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DOUBLE RAMIFICATION CYCLES AND THE n-POINT FUNCTION FOR THE MODULI SPACE OF CURVES

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Abstract. In this paper, using the formula for the integrals of the ψ-classes over the double ramification cycles found by S. Shadrin, L. Spitz, D. Zvonkine and the author, we derive a new explicit formula for the n-point function of the intersection numbers on the moduli space of curves.

1. Introduction

Consider the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex algebraic curves of genus g with n marked points. The class $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is defined as the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,n}$ formed by the cotangent lines at the i-th marked point. We define the intersection numbers by

$$\langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\psi_2^{d_2}\cdots\psi_n^{d_n}. \tag{1.1}$$

In the unstable case $2g-2+n \le 0$ we define the bracket to be equal to zero. The bracket (1.1) vanishes unless the dimension constraint

$$\sum_{i=1}^n d_i = 3g-3+n$$

is satisfied. Introduce variables $t_0, t_1, t_2, \ldots$. The celebrated conjecture of E. Witten [Wit91], proved by M. Kontsevich [Kon92], says that for the following generating function

$$F(t_0, t_1, \ldots) := \sum_{\substack{g\ge 0\\ n\ge 1}} \sum_{d_1,\ldots,d_n\ge 0} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g \frac{t_{d_1}\cdots t_{d_n}}{n!},$$

the exponent e^F is a tau-function of the KdV hierarchy in the variables $T_{2i+1} = \frac{t_i}{(2i+1)!!}$.

In this paper we discuss a different generating function for the intersection numbers (1.1). Let $n \ge 1$. Introduce variables $x_1, \ldots, x_n$. Define

$$\mathcal{F}(x_1, \ldots, x_n) := \sum_{g\ge 0} \mathcal{F}_g(x_1, \ldots, x_n),$$

where

$$\mathcal{F}_g(x_1, \ldots, x_n) := \sum_{d_1,\ldots,d_n\ge 0} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g\, x_1^{d_1}\cdots x_n^{d_n}.$$

The function $\mathcal{F}(x_1, \ldots, x_n)$ is often called the n-point function. There are several known closed formulas for it [BDY15, BH07, Oko02]. In this paper we derive a simple new formula for $\mathcal{F}(x_1, \ldots, x_n)$ using the result of [BSSZ15]. We believe that our approach can be useful for the understanding of the structure of more general Gromov-Witten invariants.

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Let us formulate our main result. Introduce variables $a_1, a_2, \ldots, a_n$. We will use the following notations.

- Let $\zeta(x) := e^{\frac{x}{2}} - e^{-\frac{x}{2}}$.
- For a permuation $\sigma \in S_n$ denote $a'_i := a_{\sigma(i)}$ and $x'_i := x_{\sigma(i)}$.
- Finally,
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We define the functions $P_n(a_1, \ldots, a_n; x_1, \ldots, x_n)$ by

$$P_1(a_1; x_1) := \frac{1}{x_1},$$

(1.2)
$$P_n(a_1, \ldots, a_n; x_1, \ldots, x_n) :=$$
$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} x'_2 \cdots x'_{n-1} \frac{\zeta\left(\begin{vmatrix} a'_1 & a'_2 \\ x'_1 & x'_2 \end{vmatrix}\right) \zeta\left(\begin{vmatrix} a'_1 + a'_2 & a'_3 \\ x'_1 + x'_2 & x'_3 \end{vmatrix}\right) \cdots \zeta\left(\begin{vmatrix} a'_1 + \cdots + a'_{n-1} & a'_n \\ x'_1 + \cdots + x'_{n-1} & x'_n \end{vmatrix}\right)}{\begin{vmatrix} a'_1 & a'_2 \\ x'_1 & x'_2 \end{vmatrix} \begin{vmatrix} a'_2 & a'_3 \\ x'_2 & x'_3 \end{vmatrix} \cdots \begin{vmatrix} a'_{n-1} & a'_n \\ x'_{n-1} & x'_n \end{vmatrix}}, \quad n \geq 2.$$

At first sight it appears that for $n \geq 2$ the function $P_n(a; x)$ has simple poles along the hyperplanes $\{a_i x_j - a_j x_i = 0\}$, but in [BSSZ15, Remark 1.6] it was shown that for $n \geq 2$ the function $P_n(a; x)$ is a power series in $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}[a_1, \ldots, a_n]$. Moreover, by [BSSZ15, Remark 1.5], the function $P_n(a; x)$ is symmetric with respect to simultaneous permutations of the variables $a_1, \ldots, a_n$ and the variables $x_1, \ldots, x_n$. Our main result is the following theorem.

Theorem 1.1. *We have*

(1.3) $$\mathcal{F}(x_1, \ldots, x_n) =$$
$$= \frac{e^{\frac{(\sum x_i)^3}{24}}}{(\sum x_i) \prod \sqrt{2\pi x_i}} \int_{\mathbb{R}^n} e^{-\sum \frac{a_i^2}{2x_i}} P_n(\sqrt{-1}a_1, \ldots, \sqrt{-1}a_n; x_1, \ldots, x_n) da - \frac{\delta_{n,1}}{x_1^2} - \frac{\delta_{n,2}}{x_1 + x_2},$$

where $da := da_1 \cdots da_n$.

