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A SIMPLE PROOF OF THE FORMULA FOR THE BETTI NUMBERS OF THE QUASIHOMOGENEOUS HILBERT SCHEMES.

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Abstract. In a recent paper the first two authors proved that the generating series of the Poincare polynomials of the quasihomogeneous Hilbert schemes of points in the plane has a simple decomposition in an infinite product. In this paper we give a very short geometrical proof of that formula.

1. INTRODUCTION

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in the plane $\mathbb{C}^2$ parametrizes ideals $I \subset \mathbb{C}[x,y]$ of colength n: $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$. It is a nonsingular, irreducible, quasiprojective algebraic variety of dimension $2n$ with a rich and much studied geometry, see [7, 13] for an introduction.

The cohomology groups of $(\mathbb{C}^2)^{[n]}$ were computed in [5], and the ring structure in the cohomology was determined independently in the papers [10] and [14].

There is a $(\mathbb{C}^*)^2$-action on $(\mathbb{C}^2)^{[n]}$ that plays a central role in this subject. The algebraic torus $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by scaling the coordinates, $(t_1,t_2)\cdot(x,y) = (t_1x, t_2y)$. This action lifts to the $(\mathbb{C}^*)^2$-action on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$.

For arbitrary non-negative integers α and β, such that $\alpha+\beta\ge 1$, let $T_{\alpha,\beta} = \{(t^\alpha, t^\beta) \in (\mathbb{C}^*)^2 | t \in \mathbb{C}^*\}$ be a one-dimensional subtorus of $(\mathbb{C}^*)^2$. If α and β are non-zero, then the fixed point set $\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}$ is called the quasihomogeneous Hilbert scheme of points on the plane $\mathbb{C}^2$.

The quasihomogeneous Hilbert scheme $\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}$ is compact and in general has many irreducible components. They were described in [6]. In the case $\alpha = 1$ the Poincare polynomials of the irreducible components were computed in [3].

The Poincare polynomial of a manifold X is defined by $P_q(X) = \sum_{i\ge 0} \dim H_i(X;\mathbb{Q}) q^{\frac{i}{2}}$. In [4] the first two authors proved the following theorem.

Theorem 1.1. *Suppose α and β are positive coprime integers, then*

$$\sum_{n\ge 0} P_q\left(\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}\right) t^n = \prod_{\substack{i\ge 1\\ (\alpha+\beta)\nmid i}} \frac{1}{1-t^i} \prod_{i\ge 1} \frac{1}{1-qt^{(\alpha+\beta)i}}.$$

In this paper we give another proof of this theorem. In [4] the large part of the proof consists of non-trivial combinatorial computations with Young diagrams. Our new proof is more geometrical and is much shorter. In fact, we prove a slightly more general statement.

Let Γ_m be the finite subgroup of $(\mathbb{C}^*)^2$ defined by

$$\Gamma_m = \left\{ (\zeta^j, \zeta^{-j}) \in (\mathbb{C}^*)^2 \middle| \zeta = \exp\left(\frac{2\pi i}{m}\right), j = 0,1,\ldots,m-1 \right\}.$$

For a manifold X let $H_*^{BM}(X;\mathbb{Q})$ denote the Borel-Moore homology group of X with rational coefficients. Let $P_q^{BM}(X) = \sum_{i\ge 0} \dim H_i^{BM}(X;\mathbb{Q}) q^{\frac{i}{2}}$.

We prove the following theorem.

Theorem 1.2. *Let α and β be any two non-negative integers, such that $\alpha+\beta \ge 1$. Then we have*

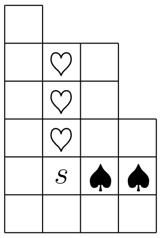
$$\sum_{n\ge 0} P_q^{BM}\left(\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}}\right) t^n = \prod_{\substack{i\ge 1\\ (\alpha+\beta)\nmid i}} \frac{1}{1-t^i} \prod_{i\ge 1} \frac{1}{1-qt^{(\alpha+\beta)i}}. \tag{1}$$

Here we use Borel-Moore homology, because the variety $\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}}$ is in general not compact, if $\alpha=0$.

If α and β are coprime, then $\Gamma_{\alpha+\beta}\subset T_{\alpha,\beta}$. Hence, Theorem 1.1 follows from Theorem 1.2.

