time evolution of a quantum system: Measurements change the state of the system in an unpredictable way rendering it useless for subsequent observation. The system and the measuring device generally do not decouple.

Hence, the predictive power of classical mechanics exceeds by far that of quantum mechanics, at least within its range of applicability. It came as a surprise then that predictability in classical mechanics—even for simple systems with a small number of degrees of freedom—has fundamental limits, too.

B. Chaotic motion

In the following, autonomous Hamiltonian systems with N degrees of freedom are considered. It turned out that all classical phase-space flows can be divided into two classes according to the way in which they permute phase-space points in the course of time. Loosely speaking, one can distinguish regular from chaotic behavior of trajectories in phase space.

Integrable systems[5] are distinguished by the fact that only regular—that is, quasiperiodic—motion occurs when the time evolution of phase-space points is considered. A global foliation of the $2N$-dimensional phase space into N-dimensional tori is associated with the existence of N global constants of the motion. Hence for any initial state the system in its time evolution is restricted to a relatively small portion of the $(2N-1)$-dimensional energy surface, namely, an N-dimensional topologically simple toroidal phase-space submanifold.

In contrast, trajectories in nonintegrable systems principally may explore the full $(2N-1)$-dimensional energy surface in phase space.[6] There are no conditions arising from the existence of constants of the motion which would confine the accessible phase space. Consequently the time evolution of such systems is much more intricate, and, typically, no simple description is available.

A useful tool in order to measure the irregularity of orbits of physical systems is the concept of algorithmic complexity originating from the information theory.[7] It can be applied likewise to time-independent and time-dependent systems. Typically—there are exceptions[8]—the algorithmic complexity of trajectories in nonintegrable systems is different from zero,[9] whereas trajectories of integrable systems in classical mechanics have vanishing complexity. Roughly speaking, the time evolution of a physical system has nonvanishing complexity if the initial conditions of an orbit have to be given with *exponential* accuracy in order to maintain *constant* numerical accuracy of the predictions for increasing time t. This concept is closely related to the notion of the Lyapunov exponent.

The definition of algorithmic complexity is based on the analysis of computer programs. A program to calculate numerically the time evolution of a physical system is composed of three different parts. (i) There is an algorithm of fixed length N_A in bits which stands for the dynamical laws of the system under consideration. (ii) A certain amount of data is needed in order to specify the initial state of the system. The length $N_D(\Delta,t)$ of this part is not a fixed number: A prescribed accuracy Δ of the output at time t requests $N_D(\Delta,t)$ bits as initial data. (iii) Finally one has to state the time t at which the program should stop. This requires roughly $\log_2 t$ bits. Algorithmic complexity of the program then is defined as its total length divided by t in the limit of arbitrary long times

$$C=\lim_{t\to\infty}\frac{1}{t}[N_A+N_D(\Delta,t)+\log_2 t]\ . \tag{1}$$

It is important to note that in the limit only $N_D(\Delta,t)$ will contribute to the complexity. A linear growth of the inaccuracy with time t is characteristic for integrable systems—the resulting *logarithmic* time dependence of $N_D(\Delta,t)$ on t does not yield positive complexity. In chaotic systems with Lyapunov exponents different from zero, i.e., in systems which are exponentially instable, the exponential growth of errors leads to a *linear* time dependence of $N_D(\Delta,t)$ and therefore to positive complexity.

III. THEORY OF QND MEASUREMENTS

A. QND observables

Consider a physical system described by a Hamiltonian $\hat{H}(\hat{\mathbf{x}},\hat{\mathbf{p}})$ which in addition may depend explicitly on time. Using the Heisenberg picture of quantum mechanics a QND observable $\hat{\mathcal{O}}(t)$ is defined by the condition[10]

$$[\hat{\mathcal{O}}(t_i),\hat{\mathcal{O}}(t_j)]=0\quad \forall t_i,t_j\in\mathcal{T}\ , \tag{2}$$

where $\mathcal{T}$ may be a discrete or continuous set of times. Commuting observables possess a common set of eigen-

states. Therefore, having forced the system into an eigenstate of the observable $\hat{O}$ by a measurement at time t_0, the outcome of a measurement of the same observable at a later time $t \in \mathcal{T}$ will be an eigenstate of both operators, $\hat{O}(t_0)$ and $\hat{O}(t)$.

