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MULTIPLICITY BOUNDS IN PRIME CHARACTERISTIC

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Dedicated to Gennady Lyubeznik on the occasion of his sixtieth birthday

Abstract. We extend a result by Huneke and Watanabe ([HW15]) bounding the multiplicity of $F$-pure local rings of prime characteristic in terms of their dimension and embedding dimensions to the case of $F$-injective, generalized Cohen-Macaulay rings. We then produce an upper bound for the multiplicity of any local Cohen-Macaulay ring of prime characteristic in terms of their dimensions, embedding dimensions and HSL numbers. Finally, we extend the upper bounds for the multiplicity of generalized Cohen-Macaulay rings in characteristic zero which have dense $F$-injective type.

1. Introduction

In [HW15], Huneke and Watanabe proved that, if $R$ is a noetherian, $F$-pure local ring of dimension $d$ and embedding dimension $v$, then $e(R) \leq \binom{v}{d}$ where $e(R)$ denotes the Hilbert-Samuel multiplicity of $R$. The following was left as an open question in [HW15, Remark 3.4]:

Question 1.1 (Huneke-Watanabe). Let $R$ be a noetherian $F$-injective local ring with dimension $d$ and embedding dimension $v$. Is it true that $e(R) \leq \binom{v}{d}$?

In this note, we answer this question in the affirmative when $R$ is generalized Cohen-Macaulay.

Theorem 1.1. Let $R$ be a $d$-dimensional noetherian $F$-injective generalized Cohen-Macaulay local ring of embedding dimension $v$. Then

$$e(R) \leq \binom{v}{d}.$$ 

Using reduction mod $p$, one can prove an analogous result for generalized Cohen-Macaulay rings of dense $F$-injective type in characteristic 0, cf. Theorem 5.2.

We also generalize these result to Cohen-Macaulay, non-$F$-injective rings as follows.

Definition 1.2 (cf. section 4 in [Lyu97]). Let $H$ be an $A$ module with Frobenius map $\theta : H \rightarrow H$ (i.e., an additive map such that $\theta(ah) = a^p\theta(h)$ for all $a \in A$ and $h \in H$). Write $\text{Nil} H = \{h \in H \mid \theta^e h = 0 \text{ for some } h \geq 0\}$. The Hartshorne-Speiser-Lyubeznik number (henceforth abbreviated HSL number) is defined as

$$\inf\{e \geq 0 \mid \theta^e \text{Nil} H = 0\}.$$ 

The HSL number of a local, Cohen-Macaulay ring $(R, \mathfrak{m})$ is defined as the HSL number of the top local cohomology module $H^\text{dim} \mathfrak{m} R(R)$ with its natural Frobenius map.

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For artinian modules over a quotient of a regular ring, HSL numbers are finite. ([Lyu97, Proposition 4.4]).

Without the $F$-injectivity assumption, we have the following upper bound in the Cohen-Macaulay case which involves the HSL number of $R$.

**Theorem 1.2** (Theorem 3.1). Assume that $(R, \mathfrak{m})$ is a reduced, Cohen-Macaulay noetherian local ring of dimension $d$ and embedding dimension $v$. Let $\eta$ be the HSL number of $R$ and write $Q = p^\eta$. Then

$$e(R) \leq Q^{v-d} \binom{v}{d}.$$

This bound is asymptotically sharp as shown in Remark 3.2.

2. Bounds on $F$-injective rings

For each commutative noetherian ring $R$, let $R^o$ denote the set of elements of $R$ that are not contained in any minimal prime ideal of $R$.

**Remark 2.1.** If $R$ is a reduced noetherian ring, then each $c \in R^o$ is a non-zero-divisor.

We begin with a Skoda-type theorem for $F$-injective rings which may be viewed as a generalization of [HW15, Theorem 3.1].