Our proof of Theorem 1.2 consists of two steps. First, we prove that the left-hand side of (1) depends only on the sum $\alpha+\beta$. We use an argument with an equivariant symplectic form that is very similar to the one that was applied by the third author in [12] (proof of Proposition 5.7). After that the case $\alpha=0$ can be done using a notion of a power structure over the Grothendieck ring of quasiprojective varieties.

In [4], as a corollary of Theorem 1.1, there was derived a combinatorial identity. In the same way Theorem 1.2 leads to a more general combinatorial identity. Denote by $\mathcal{Y}$ the set of all Young diagrams. The number of boxes in a Young diagram Y is denoted by $|Y|$. For a box $s\in Y$ we define the numbers $l_Y(s)$ and $a_Y(s)$, as it is shown on Fig. 1.



Y

$l_Y(s)=$ number of ♠
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Figure 1. Definition of the numbers $a_Y(s)$ and $l_Y(s)$

For a Young diagram Y define the number $h_{\alpha,\beta}(Y)$ by

$$h_{\alpha,\beta}(Y)=\left\{s\in Y \;\middle|\; \substack{\alpha l_Y(s)=\beta(a_Y(s)+1)\\ (\alpha+\beta)|l_Y(s)+a_Y(s)+1}\right\}.$$

The following corollary is a generalization of Theorem 1.2 from [4].

Corollary 1.3. *Let α and β be arbitrary non-negative integers, such that $\alpha+\beta\geq 1$. Then we have*

$$\sum_{Y\in\mathcal{Y}} q^{h_{\alpha,\beta}(Y)}t^{|Y|}=\prod_{\substack{i\geq 1\\ (\alpha+\beta)\nmid i}}\frac{1}{1-t^i}\prod_{i\geq 1}\frac{1}{1-qt^{(\alpha+\beta)i}}. \tag{2}$$

Proof. The proof is similar to the proof of Theorem 1.2 in [4]. We apply the results from [1, 2], in order to construct a cell decomposition of the variety $\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}}$, and show that the left-hand side of (1) is equal to the left-hand side of (2). □

We thank Ole Warnaar for suggesting this more general combinatorial identity after the paper [4] was published in arXiv.

1.1. *Organization of the paper.* In Section 2 we recall the definition of the Grothendieck ring of complex quasiprojective varieties and the properties of the natural power structure over it. Section 3 contains the proof of Theorem 1.2.

2. POWER STRUCTURE OVER THE GROTHENDIECK RING $K_0(\nu_{\mathbb{C}})$

In this section we review the definition of the Grothendieck ring of complex quasiprojective varieties and the power structure over it.

2.1. *Grothendieck ring.* The Grothendieck ring $K_0(\nu_{\mathbb{C}})$ of complex quasiprojective varieties is the abelian group generated by the classes $[X]$ of all complex quasiprojective varieties X modulo the relations:

(1) if varieties X and Y are isomorphic, then $[X]=[Y]$;
(2) if Y is a Zariski closed subvariety of X, then $[X]=[Y]+[X\backslash Y]$.

The multiplication in $K_0(\nu_{\mathbb{C}})$ is defined by the Cartesian product of varieties: $[X_1]\cdot[X_2]=[X_1\times X_2]$. The class $[\mathbb{A}^1_{\mathbb{C}}]\in K_0(\nu_{\mathbb{C}})$ of the complex affine line is denoted by $\mathbb{L}$.

We will need the following property of the Grothendieck ring $K_0(\nu_{\mathbb{C}})$. There is a natural homomorphism $\theta\colon \mathbb{Z}[z]\to K_0(\nu_{\mathbb{C}})$, defined by $z\mapsto\mathbb{L}$. This homomorphism is injective (see e.g.[11]).

2.2. *Power structure.* In [8] there was defined a notion of a power structure over a ring and there was described a natural power structure over the Grothendieck ring $K_0(\nu_{\mathbb{C}})$. This means that for a series $A(t)=1+a_1t+a_2t^2+\ldots\in 1+t\cdot K_0(\nu_{\mathbb{C}})[[t]]$ and for an element $m\in K_0(\nu_{\mathbb{C}})$ one defines a series $(A(t))^m\in 1+t\cdot K_0(\nu_{\mathbb{C}})[[t]]$, so that all the usual properties of the exponential function hold.