A simple example of a *continuous* QND observable ($\mathcal{T}$ being the full time axis) is the energy $\hat{H}(\hat{\mathbf{x}},\hat{\mathbf{p}})$ of an arbitrary autonomous system. As a constant of the motion, $d\hat{H}/dt = 0$, the Hamiltonian commutes with itself at all times

$$[\hat{H}(t_i),\hat{H}(t_j)] \equiv [\hat{H}(t_0),\hat{H}(t_0)] = 0 \ , \qquad -\infty < t_i, t_j < +\infty \ . \tag{3}$$

Obviously, having once measured the energy $\hat{H}$, another measurement of this observable at any later time t will find the system to be in the same eigenstate of the energy. It is important to note the fact that from the first measurement one can predict without any uncertainty the outcome of all later measurements of the same observable. These remarks likewise apply to any other constant of the motion.

QND observables, however, are not necessarily constants of motion. Consider, for example, the momentum operator $\hat{p}(t)$ for a one-dimensional harmonic oscillator of mass $m = 1$ and with frequency ω. Its time evolution is given by

$$\hat{p}(t) = \hat{p}(0)\cos\omega t - \omega\hat{x}(0)\sin\omega t \tag{4}$$

leading to

$$[\hat{p}(t),\hat{p}(t+\tau)] = i\hbar\omega\sin\omega\tau \ . \tag{5}$$

Consequently, for all times $\tau \in \mathcal{T} = \{t \mid t = \pi n/\omega; n \in \mathbb{Z}\}$ the commutator (5) vanishes and therefore the momentum $\hat{p}(t)$ is called a *discrete* QND observable.

Apart from the fact that in both examples the results of later measurements can be predicted without any uncertainty it is remarkable that such measurements actually can be performed on one single system without disturbing its time evolution. Here and in the following it is assumed that *ideal* measurements can be performed. In general, the state of a system is not changed by a measurement of the observable $\hat{A}$ if the system already is in an eigenstate of the operator $\hat{A}$. For this reason the class of observables described above is called "quantum nondemolition."

It is a peculiar feature of the examples mentioned above that the states present at later times $\tau \in \mathcal{T}$ coincide with those resulting from the first measurement. In principle, other eigenstates of the same operator are possible outcomes, too. This situation will be studied in some detail later on.

The notion of a "complete set of commuting observables" (CCO) denotes a set of observables the eigenstates of which span the Hilbert space of the system under consideration and which allow to label uniquely all states with the corresponding eigenvalues. If, in addition, all observables of such a set are QND observables for some set $\mathcal{T}$, this collection of observables may be called a "complete set of commuting QND observables" (CCNO). Any CCO consisting of constants of the motion is an example of a continuous CCNO. The momentum operator $\hat{p}(t)$ of the one-dimensional harmonic oscillator represents another CCNO being of discrete or stroboscopical type, however.

B. QND measurements

Besides the concept of QND observables, the notion of QND measurements turns out to be useful. It is defined as a sequence of precise measurements of a set of commuting observables $\{\hat{O}_i\}$ such that the result of each measurement is completely predictable from the result of the first measurement.[10]

QND measurements have unusual properties. The probabilistic character of quantum-mechanical predictions is suppressed when dealing with QND measurements. The deterministic time evolution of quantum states governed by the time-dependent Schrödinger equation can be monitored on the macroscopic level without any perturbation of the system. Of course, one is restricted to QND observables and possibly only stroboscopical observations. Another interesting feature is the fact that for QND measurements it is not necessary to use ensembles in order to keep track of the time evolution of the system. One single system is sufficient to perform QND measurements: the specific interaction between the system and apparatus does not introduce an unpredictable perturbation of the system.

