UNIVERSITY of York

This is a repository copy of *On generalized Thue-Morse functions and their values*.

White Rose Research Online URL for this paper:
http://eprints.whiterose.ac.uk/135616/

Version: Accepted Version

Article:

Zorin, Evgeniy orcid.org/0000-0002-3092-340X and Badziahin, Dzmitry (2019) On generalized Thue-Morse functions and their values. Journal of the australian mathematical society. ISSN 1446-7887

Reuse
Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown
If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

White Rose university consortium
Universities of Leeds, Sheffield & York

eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

ON GENERALIZED THUE-MORSE FUNCTIONS AND THEIR VALUES.

DZMITRY BADZIAHIN and EVGENIY ZORIN

(August 14, 2017)

Abstract

In this paper we extend and generalize, up to a natural bound of the method, our previous work [2], where we proved, among the other things, that the Thue-Morse constant is not badly approximable. Here we consider Laurent series defined with infinite products $f_d(x) = \prod_{n=0}^{\infty}(1 - x^{-d^n})$, $d \in \mathbb{N}$, $d \geqslant 2$, which generalize the generating function $f_2(x)$ of the Thue-Morse number, and study their continued fraction expansion. In particular, we show that the convergents of $x^{-d+1}f_d(x)$ have a regular structure. We also address the question whether the corresponding Mahler numbers $f_d(a) \in \mathbb{R}$, $a, d \in \mathbb{N}$, $a, d \geqslant 2$, are badly approximable.

1. Introduction

Our work on the well approximability of Thue-Morse constant [2] exploits the functional approximations to its generating function. Moreover, we show in [2] that the generating function of Thue-Morse constant is rationally equivalent to Laurent series with a simple continued fraction. In this work, we generalize methods from [2] to cover larger classes of numbers and functions. At the same time, this generalization exposes the internal structure of the original proof in [2] and gives better results for the original setup of the Thue-Morse constant (see Theorem 3 below).

We are going to work with the following functions, defined by an infinite product:

$$f_d(x) := \prod_{t=0}^{\infty}(1 - x^{-d^t}) \in \mathbb{Q}((x^{-1})), \quad d \in \mathbb{N}, \quad d \geqslant 2. \tag{1}$$

The class of functions (1) contains the generating function of the Thue-Morse constant, $f_2(x)$. We call the functions (1) the *generalized Thue-Morse functions.*

By expanding the brackets in the infinite product (1), the function $f_d(x)$ defines an infinite Laurent series in x^{-1} which is absolutely convergent in the disc $|x| > 1$. By substituting x^d in place of x we obtain the following functional equation,

$$f_d(x^d) = \frac{xf_d(x)}{x-1}. \tag{2}$$

Dzmitry Badziahin acknowledges the support of EPSRC Grant EP/E061613/1.
Evgeniy Zorin acknowledges the support of EPSRC Grant EP/M021858/1.

© XXXX Australian Mathematical Society 0263-6115/XX \$A2.00 + 0.00

Like in the classical case of real numbers, one can apply the continued fraction algorithm to Laurent series from $\mathbb{Q}((x^{-1}))$, we discuss this in more detail in Section 2. In particular, we can construct the continued fraction for $f_d(x)$. Its properties were investigated by van der Poorten and others in a series of papers [1, 6, 7]. They discovered quite an irregular behaviour of the sequence of partial quotients of $f_d(x)$, see [1]. In this paper we show that, on the other hand, the function $g_d(x)$,

$$g_d(x) := x^{-d+1} f_d(x), \tag{3}$$

which is rationally dependent with $f_d(x)$, has a pretty regular continued fraction expansion, see Theorems 1 and 2 below.

Note that definition (3) and functional equation (2) together give a Mahler type functional equation for $g_d(x)$:

$$g_d(x^d) = \frac{g_d(x)}{x^{d^2-2d}(x-1)}. \tag{4}$$

In [2] we established a precise recurrent formula for the sequence of partial quotients $a_i(x) \in \mathbb{Q}[x]$, $i \in \mathbb{N}$ of $g_2(x)$. Here we generalize this result to get more general properties of continued fraction of $g_d(x)$ for the other integer values $d \geqslant 3$. For instance, we manage to provide a nice description of convergents to $g_d(x)$, which is done in Theorem 1. In Section 3 we make this description completely explicit for the values d such that $f_d(x)$ is so called badly approximable, see Theorem 2.

Also, in this paper we investigate the question whether $f_d(a)$ is badly approximable for given integer values of a and d with $a, d \geqslant 2$. Recall that a number $x \in \mathbb{R}$ is said to be badly approximable if there exists a positive constant $c = c(x) > 0$ such that

$$0 < |x - p/q| \geqslant c/q^2$$

for all integers p, q with $q \neq 0$. Equivalently, the number $x \in \mathbb{R}$ is badly approximable if and only if all its partial quotients are uniformly upper bounded.

We explain in Subsection 5.1 that, for a trivial reason, $f_d(a)$ is not badly approximable for $d \geqslant 4$. In Subsections 5.2 and 5.3 we provide the results which allow to verify that for a given integer $a \geqslant 2$ the numbers $f_2(a)$ and $f_3(a)$ are not badly approximable. In particular, these results generalize the theorem from [2] concerning $f_2(a)$. They remove a principal obstacle which did not allow to apply that theorem to the whole set of integers $a \geqslant 2$, this obstacle is explained in the discussion after Corollary 16 in [2].

We would like to emphasize that this paper extends the methods used therein up to their natural limits. For instance, it would be very interesting to get similar results for functions

$$f(x) = \prod_{t=0}^{\infty} P(x^{-d^t})$$

where P belongs to as large class of polynomials (or even rational functions) as possible. Of course for many polynomials it is possible to check that $f(x)$ is not badly approximable as a Laurent series and therefore $f(n)$ is not badly approximable as a number. However in many other cases $f(x)$ is a badly approximable Laurent series and it leaves open the question whether $f(n)$ is badly approximable or not. To answer this question in full generality some new fresh ideas are needed.

2. Some definitions and preparatory results on functional continued fractions

Definition 1. *We will denote by* $\|u(x)\|$ *the* degree *of Laurent series* $u(x) \in \mathbb{Q}((x^{-1}))$*, that is the biggest degree having a non-zero coefficient in the Laurent series* $u(x)$*. In case if* $u(x)$

is a polynomial in x, this definition of degree coincides with the classical definition of degree of a polynomial.

Definition 2. *Let $p(x)/q(x)$ be a rational function and $u(x)$ be a Laurent series. We say that an integer c is* the rate of approximation *of $p(x)/q(x)$ to $u(x)$ if*

$$\|u(x) - p(x)/q(x)\| = -2\|q(x)\| - c.$$

Remark It is easy to verify with a bit of linear algebra that for any Laurent series $u(x)$ and any $n \in \mathbb{N}$ there exist polynomials $p_n(x)$ and $q_n(x)$ such that $\deg q_n \leq n$ and the rate of approximation of $p_n(x)/q_n(x)$ to $u(x)$ is at least $2n - 2\deg q_n + 1 \geqslant 1$.

In this paper, we will extensively use the apparatus of continued fractions. It is well known that Laurent series admit the continued fraction construction analogous to that for real numbers,

$$u(x) = [a_0(x); a_1(x), a_2(x), \ldots, \cdots], \tag{5}$$

where $a_i(x)$ are non-zero polynomials, $i \in \mathbb{N}$. *The n-th convergent* to $u(x)$, $n \in \mathbb{N}$, is defined to be the following rational function:

$$p_n(x)/q_n(x) = [a_0(x); a_1(x), \ldots, a_n(x)] \tag{6}$$

where the rational function $p_n(x)/q_n(x)$ is taken in its reduced form. In some situations we will need to precise Laurent series which we approximate, so we denote by $p_{n,u}(x)/q_{n,u}(x)$ the n'th convergent to Laurent series $u(x)$. Similarly, we denote by $a_{i,u}$ the ith partial quotient of Laurent series $u(x)$.

The set of convergents to Laurent series $p_n(x)/q_n(x)$, $n \in \mathbb{N}$, enjoys many nice properties similar to the properties of convergents to the real numbers. So, the rational fraction $p_n(x)/q_n(x)$ approximates $u(x)$ with the rate of approximation $\deg a_{n+1}(x)$. In other terms,

$$\left\| u(x) - \frac{p_n(x)}{q_n(x)} \right\| = -\|q_n(x)q_{n+1}(x)\| = -2\|q_n(x)\| - \|a_{n+1}(x)\|. \tag{7}$$

Also, convergents $p_n(x)/q_n(x)$ are precisely the rational fractions having strictly positive rate of approximation to $u(x)$ (see [10, Proposition 1]). Numerators and denominators of consecutive convergents enjoy the following recursive relations:

$$\begin{aligned} p_{n+1}(x) &= a_{n+1}(x)p_n(x) + p_{n-1}(x), \\ q_{n+1}(x) &= a_{n+1}(x)q_n(x) + q_{n-1}(x). \end{aligned} \tag{8}$$

We refer the reader to a nice paper [10] by van der Poorten for a more detailed account on continued fractions of formal power series.

Note that $p_n(x)$ and $q_n(x)$ are defined up to a multiplication by a non-zero constant. Sometimes for convenience we want to get the convergents $\hat{p}_n(x)/\hat{q}_n(x)$ such that $\hat{q}_n(x)$ is monic. In that case (8) should be modified to make sure that the resulting polynomial $\hat{q}_{n+1}(x)$ remains monic:

$$\begin{aligned} \hat{p}_{n+1}(x) &= \hat{a}_{n+1}(x)\hat{p}_n(x) + \beta_{n+1}\hat{p}_{n-1}(x), \\ \hat{q}_{n+1}(x) &= \hat{a}_{n+1}(x)\hat{q}_n(x) + \beta_{n+1}\hat{q}_{n-1}(x). \end{aligned} \tag{9}$$

where we define, with ρ_n denoting the leading coefficient of $q_n(x)$,

$$\hat{a}_{n+1}(x) = \frac{a_{n+1}(x)\rho_n}{\rho_{n+1}}; \quad \beta_{n+1} = \frac{\rho_{n-1}}{\rho_{n+1}}.$$

Below we prove two lemmata which provide two different sources of convergents to the function $g_d(x)$, defined by (3).