For a series $A(t)=(1-t)^{-1}$ and a quasiprojective variety M the series $A(t)^{[M]}$ coincides with the motivic zeta function $\zeta_{[M]}(t)$ introduced by M. Kapranov ([9]):

$$(1-t)^{-[M]}=1+\sum_{i\geq 1}[S^iM]t^i,$$

where S^iM is the i-th symmetric product of the variety M. There is the following generalization of this formula. Suppose that $M_1, M_2, \ldots$ and N are quasiprojective varieties. Then we have

$$\left(1+\sum_{i\geq 1}[M_i]t^i\right)^{[N]}=1+\sum_{n\geq 1}X_nt^n,\quad \text{where}$$

$$X_n=\sum_{\sum_{i\geq 1} id_i=n}\left[\left((N^{\sum d_i}\backslash\Delta)\times\left(\prod M_i^{d_i}\right)\right)\Big/\prod S_{d_i}\right]. \tag{3}$$

Here Δ is the "large diagonal" in $N^{\sum d_i}$, which consists of $(\sum d_i)$ points of N with at least two coinciding ones. The permutation group S_{d_i} acts by permuting corresponding d_i factors in $\prod N^{d_i}$ and $\prod M_i^{d_i}$ simultaneously.

We also need the following property of the power structure over $K_0(\nu_{\mathbb{C}})$. For any $i\geq 1$ and $j\geq 0$ we have

$$(1-\mathbb{L}^jt^i)^{-\mathbb{L}}=(1-\mathbb{L}^{j+1}t^i)^{-1}. \tag{4}$$

It can be derived from several statements from [8] as follows. Let a_i, $i\geq 1$, and m be from the Grothendieck ring $K_0(\nu_{\mathbb{C}})$ and $A(t)=1+\sum_{i\geq 1}a_it^i$. Then for any $s\geq 0$ we have

$$A(\mathbb{L}^st)^m=\left.(A(t)^m)\right|_{t\mapsto\mathbb{L}^st}, \tag{5}$$

$$(1-t)^{-\mathbb{L}^sm}=\left.(1-t)^{-m}\right|_{t\mapsto\mathbb{L}^st}. \tag{6}$$

Formula (5) follows from Statement 2 in [8] and equation (6) follows from Statement 3 in [8]. Also for any $s\geq 1$ we have (see [8])

$$A(t^s)^m=\left.(A(t)^m)\right|_{t\mapsto t^s}. \tag{7}$$

Obviously, formula (4) follows from (5), (6) and (7).

3. PROOF OF THEOREM 1.2

Using the $(\mathbb{C}^*)^2$-action on $(\mathbb{C}^2)^{[n]}$ and the results from [1, 2] one can easily construct a cell decomposition of $\left((\mathbb{C}^2)^{[n]}\right)^{\Gamma_{\alpha+\beta}\times T_{\alpha,\beta}}$. Thus, Theorem 1.2 is equivalent to the following formula

$$\sum_{n\geq 0}\left[\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}}\right]t^n=\prod_{\substack{i\geq 1\\(\alpha+\beta)\nmid i}}\frac{1}{1-t^i}\prod_{i\geq 1}\frac{1}{1-\mathbb{L}t^{(\alpha+\beta)i}}. \tag{8}$$

It is clear that equation (8) is a corollary of the following two lemmas.

Lemma 3.1. *For any $\alpha,\beta\geq 0$, such that $\alpha+\beta\geq 1$, we have*

$$\left[\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}}\right]=\left[\left((\mathbb{C}^2)^{[n]}\right)^{T_{0,\alpha+\beta}\times\Gamma_{\alpha+\beta}}\right].$$

Lemma 3.2. *For any $m\geq 1$ we have*

$$\sum_{n\geq 0}\left[\left((\mathbb{C}^2)^{[n]}\right)^{T_{0,m}\times\Gamma_m}\right]t^n=\prod_{\substack{i\geq 1\\ m\nmid i}}\frac{1}{1-t^i}\prod_{i\geq 1}\frac{1}{1-\mathbb{L}t^{mi}}.$$

Proof of Lemma 3.1. Let $\left((\mathbb{C}^2)^{[n]}\right)^{\Gamma_{\alpha+\beta}}=\coprod_i\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}$ be the decomposition in the irreducible components. It is sufficient to prove that

$$\left[\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}\right]=\mathbb{L}^{\frac{d_i}{2}}\left[\left(\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}\right)^{T_{\alpha,\beta}}\right],$$

where $d_i=\dim\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}$. The subvarieties $\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}$ are quiver varieties of affine type $\tilde{A}_{\alpha+\beta-1}$. We prove the above equality by using the idea in [12, Proposition 5.7].