Suppose that for a particular physical system there exists a CCNO which consists of stroboscopical QND observables only. Performing QND measurements on this system leads to a situation which strongly resembles classical mechanics. Measuring in a first experiment simultaneously all observables of the CCNO completely determines the state of the system. The results of subsequent measurements of the same set of observables are unambiguously predictable from the initial state. The actual time evolution of the single system at hand can be monitored by repeated measurements which in principle do not perturb the state of the system. Note that these statements accordingly apply to a classical system the phase-space trajectory of which is observed stroboscopically.

From one time $\tau \in \mathcal{T}$ to another time $\tau' \in \mathcal{T}$, the time evolution of a basis in Hilbert space—consisting at time τ of the eigenfunctions of a complete set of commuting QND observables—is particularly simple. It is just given by a permutation of its elements—whereas, in general, superpositions of the original states occur. This can be seen as follows. As a unitary transformation the time evolution maps any basis in Hilbert space to another basis. Denote the original CCNO by $\{\hat{O}_i\}$, the transformed set by $\{\hat{O}'_i\}$, and the associated bases by $\{|\psi\rangle_i\}$ and $\{|\psi\rangle'_i\}$, respectively. The quantum-nondemolition condition then says

$$[\{\hat{O}_i\},\{\hat{O}'_i\}] = 0 \ , \tag{6}$$

that is every element of the transformed set commutes with all elements of the original one. Consequently, the sets of operators $\{\hat{O}_i\}$ and $\{\hat{O}_i\}$ have a *common* set of eigenfunctions spanning the full Hilbert space of the system. Necessarily, the associated sets of basis elements of

both CCNO's coincide: $\{|\psi\rangle_i\} \sim \{|\psi\rangle_i'\}$. The states $\{|\psi\rangle_i'\}$, however, may be labeled according to a rule different from that one used for the original set $\{|\psi\rangle_i\}$. As a simple example consider a complete set of commuting observables consisting of two operators, $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$. If the time evolution after a time Δt effects an exchange of these operators,

$$\hat{\mathcal{O}}_1(\tau) \to \hat{\mathcal{O}}_1(\tau') \equiv \hat{\mathcal{O}}_2(\tau), \quad \hat{\mathcal{O}}_2(\tau) \to \hat{\mathcal{O}}_2(\tau') \equiv \hat{\mathcal{O}}_1(\tau) \tag{7}$$

the quantum-nondemolition condition obviously is fulfilled and the new set of eigenvectors—spanning the same Hilbert space—is obtained from a simple exchange of labels. In a time-periodic system, in the course of time the same permutation is applied again and again to the states (Schrödinger picture) or to the operators (Heisenberg picture), respectively.

Before exploiting the formal analogy between the time evolution of classical and quantum systems when described by a CCNO, the original motivation for the introduction of QND measurements is briefly presented in the next section.

C. Physics behind QND measurements

QND measurements[11] have their origin in the problem of detecting gravity waves to which the earth may be subject. The resulting classical gravitational forces on a macroscopic aluminum bar are extremely weak:[10] the required high precision of measurements at first sight seems to be unattainable when the quantum-mechanical uncertainty relation is taken into account. Performing QND measurements, however, allows one, in principle, to detect arbitrarily small classical forces—even when the measuring device has to be treated quantum mechanically.

The idea behind the application of QND measurements in this context is simple. Consider a closed system described by a Hamiltonian $\hat{H}(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ and assume $\hat{\mathcal{O}}$ to be a stroboscopical QND observable. Then a measurement of $\hat{\mathcal{O}}$ at time t_0 allows us to predict exactly the result of measuring $\hat{\mathcal{O}}(\tau)$ at a later time $\tau \in \mathcal{T}$. Any actually observed deviation from the predictions necessarily must be attributed to an external perturbation. Thus, the observed differences of prediction and experimental outcome monitor an additional "force" acting on the system. In principle the disturbance may be arbitrarily weak because the predictions pertaining to the unperturbed system are of arbitrary precision.