Lemma 1. *Let $h_d(x) := x^{-1}f_d(x)$. If $p(x)/q(x)$ is a convergent to $h_d(x)$ with the rate of approximation c then $\frac{(x-1)p(x^d)}{q(x^d)}$ is a convergent to $g_d(x)$ with the rate of approximation at least $dc-1$. Moreover, this rate of approximation is precisely $dc-1$ if and only if $(x-1)\nmid q(x)$.*

Proof. We simply use the following functional relation:

$$h_d(x^d) = x^{-d}f_d(x^d) = \frac{g_d(x)}{(x-1)}. \tag{10}$$

If

$$\left\| h_d(x) - \frac{p(x)}{q(x)} \right\| = -2\|q(x)\| - c, \tag{11}$$

then by substituting x^d in place of x in (11) and by using (10) we find:

$$\left\| h_d(x^d) - \frac{p(x^d)}{q(x^d)} \right\| = \left\| \frac{g_d(x)}{x-1} - \frac{p(x^d)}{q(x^d)} \right\| = -2d\|q(x)\| - dc, \tag{12}$$

hence, by multiplying both sides of (12) by $x-1$,

$$\left\| g_d(x) - \frac{(x-1)p(x^d)}{q(x^d)} \right\| = -2\|q(x^d)\| - dc + 1.$$

Note that the rate of approximation of the convergent $\frac{(x-1)p(x^d)}{q(x^d)}$ to $g_d(x)$ exactly equals $dc-1$ as soon as $\gcd((x-1)p(x^d), q(x^d)) = 1$. Since $p(x)$ and $q(x)$ are coprime by the definition of a convergent, this is equivalent to $(x-1)\nmid q(x)$. □

Lemma 2. *Let $u_d(x) := (1-x^{-1})f_d(x)$. If $p(x)/q(x)$ is a convergent to $u_d(x)$ with the rate of approximation c, then*

$$\frac{p^*(x)}{q^*(x)} := \frac{p(x^d)}{(1+x+x^2+\cdots+x^{d-1})q(x^d)}$$

is a convergent to $g_d(x)$ with the rate of approximation $d(c-1)+1$. Moreover, this rate of approximation is precisely $d(c-1)+1$ if and only if $(x-1)\nmid p(x)$.

Proof. The proof is very similar to the proof of Lemma 1. We firstly observe that

$$u_d(x^d) = (1-x^{-d})f_d(x^d) = (1+x+\cdots+x^{d-1})g_d(x).$$

If

$$\left\| u_d(x) - \frac{p(x)}{q(x)} \right\| = -2\|q(x)\| - c$$

then

$$\left\| \frac{u_d(x^d)}{1+x+\cdots+x^{d-1}} - \frac{p(x^d)}{(1+x+\cdots+x^{d-1})q(x^d)} \right\|$$
$$= \left\| g_d(x) - \frac{p^*(x)}{q^*(x)} \right\| = -2d\|q(x)\| - dc - (d-1).$$

Finally, the equality $\|q^*(x)\| = d\|q(x)\| + d - 1$ completes the proof of the first part of the lemma.

If $(x-1)\nmid p(x)$ then $\gcd(p(x^d), 1+x+\ldots+x^{d-1}) = 1$ and therefore $\gcd(p^*(x), q^*(x)) = 1$. Hence the rate of approximation of the convergent $p^*(x)/q^*(x)$ to $g_d(x)$ is exactly $d(c-1)+1$. □

In fact, two collections of convergents of $g_d(x)$ provided by Lemmata 1 and 2 cover the set of all the convergents of $g_d(x)$. We prove this in Theorem 1 below. Beforehand we need one more technical lemma.

Lemma 3. *Let functions $h_d(x)$ and $u_d(x)$ be as defined in Lemmata 1 and 2 respectively.*

1. *If $\frac{p(x)}{q(x)}$ approximates $u_d(x)$ with the rate of approximation c, then $\frac{p(x)}{(x-1)q(x)}$ approximates $h_d(x)$ with the rate of approximation at least $c-1$.*

2. *If $\frac{p(x)}{q(x)}$ approximates $h_d(x)$ with the rate of approximation c, then $\frac{(x-1)p(x)}{q(x)}$ approximates $u_d(x)$ with the rate of approximation at least $c-1$.*

Proof. 1. Note that $u_d(x) = (x-1)h_d(x)$. Then we have

$$\begin{aligned}\left\| h_d(x) - \frac{p(x)}{(x-1)q(x)} \right\| &= \left\| (x-1)^{-1}\left(u_d(x) - \frac{p(x)}{q(x)}\right)\right\| \\ &= -2\|q(x)\| - c - 1 \\ &= -2\|(x-1)q(x)\| - c + 1,\end{aligned} \tag{13}$$

which proves the first claim.

2. Roughly speaking, we reverse the order of calculations in (13):

$$\begin{aligned}\left\| u_d(x) - \frac{(x-1)p(x)}{q(x)} \right\| &= \left\| (x-1)\left(h_d(x) - \frac{p(x)}{q(x)}\right)\right\| \\ &= \left\| h_d(x) - \frac{p(x)}{q(x)} \right\| + 1 \\ &= -2\|q(x)\| - c + 1.\end{aligned} \tag{14}$$

This proves the second claim of the lemma, hence completes the proof.

□

Theorem 1. *Every convergent of $g_d(x)$ is either of the form given in Lemma 1 or in the form given in Lemma 2. More precisely, let $\frac{p_{m,g_d}}{q_{m,g_d}}$, $m \in \mathbb{N}$, be a convergent to $g_d(x)$. Then,*

1. *If m is odd, then there exists $t \in \mathbb{N}$ such that the t-th convergent $\frac{p_{t,u_d}}{q_{t,u_d}}$ to $u_d(x)$, where $u_d(x)$ is defined in Lemma 2, verifies*

$$\frac{p_{m,g_d}}{q_{m,g_d}} = \frac{p_{t,u_d}(x^d)}{(1+x+\cdots+x^{d-1})q_{t,u_d}(x^d)} \quad \text{with } (x-1) \nmid p_{t,u_d}(x) \tag{15}$$

2. *If m is even, then there exists $s \in \mathbb{N}$ such that the s-th convergent $\frac{p_{s,h_d}}{q_{s,h_d}}$ to $h_d(x)$, where $h_d(x)$ is defined in Lemma 1, verifies*

$$\frac{p_{m,g_d}}{q_{m,g_d}} = \frac{(x-1)p_{s,h_d}(x^d)}{q_{s,h_d}(x^d)} \quad \text{with } (x-1) \nmid q_{s,h_d}(x) \tag{16}$$

Proof. We prove by induction. One can readily verify that the first two convergents of $g_d(x)$ are $p_{0,g_d}(x)/q_{0,g_d}(x) = 0/1$ and $p_{1,g_d}(x)/q_{1,g_d}(x) = 1/(1+x+\cdots+x^{d-1})$. The first one is generated by Lemma 1 from the convergent $0/1$ to $h_d(x)$ and the second one is generated by Lemma 2 from the convergent $1/1$ to $u_d(x)$. Therefore the zeroth and the first convergents of $g_d(x)$ satisfy (16) and (15) respectively.

Assume that we have proved the claim of the theorem up to an odd $m \in \mathbb{N}$, so we have (15) for this $m \in \mathbb{N}$. We are going to prove that the claim of the theorem holds true for $m+1$. To this end, consider (15), take the index $t \in \mathbb{N}$ given by this equality and denote by c the rate of approximation of $u_d(x)$ by $p_{t,u_d}(x)/q_{t,u_d}(x)$. Then by Lemma 2 the rate of approximation of $g_d(x)$ by $\frac{p_{m,g_d}}{q_{m,g_d}}$ is $d(c-1)+1$. Moreover, by the general property of continued fractions this rate of approximation also equals to $\|a_{m+1,g_d}\|$, hence $\|a_{m+1,g_d}\| = d(c-1)+1$. Further, (8) implies

$$\|q_{m+1,g_d}(x)\| = \|q_{m,g_d}(x)\| + \|a_{m+1,g_d}(x)\| = \|q_{m,g_d}(x)\| + d(c-1)+1 = d\|q_{t,u_d}(x)\| + dc. \quad (17)$$

Also, (8) together with the definition of c imply

$$\|q_{t+1,u_d}(x)\| = \|q_{t,u_d}(x)\| + \|a_{t+1,u_d}(x)\| = \|q_{t,u_d}(x)\| + c. \quad (18)$$

Now consider two cases: $c \geqslant 2$ and $c = 1$.

Case 1. $c \geqslant 2$. By Lemma 3, $\frac{p_{t,u_d}(x)}{(x-1)q_{t,u_d}(x)}$ is a convergent to $h_d(x)$, let us say it is the s-th convergent to $h_d(x)$,

$$\frac{p_{s,h_d}(x)}{q_{s,h_d}(x)} = \frac{p_{t,u_d}(x)}{(x-1)q_{t,u_d}(x)}. \quad (19)$$

Moreover, the numerator and denominator of $\frac{p_{t,u_d}(x)}{(x-1)q_{t,u_d}(x)}$ are coprime, since $(x-1) \nmid p_{t,u_d}(x)$. Therefore the rate of approximation of $h_d(x)$ by s-th convergent equals $c-1$. The latter fact implies that the next convergent $\frac{p_{s+1,h_d}(x)}{q_{s+1,h_d}(x)}$ to $h_d(x)$ satisfies

$$\|q_{s+1,h_d}(x)\| = \|q_{s,h_d}(x)\| + c - 1 = \|q_{t,u_d}(x)\| + c. \quad (20)$$

Note that $(x-1) \nmid q_{s+1,h_d}(x)$. Indeed, (20) implies that $q_{s,h_d}(x)$ is divisible by $x-1$ and by a general property of continued fractions $\gcd(q_{s+1,h_d}(x), q_{s,h_d}(x)) = 1$.

Finally, we have by Lemma 1 that $\frac{(x-1)p_{s+1,h_d}(x^d)}{q_{s+1,h_d}(x^d)}$ is a convergent to $g_d(x)$. Note that

$$\|q_{s+1,h_d}(x^d)\| = d\|q_{t,u_d}(x)\| + dc = \|q_{m+1,g_d}(x)\|,$$

because of (20) and (17). Therefore $p_{m+1,g_d}/q_{m+1,g_d}$ is generated by Lemma 1 from $p_{s+1,h_d}(x)/q_{s+1,h_d}(x)$ and $(x-1) \nmid q_{s+1,h_d}(x)$. The formula (16) is verified for the value $m+1$.