**Theorem 2.2.** Let $(R, \mathfrak{m})$ be a commutative noetherian ring of characteristic $p$ and let $\mathfrak{a}$ be an ideal that can be generated by $\ell$ elements. Assume that each $c \in R^o$ is a non-zero-divisor. Then

$$\mathfrak{a}^{\ell+1} \subseteq \mathfrak{a}^F,$$

where $\mathfrak{a}^{\ell+1}$ is the integral closure of $\mathfrak{a}^{\ell+1}$ and $\mathfrak{a}^F$ the Frobenius closure of $\mathfrak{a}$.

**Proof.** For each $x \in \mathfrak{a}^{\ell+1}$ pick $c \in R^o$ such that for $N \gg 1$, $cx^N \in \mathfrak{a}^{(\ell+1)N}$ ([HS06, Corollary 6.8.12]). Note that $c$ is a non-zero-divisor by our assumptions. We have $cx^N \in c(\mathfrak{a}^{(\ell+1)N} : c) \subseteq cR \cap \mathfrak{a}^{(\ell+1)N}$. An application of the Artin-Rees Lemma given $a$ such that $cx^N \in c\mathfrak{a}^{(\ell+1)N-k}$ for all large $N$, and so $x^N \in \mathfrak{a}^{(\ell+1)N-k}$ for all large $N$. For any large enough $N = p^r$ we have $x^F \in \mathfrak{a}^{p^r}$, i.e., $x$, and hence $\mathfrak{a}^{\ell+1}$ is in the Frobenius closure of $\mathfrak{a}$. □

**Corollary 2.3.** Let $(R, \mathfrak{m})$ be a $d$-dimensional noetherian local ring of characteristic $p$. Assume that $\mathfrak{m}$ admits a minimal reduction $J$. Then

(a) $\mathfrak{m}^{d+1} \subseteq \overline{\mathfrak{m}^{d+1}} = \overline{J^{d+1}} \subseteq J^F$, and
(b) $\ell(R) \leq \binom{v}{d} + \ell(J^F / J)$.

**Proof.** Since $\mathfrak{m}^{d+1} \subseteq \overline{\mathfrak{m}^{d+1}} = \overline{J^{d+1}}$, (a) follows from Theorem 2.2.

For part (b), since $\overline{J^{d+1}} \subseteq J^F$ and $J$ is generated by $d$ elements, we have $\ell(R/J^F) \leq \binom{v}{d}$ (as in the proof of [HW15, Theorem 3.1]). Then

$$\ell(R) \leq \ell(R/J) + \ell(J^F / J) \leq \binom{v}{d} + \ell(J^F / J).$$

□
Proof of Theorem 3.1. Let \( \hat{R} \) denote the completion of \( R \). Then \( R \) is \( F \)-injective and generalized Cohen-Macaulay if and only if \( \hat{R} \) is so, and \( e(R) = e(\hat{R}) \). Hence we may assume that \( R \) is complete. Since \( R \) is \( F \)-injective, it is reduced ([SZ13 Remark 2.6]) and hence each \( c \in R^n \) is a non-zero-divisor by Remark 2.7. It is proved in [Ma15 Theorem 1.1] that a generalized Cohen-Macaulay local ring is \( F \)-injective if and only if every parameter ideal is Frobenius closed. Let \( J \) denote a minimal reduction of \( \mathfrak{m} \), then \( J^F = J \). Our corollary follows immediately from Corollary 2.3.

3. Bounds on multiplicity using HSL numbers

**Theorem 3.1.** Assume that \((R, \mathfrak{m})\) is a reduced, Cohen-Macaulay noetherian local ring of dimension \( d \) and embedding dimension \( v \). Let \( \eta \) be the HSL number of \( R \) and write \( \eta = \binom{v}{d} \).