Caves *et al.*[12] discuss at length models of "antennas" for the detection of gravity waves and actual realizations thereof. Clearly, the accuracy of experiments is limited because of the fact that "ideal" measurements cannot be performed. Typically the antenna consists of an aluminum bar of several tons which is treated as a quantum-mechanical oscillator. The coupling of the bar to the classical gravity waves is assumed to be linear making a thorough analysis of the problem feasible.[13]

IV. QND MEASUREMENTS AND CHAOS

A. General relationship

As shown above measurements performed with a complete set of QND observables lead to a situation which is similar to the observation of classical objects. Not only the time evolution of the quantum state is deterministic but also the outcomes of measurements are unambiguously predictable. Even for one single system a sequence of measurements can be carried out—it is not necessary to use ensembles of equivalently prepared systems in order to compare theoretical predictions with actual experiments.

This particular quantum-mechanical situation where at first sight determinism prevails is suitable for investigating the occurrence of deterministic randomness in quantum mechanics. The statistical properties of quantum mechanics are suppressed, and one can fully concentrate on the question whether chaotic behavior may emerge from the underlying deterministic time evolution in Hilbert space. It should be clear at this point that, in principle, the door is open to both types of behavior, regular and irregular.

In the following, discrete QND observables are assumed for simplicity. Furthermore, only the deterministic transformation of the attached *basis* of eigenvectors in Hilbert space is considered—this, of course, is sufficient to construct the evolution of any state.

As a first example assume the set of eigenvectors of a CCNO to have a finite number of elements. Any repeated automorphism of this set will provide a periodic time evolution leaving no room for chaotic behavior.

However, a continuously labeled set of (generalized) basis vectors may give rise to irregular behavior and, consequently, to a limitation of long-time predictions as the following arguments indicate. Assume the labels which are necessary to mark uniquely all elements of the basis to vary over a compact manifold Γ endowed with a natural metric which should be clearly distinguished from the familiar Hilbert-space metric (cf. below). The deterministic time evolution from one time $\tau \in \mathcal{T}$ to the next $\tau' \in \mathcal{T}$ induces an automorphism of the set of basis elements onto itself as was shown in Sec. III B. If the underlying manifold Γ of labels effectively is "stretched and folded" in the course of time, arbitrarily fine structures may evolve, a process which typically is accompanied by positive algorithmic complexity of the time evolution. But this is just another way of saying that tremendous numerical efforts have to be made in order to predict reliably the long-time behavior of the system. "Sensitive dependence on initial conditions" will manifest itself in the quantum system as follows: after a few iterations of the automorphism basis vectors with initially close labels are mapped (according to the transformation of the labels) to other "distant" states—separated on the manifold Γ with respect to the metric of the labels. Therefore, states initially localized with respect to the labels on the manifold Γ will quickly be spread over a large portion of Γ, that is over the basis vectors of the CCNO. This phenomenon is closely analogous to the exponentially quick distribution

of an initially localized phase-space density in classical mechanics.

The intermediate case of an infinite but countable number of basis elements is of physical importance but a general discussion seems to be quite subtle. Here only some arguments pertaining to this case will be sketched. Assume a time evolution which periodically maps an enumerable Hilbert-space basis onto itself. Formally this process is equivalent to a periodically repeated permutation of the positive integers, $\mathbb{N}_+$. The specification of a particular initial state does not require initial data $N_D(\Delta,t)$ [cf. Eq. (1)] increasing exponentially in time t. Therefore, in systems with a countable number of basis elements, this term does not give rise to positive algorithmic complexity. But in such systems algorithmic complexity may arise from encoding the time evolution. Although there is no evidence for dynamical systems possessing this property, a short discussion of this situation is given for the sake of completeness.

The number of bits needed for the specification of the dynamical law N_A is not necessarily a fixed number as was assumed above (cf. Sec. II B). Indeed, in order to prove that this term actually may give rise to positive algorithmic complexity of the underlying time evolution it is sufficient to show that there exist permutations of $\mathbb{N}_+$ the specification of which requires a program with positive algorithmic complexity. This can be done as follows. The number of permutations of $\mathbb{N}_+$ may be shown to be uncountable by using essentially Cantor's proof of the nondenumerability of the real numbers. The set of real numbers with zero complexity, however, is countable.[7] Consequently, there are permutations of the set $\mathbb{N}_+$ which cannot be compressed algorithmically, and this in turn may lead to positive algorithmic complexity of the time evolution of the underlying physical system. As was mentioned above no physical system is known for which algorithmic complexity stems from encoding the time evolution.