Let us make a few comments about the right-hand side of equation (1.3) and, in particular, show that it is a power series in $x_1, \ldots, x_n$. Note that for $x > 0$ and $d \geq 0$ we have

$$\frac{1}{\sqrt{2\pi x}} \int_{\mathbb{R}} a^d e^{-\frac{a^2}{2x}} da = \begin{cases} (d-1)!! x^{\frac{d}{2}}, & \text{if } d \text{ is even}, \\ 0, & \text{if } d \text{ is odd}. \end{cases} \tag{1.4}$$

Therefore, the transformation

$$\frac{1}{\prod \sqrt{2\pi x_i}} \int_{\mathbb{R}^n} e^{-\sum \frac{a_i^2}{2x_i}} P_n(\sqrt{-1}a; x) da \tag{1.5}$$

just means that we replace each monomial $a_1^{d_1} \cdots a_n^{d_n}$ in $P_n(a; x)$ by

$$(-1)^{\frac{1}{2}\sum d_i} \prod \left((d_i - 1)!! x_i^{\frac{d_i}{2}}\right),$$

if all exponents d_i are even and by zero otherwise. The transformation (1.5) can also be interpreted as the Laplace transformation. For $n \geq 3$, in [BSSZ15, Remark 1.6] it was shown that the function $\frac{P_n(a;x)}{x_1 + \cdots + x_n}$ is a power series in $x_1, \ldots, x_n$ with coefficients from $\mathbb{C}[a_1, \ldots, a_n]$. Therefore, the right-hand side of (1.3) is a power series in $x_1, \ldots, x_n$. In the case $n = 2$, again from [BSSZ15, Remark 1.6] we know that $\frac{P_2(a_1, a_2; x_1, x_2)}{x_1 + x_2} - \frac{1}{x_1 + x_2}$ is a power series in x_1, x_2 with coefficients from $\mathbb{C}[a_1, a_2]$. This implies that the right-hand side of (1.3) is a power series in x_1

and x_2. For $n=1$ we immediately see that the right-hand side of (1.3) is equal to $\frac{e^{\frac{x_1^3}{24}}-1}{x_1^2}$ that is of course a power series in x_1.

Let us describe briefly the plan of the proof of Theorem 1.1. The double ramification cycle $\mathrm{DR}_g(a_1,\dots,a_n)$ is a cohomology class in $H^{2g}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$. It depends on a list of integers $a_1,\dots,a_n$ satisfying $\sum a_i=0$. In [BSSZ15] the authors derived an explicit formula for the generating series

$$\sum_{\substack{g\ge 0\\ 2g-2+n>0}}\sum_{d_1,\dots,d_n\ge 0}\left(\int_{\overline{\mathcal{M}}_{g,n}}\mathrm{DR}_g(a_1,\dots,a_n)\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}.$$

We recall it in Section 2. Denote by $\pi_m\colon\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n-m}$ the forgetful map that forgets the last m marked points. Let $a_1,\dots,a_n$ and $b_1,\dots,b_g$ be arbitrary integers and let $A:=\sum a_i$ and $B:=\sum b_j$. The push-forward $\pi_{g*}\mathrm{DR}_g(a_1,\dots,a_n,-A-B,b_1,\dots,b_g)$ is a cohomology class of degree 0 and it is equal to $g!b_1^2b_2^2\cdots b_g^2$ times the unit. This observation together with the string equation for the intersection numbers (1.1) implies that

$$\begin{aligned}
&g!b_1^2\cdots b_g^2\left(\sum x_i\right)\mathcal{F}_g(x_1,\dots,x_n)=\\
&\qquad=\sum_{d_1,\dots,d_n\ge 0}\left(\int_{\overline{\mathcal{M}}_{g,n+g+1}}\pi_{g*}\mathrm{DR}_g(a_1,\dots,a_n,-A-B,b_1,\dots,b_g)\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}.
\end{aligned}$$

Moreover, this polynomial is the term of the lowest degree $3g-2+n$ with respect to the x-variables in the infinite series

$$\sum_{m\ge 0}\sum_{d_1,\dots,d_n\ge 0}\left(\int_{\overline{\mathcal{M}}_{g,n+g+1}}\pi_{g*}\mathrm{DR}_m(a_1,\dots,a_n,-A-B,b_1,\dots,b_g)\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}.$$

In Section 2 we derive an explicit formula for this series and then in Section 3 prove Theorem 1.1.

In Section 4 we do an explicit computation of the one-point and the two-point functions using our general formula.