Case 2. $c = 1$. We firstly show that there exists a convergent $p_{s,h_d}(x)/q_{s,h_d}(x)$ to $h_d(x)$ with

$$\|q_{s,h_d}(x)\| = \|q_{t,u_d}(x)\| + 1. \quad (21)$$

Consider the convergent $p_{w,h_d}(x)/q_{w,h_d}(x)$ to $h_d(x)$ where the index $w \in \mathbb{N}$ is the biggest possible such that the degree of q_{w,h_d} does not exceed $\|q_{t,u_d}(x)\|$. Denote by $\tilde{c}$ the rate of approximation of $h_d(x)$ by $p_{w,h_d}(x)/q_{w,h_d}(x)$. It follows from (7) and (8) that

$$\tilde{c} > \|q_{t,u_d}(x)\| - \|q_{w,h_d}(x)\|. \quad (22)$$

Indeed, by (7) we have that $\tilde{c}$ is equal to the degree of the $w+1$-st partial quotient of $h_d(x)$. At the same time, (8) implies that the next convergent to $h_d(x)$, following $p_{w,h_d}(x)/q_{w,h_d}(x)$, has the denominator of degree equal to

$$\|q_{w+1,h_d}(x)\| = \|q_{w,h_d}(x)\| + \tilde{c}. \quad (23)$$

Recall that by the definition of w, the denominator $\|q_{w+1,h_d}(x)\|$ has to be strictly greater than $\|q_{t,u_d}(x)\|$, hence

$$\|q_{w,h_d}(x)\| + \tilde{c} > \|q_{t,u_d}(x)\|$$

and (22) readily follows.

By (22), we have $\tilde{c} \geqslant \|q_{t,u_d}(x)\| - \|q_{w,h_d}(x)\| + 1$. Note that in case if $\|q_{w,h_d}(x)\| + \tilde{c} = \|q_{t,u_d}(x)\| + 1$, then (by recalling (23)) we readily have (21) with $s = w+1$. So, in order to prove (21), it remains us to consider the subcase

$$\tilde{c} \geqslant \|q_{t,u_d}(x)\| - \|q_{w,h_d}(x)\| + 2. \tag{24}$$

We deduce with Lemma 3 (point 2) that $(x-1)p_{w,h_d}(x)/q_{w,h_d}(x)$ is a convergent to $u_d(x)$ and, by taking into account (24),

$$\begin{aligned} \left\| u_d(x) - \frac{(x-1)p_{w,h_d}(x)}{q_{w,h_d}(x)} \right\| &\leqslant -2\|q_{w,h_d}(x)\| - \tilde{c} + 1 \\ &\leqslant -2\|q_{w,h_d}(x)\| - \|q_{t,u_d}(x)\| + \|q_{w,h_d}(x)\| - 1. \end{aligned} \tag{25}$$

The rational function $(x-1)p_{w,h_d}(x)/q_{w,h_d}(x)$ does not coincide with $p_{t,u_d}(x)/q_{t,u_d}(x)$, because otherwise we must have $\|q_{w,h_d}(x)\| = \|q_{t,u_d}(x)\|$ and $(x-1) \mid p_{t,u_d}(x)$. The last condition contradicts (15) (recall that we have (15) for the index m and the implied index $t \in \mathbb{N}$ by the hypothesis of recurrence).

Therefore we have $\|q_{w,h_d}(x)\| < \|q_{t,u_d}(x)\|$. At the same time, it follows from (25) (as well as the general properties of the continued fractions (7) and (8)), that the next convergent of $u_d(x)$ after $\frac{(x-1)p_{w,h_d}(x)}{q_{w,h_d}(x)}$ has the degree of the denominator at least $\|q_{w,h_d}(x)\| + \|q_{t,u_d}(x)\| - \|q_{w,h_d}(x)\| + 1 > \|q_{t,u_d}(x)\|$. So, necessarily $\|q_{t,u_d}\|$ is contained strictly between the degrees of denominators of two consecutive convergents of u_d. Therefore q_{t,u_d} itself can not be a denominator of a convergent to u_d, which is absurd. The last contradiction shows that there exists $s \in \mathbb{N}$ verifying (21), and this completes the proof of (21).

Further, we consider the rational fraction $\frac{p_{s,h_d}(x)}{q_{s,h_d}(x)}$ given by (21). We claim that $(x-1) \nmid q_{s,h_d}(x)$. Indeed, assume this is not the case. Then by Lemma 3 we have that $\frac{p_{s,h_d}(x)}{q_{s,h_d}(x)(x-1)^{-1}}$ is a convergent to $u_d(x)$, moreover its rate of convergence to u_d is at least 2 (because $\frac{(x-1)p_{s,h_d}(x)}{q_{s,h_d}(x)}$ has the rate of convergence at least zero). Then use (21) to compare the degrees of numerators and denominators to find

$$\frac{p_{t,u_d}(x)}{q_{t,u_d}(x)} = \frac{p_{s,h_d}(x)}{q_{s,h_d}(x)(x-1)^{-1}}.$$

This is a contradiction, because we consider the case when the rate of convergence of $\frac{p_{t,u_d}(x)}{q_{t,u_d}(x)}$ to $u_d(x)$ is $c = 1$.

Finally, by Lemma 1, $\frac{(x-1)p_{s,h_d}(x^d)}{q_{s,h_d}(x^d)}$ is a convergent to $g_d(x)$ verifying, in view of (21) and (17),

$$\|q_{s,h_d}(x^d)\| \overset{(21)}{=} d\|q_{t,u_d}(x)\| + d \overset{(17)}{=} \|q_{m+1,g_d}(x)\|.$$

Therefore in both Case 1 and Case 2 we have that $p_{m+1,g_d}(x)/q_{m+1,g_d}(x)$ is generated by Lemma 1 from a convergent $p_{s,h_d}(x)/q_{s,h_d}(x)$, for some $s \in \mathbb{N}$, to $h_d(x)$ such that $(x-1) \nmid q_{s,h_d}(x)$. In other words, (16) is verified for the value $m+1$.

To complete the induction, we need to assume that (16) is verified for an even m, and to check (15) for $m+1$. Arguments here are analogous to those for an odd m, which was treated in the first part of this proof, therefore for the case of an even m we will provide an outline of the arguments and skip some of the details.

Denote by c the rate of approximation of $h_d(x)$ by $p_{s,h_d}(x)/q_{s,h_d}(x)$, where s is taken from (16). By Lemma 1, we have

$$\|q_{m+1,g_d}(x)\| = \|q_{m,g_d}(x)\| + dc - 1 = d\|q_{s,h_d}(x)\| + dc - 1.$$

Also, (8) implies

$$\|q_{s+1,h_d}(x)\| = \|q_{s,h_d}(x)\| + c.$$

We consider two cases.

Case 1. $c \geqslant 2$. By Lemma 3, $\frac{(x-1)p_{s,h_d}(x)}{q_{s,h_d}(x)}$ is a convergent to $u_d(x)$, let us say it is the t-th convergent to $u_d(x)$. Moreover, its numerator and denominator are coprime, since $(x-1) \nmid q_{s,h_d}(x)$. Therefore it approximates $u_d(x)$ with the rate $c-1$, that is

$$\|q_{t+1,u_d}(x)\| = \|q_{t,u_d}(x)\| + c - 1 = \|q_{s,h_d}(x)\| + c - 1.$$

Since $p_{t+1,u_d}(x)$ is coprime with $p_{t,u_d}(x)$ and the latter is divisible by $(x-1)$, we have $(x-1) \nmid p_{t+1,u_d}(x)$. Finally, by Lemma 2, $\frac{p_{t+1,u_d}(x^d)}{(1+x+\ldots+x^{d-1})q_{t+1,u_d}(x^d)}$ is a convergent of $g_d(x)$ and

$$\|(1+x+\ldots+x^{d-1})q_{t+1,u_d}(x^d)\| = d\|q_{s,h_d}(x)\| + dc - 1 = \|g_{m+1,g_d}(x)\|.$$

The last equation confirms (15) for the value $m+1$.

Case 2. $c = 1$. We firstly show that there exists a convergent $p_{t,u_d}(x)/q_{t,u_d}(x)$ to $u_d(x)$ such that

$$||q_{t,u_d}(x)|| = ||q_{s,h_d}(x)||. \tag{26}$$

Consider the convergent $p_{w,u_d}(x)/q_{w,u_d}(x)$ to $u_d(x)$ where the index $w \in \mathbb{N}$ is the biggest possible such that $\|q_{w,u_d}(x)\| < \|q_{s,h_d}(x)\|$. If its rate of approximation $\tilde{c}$ equals $\|q_{s,h_d}(x)\| - \|q_{w,u_d}(x)\|$ then $\|q_{w+1,u_d}(x)\| = \|q_{s,h_d}(x)\|$ and (26) is verified. Otherwise we have

$$\tilde{c} \geqslant \|q_{s,h_d}(x)\| - \|q_{w,u_d}(x)\| + 1 \geqslant 2.$$

We are going to get a contradiction. By Lemma 3, $p_{w,u_d}(x)/(x-1)q_{w,u_d}(x)$ is a convergent of $h_d(x)$. $(x-1)q_{w,u_d}(x)$ does not coincide with $q_{s,h_d}(x)$ because by the inductional assumption (15), $(x-1) \nmid q_{s,h_d}(x)$. Therefore $\|(x-1)q_{w,u_d}(x)\| < \|q_{s,h_d}(x)\|$. At the same time, the next convergent of $h_d(x)$ after $p_{w,u_d}(x)/(x-1)q_{w,u_d}(x)$ has the degree of the denominator at least

$$\|(x-1)q_{w,u_d}(x)\| + \tilde{c} - 1 > \|q_{s,h_d}(x)\|.$$

So, necessarily, $\|q_{s,h_d}(x)\|$ is strictly between the degrees of denominators of two consecutive convergents of $h_d(x)$, which is absurd. This finishes the verification of (26).

Consider the rational fraction $\frac{p_{t,u_d}(x)}{q_{t,u_d}(x)}$ given by (26). We claim that $(x-1) \nmid p_{t,u_d}(x)$. Indeed, otherwise, by Lemma 3 we have that

$$\frac{p_{t,u_d}(x)(x-1)^{-1}}{q_{t,u_d}(x)} = \frac{p_{s,h_d}(x)}{q_{s,h_d}(x)}$$

is a convergent to $h_d(x)$ with the rate of approximation at least 2. This is a contradiction, because we consider the case when the rate of convergence of $\frac{p_{s,h_d}(x)}{q_{s,h_d}(x)}$ is $c = 1$.