**Proof.** We may assume that \( R \) is complete since \( e(R) = e(\hat{R}) \). Hence \( \mathfrak{m} \) admits a minimal reduction \( J \) (generated by \( d \) elements). We have \( e(R) = \ell(R/J) \), and Theorem 2.2 shows that \( \mathfrak{m}^{d+1} \subseteq J^F \). Now \( (J^F)^[Q] = J^{[Q]} \) for \( Q = p^n \) hence \( (\mathfrak{m}^{d+1})^{[Q]} \subseteq J^{[Q]} \).

Extend a set of minimal generators \( x_1, \ldots, x_d \) of \( J \) to a minimal set of generators \( x_1, \ldots, x_d, y_1, \ldots y_{v-d} \) of \( \mathfrak{m} \). Now \( R/J^{[Q]} \) is spanned by monomials

\[
x_1^{\gamma_1} \cdots x_d^{\gamma_d} y_1^{\alpha_1 Q + \beta_1} \cdots y_{v-d}^{\alpha_{v-d} Q + \beta_{v-d}}
\]

where \( 0 \leq \gamma_1, \ldots, \gamma_d, \beta_1, \ldots, \beta_{v-d} < Q \) and \( 0 \leq \alpha_1 + \cdots + \alpha_{v-d} < d + 1 \). The number of such monomials is \( Q^d \binom{v}{d} \) and so \( \ell(R/J^{[Q]}) \leq Q^d \binom{v}{d} \).

Note that as \( J \) is generated by a regular sequence, \( \ell(R/J^{[Q]}) = Q^d \ell(R/J) \) and we conclude that

\[
\ell(R/J) = \frac{\ell(R/J^{[Q]})}{Q^d} \leq Q^{v-d} \binom{v}{d}.
\]

**Remark 3.2.** The next family of examples shows that the bound in Theorem 3.1 is asymptotically sharp.

Let \( \mathbb{F} \) be a field of prime characteristic \( p \), let \( n \geq 2 \), and let \( S = \mathbb{F}[x_1, \ldots, x_n] \). Let \( E \) denote the injective hull of the residue field of \( S_m \).

Define \( f = \sum_{i=1}^n x_i^p \cdots x_{i-1}^p x_{i+1}^p \cdots x_n^p \) and \( h = x_1 \ldots x_{n-1} \). We claim that \( f \) is square-free: if this is not the case write \( f = r^a s \) where \( r \) is irreducible of positive degree, and \( \alpha \geq 2 \). Let \( \partial \) denote the partial derivative with respect to \( x_n \). Note that \( \partial f = h^p \) and so

\[
h^p = \alpha r^{a-1} (\partial r)s + r^a (\partial s) = r^{a-1} (\alpha (\partial r)s + r (\partial s)).
\]

We deduce that \( r \) divides \( h \), but this would imply that \( x_i^2 \) divides all terms of \( f \) for some \( 1 \leq i \leq n - 1 \), which is false. We conclude that \( S/fS \) is reduced.

Let \( R \) be the localization of \( S/fS \) at \( m = (x_1, \ldots, x_n)S \). We compute next the HSL number \( \eta \) of \( R \) using the method described in sections 4 and 5 in [Kat08]. It is not hard to show that \( H_{\mathfrak{m}_R}^{-1}(R) \cong \text{ann}_E f \) where \( E = H_{\mathfrak{m}_S}^{-1}(S) \), and that, after identifying these, the natural Frobenius action on \( \text{ann}_E f \) is given by \( f^{p-1}T \) where \( T \) is the natural Frobenius action on \( E \).
To find the HSL number $\eta$ of $H^m_n(R)$ we readily compute $I_1(f)$ to be the ideal generated by \{\$x_1\ldots x_{i-1}x_{i+1}\ldots x_n | 1 \leq i \leq n$\} and 

\[
I_2(f^{p+1}) = I_1(f I_1(f)) = \sum_{i=1}^{n} I_1 \left( \sum_{j=1}^{i-1} x_i^{p+1} \ldots x_j^{p+1} x_{i-1}^{p+1} x_{i+1}^{p+1} \ldots x_n^{p+1} \right) + x_1^{p+1} \ldots x_i^{p+1} x_{i+1}^{p+1} \ldots x_n^{p+1} + \sum_{j=i+1}^{n} x_1^{p+1} \ldots x_{i-1}^{p+1} x_i^{p+1} \ldots x_{j-1}^{p+1} x_j^{p+1} \ldots x_n^{p+1} \right) = I_1(f)
\]

and we deduce that $\eta = 1$.