The basis vectors of position operators usually carry labels defining a compact manifold if the configuration space of the system is bounded. The natural metric is then given by the spatial distance of points in configuration space which unambiguously correspond to the eigenstates under consideration. If, in addition, for a particular system the position operators represent a complete set of commuting quantum-nondemolition observables, they are suited to demonstrate the presence or absence of deterministic randomness in the time evolution of the system.

Such behavior actually exists. In the next section it is demonstrated that the scheme of QND measurements is useful in order to discuss the configurational quantum-cat map, a quantum system that was shown to have a time evolution with positive algorithmic complexity. It is interesting to note that the model to be presented is explicitly time-dependent—dynamic evolution arising from "natural" autonomous Hamilton operators apparently does not permute basis elements in such a way that the associated map of the manifold of labels undergoes stretching and folding.

It is important to be aware of the fact that QND measurements are in no way related to the *formation* of deterministic randomness. They only provide a framework where quantum-mechanical statements apply to reality without introducing a statistical element, thereby allowing one to isolate the phenomenon to study. Having uncovered the role of deterministic randomness, its consequences may be investigated in the general situation, that is in the presence of typical quantum-mechanical uncertainties. It is plausible to expect a simple superposition of both effects.

Two points are worth mentioning before dealing with a specific example. First, the general scheme introduced above represents one possible scenario of how deterministic randomness may enter quantum mechanics. Further investigations have to clarify its relevance for typical systems. The examples in the following section demonstrate that the formation of arbitrarily fine structures is accompanied by an exponential growth of the momenta and, hence, the energy. Secondly, an immediate physical consequence of deterministic randomness in quantum mechanics becomes apparent: Not all systems will be suited *a priori* for monitoring small forces with QND measurements. An exact determination of the initial state is impossible; any initial inaccuracy quickly will predominate all subsequent measurements if the underlying dynamics shows sensitive dependence on initial conditions. In addition, any external perturbation will strongly influence the actual time evolution because of the exponentially instable motion.

B. Examples

The configurational quantum-cat map[3] provides a nontrivial example where chaotic behavior can be observed in QND measurements. The physical system underlying the configurational quantum-cat map consists of a charged particle constrained to move in a unit square of the (x_1,x_2) plane with periodic-boundary conditions (period 1) under the influence of time-dependent electromagnetic fields. The Hamiltonian reads

$$H(\mathbf{x},\mathbf{p},t)=\tfrac{1}{2}\mathbf{p}^2+\tfrac{1}{2}(\mathbf{p}\cdot\underline{V}\cdot\mathbf{x}+\mathbf{x}\cdot\tilde{\underline{V}}\cdot\mathbf{p})\Delta_{T,\epsilon}(t)\ . \tag{8}$$

Here, $\Delta_{T,\epsilon}(t)$ is a sequence of smooth kicks of period T, duration ϵ, and height $1/\epsilon$ with $\epsilon \ll T$. $\underline{V}$ is a constant 2×2 matrix such that $\underline{C}=\exp(\underline{V})$ is integer hyperbolic and has determinant 1. The transpose of $\underline{V}$ is denoted by $\tilde{\underline{V}}$. Comparing (8) with

$$H'(\mathbf{x},\mathbf{p},t)=\frac{1}{2}\left[\mathbf{p}-\frac{e}{c}\,\mathbf{A}(\mathbf{x},t)\right]^2+e\phi(\mathbf{x},t) \tag{9}$$

one can determine the vector potential $\mathbf{A}(\mathbf{x},t)$ and the scalar potential $\phi(\mathbf{x},t)$. It turns out[14] that the associated magnetic field $\mathbf{B}(\mathbf{x},t)$ is spatially uniform and directed along the x_3 axis, whereas the electric field $\mathbf{E}(\mathbf{x},t)$ has components in the (x_1,x_2) plane only. Although it contains terms proportional to the *square* and the *derivative* of the kick function $\Delta_{T,\epsilon}(t)$, such an electric field can, in principle, be realized for any *finite* kick width ϵ. Due to the bilinearity of the Hamiltonian $H(\mathbf{x},\mathbf{p},t)$ the classical as well as the quantum-mechanical model can be discussed in full detail.[3,14]