Finally, by Lemma 2, $\frac{p_{t,u_d}(x^d)}{(1+x+\ldots+x^{d-1})q_{t,u_d}(x^d)}$ is a convergent to $g_d(x)$ and

$$\|(1+x+\ldots+x^{d-1})q_{t,u_d}(x^d)\| = d\|q_{s,h_d}(x)\| + d - 1 = \|q_{m+1,g_d}(x)\|.$$

Therefore in both Case 1 and Case 2 we have that $p_{m+1,g_d}(x)/q_{m+1,g_d}(x)$ is generated by Lemma 2 from a convergent $p_{t,u_d}(x)/q_{t,u_d}(x)$, for some $t \in \mathbb{N}$ such that $(x-1) \nmid p_{t,u_d}(x)$. This completes the proof by induction.

□

Theorem 1 shows that all convergents of $g_d(x)$ are of a very special form. That form allows us to compute the precise formula for convergents of $g_d(x)$ in case $f_d(x)$ is badly approximable.

3. Badly approximable Laurent series

As in the classical case of real numbers, we say that $f(x) \in \mathbb{Q}((x^{-1}))$ is badly approximable if the degree of every its partial quotient is bounded from above by an absolute constant. Otherwise we say that $f(x)$ is well approximable. In other terms, $f(x)$ is well approximable if its continued fraction expansion contains partial quotients of arbitrary large degree.

We recall one standard result about well (badly) approximable series, which counterpart in $\mathbb{R}$ is classical.

Proposition 1. *Let $f(x) \in \mathbb{Q}((x^{-1}))$, $a(x), b(x) \in \mathbb{Q}[x]\backslash\{0\}$. Then $f(x)$ is well (respectively badly) approximable if and only if $g(x) := \frac{a(x)}{b(x)}f(x)$ is well (respectively badly) approximable.*

Proof. If $f(x)$ is well approximable then $\forall c > 0$ there exists $p(x)/q(x)$ such that

$$\left\| f(x) - \frac{p(x)}{q(x)} \right\| < -2\|q(x)\| - c.$$

Therefore

$$\left\| g(x) - \frac{a(x)p(x)}{b(x)q(x)} \right\| < -2\|b(x)q(x)\| - c + \|a(x)\| + \|b(x)\|.$$

Since $\|a(x)\|$ and $\|b(x)\|$ are fixed and c can be made arbitrarily large, $g(x)$ is also well approximable. The inverse statement can be proved analogously by noting that $f(x) = \frac{b(x)}{a(x)}g(x)$. □

The next lemma shows that the continued fraction of $g_d(x)$ verifies the following very special dichotomy: either the degrees of its partial quotients are unbounded, or, if not, all these degrees are upper bounded by $d-1$.

Lemma 4. *If $g_d(x)$ has at least one partial quotient of degree at least d then $g_d(x)$ is well approximable.*

Proof. Assume that there exists a partial quotient of $g_d(x)$ of degree $c \geqslant d$. Then there exists a convergent $p(x)/q(x)$ to $g_d(x)$ with the rate of approximation equals c:

$$\left|\left| g_d(x) - \frac{p(x)}{q(x)} \right|\right| = -2||q(x)|| - c. \tag{27}$$

The idea is to find another convergent $p^+(x)/q^+(x)$ to $g_d(x)$ which has the rate of approximation $c^+ > c$. If we are able to do this, then we apply this construction recursively to find that there exist convergents of $g_d(x)$ with arbitrarily large rate of approximation which in turn implies that $g_d(x)$ is well approximable.

Substitute x^d in place of x to (27) to get

$$\left|\left|g_d(x^d) - \frac{p(x^d)}{q(x^d)}\right|\right| = -2||q(x^d)|| - dc. \tag{28}$$

Further, substitute the right hand side of (4) in place of $g_d(x^d)$ into (28) and multiply both sides by $x^{d^2-2d}(x-1)$:

$$\left|\left|g_d(x) - \frac{x^{d^2-2d}(x-1)p(x^d)}{q(x^d)}\right|\right| = -2||q(x^d)|| - dc + 1 + d^2 - 2d = -2||q^+(x)|| - c^+,$$

where $q^+(x) = q(x^d)$ and $c^+ = dc + 2d - 1 - d^2$. One can easily check that for $c \geqslant d$ we have $c^+ > c$. This finishes the proof of the lemma. $\square$

Lemma 5. *If $f_d(x)$ is badly approximable then all the partial quotients of $h_d(x)$ and $u_d(x)$ are linear.*

Proof. Assume that $h_d(x)$ has a partial quotient of degree at least 2. In this case there exists a convergent $p(x)/q(x)$ to $h_d(x)$ with the rate of approximation at least 2. Then Lemma 1 gives

$$\left|\left|g_d(x) - \frac{(x-1)p(x^d)}{q(x^d)}\right|\right| \leqslant -2||q(x^d)|| - 2d + 1.$$

Since $2d - 1 \geqslant d$, we have by Lemma 4 that $g_d(x)$ is well approximable. Then Proposition 1 implies that $f_d(x)$ is well approximable as well.

Similar considerations work in the case of $u_d(x)$. If there exists a convergent $p(x)/q(x)$ of $u_d(x)$ with the rate of approximation at least 2 then we use Lemma 2 to get

$$\left|\left|g_d(x) - \frac{p^*(x)}{q^*(x)}\right|\right| \leqslant -2||q^*(x)|| - d - 1.$$

Again, we have $d + 1 > d$ and therefore $g_d(x)$ together with $f_d(x)$ are well approximable. $\square$

If we know that $f_d(x)$ is badly approximable then with the help of Lemma 5 we can find the recurrent formula for the convergents of $g_d(x)$. The following theorem generalizes Proposition 3.2 from [2].

Theorem 2. *If $f_d(x)$ is badly approximable then the monic denominators $q_{n,g_d}(x)$ of the convergents of $g_d(x)$ satisfy the following recurrent equations*

$$q_{1,g_d}(x) = x^{d-1} + \cdots + x + 1; \quad q_{2,g_d}(x) = x^d + 1;$$
$$q_{2k+1,g_d}(x) = (x^{d-1} + \cdots + x + 1)q_{2k,g_d}(x) + \beta_{2k+1}q_{2k-1,g_d}(x); \quad k \in \mathbb{N} \tag{29}$$
$$q_{2k+2,g_d}(x) = (x-1)q_{2k+1,g_d}(x) + \beta_{2k+2}q_{2k,g_d}(x), \tag{30}$$

where β_k are some rational numbers.

In other words Theorem 2 almost completely describes the continued fraction expansion of badly approximable functions $g_d(x)$, up to determination of rational parameters β_k.

Proof of Theorem 2. We assume that $f_d(x)$ is badly approximable, so by Lemma 5 we have that the partial quotients $a_{n,h_d}(x)$ of $h_d(x)$ are linear which in turn implies that the nth convergent $\frac{p_{n,h_d}(x)}{q_{n,h_d}(x)}$ to $h_d(x)$ has denominator of degree n with the rate of approximation 1. Hence by Lemma 1,

$$\frac{(x-1)p_{n,h_d}(x^d)}{q_{n,h_d}(x^d)} \tag{31}$$

is a convergent to $g_d(x)$ with the rate of convergence at least $d-1$. The polynomial $x-1$ does not divide $q_{n,h_d}(x)$ because otherwise $\frac{p_{n,h_d}(x^d)}{q_{n,h_d}(x^d)(x-1)^{-1}}$ is a convergent to $g_d(x)$ with the rate of approximation at least d and therefore by Lemma 4, $g_d(x)$ is well approximable, which is not true.

Also, Lemma 5 implies that the n-th convergent $\frac{p_{n,u_d}(x)}{q_{n,u_d}(x)}$ has denominator of degree n and the rate of convergence 1. Then we infer with Lemma 2 that

$$\frac{p^*_{n,u_d}(x)}{q^*_{n,u_d}(x)} = \frac{p_{n,u_d}(x^d)}{(x^{d-1}+\cdots+x+1)q_{n,u_d}(x^d)} \tag{32}$$

is a convergent of $g_d(x)$ with the rate of convergence 1. As before, $x-1$ does not divide $p_{n,u_d}(x)$ because otherwise

$$\frac{(x^{d-1}+\ldots+1)^{-1}p_{n,u_d}(x^d)}{q_{n,u_d}(x^d)}$$

is the convergent of $g_d(x)$ with the rate of approximation at least d which is impossible. Therefore, $p_{n,u_d}(x^d)$ and $(1+x+\ldots+x^{d-1})q_{n,u_d}(x^d)$ are coprime.

So for each $k \in \mathbb{Z}_{\geqslant 0}$ there exists a convergent of $g_d(x)$ of the form (31), with the denominator of degree kd, and another one of the form (32), with the denominator of degree $kd+d-1$. By Theorem 1 no other convergents of $g_d(x)$ exist. This allows us to construct $q_{1,g_d}(x)$ and $q_{2,g_d}(x)$:

$$q_{1,g_d}(x) = (x^{d-1}+\ldots+x+1)q_{0,u_d}(x) = x^{d-1}+\ldots+x+1;$$

$$q_{2,g_d}(x) = g_{1,h_d}(x^d) = x^d+1.$$

The second line above readily follows from the fact that $\frac{1}{x+1}$ is the first convergent of $h_d(x)$.

For the general denominators $q_{n,g_d}(x)$ we have the following formula

$$q_{2k+1,g_d}(x) = (x^{d-1}+\cdots+x+1)q_{k,u_d}(x^d) \quad \text{and} \quad q_{2k,g_d}(x) = q_{k,h_d}(x^d).$$

Using the formula (9) for the monic convergents of $g_d(x)$ we have

$$q_{2k+1,g_d}(x) = a_{2k+1}(x)q_{2k,g_d}(x) + \beta_{2k+1}q_{2k-1,g_d}(x) \tag{33}$$

where $a_{2k+1} \in \mathbb{Q}[x]$ is monic and $\beta_{2k+1} \in \mathbb{Q}$. By comparing the degrees of both sides of this equation we find $\|a_{2k+1}(x)\| = d-1$. We also have $x^{d-1}+\ldots+x+1 \mid q_{2k+1,g_d}(x), q_{2k-1,g_d}(x)$ and

$$\gcd(x^{d-1}+\cdots+x+1, q_{2k,g_d}(x)) \mid \gcd(q_{2k-1,g_d}(x), q_{2k,g_d}(x)) = 1.$$

Therefore $x^{d-1}+\cdots+x+1 \mid a_{2k+1}(x)$. Since the degrees of these two polynomials coincide and both of them are monic we conclude $a_{2k+1}(x) = x^{d-1}+\cdots+x+1$.