2. Double ramification cycles

Let $n\ge 2$ and let $a_1,\dots,a_n$ be a list of integers satisfying $\sum a_i=0$. The double ramification cycle $\mathrm{DR}_g(a_1,\dots,a_n)$ is a cohomology class in $H^{2g}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$. If not all of a_i's are equal to zero, then the restriction

$$\mathrm{DR}_g(a_1,\dots,a_n)|_{\mathcal{M}_{g,n}}$$

to the moduli space of smooth curves $\mathcal{M}_{g,n}\subset\overline{\mathcal{M}}_{g,n}$ can be defined as the Poincaré dual to the locus of pointed smooth curves $[C,p_1,\dots,p_n]$ satisfying $\mathcal{O}_C\left(\sum a_ip_i\right)\cong\mathcal{O}_C$. We refer the reader, for example, to [BSSZ15] for the general definition of the double ramification cycle on the whole moduli space $\overline{\mathcal{M}}_{g,n}$. We will often consider the Poincaré dual to the double ramification cycle $\mathrm{DR}_g(a_1,\dots,a_n)$. It is an element of $H_{2(2g-3+n)}(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ and, abusing our notation a little bit, it will also be denoted by $\mathrm{DR}_g(a_1,\dots,a_n)$. An explicit formula for the double ramification cycle $\mathrm{DR}_g(a_1,\dots,a_n)$ in terms of tautological classes in the cohomology $H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ was recently obtained in [JPPZ16]. In particular, it implies that the double ramification cycle $\mathrm{DR}_g(a_1,\dots,a_n)$ depends polynomially on the parameters $a_1,\dots,a_n$.

Let us now formulate the main result of [BSSZ15]. Let $d_1,\dots,d_n$ be non-negative integers satisfying $\sum d_i=2g-3+n$. Then the integral

$$\int_{\mathrm{DR}_g(a_1,\dots,a_n)}\psi_1^{d_1}\cdots\psi_n^{d_n}\tag{2.1}$$

is equal to the coefficient of $x_1^{d_1}\cdots x_n^{d_n}$ in the generating function

$$\frac{P_n(a_1,\ldots,a_n;x_1,\ldots,x_n)}{\zeta(x_1+\cdots+x_n)}-\frac{\delta_{n,2}}{x_1+x_2}. \tag{2.2}$$

As it was discussed in the introduction, this generating series is a power series in $x_1,\ldots,x_n$ with coefficients from $\mathbb{C}[a_1,\ldots,a_n]$. We see that the integral (2.1) is a polynomial in $a_1,\ldots,a_n$. Note that $\zeta(x)=\sum_{i\ge 0}\frac{x^{2i+1}}{2^{2i}(2i+1)!}$, then from the definition (1.2) it is easy to see that the series (2.2) has the form

$$\frac{P_n(a_1,\ldots,a_n;x_1,\ldots,x_n)}{\zeta(x_1+\cdots+x_n)}-\frac{\delta_{n,2}}{x_1+x_2}=\sum_{\substack{j\ge 0\\ 2j-2+n>0}}\sum_i f_{i,j}(a)g_{i,j}(x),$$

where $g_{i,j}(x)\in\mathbb{C}[x_1,\ldots,x_n]$ is a polynomial of degree $2j-3+n$, $f_{i,j}(a)\in\mathbb{C}[a_1,\ldots,a_n]$ and for any fixed j the second summation is finite. From this we conclude that all terms in the series (2.2) have the geometrical meaning and, thus,

(2.3)
$$\sum_{\substack{g\ge 0\\ 2g-2+n>0}}\sum_{d_1,\ldots,d_n\ge 0}\left(\int_{\mathrm{DR}_g(a_1,\ldots,a_n)}\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}=\frac{P(a_1,\ldots,a_n;x_1,\ldots,x_n)}{\zeta(x_1+\cdots+x_n)}-\frac{\delta_{n,2}}{x_1+x_2}.$$

Now we want to generalize formula (2.3) for the integrals over the double ramification cycles with forgotten points. We begin with the following lemma.

Lemma 2.1. *For $n\ge 2$ we have*

$$P_n(a_1,\ldots,a_n;x_1,\ldots,x_n)|_{x_n=0}=\frac{\zeta(a_n(x_1+\cdots+x_{n-1}))}{a_n}P_{n-1}(a_1,\ldots,a_{n-1};x_1,\ldots,x_{n-1}).$$

Proof. For $n=2$ we have

$$P_2(a_1,a_2;x_1,x_2)=\frac{\zeta(a_1x_2-a_2x_1)}{a_1x_2-a_2x_1}.$$

Therefore,

$$P_2(a_1,a_2;x_1,x_2)|_{x_2=0}=\frac{\zeta(a_2x_1)}{a_2x_1}=\frac{\zeta(a_2x_1)}{a_2}P_1(a_1;x_1).$$

Suppose $n\ge 3$. Note that if we set $x_n=0$, then the product $x_2'\cdots x_{n-1}'$ on the right-hand side of (1.2) vanishes unless $\sigma(n)=n$. Therefore, we obtain