We now compute 

\[
\Gamma_{n,p} := \frac{\deg f}{(n-1)p^n} = \frac{(n-1)p + 1}{np}.
\]

We have $\lim_{n \to \infty} \Gamma_{n,p} = 1$ and $\lim_{p \to \infty} \Gamma_{n,p} = (n-1)/n$, so we can find values of $\Gamma_{n,p}$ arbitrarily close to 1.

### 4. Examples

The injectivity of the natural Frobenius action on the top local cohomology $H^d_m(R)$ does not imply $\epsilon(R) \leq \binom{\ell}{d}$ as shown by the following example.

**Example 4.1.** Let $S = \mathbb{Z}/2\mathbb{Z}[x, y, u, v]$, let $m$ be its ideal generated by the variables, define $I = (v, x) \cap (u, x) \cap (v, y) \cap (u, y) \cap (v, u) \cap (y - u, x - v) = (vx(y - u), yu(x - v), yuv(y - u), xuv(x - v))$, and let $R = S/I$: this is a reduced 2-dimensional ring.

We compute the following graded $S$-free resolution of $I$

\[
0 \longrightarrow S(-6) \overset{B}{\longrightarrow} S^4(-5) \overset{A}{\longrightarrow} S^2(-3) \oplus S^2(-4) \longrightarrow I \longrightarrow 0
\]

where

\[
A = \begin{bmatrix} u(x - v) & yu & 0 & 0 \\ 0 & 0 & xv & v(y - u) \\ 0 & -x & 0 & v - x \\ u - y & 0 & -y & 0 \end{bmatrix}, \quad B = \begin{bmatrix} y \\ v - x \\ u - y \\ x \end{bmatrix}
\]

and note that $R$ has projective dimension 3, hence depth 1 and so it is not Cohen-Macaulay. Also, we can read the Hilbert series of $R$ from its graded resolution and we obtain

\[
\frac{1 - 2t^3 - 2t^4 + 4t^5 - t^6}{(1 - t)^4} = \frac{1 + 2t + 3t^2 + 2t^3 - t^4}{(1 - t^2)}
\]

and so the multiplicity of $R$ is $1 + 2 + 3 + 2 - 1 = 7$ exceeding $\binom{4}{2} = 6$ (cf. [HHIT1\ §6.1.2].)

Note that $R$ is not $F$-injective, but the natural Frobenius action on the top local cohomology module is injective.
From the proof of Theorem 1.1, we can see that if a minimal reduction of the maximal ideal in an $F$-injective local ring $R$ is Frobenius closed then the bound $e(R) \leq \binom{\text{dim} R}{2}$ will hold. Hence we may ask whether minimal reductions would be Frobenius-closed in such rings (cf. Theorem 6.5 and Problem 3 in [QS17]). However, the following example shows this not to be the case.

**Example 4.2.** Let $S = \mathbb{Z}/2\mathbb{Z}[x, y, u, v, w]$, let $\mathfrak{m}$ be its ideal generated by the variables and let $I_1 = (x, y)\cap(x+y, u+w, v+w)$, $I_2 = (u, v, w)\cap(x, u, v)\cap(y, u, v) = (u, v, xyw)$, and $I = I_1 \cap I_2$. Fedder’s Criterion [Fed83, Proposition 1.7] shows that $S/I_1$, $S/I_2$ and $S/(I_1 + I_2)$ are $F$-pure, and [QS17, Theorem 5.6] implies that $S/I$ is $F$-injective. Also, $S/I$ is almost Cohen-Macaulay: it is 3-dimensional and its localization at $\mathfrak{m}$ has depth 2.