Next,

$$q_{2k+2,g_d}(x) = a_{2k+2}(x)q_{2k+1,g_d}(x) + \beta_{2k+2}q_{2k,g_d}(x) \tag{34}$$

where $a_{2k+2}(x) \in \mathbb{Q}[x]$ is monic and $\beta_{2k+2} \in \mathbb{Q}$. Degree comparing gives us that $a_{2k+2}(x)$ is linear. Also we have

$$a_{2k+2}(x) \cdot (x^{d-1} + \cdots + x + 1) q_{k,u_d}(x^d) = q_{k+1,h_d}(x^d) - q_{k,h_d}(x^d)$$

Therefore $a_{2k+2}(x) \cdot (x^{d-1} + \cdots + x + 1)$ is a polynomial in x^d. This is only possible if $a_{2k+2} = x - 1$. □

Remark. The formulae for $q_{1,g_d}(x)$ and $q_{2,g_d}(x)$ do not require $f_d(x)$ to be badly approximable. So this part of Theorem 2 is satisfied for all values d.

Unfortunately, Theorem 2 does not cover too many cases of functions $g_d(x)$. In fact $f_d(x)$ is badly approximable only for $d = 2$ and $d = 3$. For $d = 2$ it is shown in [2] and for $d = 3$ it is shown in [1]. It is not too difficult to show that $f_d(x)$ is well approximable for $d \geqslant 4$, however for the sake of completeness we provide the proof here.

Proposition 2. *Let $d \in \mathbb{N}$, $d \geqslant 4$. The function $f_d(x)$ is well approximable.*

Proof. The finite products $r_k(x) = \prod_{t=0}^{k}(1 - x^{-d^t})$ provide approximations to $f_d(x)$ good enough to conclude that it is well approximable. Indeed,

$$||f_d(x) - r_k(x)|| = \left|\left|\prod_{t=0}^{k}(1 - x^{-d^t}) \cdot \left(\prod_{t=1}^{\infty}(1 - x^{-d^{t+k}}) - 1\right)\right|\right| = -d^{k+1}.$$

$r_k(x)$ is a rational function with denominator

$$x^{\sum_{t=0}^{k} d^t} = x^{\frac{d^{k+1}-1}{d-1}}.$$

Finally we have that for $d \geqslant 4$, $d^{k+1} - 2\frac{d^{k+1}-1}{d-1} \to \infty$ as k tends to infinity. Therefore the rational functions $r_k(x)$ provide approximations to $f_d(x)$ with an arbitrarily large rate. This completes the proof of the proposition. □

4. Computing the values of β_k

To find the precise formula for the continued fraction of $g_d(x)$ we still need to compute the values of the parameters β_k in (29) and (30). For $d = 2$ this has already been done in [2].

Theorem BZ1 . *In the case $d = 2$ the values β_k in* (29) *and* (30) *can be computed by the following recurrent formulae*

$$\begin{aligned}
&\beta_3 = -1, \quad \beta_4 = 1,\\
&\beta_{2k+1} = -\frac{\beta_{k+1}}{\beta_{2k}},\\
&\beta_{2k+2} = 1 + (-1)^k - \beta_{2k+1} \quad \textit{for } k \geqslant 2.
\end{aligned}$$

In the case $d = 3$ the formulae for β_k are more complicated. In this section we will get several equations between different values of the sequence β_k and will explain how to get other equations which will finally enable us to provide the complete recurrent formula for β_k.

From now on we will most often speak about the convergents of $g_3(x)$ therefore for convenience instead of $p_{n,g_3}(x)$ and $q_{n,g_3}(x)$ we will just write $p_n(x)$ and $q_n(x)$ respectively.

Lemma 6. *For all $k \in \mathbb{N}$ the convergents to $g_3(x)$ satisfy the following formula:*

$$q_{6k}(x) = q_{2k}(x^3); \quad p_{6k}(x) = x^3(x-1)p_{2k}(x^3).$$

Proof. Note that by Theorem 2 the degrees of $q_k(x)$ exhaust all positive integers congruent to 0 or 2 modulo 3. Therefore $\|q_{2k}(x)\| = 3k$ and $\|q_{2k+1}(x)\| = 3k+2$, $k \in \mathbb{N}$.

From Theorem 2 we also have that

$$\left\| g_3(x) - \frac{p_{2k}(x)}{q_{2k}(x)} \right\| = -2||q_{2k}(x)|| - ||a_{2k+1}(x)|| = -2||q_{2k}(x)|| - 2. \tag{35}$$

Recall that $g_d(x)$ satisfies the functional equation (4). In particular, $g_3(x^3) = \frac{g_3(x)}{x^3(x-1)}$. Then the equation (35) with x^3 substituted in place of x gives us

$$\left\| g_3(x^3) - \frac{p_{2k}(x^3)}{q_{2k}(x^3)} \right\| = \left\| \frac{g_3(x)}{x^3(x-1)} - \frac{p_{2k}(x^3)}{q_{2k}(x^3)} \right\| = -6\|q_{2k}(x)\| - 6.$$

Multiply both sides of this equation by $x^3(x-1)$ to get

$$\left\| g_3(x) - \frac{p_{2k}(x^3)x^3(x-1)}{q_{2k}(x^3)} \right\| = -2||q_{2k}(x^3)|| - 2.$$

This shows that $\frac{p_{2k}(x^3)x^3(x-1)}{q_{2k}(x^3)}$ is a convergent to $g_d(x)$. Note that the fraction $\frac{p_{2k}(x^3)x^3(x-1)}{q_{2k}(x^3)}$ is irreducible, because in the opposite case the rate of its convergence to $g_3(x)$ would be at least 3, hence Lemma 4 would have implied that $g_3(x)$ is not badly approximable, which is not the case [1]. Finally, by calculating the degree of denominator of this convergent we conclude the proof. □

Since $q_0(x) = 1$, formulae for $q_1(x)$ and $q_2(x)$ allow us to conclude that $\beta_2 = 2$.

Proposition 3. *For each integer $k \geqslant 0$ we have*

$$\beta_{6k+6}\beta_{6k+4}\beta_{6k+2} = \beta_{2k+2}. \tag{36}$$

Proof. By (30) we have

$$q_{2k+2}(x) = (x-1)q_{2k+1}(x) + \beta_{2k+2}q_{2k}(x).$$

We substitute x^3 in place of x, use Lemma 6 and consider the resulting equation modulo $x-1$ to get

$$q_{6k+6}(x) \equiv \beta_{2k+2}q_{6k}(x) \pmod{x-1}. \tag{37}$$

Consequent usage of formula (30) for $q_{6k+6}(x)$ down to $q_{6k+2}(x)$ leads to

$$\begin{aligned} q_{6k+6}(x) &= (x-1)q_{6k+5}(x) + \beta_{6k+6}q_{6k+4}(x) \equiv \beta_{6k+6}q_{6k+4}(x) \\ &= \beta_{6k+6}\beta_{6k+4}q_{6k+2}(x) \equiv \beta_{6k+6}\beta_{6k+4}\beta_{6k+2}q_{6k}(x) \pmod{x-1}. \end{aligned}$$

Hence we get $\beta_{2k+2}q_{6k}(x) \equiv \beta_{6k+6}\beta_{6k+4}\beta_{6k+2}q_{6k}(x) \pmod{x-1}$. Finally from Lemma 6, $\gcd(x-1, q_{6k}(x)) \mid \gcd(p_{6k}(x), q_{6k}(x)) = 1$, therefore we can divide the congruence by $q_{6k}(x)$. This finishes the proof of the proposition. □

More relations between values of β can be derived by considering coefficients with the highest degrees of x in $q_k(x)$. More exactly, write the polynomials $q_k(x)$ in the following form (recall Theorem 1)

$$\begin{aligned} &q_{2k}(x) = q_{k,h_d}(x^3) = x^{3k} + a_{2k}x^{3k-3} + b_{2k}x^{3k-6} + \ldots; \\ &q_{2k+1}(x) = (x^2+x+1)q_{k,u_d}(x^3) = (x^2+x+1)(x^{3k} + a_{2k+1}x^{3k-3} + b_{2k+1}x^{3k-6} + \ldots). \end{aligned}$$

Proposition 4. *Coefficients a_k and β_k, $k \in \mathbb{N}$, are related by the following equations*

$$a_{6k} = 0, \quad a_{2k} - a_{2k-1} = \beta_{2k} - 1, \quad a_{2k+1} - a_{2k} = \beta_{2k+1}.$$

In particular, these equations imply

$$\sum_{i=1}^{6} \beta_{6k+i} = 3. \tag{38}$$

Proof. Firstly, by Lemma 6 and Theorem 1, $q_{6k}(x) = q_{2k}(x^3) = q_{k,h_d}(x^9)$, therefore the coefficient at x^{9k-3} in $q_{6k}(x)$ is zero.

Secondly, we compare the coefficients at several leading degrees of x in Equation (34).

$$x^{3k} + a_{2k}x^{3k-3} + \ldots = q_{2k}(x) = (x^3 - 1)(x^{3k-3} + a_{2k-1}x^{3k-6} + \ldots) + \beta_{2k}(x^{3k-3} + \ldots).$$

Comparison of the coefficients at x^{3k-3} gives us the equation $a_{2k} - a_{2k-1} = \beta_{2k} - 1$.

Thirdly, for $q_{2k+1}(x)$ we have, by using Equation (33),

$$(x^2 + x + 1)(x^{3k} + a_{2k+1}x^{3k-3} + \ldots) = q_{2k+1}(x) = (x^2 + x + 1)(x^{3k} + a_{2k}x^{3k-3} + \ldots)$$

$$+\beta_{2k+1}(x^2 + x + 1)(x^{3k-3} + \ldots)$$

Then dividing by $x^2 + x + 1$ and comparing the coefficients at x^{3k-3} gives us $a_{2k+1} - a_{2k} = \beta_{2k+1}$.

Finally we sum up six equations of the above form to get

$$0 = a_{6k+6} - a_{6k} = \sum_{i=1}^{6}(a_{6k+i} - a_{6k+i-1}) = \sum_{i=1}^{6} \beta_{6k+i} - 3.$$

□

One can compare the coefficients at the preceding powers of x in the formulae (29) and (30) for $q_k(x)$ to get the relations between b_k, a_k and β_k. The result is presented in Proposition 5 below. Its proof does not involve any new idea in addition to those from Proposition 4. Therefore we leave this result without proof.