$$P_n(a_1,\ldots,a_n;x_1,\ldots,x_n)|_{x_n=0}=\sum_{\substack{\sigma\in S_n\\ \sigma(1)=1,\,\sigma(n)=n}}x_2'\cdots x_{n-1}'\times$$
$$\times\frac{\zeta\left(\begin{vmatrix}a_1' & a_2'\\ x_1' & x_2'\end{vmatrix}\right)\cdots\zeta\left(\begin{vmatrix}a_1'+a_2'+\cdots+a_{n-2}' & a_{n-1}'\\ x_1'+x_2'+\cdots+x_{n-2}' & x_{n-1}'\end{vmatrix}\right)\zeta\left(\begin{vmatrix}a_1'+\cdots+a_{n-1}' & a_n\\ x_1'+\cdots+x_{n-1}' & 0\end{vmatrix}\right)}{\begin{vmatrix}a_1' & a_2'\\ x_1' & x_2'\end{vmatrix}\cdots\begin{vmatrix}a_{n-2}' & a_{n-1}'\\ x_{n-2}' & x_{n-1}'\end{vmatrix}\begin{vmatrix}a_{n-1}' & a_n\\ x_{n-1}' & 0\end{vmatrix}}=$$
$$=\frac{\zeta(a_n(x_1+\cdots+x_{n-1}))}{a_n}\sum_{\substack{\sigma\in S_{n-1}\\ \sigma(1)=1}}x_2'\ldots x_{n-2}'\frac{\zeta\left(\begin{vmatrix}a_1' & a_2'\\ x_1' & x_2'\end{vmatrix}\right)\cdots\zeta\left(\begin{vmatrix}a_1'+a_2'+\cdots+a_{n-2}' & a_{n-1}'\\ x_1'+x_2'+\cdots+x_{n-2}' & x_{n-1}'\end{vmatrix}\right)}{\begin{vmatrix}a_1' & a_2'\\ x_1' & x_2'\end{vmatrix}\cdots\begin{vmatrix}a_{n-2}' & a_{n-1}'\\ x_{n-2}' & x_{n-1}'\end{vmatrix}}=$$
$$=\frac{\zeta(a_n(x_1+\cdots+x_{n-1}))}{a_n}P_{n-1}(a_1,\ldots,a_{n-1};x_1,\ldots,x_{n-1}).$$

The lemma is proved. □

Introduce the series $S(x):=\frac{\zeta(x)}{x}=1+\sum_{i\ge 1}\frac{x^{2i}}{2^{2i}(2i+1)!}$.

Lemma 2.2. *Let $n\ge 3$, $m\ge 0$ and $a_1,\dots,a_n,b_1,\dots,b_m$ be integers satisfying $\sum a_i+\sum b_j=0$. Then we have*

$$\sum_{g\ge 0}\sum_{d_1,\dots,d_n\ge 0}\left(\int_{\mathrm{DR}_g(a_1,\dots,a_n,b_1,\dots,b_m)}\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}=$$
$$=\frac{\prod_{i=1}^m S(b_iX)}{S(X)}X^{m-1}P_n(a_1,\dots,a_n;x_1,\dots,x_n),$$

where $X:=\sum x_i$.

Proof. Using equation (2.3) and Lemma 2.1, we compute

$$\begin{aligned}\sum_{g\ge 0}\sum_{d_1,\dots,d_n\ge 0}&\left(\int_{\mathrm{DR}_g(a_1,\dots,a_n,b_1,\dots,b_m)}\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}=\\
&=\left.\frac{P_{n+m}(a_1,\dots,a_n,b_1,\dots,b_m;x_1,\dots,x_{n+m})}{\zeta(x_1+\dots+x_{n+m})}\right|_{x_{n+1}=\dots=x_{n+m}=0}=\\
&=\frac{1}{\zeta(X)}\left(\prod_{i=1}^m\frac{\zeta(b_iX)}{b_i}\right)P_n(a_1,\dots,a_n;x_1,\dots,x_n)=\\
&=\frac{\prod_{i=1}^m S(b_iX)}{S(X)}X^{m-1}P_n(a_1,\dots,a_n;x_1,\dots,x_n).\end{aligned}$$

The lemma is proved. ∎

We keep the assumptions of the last lemma. Recall that by $\pi_m\colon\overline{\mathcal{M}}_{g,n+m}\to\overline{\mathcal{M}}_{g,n}$ we denote the forgetful map that forgets the last m marked points. For a subset $I\subset\{1,\dots,m\}$ let

$$B_I:=\sum_{i\in I}b_i.$$

The following proposition generalizes formula (2.3).

Proposition 2.3. *We have*

$$\begin{aligned}&\sum_{g\ge 0}\sum_{d_1,\dots,d_n\ge 0}\left(\int_{\pi_{m*}\mathrm{DR}_g(a_1,\dots,a_n,b_1,\dots,b_m)}\psi_1^{d_1}\cdots\psi_n^{d_n}\right)x_1^{d_1}\cdots x_n^{d_n}=\\
&=\frac{1}{S(X)}\sum_{I_0\sqcup I_1\sqcup\ldots\sqcup I_n=\{1,\dots,m\}}X^{|I_0|-1}\prod_{i=1}^n(-x_i)^{|I_i|}\prod_{i\in I_0}S(b_iX)P_n\left(a_1+B_{I_1},\dots,a_n+B_{I_n};x\right).\end{aligned}\tag{2.4}$$

Proof. For a subset $I=\{i_1,i_2,\dots,i_{|I|}\}\subset\{1,\dots,m\}$, where $i_1<i_2<\dots<i_{|I|}$, we denote by $\overline{b}_I$ the string $b_{i_1},b_{i_2},\dots,b_{i_{|I|}}$. Clearly, formula (2.4) follows from Lemma 2.2 and the equation

$$\int_{\pi_{m*}\mathrm{DR}_g(a_1,\dots,a_n,b_1,\dots,b_m)}\psi_1^{d_1}\cdots\psi_n^{d_n}=\sum_{\substack{\coprod_{i=0}^n I_i=\{1,\dots,m\}\\|I_i|\le d_i}}(-1)^{m-|I_0|}\int_{\mathrm{DR}_g\left(a_1+B_{I_1},\dots,a_n+B_{I_n},\overline{b}_{I_0}\right)}\prod_{i=1}^n\psi_i^{d_i-|I_i|}.\tag{2.5}$$