Let $\left(\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}\right)^{T_{\alpha,\beta}}=\coprod_j\left((\mathbb{C}^2)^{[n]}\right)_{i,j}^{\Gamma_{\alpha+\beta}\times T_{\alpha,\beta}}$ be the decomposition in the irreducible components. Consider the $\mathbb{C}^*$-action on $(\mathbb{C}^2)^{[n]}$ induced by the homomorphism $\mathbb{C}^*\to(\mathbb{C}^*)^2, t\mapsto(t^\alpha,t^\beta)$. Define the sets $C_{i,j}$ by

$$C_{i,j}=\left\{z\in\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}\middle|\lim_{t\to 0,t\in\mathbb{C}^*}t\cdot z\in\left((\mathbb{C}^2)^{[n]}\right)_{i,j}^{\Gamma_{\alpha+\beta}\times T_{\alpha,\beta}}\right\}.$$

From [1, 2] it follows that the set $C_{i,j}$ is a locally trivial fiber bundle over $\left((\mathbb{C}^2)^{[n]}\right)_{i,j}^{\Gamma_{\alpha+\beta}\times T_{\alpha,\beta}}$ with an affine space as a fiber. Let us denote by $d_{i,j}$ the dimension of a fiber. For $p\in\left((\mathbb{C}^2)^{[n]}\right)_{i,j}^{\Gamma_{\alpha+\beta}\times T_{\alpha,\beta}}$ the tangent space $T_p\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}$ is a $\mathbb{C}^*$-module. Let

$$T_p\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}=\sum_{m\in\mathbb{Z}}H(m)$$

be the weight decomposition. It is clear that $d_{i,j}=\dim\left(\bigoplus_{m\geq 1}H(m)\right)$.

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ has the canonical symplectic form ω that is induced from the symplectic form $dx\wedge dy$ on $\mathbb{C}^2$ (see e.g.[13]). The form ω has weight $-\alpha-\beta$ with respect to the $\mathbb{C}^*$-action on $(\mathbb{C}^2)^{[n]}$. The restriction $\omega|_{\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}}$ is the canonical symplectic form on the quiver variety (see [12]). Therefore, the spaces $\bigoplus_{m\leq 0}H(m)$ and $\bigoplus_{m\geq\alpha+\beta}H(m)$ are dual with respect to this form. Obviously, the $(\alpha+\beta)$-th root of unity $\sqrt[\alpha+\beta]{1}$ acts trivially on $\left((\mathbb{C}^2)^{[n]}\right)_i^{\Gamma_{\alpha+\beta}}$, thus, $H(m)=0$, if $(\alpha+\beta)\nmid m$. We get $\bigoplus_{m\geq\alpha+\beta}H(m)=\bigoplus_{m\geq 1}H(m)$ and $d_{i,j}=\dim\left(\bigoplus_{m\geq 1}H(m)\right)=\frac{d_i}{2}$. This completes the proof of the lemma. □

Proof of Lemma 3.2. Obviously, we have $\left((\mathbb{C}^2)^{[n]}\right)^{T_{0,m}}=\left((\mathbb{C}^2)^{[n]}\right)^{T_{0,1}}$. For a partition $\lambda=(\lambda_1,\ldots,\lambda_l)$, $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_l\geq 1$, and a point $x_0\in\mathbb{C}$ define the ideal $I_{\lambda,x_0}\subset\mathbb{C}[x,y]$ by

$$I_{\lambda,x_0}=(y^{\lambda_1},(x-x_0)y^{\lambda_2},\ldots,(x-x_0)^{l-1}y^{\lambda_l},(x-x_0)^l).$$

In [13] it is proved that each element $I \in \left((\mathbb{C}^2)^{[n]}\right)^{T_{0,1}}$ can be uniquely expressed as

$$I = I_{\lambda^1,x_1} \cap \ldots \cap I_{\lambda^k,x_k}$$

for some distinct points $x_1,\ldots,x_k \in \mathbb{C}$ and for some partitions $\lambda^1,\ldots,\lambda^k$ satisfying $\sum_{i=1}^k |\lambda^i| = n$.