Proposition 5. *Coefficients b_k, a_k and β_k are related by the following equations*

$$b_{6k} = 0, \quad b_{2k} - b_{2k-1} = \beta_{2k}a_{2k-2} - a_{2k-1}, \quad b_{2k+1} - b_{2k} = \beta_{2k+1}a_{2k-1}.$$

In particular, these equations imply

$$\sum_{\substack{1 \leqslant i,j \leqslant 6 \\ j-i>1}} \beta_{6k+i}\beta_{6k+j} = 3 + \beta_{6k}\beta_{6k+1}. \tag{39}$$

By considering more coefficients we can get more equations relating values $\beta_{6k+1}, \ldots, \beta_{6k+6}$ with the previous values of β_i, $i \leq 6k$. However they become overwhelmingly complicated. Perhaps one can use some tricks similar to those in Proposition 3 to find simpler relations between different values of β_k. It would be very interesting to discover such relations.

5. Mahler numbers

In this section we will consider the Mahler numbers $f_d(a)$, where $a \geqslant 2$ is an integer and $f_d(x)$ is the Laurent series defined by (1). It appears that some of approximation properties of these numbers can be derived from the study of the continued fraction of the function $f_d(x)$. In this section, we investigate the following problem:

Problem A. *Given $a, d \in \mathbb{Z}, a, d \geqslant 2$, determine whether $f_d(a)$ is badly approximable.*

5.1. The case $d \geqslant 4$ Problem A is relatively easy in the case $d \geqslant 4$. For this case the answer follows from a simple Proposition 6 below.

Recall that the exponent of irrationality of $x \in \mathbb{R}$ is defined to be the supremum of all positive real numbers τ such that the inequality $\left|x - \frac{p}{q}\right| < q^{-\tau}$ has infinitely many integer solutions p, q with $q \neq 0$. It is easy to verify with the definitions that the irrationality exponent of a badly approximable number necessarily equals two.

Proposition 6. *Let $d \in \mathbb{N}$, $d \geqslant 4$ and let $a \in \mathbb{N}$, $a \geqslant 2$. Then the number $f_d(a)$ is well approximable. Moreover, the exponent of irrationality of $f_d(a)$ is at least $d - 1$.*

Proof. The proof is very much similar to the proof of Proposition 2. With the reference to the notation of the proof of Proposition 2, note that the coefficient with the highest degree in the series $f_d(x) - r_k(x)$ is 1. So substituting a in place of x we find

$$|f_d(a) - r_k(a)| \leqslant \sum_{t=d^{k+1}}^{\infty} a^{-t} \leqslant \frac{2}{a^{d^{k+1}}},$$

whilst the denominator q_k of the rational fraction $r_k(a)$ is at most $a^{\frac{d^{k+1}-1}{d-1}}$. The estimate $q_k^{-(d-1)} = a^{-(d^{k+1}-1)} \geqslant 2a^{-d^{k+1}}$ proves that $f_d(a)$ has exponent of irrationality at least $d - 1 \geqslant 3$ and so $f_d(a)$ is not badly approximable. □

Proposition 6 immediately tells us that for $d \geqslant 4$, $f_d(a)$ are not badly approximable for any integer $a \geqslant 2$. So it remains to study Problem A for the cases $d = 2$ and $d = 3$. In these two cases the exponent of irrationality of $f_d(a)$, $a \in \mathbb{N}$, $a \geqslant 2$, $d = 2, 3$, is 2. For $d = 2$ this is proved in [3] and for $d = 3$ it follows from [4, Theorem 2.5]. Therefore the solution to Problem A in the cases $d = 2, 3$ needs more subtle considerations.

5.2. The case $d = 2$ The case $d = 2$ is studied in [2, Theorem 5.1] where the following theorem is proved. Recall that $a \,||\, b$ means that a divides b but a^2 does not.

Theorem BZ2 . *Let $p_t(x)/q_t(x)$ be the convergents of the series $g_2(x)$. Assume that there exist positive integers n, t, p such that*

1. *p is a prime number and $p \,||\, a^{2^n} - 1$;*
2. *2 is a primitive root modulo p^2.*
3. *$p \,||\, q_t(1)$;*
4. *$q_t'(1) \not\equiv 0 \pmod{p}$.*

Then $f_2(a)$ is not badly approximable.

This theorem allows us to show that $f_2(a)$ is not badly approximable for many integer values of a. However, as explained in [2], there are some integer values a that can not be covered by Theorem BZ2. The smallest uncovered integer is $a = 15$.

Here we provide a stronger version of Theorem BZ2, which covers the case $a = 15$ as well as many other extra values of a. For this stronger statement, Theorem 3, we did not detect any constraints which prevent Theorem 3 to be applied to any integer $a \geqslant 2$. So we believe that this theorem allows to prove that $f_2(a)$ is not badly approximable for all $a \geqslant 2$. On the other hand the conditions in Theorem 3 depend on several parameters and we do not know a general procedure which provides these parameters for a generic a.

In what follows, we denote by $\Gamma(a, p^k)$, $k \in \mathbb{N}$, $a \in \mathbb{Z}$, the multiplicative subgroup of $\mathbb{Z}/p^k\mathbb{Z}$ generated by a.

Theorem 3. *Let $p_t(x)/q_t(x)$ be the convergents of the series $g_2(x)$. Assume that there exist positive integers n_0, t, p such that*

1. *p is an odd prime number and $p \,||\, a^{2^{n_0}} - 1$;*
2. *$|\Gamma(2, p^2)| = p|\Gamma(2, p)|$;*
3. *$q_t(a^{2^{n_0}}) \equiv 0 \pmod{p^2}$;*
4. *$q_t'(1) \not\equiv 0 \pmod{p}$.*

Then $f_2(a)$ is not badly approximable.

Remark 1. Condition 2 of Theorem 3 is satisfied for the most of primes we know of. More precisely, the only primes which do not satisfy this condition are the so called Weiferich primes, i.e. the primes p such that p^2 divides $2^{p-1} - 1$. Indeed, if p is a non-Weiferich prime then property 2 of Theorem 3 follows from Lemma 7 below. Weiferich primes were rigorously studied. Currently only two of them are known: 1093 and 3511, and no more Weiferich primes exist [5] below 3×10^{15}.

Lemma 7. *Let p be an odd prime and assume that*

$$2^{p-1} \not\equiv 1 \pmod{p^2}. \tag{40}$$

Then $|\Gamma(2, p^2)| = p|\Gamma(2, p)|$.

Proof. The multiplicative subgroup H of $\mathbb{Z}/p^2\mathbb{Z}$ of all elements $a \equiv 1 \pmod{p}$ has order p. Because of the small Fermat's theorem, $\Gamma(2^{p-1}, p^2) < H$. Hence Assumption (40) implies that $\left|\Gamma(2^{p-1}, p^2)\right| = p$.

Note that, by definition, $\Gamma(2^{p-1}, p^2) \subset \Gamma(2, p^2)$, hence $p \mid \left|\Gamma(2, p^2)\right|$. At the same time, by a reduction modulo p the group $\Gamma(2, p^2)$ is mapped onto the group $\Gamma(2, p)$, hence $|\Gamma(2, p)|$ divides $\left|\Gamma(2, p^2)\right|$.

By the small Fermat's theorem $|\Gamma(2, p)| \mid p - 1$, so $\gcd(|\Gamma(2, p)|, p) = 1$. We readily infer that $p\,|\Gamma(2, p)|$ divides $\left|\Gamma(2, p^2)\right|$ and so $\left|\Gamma(2, p^2)\right| \geqslant p\,|\Gamma(2, p)|$.

On the other hand, the reduction modulo p sends $\Gamma(2, p^2)$ onto $\Gamma(2, p)$ and under this map each element in $\Gamma(2, p)$ has at most p preimages. We conclude that $\left|\Gamma(2, p^2)\right| = p\,|\Gamma(2, p)|$ and this completes the proof of the lemma. □

The big part of the proof of Theorem 3 is the same as for Theorem BZ2. Therefore it will be just briefly outlined here and we refer the reader to [2] for the details. In this paper we mainly focus on the part of the proof which is specific to Theorem 3.

In the proof of Theorem 3 we will need the following lemma.

Lemma 8. *Let $a \in \mathbb{Z} \setminus \{0\}$, and let p be an odd prime number. If*

$$|\Gamma(a, p^2)| = p\,|\Gamma(a, p)|, \tag{41}$$

then for each $m \in \mathbb{N}$,

$$|\Gamma(a, p^{m+1})| = p^m |\Gamma(a, p)|.$$

Proof. We will show that for any $m \geqslant 2$,

$$|\Gamma(a, p^{m+1})| = p|\Gamma(a, p^m)|, \tag{42}$$

then the lemma readily follows by induction.

Fix $m \in \mathbb{N}$. To simplify the notation, we write

$$h := |\Gamma(a, p^{m+1})|.$$

Then, $a^h \equiv 1 \bmod p^{m+1}$. By reducing modulo p^m we get $a^h \equiv 1 \bmod p^m$, hence

$$h = s|\Gamma(a, p^m)| \tag{43}$$

for some $s \in \mathbb{N}$. At the same time, we have

$$a^{|\Gamma(a,p^m)|} \equiv 1 + tp^m \bmod p^{m+1}.$$

By raising both sides of this congruence to the power s and applying (43), we find

$$1 \equiv (1 + tp^m)^s \bmod p^{m+1}. \tag{44}$$

By expanding brackets on the right hand side of (44), we find

$$1 \equiv 1 + stp^m \bmod p^{m+1},$$

hence p divides either s or t (or both).

If p divides s then (43) implies $h \geqslant p|\Gamma(a, p^m)|$. On the other hand the reduction modulo p^m sends $\Gamma(a, p^{m+1})$ onto $\Gamma(a, p^m)$ and under this map each element in $\Gamma(a, p^m)$ has at most p preimages. Therefore

$$h = p|\Gamma(a, p^m)|,$$

and this is precisely (42).

Now suppose that p divides t. In this case we have

$$h = |\Gamma(a, p^{m+1})| = |\Gamma(a, p^m)|. \tag{45}$$

Consider the polynomial congruence

$$f(x) \equiv 0 \bmod p^{m+1}, \tag{46}$$

where $f(x) = x^{|\Gamma(a,p^m)|} - 1$.

Because of (45) the solutions to the congruence (46) are precisely the elements of $\Gamma(a, p^{m+1})$ and these solutions are congruent modulo p^{m+1} to

$$1, a, \ldots, a^{|\Gamma(a,p^m)|-1}. \tag{47}$$

Note that, as the representatives of $\Gamma(a, p^m)$, the elements of the list (47) are pairwise distinct modulo p^m.

At the same time, we easily calculate

$$f'(x) = |\Gamma(a,p^m)|x^{|\Gamma(a,p^m)|-1}$$

The assumption (41) readily implies that $|\Gamma(a,p^m)|$ is divisible by p for any $m \geqslant 2$, so for any integer value x we have

$$f'(x) \equiv 0 \bmod p.$$

Then Hensel's lemma implies that for any integer u that verifies (46) and any $\theta = 0,\dots,p-1$ the integer $u + \theta p^m$ is also a solution to (46).