The proof of this equation is very similar to the proof of Proposition 2.3 in [BS11]. Let us first prove that for $m\ge 1$ we have

(2.6)
$$\int_{\pi_{m*}\mathrm{DR}_g(a_1,\ldots,a_n,b_1,\ldots,b_m)}\psi_1^{d_1}\cdots\psi_n^{d_n}=\int_{(\pi_{m-1})_*\mathrm{DR}_g(a_1,\ldots,a_n,b_m,b_1,\ldots,b_{m-1})}\psi_1^{d_1}\cdots\psi_n^{d_n}\\ -\sum_{\substack{1\le i\le n\\ d_i>0}}\int_{(\pi_{m-1})_*\mathrm{DR}_g(a_1,\ldots,a_i+b_m,\ldots,a_n,b_1,\ldots,b_{m-1})}\psi_i^{d_i-1}\prod_{j\ne i}\psi_j^{d_j}.$$

For this we write $\pi_m=\pi_1\circ\pi_{m-1}$, where $\pi_{m-1}\colon\overline{\mathcal{M}}_{g,n+m}\to\overline{\mathcal{M}}_{g,n+1}$ and $\pi_1\colon\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$. Then we have
$$\int_{\pi_{m*}\mathrm{DR}_g(a_1,\ldots,a_n,b_1,\ldots,b_m)}\psi_1^{d_1}\cdots\psi_n^{d_n}=\int_{(\pi_{m-1})_*\mathrm{DR}_g(a_1,\ldots,a_n,b_m,b_1,\ldots,b_{m-1})}\pi_1^*\left(\psi_1^{d_1}\cdots\psi_n^{d_n}\right).$$
For an integer $N\ge 2$ and a subset $J\in\{1,\ldots,N\}$, $|J|\ge 2$, denote by δ_0^J the cohomology class in $H^2(\overline{\mathcal{M}}_{g,N},\mathbb{Q})$ that is Poincaré dual to the divisor whose generic point is a nodal curve made of one smooth component of genus 0 with the marked points labeled by the set J and of another smooth component of genus g with the remaining marked points, joined at a separating node. If $1\le i\le n$ and $d\ge 1$, then we have $\pi_1^*\psi_i^d=\psi_i^d-\delta_0^{\{i,n+1\}}\pi_1^*\psi_i^{d-1}$. Therefore,
$$\pi_1^*\left(\psi_1^{d_1}\cdots\psi_n^{d_n}\right)=\psi_1^{d_1}\cdots\psi_n^{d_n}-\sum_{\substack{1\le i\le n\\ d_i>0}}\delta_0^{\{i,n+1\}}\pi_1^*\left(\psi_i^{d_i-1}\prod_{j\ne i}\psi_j^{d_j}\right).$$
We have (see [BSSZ15] and, in particular, Section 2.1 there for the explanation of the notation $\boxtimes$)
$$\delta_0^{\{i,n+1\}}\cdot(\pi_{m-1})_*\mathrm{DR}_g(a_1,\ldots,a_n,b_m,b_1,\ldots,b_{m-1})=\\ =(\pi_{m-1})_*\left(\mathrm{DR}_0(a_i,b_m,-a_i-b_m)\boxtimes\mathrm{DR}_g(a_1,\ldots,\widehat{a_i},\ldots,a_n,b_1,\ldots,b_{m-1},a_i+b_m)\right).$$
Therefore,
$$\int_{(\pi_{m-1})_*\mathrm{DR}_g(a_1,\ldots,a_n,b_m,b_1,\ldots,b_{m-1})}\delta_0^{\{i,n+1\}}\pi_1^*\left(\psi_i^{d_i-1}\prod_{j\ne i}\psi_j^{d_j}\right)=\\ =\int_{(\pi_{m-1})_*\mathrm{DR}_g(a_1,\ldots,a_i+b_m,\ldots,a_n,b_1,\ldots,b_{m-1})}\psi_i^{d_i-1}\prod_{j\ne i}\psi_j^{d_j}.$$
This completes the proof of equation (2.6).

It is easy to see that, applying formula (2.6) m times, we get equation (2.5). The proposition is proved. □

Finally, let us formulate an important geometric property of the double ramification cycle that we will use in the next section. Let $g\ge 0$, $n\ge 3$ and $a_1,\ldots,a_n,b_1,\ldots,b_g$ be integers satisfying $\sum a_i+\sum b_j=0$. Then we have (see [BSSZ15, Example 3.7])

(2.7) $$\pi_{g*}\mathrm{DR}_g(a_1,\ldots,a_n,b_1,\ldots,b_g)=g!b_1^2\cdots b_g^2[\overline{\mathcal{M}}_{g,n}],$$

(2.8) $$\pi_{g*}\mathrm{DR}_m(a_1,\ldots,a_n,b_1,\ldots,b_g)=0,\qquad m<g.$$

The last property implies that the coefficient of a monomial $x_1^{d_1}\cdots x_n^{d_n}$ in the generating series (2.4) is zero, if $\sum d_i<3m-3+n$. We will use this observation in the next section.