Denote by $\mathbb{C}_x$ the x-axis in the plane $\mathbb{C}^2$. Consider the map $\pi_n\colon \left((\mathbb{C}^2)^{[n]}\right)^{T_{0,1}} \to S^n\mathbb{C}_x$ defined by

$$\pi_n\left(I_{\lambda^1,x_1} \cap \ldots \cap I_{\lambda^k,x_k}\right) = \sum_{i=1}^k |\lambda^i|[x_i].$$

Suppose Z is an open subset of $\mathbb{C}_x$. From (3) it follows that

$$\sum_{n\ge 0}\left[\pi_n^{-1}\left(S^n Z\right)\right]t^n = \left(\prod_{i\ge 1}\frac{1}{1-t^i}\right)^{[Z]}.$$

The Γ_m-action on $\mathbb{C}_x\backslash\{0\}$ is free and $(\mathbb{C}_x\backslash\{0\})/\Gamma_m \cong \mathbb{C}_x\backslash\{0\}$, therefore,

$$\left(\pi_n^{-1}\left(S^n(\mathbb{C}_x\backslash\{0\})\right)\right)^{\Gamma_m} \cong \begin{cases}\emptyset, & \text{if } m\nmid n,\\ \pi_l^{-1}\left(S^l(\mathbb{C}_x\backslash\{0\})\right), & \text{if } n = ml.\end{cases}$$

We obtain

$$\sum_{n\ge 0}\left[\left(\pi_n^{-1}\left(S^n(\mathbb{C}_x\backslash\{0\})\right)\right)^{\Gamma_m}\right]t^n = \left(\prod_{i\ge 1}\frac{1}{1-t^{mi}}\right)^{\mathbb{L}-1}.$$

Therefore, we get

$$\sum_{n\ge 0}\left[\left((\mathbb{C}^2)^{[n]}\right)^{T_{0,1}\times\Gamma_m}\right]t^n = \left(\sum_{n\ge 0}[\pi_n^{-1}(n[0])]t^n\right)\left(\sum_{n\ge 0}\left[\left(\pi_n^{-1}\left(S^n(\mathbb{C}_x\backslash\{0\})\right)\right)^{\Gamma_m}\right]t^n\right) =$$

$$= \left(\prod_{i\ge 1}\frac{1}{1-t^i}\right)\left(\prod_{i\ge 1}\frac{1}{1-t^{mi}}\right)^{\mathbb{L}-1} = \prod_{\substack{i\ge 1\\ m\nmid i}}\frac{1}{1-t^i}\prod_{i\ge 1}\frac{1}{1-\mathbb{L}t^{mi}}.$$

The lemma is proved. □

The theorem is proved.

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References

[1] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. *Annals of Mathematics (2)* **98** (1973) 480-497.

[2] A. Bialynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bulletin de l'Academie Polonaise des Sciences, Serie des Sciences Mathematiques, Astronomiques et Physiques* **24** (1976), no. 9, 667-674.

[3] A. Buryak. The classes of the quasihomogeneous Hilbert schemes of points on the plane. *Moscow Mathematical Journal* **12** (2012), no. 1, 21-36.

[4] A. Buryak, B. L. Feigin. Generating series of the Poincare polynomials of quasihomogeneous Hilbert schemes. Symmetries, Integrable Systems and Representations, 15-33, *Springer Proceedings in Mathematics and Statistics* **40** (2013).

[5] G. Ellingsrud, S. A. Stromme. On the homology of the Hilbert scheme of points in the plane. *Inventiones Mathematicae* **87** (1987), no. 2, 343-352.

[6] L. Evain. Irreducible components of the equivariant punctual Hilbert schemes. *Advances in Mathematics* **185** (2004), no. 2, 328-346.

[7] L. Gottsche. Hilbert schemes of points on surfaces. ICM Proceedings, Vol. II (Beijing, 2002), 483-494.

[8] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernandez. A power structure over the Grothendieck ring of varieties. *Mathematical Research Letters* **11** (2004), no. 1, 49-57.

[9] M. Kapranov. The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups. *arXiv:math/0001005.*

[10] M. Lehn and C. Sorger. Symmetric groups and the cup product on the cohomology of Hilbert schemes. *Duke Mathematical Journal* **110** (2001), no. 2, 345-357.

[11] E. Looijenga. Motivic measures. Seminaire Bourbaki, 1999-2000 (874), 2000.

[12] H. Nakajima. Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Mathematical Journal* **76** (1994), no. 2, 365-416.

[13] H. Nakajima. Lectures on Hilbert schemes of points on surfaces. AMS, Providence, RI, 1999.

[14] E. Vasserot. Sur lanneau de cohomologie du schema de Hilbert de $\mathbb{C}^2$. *Comptes Rendus de l'Academie des Sciences - Series I - Mathematics* **332** (2001), no. 1, 7-12.

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