For any $\theta = 0,\dots,p-1$ the number $a + \theta p^m$ is not congruent modulo p^{m+1} to any element of the list (47), because all representatives there are distinct modulo p^m. However it contradicts the fact that all the residues modulo p^{m+1} verifying (46) are given in (47). This contradiction proves the lemma. □

Proof of Theorem 3. Firstly, since $f_2(a)$ and $g_2(a)$ are rationally dependent, to prove the theorem it is enough to show that $g_2(a)$ is not badly approximable.

Secondly, for each convergent $p(x)/q(x)$ of $g_2(x)$ we provide the series of convergents $\tilde{p}_n(x)/\tilde{q}_n(x)$ of $g_2(x)$ such that

$$\tilde{p}_n(x) = \prod_{t=0}^{n-1}(x^{2^t} - 1)p(x^{2^n}); \quad \tilde{q}_n(x) = q(x^{2^n}). \tag{48}$$

By multiplying both $p(x)$ and $q(x)$ by some integer constant, we can always guarantee that $p(x)$, $q(x)$ and in turn $\tilde{p}_n(x), \tilde{q}_n(x)$ are all in $\mathbb{Z}[x]$. Moreover (see [2, Lemma 4.3]), values $\tilde{p}_n(a)/\tilde{q}_n(a)$ provide very good (but probably not the best) approximations to $g_2(a)$. Namely, there exists a constant C which does not depend on n, such that

$$\left|g_2(a) - \frac{\tilde{p}_n(a)}{\tilde{q}_n(a)}\right| \leqslant \frac{C}{(\tilde{q}_n(a))^2}.$$

Hence, to show that $g_2(a)$ is not badly approximable, it is sufficient to find the initial convergent $p(x)/q(x)$ and $n \in \mathbb{N}$ such that $\tilde{p}_n(a)$ and $\tilde{q}_n(a)$ have an arbitrarily large common integer factor. By (48) and the first condition of the theorem we already have that $p^{n-n_0} \mid \tilde{p}_n(a)$. So we only need to show that the sequence $\tilde{q}_n(a)$, $n \in \mathbb{N}$, contains elements which are divisible by arbitrarily large powers of p.

For the initial convergent we choose $p_t(x)/q_t(x)$. The aim now is to show that for each $m \in \mathbb{N}$ one can find $n \in \mathbb{N}$ such that $q_t(a^{2^n})$ is divisible by p^m. Conditions 3 and 4 and Hensel's lemma imply that the equation $q_t(x) = 0$ has a solution $x \in \mathbb{Z}_p$ such that

$$x \equiv a^{2^{n_0}} \pmod{p^2}. \tag{49}$$

In particular, $x \equiv 1 \pmod p$. We want to show that for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that

$$a^{2^n} \equiv x \pmod{p^m}, \tag{50}$$

which will immediately imply that $p^m \mid \tilde{q}_n(a)$.

For every $m \in \mathbb{N}$ the multiplicative group $\mathcal{R}^*_{p^m} := (\mathbb{Z}/p^m\mathbb{Z})^*$ of residues modulo p^m has the order $(p-1)p^{m-1}$. As the element $a^{2^{n_0}}$ is congruent to one modulo p, it lies in the kernel of the canonical projection $\mathcal{R}^*_{p^m} \to \mathcal{R}^*_p$. The multiplicative group $\mathcal{R}^*_p$ of residues modulo p has the order $p-1$, so the residue $a^{2^{n_0}}$ has the order p^l in $\mathcal{R}^*_{p^m}$, for some $l \le m-1$. If the value l is strictly smaller than $m-1$, then we necessarily have $a^{2^{n_0}} \equiv 1 \bmod p^2$, which contradicts the first condition of the theorem, hence the multiplicative order of $a^{2^{n_0}}$ modulo p^m is exactly p^{m-1} and thus the set of residues $\{a^{2^{n_0}\cdot s} \bmod p^m : s \in \mathbb{N}, \gcd(s,p) = 1\}$ coincides with the set of residues modulo p^m congruent to 1 modulo p but not congruent to 1 modulo p^2. So, there is an $s \in \mathbb{N}$ such that

$$a^{2^{n_0}\cdot s} \equiv x \bmod p^m \tag{51}$$

and $s \not\equiv 0 \bmod p$. Moreover, because of the congruence (49) we have

$$s \equiv 1 \bmod p \tag{52}$$

The congruence (52) implies that the residue of $\theta = 2^{n_0}s$ modulo p lies in $\Gamma(2,p)$.

Because of Condition 2 we can apply Lemma 8. It implies that for any $m \in \mathbb{N}$ the group $\Gamma(2,p^m)$ coincides with the full preimage of $\Gamma(2,p)$ under the canonical projection $\mathcal{R}^*_{p^m} \to \mathcal{R}^*_p$. In particular, there exists $t_m \in \mathbb{N}$ such that

$$2^{t_m} \equiv 2^{n_0}s \bmod p^{m-1}.$$

For this t_m, we have

$$2^{2^{t_m}} \equiv 2^{2^{n_0}s} \bmod p^m \tag{53}$$

(recall that $2^{2^{n_0}s}$ has order p^{m-1} in $\mathcal{R}^*_{p^m}$, because $2^{2^{n_0}}$ has order p^{m-1} and s is coprime to p). Taking (51) and (53) together we conclude

$$2^{2^{t_m}} \equiv x \bmod p^m,$$

which is precisely (50). This finishes the proof.

□

Theorem 3 provides an algorithm for showing that $f_n(a)$ is not badly approximable for a given a. We firstly find p such that Conditions 1 and 2 of the theorem are satisfied. Then we try to find the denominator of a convergent $q_t(x)$ which satisfies Conditions 3 and 4.

Let's use Theorem 3 for some small prime values p. For $p = 3$ Condition 1 is satisfied for all a except $a \equiv 0 \pmod 3$ and $a \equiv \pm 1 \pmod 9$. Condition 2 can be easily checked. With help of Theorem BZ1 we find

$$q_9(x) = (x+1)(x^8 - x^6 + x^2 + 2)$$

which satisfies $q_9(7) \equiv 0 \pmod 9$ and $q_9'(1) \not\equiv 0 \pmod 3$. It is not difficult to show that for $a \not\equiv 0, 3, 6, \pm 1 \pmod 9$ one can always find n_0 such that $a^{2^{n_0}} \equiv 7 \pmod 9$. Therefore Theorem 3 states that $f_2(a)$ is not badly approximable for all $a \not\equiv 0, 3, 6, \pm 1 \pmod 9$.

Remark 2. In [2] we were too brave, stating that $f_2(a)$ is not badly approximable for all a coprime with 3. Unfortunately we forgot about the case $a \equiv \pm 1 \pmod 9$ which violates the first condition of Theorem BZ2.

Using $p = 5$ and $q_{11}(x)$ we can show that $f(a)$ is not badly approximable for all $a \in \mathbb{N}$ such that $a \not\equiv 0 \pmod 5$ and $a \not\equiv \pm 1, \pm 7 \pmod{25}$.

We conducted this procedure (using a small computer program) for some other small primes p. The results are presented in the following table, where the column $x \bmod p^2$ specifies the solution to the congruence $q_t(x) \equiv 1 \pmod{p^2}$.

p	$q_t(x)$	$x \bmod p^2$	values a which pass Conditions 1 and 3
3	$q_9(x)$	7	$a \equiv \pm 2, \pm 4 \pmod 9$
5	$q_{11}(x)$	11	$a \not\equiv 0 \pmod 5, a \not\equiv \pm 1, \pm 7 \pmod{25}$
7	$q_{41}(x)$ $q_{187}(x)$	15 43	$a \equiv \pm 1 \pmod 7, a \not\equiv \pm 1 \pmod{49}$
11	$q_{43}(x)$	34	$a \equiv \pm 1 \pmod{11}, a \not\equiv \pm 1 \pmod{11^2}$
13	$q_{33}(x)$	14	$a \equiv \pm 1, \pm 5 \pmod{13}, a \not\equiv \pm 1, \pm 70 \pmod{13^2}$
17	$q_{13}(x)$ $q_{157}(x)$	69 86	$a^{16} \equiv 1 \pmod{17}$, $a^{16} \not\equiv 1 \pmod{17^2}$
19	$q_{19}(x)$	210	$a \equiv \pm 1 \pmod{19}, a \not\equiv \pm 1 \pmod{19^2}$
23	$q_{79}(x)$ $q_{187}(x)$	277 254	$a \equiv \pm 1 \pmod{23}, a \not\equiv \pm 1 \pmod{23^2}$
29	$q_{35}(x)$	117	$a \equiv \pm 1, \pm 12 \pmod{29}, a \not\equiv \pm 1, \pm 41 \pmod{29^2}$
31	$q_{29}(x)$	156	$a \equiv \pm 156, \pm 280, \pm 311, \pm 340, \pm 402 \pmod{31^2}$
37	$q_{21}(x)$	408	$a \equiv \pm 1, \pm 6 \pmod{37}, a \not\equiv \pm 1, \pm 117 \pmod{37^2}$

There are only two values of a below 100 which are not covered by this table: $a = 26$ and $a = 82$.

For $a = 26$ we can take $p = 677 = a^2 + 1$. Then Conditions 1 and 2 are satisfied. Further, Conditions 3 and 4 are satisfied for $q_{319}(x)$ which has root $x \equiv 291111 \equiv 26^{2^{204}} \pmod{677^2}$ in $\mathbb{Q}_{677}$ and so Theorem 3 implies that $f_2(26)$ is not badly approximable.

For $a = 82$ we can take $p = 83 = a + 1$. Then Conditions 1 and 2 are satisfied. Further, Conditions 3 and 4 are satisfied for $q_{91}(x)$ which has root $x \equiv 5479 \equiv 82^{2^{56}} \pmod{83^2}$ in $\mathbb{Z}_{83}$ and so Theorem 3 implies that $f_2(82)$ is not badly approximable.

We believe that for each a we can carefully choose p and $q_t(x)$ such that Conditions 1 – 4 of Theorem 3 are satisfied.

5.3. The case $d = 3$ In the case $d = 3$ we can use methods very similar to those for the case $d = 2$. However not every convergent $p(x)/q(x)$ to $g_3(x)$ produces a nice infinite sequence of convergents to $g_3(x)$. On the other hand some of them do, as it is shown in Lemma 9 below.