3. Proof of Theorem 1.1

We divide this section in two parts. In Section 3.1 we use the string equation for the intersection numbers (1.1) in order to reduce Theorem 1.1 to the case $n\ge 3$. We treat this case in Section 3.2.

3.1. **Reduction to the case $n \geq 3$.** The intersection numbers $\left\langle \tau_{d_1} \cdots \tau_{d_n} \right\rangle_g$ satisfy the string equation

$$\left\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \right\rangle_g = \sum_{\substack{1 \le i \le n \\ d_i > 0}} \left\langle \tau_{d_i - 1} \prod_{j \ne i} \tau_{d_j} \right\rangle_g ,$$

with one exceptional case $\left\langle \tau_0^3 \right\rangle_0 = 1$. The string equation implies that for $n \geq 2$ we have

$$\mathcal{F}(x_1, \ldots, x_n)|_{x_n = 0} = (x_1 + \cdots + x_{n-1}) \mathcal{F}(x_1, \ldots, x_{n-1}) + \delta_{n,3}. \tag{3.1}$$

Denote the right-hand side of (1.3) by $\mathcal{G}(x_1, \ldots, x_n)$. Let us show that the function $\mathcal{G}$ also satisfies equation (3.1). We compute

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^n \sqrt{2\pi x_i}} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n \frac{a_i^2}{2x_i}} P_n\left(\sqrt{-1} a_1, \ldots, \sqrt{-1} a_n; x_1, \ldots, x_n\right) da_1 \cdots da_n \Big|_{x_n = 0} = \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n \frac{a_i^2}{2}} P_n\left(\sqrt{-x_1} a_1, \ldots, \sqrt{-x_n} a_n; x_1, \ldots, x_n\right) da_1 \cdots da_n \Big|_{x_n = 0} \overset{\text{by Lemma 2.1}}{=} \\
&= \frac{x_1 + \cdots + x_{n-1}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n \frac{a_i^2}{2}} P_{n-1}\left(\sqrt{-x_1} a_1, \ldots, \sqrt{-x_{n-1}} a_{n-1}; x_1, \ldots, x_{n-1}\right) da_1 \cdots da_n = \\
&= \frac{x_1 + \cdots + x_{n-1}}{\prod_{i=1}^{n-1} \sqrt{2\pi x_i}} \int_{\mathbb{R}^{n-1}} e^{-\sum_{i=1}^{n-1} \frac{a_i^2}{2x_i}} P_{n-1}\left(\sqrt{-1} a_1, \ldots, \sqrt{-1} a_{n-1}; x_1, \ldots, x_{n-1}\right) da_1 \cdots da_{n-1}.
\end{aligned}$$

Thus, we obtain

$$\mathcal{G}(x_1, \ldots, x_n)|_{x_n = 0} = (x_1 + \cdots + x_{n-1}) \mathcal{G}(x_1, \ldots, x_{n-1}) + \delta_{n,3}. \tag{3.2}$$

We see that if equation (1.3) holds for $n = n_0$, then it holds for all $n \leq n_0$. Hence, it is sufficient to prove equation (1.3) for $n \geq 3$.

3.2. **Case $n \geq 3$.** We assume that $n \geq 3$. Let $g \geq 0$ and consider arbitrary integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_g$. Let

$$A := \sum_{i=1}^n a_i, \qquad B := \sum_{i=1}^g b_i, \qquad X := \sum_{i=1}^n x_i.$$

Equations (2.7) and (3.1) imply that

$$\begin{aligned}
g! b_1^2 \cdots b_g^2 X \mathcal{F}_g(x_1, \ldots, x_n) &= g! b_1^2 \cdots b_g^2 \, \mathcal{F}_g(x_1, \ldots, x_n, x_{n+1})|_{x_{n+1} = 0} = \\
&= \sum_{d_1, \ldots, d_n \ge 0} \left(\int_{\pi_{g*} \mathrm{DR}_g(a_1, \ldots, a_n, -A-B, b_1, \ldots, b_g)} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) x_1^{d_1} \cdots x_n^{d_n}
\end{aligned} \tag{3.3}$$

independently of $a_1, \ldots, a_n$. Moreover, by the remark at the end of Section 2, expression (3.3) is equal to the term of the lowest degree $3g - 2 + n$ with respect to the x-variables in the series

$$\sum_{m \ge 0} \sum_{d_1, \ldots, d_n \ge 0} \left(\int_{\pi_{g*} \mathrm{DR}_m(a_1, \ldots, a_n, -A-B, b_1, \ldots, b_g)} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) x_1^{d_1} \cdots x_n^{d_n}. \tag{3.4}$$

Proposition 2.3 and Lemma 2.1 imply that

$$\begin{aligned}
&\sum_{m \ge 0} \sum_{d_1, \ldots, d_n \ge 0} \left(\int_{\pi_{g*} \mathrm{DR}_m(a_1, \ldots, a_n, -A-B, b_1, \ldots, b_g)} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) x_1^{d_1} \cdots x_n^{d_n} = \\
&= \frac{S((-A-B)X)}{S(X)} \sum_{\coprod_{i=0}^n I_i = \{1, \ldots, g\}} X^{|I_0|} \prod_{i=1}^n (-x_i)^{|I_i|} \prod_{i \in I_0} S(b_i X) P_n\left(a_1 + B_{I_1}, \ldots, a_n + B_{I_n}; x\right).
\end{aligned} \tag{3.5}$$