Its not hard to check that the ideal $J$ generated by the images in $S$ of $u, v, w$ is a minimal reduction. However $J^F \neq J$: while $v^2 \notin J$, we have

$$v^4 = x y w^2 + v^2(y + v)^2 + y e w(x + y) + (v + w)(y^2 v + x y w),$$

hence $v^2 \in J^F \setminus J$.

5. **Bounds in Characteristic zero**

Throughout this section $K$ will denote a field of characteristic zero, $T = K[x_1, \ldots, x_n]$, $R$ will denote the finitely generated $K$-algebra $R = T/I$ for some ideal $I \subseteq T$, and $\mathfrak{m} = (x_1, \ldots, x_n)R$; $d$ and $v$ will denote the dimension and embedding dimension, respectively, of $R_\mathfrak{m}$. We also choose $y = y_1, \ldots, y_d \in \mathfrak{m}$ whose images in $R_\mathfrak{m}$ form a minimal reduction of $\mathfrak{m}R_\mathfrak{m}$.

We may, and do assume that the only maximal ideal containing $y$ is $\mathfrak{m}$. Otherwise, if $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are all the maximal ideals distinct from $\mathfrak{m}$ which contain $y$, we can pick $f \in (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r) \setminus \mathfrak{m}$, and now the only maximal ideal containing $y$ in $R_f$ is $\mathfrak{m}R_f$. We may now replace $R$ with $R' = K[x_1, \ldots, x_n, x_{n+1}] / I + \langle x_{n+1} f - 1 \rangle \cong R_f$ and since $R_\mathfrak{m} = (R_f)_{\mathfrak{m}}$ we are not affecting any local issues.

The main tool used in this section descent techniques described in [HH06]. We start by introducing a flavour of it useful for our purposes.

**Definition 5.1.** By descent objects we mean

1. a finitely generated $K$-algebra $R$ as above,
2. a finite set of finitely generated $T$-modules,
3. a finite set of $T$ linear maps between $T$-modules in (2),
4. a finite set of finite complexes involving maps in (3),

By descent data for these descent objects we mean

(a) A finitely generated $\mathbb{Z}$-subalgebra $A$ of $K$, $T_A = A[x_1, \ldots, x_n]$, $I_A \subseteq T_A$ such that with $R_A = T_A/I_A$

   - $R_A \subseteq R$ induces an isomorphism $R_A \otimes_A K \cong R \otimes_A K = R$, and
   - $R_A$ is $A$-free.

(b) For each $M$ in (2), a finitely generated free $A$-submodule $M_A \subseteq M$ such that this inclusion induces an isomorphism $M_A \otimes_A K \cong M \otimes_A K = M$.

(c) For every $\phi : M \to N$ in (3) an $A$ linear map $\phi_A : M_A \to N_A$ such that

   - $\phi_A \otimes 1 : M_A \otimes_K K \to N_A \otimes_K K$ is the map $\phi$, and
   - $\text{Im} \phi$, $\text{Ker} \phi$ and $\text{Coker} \phi$ are $A$-free.
(d) For every homological complex
\[ \mathcal{C}_\bullet = \cdots \to \partial_{i+2} C_{i+1} \to \partial_{i+1} C_i \to \partial_i \to \cdots \]
in (4), an homological complex
\[ \mathcal{C}_{A\bullet} = \cdots \to (\partial_{i+2})_{A\lambda} C_{i+1} A \to (\partial_{i+1})_{A\lambda} C_i A \to (\partial_i)_{A\lambda} \to \cdots \]
such that \( H_i(\mathcal{C}_{A\lambda} \otimes_A K) = H_i(\mathcal{C}_A) \otimes_A K \). For every cohomological complex in (4), a similar corresponding contraction.