Lemma 9. *Let $p_t(x)/q_t(x)$ be the sequence of the convergents of $g_3(x)$ and d_t be the least common multiple of the denominators of all rational coefficients of $p_t(x)$ and $q_t(x)$. Then for each even t the rational functions $\tilde{p}_{t,n}(x)/\tilde{q}_{t,n}(x)$ where*

$$\tilde{p}_{t,n}(x) := \prod_{k=0}^{n-1} (x^{3^{k+1}}(x^{3^k} - 1)) p_t(x^{3^n}) \quad \text{and} \quad \tilde{q}_{t,n}(x) := q_t(x^{3^n}), \tag{54}$$

are all convergents of $g_3(x)$. Moreover for each positive integer $a > 1$ there exists a constant C independent of n such that

$$\left| g_3(a) - \frac{d_t \tilde{p}_{t,n}(a)}{d_t \tilde{q}_{t,n}(a)} \right| \leqslant \frac{C}{(d_t \tilde{q}_{t,n}(a))^2}.$$

In other words Lemma 9 is an analogue of Lemma 4.3 from [2] and it says that $d_t \tilde{p}_{t,n}(a)/d_t \tilde{q}_{t,n}(a)$ is almost the best rational approximation of $g_3(a)$.

Proof. The first statement of the lemma follows from the successive application of Lemma 6. We proceed with the proof of the second statement.

Denote $G(x) := g_3(x) - p_t(x)/q_t(x)$, an infinite series in x^{-1}. Since $t = 2t_0$ is even, Theorem 2 implies that $G(x)$ starts with the term $c_1x^{-6t_0-2}$ where c_1 is some integer constant. Take a compact disc $\mathbf{D} \subset \{x \in \mathbb{C} : |x| > 1\}$ with the center at infinity inside the set of convergence of $G(x)$ which contains the value a. For the sake of concretness we can take $\mathbf{D} = \{x \in \mathbb{C} : |x| > \frac{1+a}{2}\}$. Then there exists a constant c such that for each $x \in \mathbf{D}$, $G(x) \leqslant cx^{-6t_0-2}$. Consider $|G(x^{3^n})|\prod_{k=0}^{n-1}(x^{3^{k+1}}(x^{3^k}-1))$ where $n \in \mathbb{N}$. Surely x^{3^n} also belongs to $\mathbf{D}$ therefore, taking into account the functional relations (4) for $g_3(x)$, we find

$$|G(x^{3^n})|\prod_{k=0}^{n-1}(x^{3^{k+1}}(x^{3^k}-1)) = \left|g_3(x) - \frac{p_t(x^{3^{n+1}})\prod_{k=0}^{n-1}(x^{3^{k+1}}(x^{3^k}-1))}{q_t(x^{3^{n+1}})}\right| \tag{55}$$

$$\leqslant \frac{c\prod_{k=0}^{n-1}(x^{3^{k+1}}(x^{3^k}-1))}{x^{3^n(6t_0+2)}}$$

By noticing that $x^{3^k} - 1 \leqslant x^{3^k}$ and comparing the powers of x at the numerator and the denominator we get that the right hand side of this inequality is bounded above by

$$\frac{c\prod_{k=0}^{n-1}(x^{3^{k+1}}(x^{3^k}-1))}{x^{3^n(6t_0+2)}} \leqslant \frac{c}{x^{2\cdot 3^{n+1}t_0+2}}.$$

By substituting (54) into the inequality (55) we get

$$\left|g_3(x) - \frac{\tilde{p}_{t,n}(x)}{\tilde{q}_{t,n}(x)}\right| \leqslant \frac{c}{x^{2\cdot 3^{n+1}t_0+2}}.$$

The degree of the polynomial $q_t(x)$ is $3t_0$. Thus there exists an absolute constant c_2 such that for each $x \in \mathbf{D}$, $|q_t(x)| \leqslant c_2x^{3t_0}$ which in turn implies that $|\tilde{q}_{t,n}(x)| = |q_t(x^{3^n})| \leqslant c_2x^{3^{n+1}t_0}$. Therefore

$$\left|g_3(x) - \frac{d_t\tilde{p}_{t,n}(x)}{d_t\tilde{q}_{t,n}(x)}\right| \leqslant \frac{c\cdot c_2^2x^{-2}d_t^2}{(d_t\tilde{q}_{t,n}(x))^2}$$

This implies the second statement of the lemma with $C = c\cdot c_2^2a^{-2}d_t^2$. □

Lemma 9 suggests an analogous method for checking whether $f_3(a)$ is badly approximable as in Theorem 3. As soon as we have $p \mid a^{3^{n_0}} - 1$ for some prime p, we immediately have from the formulae (54) that $p^{n-n_0} \mid \tilde{p}_{t,n}(a)$ for all integer $n \geqslant n_0$ and all even t. Then if we are able to show that for some fixed even t the sequence $\tilde{q}_{t,n}(a)$ contains elements which are divisible by an arbitrarily large power of p then $g_3(a)$ and in turn $f_3(a)$ are not badly approximable. We conclude this idea in the following theorem. Since its proof mostly repeats the steps of Theorem 3 we leave it for an enthusiastic reader.

Theorem 4. *Let, as before, $p_t(x)/q_t(x)$ be the convergents of the series $g_3(x)$. Assume that there exist positive integers n_0, t, p such that*

1. *$p \geqslant 5$ is a prime number and $p \,||\, a^{3^{n_0}} - 1$;*
2. *$|\Gamma(3,p^2)| = p|\Gamma(3,p)|$;*
3. *t is even and $q_t(a^{3^{n_0}}) \equiv 0 \pmod{p^2}$;*
4. *$q_t'(1) \not\equiv 0 \pmod{p}$.*

Then $f_3(a)$ is not badly approximable.

Remark 3. Similarly to the remark to Theorem 3, we can note that condition 2 of Theorem 4 holds true for all the primes verifying

$$3^{p-1} \not\equiv 1 \pmod{p^2}. \tag{56}$$

As far as the authors are aware, currently they know only two primes failing (56), 11 and 1006003. It is also known that all the other primes in the range $5 \le p < 2^{32}$ verify (56), see [8]. So, for all primes in the range $5 \le p < 2^{32}$ different from 11 and 1006003, condition 2 of Theorem 4 holds true.

Corollary 1. *The number $f_3(2)$ is not badly approximable. Moreover, for any integer a congruent modulo 49 to any number from the set*

$$\{2, 4, 8, 9, 11, 15, 16, 22, 23, 25, 29, 32, 36, 37, 39, 43, 44, 46\}$$

the number $f_3(a)$ is not badly approximable.

Proof. With a bit of computational efforts we can find that Theorem 4 is applicable with the parameters $n_0 = 2$, $t = 4$ and $p = 7$. Indeed, in this case

$$2^{3^2} - 1 \equiv 21 \bmod 7^2,$$

so Condition 1 of Theorem 4 is satisfied. Further, Condition 2 of Theorem 4 is satisfied as well because of the Remark (or alternatively it is easy to check straightforwardly that 3 is a primitive root modulo 7^2). Finally, $p_8(x)/q_8(x)$ is the convergent to $g_3(x)$ with

$$q_8(x) = 1 + x^3 + x^6 + 2x^9 + 2x^{12}, \tag{57}$$

hence

$$q_8\left(2^{3^2}\right) \equiv 0 \bmod 7^2$$

and

$$q_8'(1) \equiv 3 \bmod 7,$$

thus Conditions 3 and 4 of Theorem 4 are satisfied as well and we conclude that $f_3(2)$ is not badly approximable. This proves the first part of the corollary.

To prove the second part of the corollary, we also choose $t = 8$, $p = 7$ and choose n_0 according to the following table

a	2	4	8	9	11	15	16	22	23	25	29	32	36	37	39	43	44	46
n_0	3	1	2	6	3	6	5	1	1	6	4	4	3	4	5	5	2	2

Then, verification of conditions of Theorem 4 goes in the same way as in the first part of this proof. It is easy to verify by straightforward computations that $2^{3^n} - 1$ is never divisible by 49 for any integer n in the range $1 \le n \le 6$ (actually, with a bit more of computations and a use of Euler's Theorem one can verify that $2^{3^n} - 1$ is never divisible by 49 for any $n \in \mathbb{N}$). Also, $2^3 \equiv 1 \bmod 7$, so for any $n \in \mathbb{N}$ we have $2^{3^n} - 1 \equiv 0 \bmod 7$, and we readily have condition 1 of Theorem 4. Condition 2 of Theorem 4 holds true because of Remark 3.

Verification of conditions 3 and 4 of Theorem 4, with the polynomial $q_8(x)$ given by (57), is just a simple routine computation, so we leave it to the interested reader.

As far as all the conditions of Theorem 4 are verified, its conclusion is that $f_3(a)$ is not badly approximable, for the corresponsing values of a, and this proves the second part of the corollary. □

References

[1] J. P. Allouche, M. Mendès France, A. J. van der Poorten, "An infinite product with bounded partial quotients". Acta Arith. 59 (1991), 171–182.

[2] D. A. Badziahin, E. Zorin, "Thue-Morse constant is not badly approximable". Int. Math. Research. Not. (2014), doi: 10.1093/imrn/rnu238.

[3] Y. Bugeaud, "On the irrationality exponent of the Thue-Morse-Mahler numbers". Ann. Inst. Fourier (Grenoble), 61 (2011), no. 5, 2065–2076.

[4] Y. Bugeaud, G.-N. Han, Z.-Y. Wen, J.-Y. Yao, "Hankel determinants, Pade approximations, and irrationality exponents", preprint, http://arxiv.org/abs/1503.02797.

[5] P. Ribenboim , "Die Welt der Primzahlen: Geheimnisse und Rekorde" (in German), 2004, New York: Springer, p. 237.

[6] M. Mendès France, A. J. van der Poorten, "From geometry to Euler identities", Theoret. Comput. Sci., Vol. 65 (2), 1989, pp. 213-220

[7] M. Mendès France, A. J. van der Poorten, "Some explicit continued fraction expansions", Mathematika, Vol. 38 (1), 1991, pp. 1-9

[8] P. L. Montgomery, New solutions of $a^{p-1} \equiv 1 \pmod{p^2}$, Math. Comp. 61 (1993), 361-363, MR 94d:11003

[9] I. Niven, H. S. Zuckerman, and H. L. Montgomery, "An Introduction to the Theory of Numbers", 5th Ed., John Wiley and Sons, New York, 1991.

[10] A. J. van der Poorten, "Formal Power Series and their Continued Fraction Expansion", Algorithmic Number Theory. Lecture Notes in Computer Science 1423, 1998, 358–371.