The quantum-mechanical time-evolution operator $\hat{U}(T)$ over one period, the Floquet operator, consists of two parts: $\hat{U}(T)=\hat{U}_F(T)\hat{U}_K$, representing free-particle motion during time intervals of length T and the kick transformation, respectively. Explicit calculations reveal that in this model the unit square, representing the compact manifold Γ defined by the labels of the position eigenstates, undergoes a transformation which is known to have algorithmic complexity. In this process both momenta and energy generically grow at an exponential rate. Furthermore, the spectrum of quasienergies associated to the Floquet operator is found to be absolutely continuous.

The connection with QND measurements can be established in the following way. One complete set of observables used to describe the system consists of the position operators in the Heisenberg picture, $\hat{x}_1(t)$ and $\hat{x}_2(t)$, which for all times do commute

$$[\hat{x}_1(t),\hat{x}_2(t')]=0 \quad \forall t,t' \ . \tag{10}$$

For particular values of the kicking period, $T=2n/h, n\in\mathbb{Z}$, the operator $\hat{U}_F(T)$ becomes equal to the identity and the components of the operator $\hat{\mathbf{x}}(t)$ commute with themselves periodically $(r,s=1,2)$:

$$\begin{aligned}
&[\hat{x}_r(nT^-),\hat{x}_s(mT^-)] \\
&\quad=[\hat{U}(nT)\hat{x}_r(0^-)\hat{U}^+(nT),\hat{U}(mT)\hat{x}_s(0^-)\hat{U}^+(mT)] \\
&\quad=[\hat{U}_K^n\hat{x}_r(0^-)(\hat{U}_K^+)^n,\hat{U}_K^m\hat{x}_s(0^-)(\hat{U}_K^+)^m] \\
&\quad=[(\underline{C}^n\cdot\hat{x}(0^-))_r,(\underline{C}^m\cdot\hat{x}(0^-))_s] \\
&\quad=(\underline{C}^n)_{r\rho}(\underline{C}^m)_{s\sigma}[\hat{x}_\rho(0^-),\hat{x}_\sigma(0^-)]=0 \ .
\end{aligned} \tag{11}$$

This equation establishes that the position operators $\hat{\mathbf{x}}(t)$ build a CCO of stroboscopical QND observables. The set of times $\mathcal{T}$ can be taken as $\mathcal{T}=\{t|t=(lT)^-,l\in\mathbb{Z}\}$ corresponding to all times just before the application of the kicks. This approach naturally explains the importance of "quantum resonances," first observed by Casati *et al.*[15]

The apparent suppression of the free-time evolution, that is $\hat{U}_F(T)\equiv 1$, and the map of the position basis onto itself can be seen clearly in the Schrödinger picture. Each state is mapped according to a two-dimensional hyperbolic map to another state in the unit square

$$\hat{U}_K|\mathbf{x}\rangle=|(\underline{C}\cdot\mathbf{x})\mathrm{mod}1\rangle \ . \tag{12}$$

The algorithmic complexity of the quantum motion is a direct consequence of this relation. In order to specify the time evolution of any position eigenstate one has to perform exactly those calculations which are necessary to determine orbits for the Arnold cat map[16] where the unit square represents the phase space of a classical one-dimensional system. Therefore, the quantum motion of almost all position eigenstates and, *a fortiori,* of linear superpositions is algorithmically complex. It has been shown[3,14] that this system exhibits a number of features typical for classically irregular motion. For example, the exponential instability in the mapping of the configuration space onto itself indeed entails the evolution of arbitrary fine structure of the wave function in the configuration representation. Consequently the long-time behavior of the wave function—and hence of all physical properties of the system—becomes extremely sensitive to an inaccurate determination of the initial state and to external perturbations, both of which are inevitable for realistic systems.