Descent data exist: see [HH06 Chapter 2].

Notice that for any maximal ideal \( \mathfrak{p} \subset A \), the fiber \( \kappa(\mathfrak{p}) = A/\mathfrak{p} \) is a finite field. Given any property \( \mathcal{P} \) of rings of prime characteristic, we say that \( R \) as in the definition above as dense \( \mathcal{P} \) type if there exists descent data \((A, R_A)\) and such that for all maximal ideals \( \mathfrak{p} \subset A R_A \otimes_A \kappa(\mathfrak{p}) \) has property \( \mathcal{P} \).

Notice also that for any maximal ideal \( \mathfrak{p} \subset A R_A \otimes_A \kappa(\mathfrak{p}) \) has property \( \mathcal{P} \).

The main result in this section is the following theorem.

**Theorem 5.2.** If \( R_m \) is Cohen-Macaulay on the punctured spectrum and has dense F-injective type, then \( e(R_m) \leq \binom{n}{d} \).

**Lemma 5.3.** There exists descent data \((A, R_A)\) for \( R \) with the following properties.

(a) \( y_1, \ldots, y_d \in R_A \),

(b) for all maximal ideals \( \mathfrak{p} \subset A \) the images of \( y_1, \ldots, y_d \) in \( R_\kappa(\mathfrak{p}) \) are a minimal reduction of \( \mathfrak{m} R_\kappa(\mathfrak{p}) \),

(c) if \( R_m \) is Cohen-Macaulay on its punctured spectrum, so is \( R_{\kappa(\mathfrak{p})} \) for all maximal ideals \( \mathfrak{p} \subset A \).

(d) if \( R_m \) is unmixed, so is \( R_\mathfrak{p} \) for all maximal ideals \( \mathfrak{p} \subset A \).

**Proof.** Start with some descent data \((A, R_A)\) where \( A \) contains all K-coefficients among a set of generators \( g_1, \ldots, g_d \) of \( I \), \( I_A \) is the ideal of \( A[x_1, \ldots, x_n] \) generated by \( g_1, \ldots, g_d \) and \( R_A = A[x_1, \ldots, x_n]/I_A \). Let \( x \) and \( x \) denote \( (x_1, \ldots, x_n) \). For (a) write \( y_i = Q_i(x_1, \ldots, x_n) + I \) for all \( 1 \leq i \leq d \) and extend \( A \) to include all the K-coefficients in \( Q_1, \ldots, Q_d \).

Assume that \( \mathfrak{m}^{s+1} \subseteq y \mathfrak{m}^s \) for some \( s \). Write each monomial of degree \( s + 1 \) in the form \( r_1(x)Q_1(x) + \cdots + r_d(x)Q_d(x) + a(x) \) where \( r_1, \ldots, r_d \) are polynomials of degrees at least \( s \) and \( a(x) \in I \); enlarge \( A \) to include all the K-coefficients of \( r_1, \ldots, r_d, a \).

With this enlarged \( A \) we have \((xR_A)^{s+1} \subseteq (yR_A)(xR_A)^s \) and tensoring with any \( \kappa(\mathfrak{p}) \) gives \((xR_\kappa(\mathfrak{p}))^{s+1} \subseteq (yR_\kappa(\mathfrak{p}))(xR_\kappa(\mathfrak{p}))^s \).