In the last expression the multiplication by $\frac{S((-A-B)X)}{S(X)}$ doesn't change the lowest degree term with respect to the x-variables. Therefore, we obtain

$$
\begin{aligned}
&g!b_1^2\cdots b_g^2 X\mathcal{F}_g(x_1,\ldots,x_n)=\\
&=\left[\sum_{\coprod_{i=0}^n I_i=\{1,\ldots,g\}} X^{|I_0|}\prod_{i=1}^n(-x_i)^{|I_i|}\prod_{i\in I_0}S(b_iX)P_n\left(a_1+B_{I_1},\ldots,a_n+B_{I_n};x\right)\right]_{3g-2+n}=\\
&=\left[\sum_{\coprod_{i=0}^n I_i=\{1,\ldots,g\}} X^{|I_0|}\prod_{i=1}^n(-x_i)^{|I_i|}\prod_{i\in I_0}S(b_iX)P_n\left(B_{I_1},\ldots,B_{I_n};x\right)\right]_{3g-2+n},
\end{aligned}
$$

where by $[\cdot]_d$ we denote the degree d part with respect to the x-variables. Equivalently, we can write

$$
\begin{aligned}
(3.6)\quad & g!X\mathcal{F}_g(x_1,\ldots,x_n)=\\
&=\left[\mathrm{Coef}_{b_1^2\cdots b_g^2}\sum_{\coprod_{i=0}^n I_i=\{1,\ldots,g\}} X^{|I_0|}\prod_{i=1}^n(-x_i)^{|I_i|}\prod_{i\in I_0}S(b_iX)P_n\left(B_{I_1},\ldots,B_{I_n};x\right)\right]_{3g-2+n}.
\end{aligned}
$$

From the definition (1.2) we see that the series $P_n(a_1,\ldots,a_n;x_1,\ldots,x_n)$ satisfies the following homogeneity property:

$$
P_n(\lambda^{-1}a_1,\ldots,\lambda^{-1}a_n;\lambda x_1,\ldots,\lambda x_n)=\lambda^{n-2}P_n(a_1,\ldots,a_n;x_1,\ldots,x_n).
$$

Therefore, the expression in the square brackets on the right-hand side of (3.6) has automatically degree $3g-2+n$ in the x-variables. Since $S(z)=1+\frac{z^2}{24}+\ldots$, we have

$$
\mathrm{Coef}_{b_1^2\cdots b_g^2}\left(\prod_{i\in I_0}S(b_iX)P_n\left(B_{I_1},\ldots,B_{I_n};x\right)\right)=\frac{X^{2|I_0|}}{24^{|I_0|}}\mathrm{Coef}_{\prod_{i\notin I_0}b_i^2}P_n\left(B_{I_1},\ldots,B_{I_n};x\right).
$$

Note that

$$
\mathrm{Coef}_{\prod_{i\notin I_0}b_i^2}\prod_{i=1}^n B_{I_i}^{d_i}=\begin{cases}\prod_{i=1}^n\frac{d_i!}{2^{d_i/2}}, & \text{if } d_i=2|I_i|,\\ 0, & \text{otherwise.}\end{cases}
$$

Therefore, we get

$$
\begin{aligned}
&\mathrm{Coef}_{b_1^2\cdots b_g^2}\sum_{\coprod_{i=0}^n I_i=\{1,\ldots,g\}} X^{|I_0|}\prod_{i=1}^n(-x_i)^{|I_i|}\prod_{i\in I_0}S(b_iX)P_n\left(B_{I_1},\ldots,B_{I_n};x\right)=\\
&=\sum_{\coprod_{i=0}^n I_i=\{1,\ldots,g\}}\frac{X^{3|I_0|}}{24^{|I_0|}}\prod_{i=1}^n\left((-x_i)^{|I_i|}\frac{(2|I_i|)!}{2^{|I_i|}}\right)\mathrm{Coef}_{\prod_{i=1}^n a_i^{2|I_i|}}P_n(a;x)=\\
&=\sum_{\substack{m_0,m_1,\ldots,m_n\geq 0\\ m_0+\cdots+m_n=g}}\frac{g!}{m_0!\cdots m_n!}\left(\frac{X^3}{24}\right)^{m_0}\prod_{i=1}^n\left((-x_i)^{m_i}\frac{(2m_i)!}{2^{m_i}}\right)\mathrm{Coef}_{\prod_{i=1}^n a_i^{2m_i}}P_n(a;x)=\\
&=\sum_{\substack{m_0,m_1,\ldots,m_n\geq 0\\ m_0+\cdots+m_n=g}}\frac{g!}{m_0!}\left(\frac{X^3}{24}\right)^{m_0}\prod_{i=1}^n\left((-x_i)^{m_i}(2m_i-1)!!\right)\mathrm{Coef}_{\prod_{i=1}^n a_i^{2m_i}}P_n(a;x).
\end{aligned}
$$