The nonresonant time evolution, i.e., $T\neq 2n/h$, $n\in\mathbb{Z}$, is more complicated. In addition to the permutation of basis vectors rotations in Hilbert space do occur stemming from the free-time evolution of the particle. Consequently, from one time just before a kick to the next one has to consider the mapping of the total Hilbert space onto itself—basis elements are no longer permuted among themselves. The possibility of making QND measurements is lost; the mechanism giving rise to deterministic randomness, however, still is present in the time evolution of the system, namely, the periodically applied kick transformation. This illustrates the fact that the possibility of QND measurements is not connected to the (regular or irregular) time evolution but just represents a convenient way of tackling the problem.

There is another example where QND measurements have been exploited in order to investigate the occurrence of deterministic randomness in quantum mechanics. Chirikov, Izraelev, and Shepelyansky[17] introduced an abstract Hamiltonian system with three degrees of freedom showing "true quantum chaos." The Hamiltonian is constructed in such a way that the time evolution of the wave function in configuration space—more precisely, of the quantum-mechanical probability distribution—exactly parallels the time evolution of the classical density in configuration space. Again, at the expense of growing momenta the classical density develops arbitrarily fine structures entailing sensitive dependence on initial conditions. Hence, these properties also have to be attributed to the quantum system. In this example, all three position operators constitute a CCO consisting of continuous QND observables. It is possible to follow the time evolution of a position eigenstate by (continuously) repeated measurements of the particle position without disturbing the system.

In a paper on measurements in quantum theory Sudarshan[18] presents a theory having some aspects in common with the approach developed here. Yet, the problem of deterministic randomness is not touched upon. Embedding a classical system with N degrees of freedom into a quantum system with $2N$ degrees of freedom, it is shown that the N pairs of canonically conjugate classical variables formally may be considered as $2N$ commuting position operators of an abstract quantum system. These $2N$ commuting observables may be measured simultaneously for all times. In certain circumstances—which translate into the condition of dealing with continuous QND observables—the time evolution of an associated eigenstate can be followed by repeated measurements. Consequently, the time evolution of such a state related to a CCNO with $2N$ elements may be compared with the motion of a phase-space point of an N-dimensional classical system. This fact actually is exploited in the configurational quantum-cat map: The time evolution of

a *two*-dimensional quantum system in configuration space parallels that of a *one*-dimensional classical system in phase space. In contrast to Sudarshan's theory where the Hamilton operator is not an observable, the particular quantum model treated here has a natural physical interpretation.

V. SUMMARY

In this paper a method is presented allowing for particular systems to separate in a physically realizable way the processes which determine the time evolution of quantum systems, namely, the deterministic evolution of states in Hilbert space according to Schrödinger's equation on the one hand and the process of measurement on the other. By performing measurements of a complete set of commuting QND observables it is possible to create a situation in which the intrinsic probabilistic character of quantum-mechanical statements is completely absent. This setting is appropriate to investigate the occurrence of deterministic randomness in the deterministic quantum-mechanical time evolution. As a result particular flows containing maps of uncountable sets of basis elements onto themselves are found to be capable of generating algorithmically complex quantum motions. Equivalently one may speak of the quantum motion to be exponentially instable. This is demonstrated explicitly for the configurational quantum-cat map, the time evolution of which in configuration space formally parallels the motion of Arnold's cat map known from classical mechanics.

The approach introduced here indicates that a more general characterization of the types of deterministic motion in quantum systems and the conditions for their occurrence may possibly be achieved by investigating systematically abstract one-to-one maps of Hilbert-space bases onto themselves, i.e., permutations of the basis vectors. Such a characterization should be compared with the hierarchy of statistical behavior for classical systems and may shed light on the foundations of statistical mechanics.

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