If \( R_m \) is Cohen-Macaulay on its punctured spectrum, then we can find a localization of \( R \) at one element whose only point at which it can fail to be non-Cohen-Macaulay is \( \mathfrak{m} \). After adding a new variable to \( R \) as at the beginning of this section, we may assume that the non-Cohen-Macaulay locus of \( R \) is contained in \( \{\mathfrak{m}\} \). The hypothesis in (c) is now equivalent to the existence of a \( k \geq 1 \) such that \( \mathfrak{m}^k \text{Ext}_{A^k}^i(R, T) = 0 \) for all \( h I < i \leq n \). Let \( \mathcal{F} \) be a free \( T \)-resolution of \( R \). Include \( \mathfrak{m}, \mathcal{F} \) and \( \mathcal{C} = \text{Hom}(\mathcal{F}, T) \) in the descent objects. Now, with the corresponding descent data, \( \mathcal{F}_A \) is a \( T_A \)-free resolution of \( R_A \). Localize \( A \) at one element, if necessary, so that \( \mathfrak{m}_A^k \text{Ext}_{T_A}^i(R_A, T_A) \) is \( A \)-free for all \( h I < i \leq n \). Fix any \( h I < i \leq n \);
we have
\[ \text{Ext}^i_{T^A}(R_A, T_A) \otimes_A K = H^i(\text{Hom}(T^A, T_A)) \otimes_A K = H^i(\mathcal{C}_A) \otimes_A K = H^i(\mathcal{C}) \]
and hence \( m^k \text{Ext}^i_{T^A}(R_A, T_A) \otimes_A K = 0 \) so \( m^k \text{Ext}^i_{T^A}(R_A, T_A) = 0 \). Now for any maximal ideal \( p \subset A \), \( m^k \text{Ext}^i_{T^A}(R_{\kappa(p)}, T_{\kappa(p)}) = 0 \), and hence \( R_{\kappa(p)} \) is Cohen-Macaulay on its punctured spectrum.

The last statement is [HH06] Theorem 2.3.9.

\[ \Box \]

Proof of Theorem 5.2 Using [BH93] Theorem 4.6.4 we write \( e(R_m) = \chi(y; R_m) \), and using the fact that \( R \) was constructed so that \( m \) is the only maximal ideal containing \( y \), we deduce that \( e(R_m) = \chi(y; R) = \sum_{i=0}^d (-1)^i \ell_R H_i(y, R) \). We add to the descent objects in Lemma 5.3 the Koszul complex \( \mathfrak{K}(y; R) \) and extend the descent data in Lemma 5.3 to cater for these.

For all \( 0 \leq i \leq d \) we have \( H_i(y; R) \cong H_i(y; R_A) \otimes_A K \) and \( \ell(H_i(y; R)) = \text{rank } H_i(y; R_A) \).

Pick any maximal ideal \( p \subset A \). We have \( H_i(y; R_A) \otimes_A \kappa(p) \cong H_i(y; R_{\kappa(p)}) \).

Note that \( H_i(y; R_{\kappa(p)}) \) is only supported at \( m R_{\kappa(p)} \). Otherwise, we can find an \( x \in m R_{\kappa(p)} \) such that \( 0 \neq H_i(y; R_{\kappa(p)})_x \cong H_i(y; R_A)_x \otimes_A \kappa(p) \), hence \( H_i(y; R_A)_x \neq 0 \) and \( (H_i(y; R_A) \otimes_A K)_x \cong H_i(y; R)_x = 0 \), contradicting the fact that \( \text{Supp } H_i(y; R) \subseteq \{ m \} \).

Now
\[ e((R_{\kappa(p)})_m) = \chi(y; (R_{\kappa(p)})_m) = \chi(y; R_{\kappa(p)}) = \sum_{i=0}^d (-1)^i \ell_R H_i(y, R_{\kappa(p)}) = \sum_{i=0}^d (-1)^i \text{rank } H_i(y, R_A) \]
and so Theorem 1.1 implies that \( e(R_m) = e((R_{\kappa(p)})_m) \leq \binom{n}{d} \).

\[ \Box \]

Remark 5.4. In [Sch09] it is conjectured that being a \( K \)-algebra with dense \( F \)-injective type is equivalent to being a Du Bois singularity. Recently, the multiplicity of Cohen-Macaulay Du Bois singularities has been bounded by \( \binom{n}{d} \) (see [Shi17]) and hence the results of this section provide further evidence for the conjecture above.

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