Finally, we obtain

$$X\mathcal{F}(x_1,\dots,x_n) = \sum_{g\geq 0} X\mathcal{F}_g(x_1,\dots,x_n) =$$

$$= \sum_{m_0,m_1,\dots,m_n\geq 0} \frac{1}{m_0!}\left(\frac{X^3}{24}\right)^{m_0} \prod_{i=1}^n ((-x_i)^{m_i}(2m_i-1)!!)\,\mathrm{Coef}_{\prod_{i=1}^n a_i^{2m_i}} P_n(a;x) \overset{\text{by (1.4)}}{=}$$

$$= \frac{e^{\frac{X^3}{24}}}{\prod\sqrt{2\pi x_i}} \int_{\mathbb{R}^n} e^{-\sum \frac{a_i^2}{2x_i}} P_n\left(\sqrt{-1}a;x\right) da.$$

This completes the proof of the theorem.

4. Examples

For $n=1$ Theorem 1.1 immediately gives the well-known formula (see e.g. [FP00])

$$\mathcal{F}(x_1) = \frac{e^{\frac{x_1^3}{24}}-1}{x_1^2}.$$

Suppose $n=2$. We have

$$P_2(a_1,a_2;x_1,x_2) = \frac{\zeta(a_1x_2-a_2x_1)}{a_1x_2-a_2x_1} = S(a_1x_2-a_2x_1) = \sum_{n\geq 0}\frac{(a_1x_2-a_2x_1)^{2n}}{(2n+1)!2^{2n}} =$$

$$= \sum_{n\geq 0}\sum_{m_1+m_2=2n} \frac{(a_1x_2)^{m_1}(-a_2x_1)^{m_2}}{(2n+1)2^{2n}m_1!m_2!}.$$

Therefore, we obtain

$$\frac{e^{\frac{(x_1+x_2)^3}{24}}}{2\pi\sqrt{x_1x_2}} \int_{\mathbb{R}^2} e^{-\frac{a_1^2}{2x_1}-\frac{a_2^2}{2x_2}} P_2(\sqrt{-1}a_1,\sqrt{-1}a_2;x_1,x_2)da =$$

$$= \frac{e^{\frac{(x_1+x_2)^3}{24}}}{2\pi\sqrt{x_1x_2}} \sum_{n\geq 0}\sum_{m_1+m_2=n} \int_{\mathbb{R}^2} e^{-\frac{a_1^2}{2x_1}-\frac{a_2^2}{2x_2}} \left(\frac{(-1)^n a_2^{2m_1}a_1^{2m_2}x_1^{2m_1}x_2^{2m_2}}{2^{2n}(2n+1)(2m_1)!(2m_2)!}\right) da =$$

$$= e^{\frac{(x_1+x_2)^3}{24}} \sum_{n\geq 0}\sum_{m_1+m_2=n} \frac{(-1)^n x_2^{m_1}x_1^{m_2}x_1^{2m_1}x_2^{2m_2}}{2^{3n}(2n+1)m_1!m_2!} =$$

$$= e^{\frac{x_1^3+x_2^3}{24}} e^{\frac{x_1x_2(x_1+x_2)}{8}} \sum_{n\geq 0} \frac{(-1)^n(x_1x_2(x_1+x_2))^n}{8^n(2n+1)n!} =$$

$$= e^{\frac{x_1^3+x_2^3}{24}} \sum_{n\geq 0}\left(\frac{x_1x_2(x_1+x_2)}{8}\right)^n \sum_{m_1+m_2=n}\frac{(-1)^{m_2}}{m_1!m_2!(2m_2+1)}.$$

Let $C_n := \sum_{m_1+m_2=n}\frac{(-1)^{m_2}}{m_1!m_2!(2m_2+1)}$. For $n\geq 1$ we compute

$$C_n = \frac{1}{n!}\int_0^1 (1-y^2)^n dy = \frac{2}{(n-1)!}\int_0^1 y^2(1-y^2)^{n-1}dy = -2nC_n + 2C_{n-1}.$$

Therefore, $C_n = \frac{2}{2n+1}C_{n-1}$, and, since $C_0=1$, we get $C_n = \frac{2^n}{(2n+1)!!}$. Hence, we obtain

$$\frac{e^{\frac{(x_1+x_2)^3}{24}}}{2\pi\sqrt{x_1x_2}} \int_{\mathbb{R}^2} e^{-\frac{a_1^2}{2x_1}-\frac{a_2^2}{2x_2}} P_2(\sqrt{-1}a_1,\sqrt{-1}a_2;x_1,x_2)da = e^{\frac{x_1^3+x_2^3}{24}} \sum_{n\geq 0}\frac{n!}{(2n+1)!}\left(\frac{x_1x_2(x_1+x_2)}{2}\right)^n.$$

As a result, we get the following formula for the two-point function:

$$\mathcal{F}(x_1,x_2)=\frac{e^{\frac{x_1^3+x_2^3}{24}}}{(x_1+x_2)}\sum_{n\ge 0}\frac{n!}{(2n+1)!}\left(\frac{x_1x_2(x_1+x_2)}{2}\right)^n-\frac{1}{x_1+x_2}.$$

This formula was found by R. Dijkgraaf (see e.g. [FP00]).

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