Note on the spectrum of classical and uniform exponents of Diophantine approximation

Antoine MARNAT*
antoine.marnat@york.ac.uk

Abstract

Using the Parametric Geometry of Numbers introduced recently by W.M. Schmidt and L. Summerer [13, 14] and results by D. Roy [10, 11], we establish that the $2n$ exponents of Diophantine approximation in dimension $n \geq 3$ are algebraically independent.

1 Introduction

Throughout this paper, the integer $n \geq 1$ denotes the dimension of the ambient space $\mathbb{R}^n$ endowed with its Euclidean norm and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ denotes an n-tuple of real numbers such that $1, \theta_1, \dots, \theta_n$ are $\mathbb{Q}$-linearly independent.

Let d be an integer with $0 \leq d \leq n-1$. We define the exponent $\omega_d(\boldsymbol{\theta})$ (resp. the uniform exponent $\hat{\omega}_d(\boldsymbol{\theta})$) as the supremum of the real numbers ω for which there exist rational affine subspaces $L \subset \mathbb{R}^n$ such that

$$\dim(L) = d\ ,\ H(L) \leq H \ \text{ and } \ H(L)d(\boldsymbol{\theta}, L) \leq H^{-\omega}$$

for arbitrarily large real numbers H (resp. for every sufficiently large real number H). Here $H(L)$ denotes the exponential height of L (see [12] for more details), and $d(\boldsymbol{\theta}, L) = \min_{P \in L} d(\boldsymbol{\theta}, P)$ is the minimal distance between $\boldsymbol{\theta}$ and a point of L. Note that this definition is independent of the choice of a norm on $\mathbb{R}^n$.

These exponents were introduced originally by M. Laurent [7]. They interpolate between the classical exponents $\omega(\boldsymbol{\theta}) = \omega_{n-1}(\boldsymbol{\theta})$ and $\lambda(\boldsymbol{\theta}) = \omega_0(\boldsymbol{\theta})$ (resp. $\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega}_{n-1}(\boldsymbol{\theta})$ and $\hat{\lambda}(\boldsymbol{\theta}) = \hat{\omega}_0(\boldsymbol{\theta})$) that were introduced by A. Khintchine [4, 5], V. Jarník [3] and Y. Bugeaud

*supported by the Austrian Science Fund (FWF), Project F5510-N26, and FWF START project Y-901 and EPSRC Programme Grant EP/J018260/1

and M. Laurent [1, 2].

We have the relations

$$\omega_0(\boldsymbol{\theta}) \le \omega_1(\boldsymbol{\theta}) \le \cdots \le \omega_{n-1}(\boldsymbol{\theta}),$$
$$\hat{\omega}_0(\boldsymbol{\theta}) \le \hat{\omega}_1(\boldsymbol{\theta}) \le \cdots \le \hat{\omega}_{n-1}(\boldsymbol{\theta}),$$

and Minkowski's First Convex Body Theorem [9] and Mahler's compound convex bodies theory provide the lower bounds

$$\omega_d(\boldsymbol{\theta}) \ge \hat{\omega}_d(\boldsymbol{\theta}) \ge \frac{d+1}{n-d}, \quad \text{for } 0 \le d \le n-1.$$

These $2n$ exponents happen to be related as was first noticed by Khinchin with his transference theorem [5]. We use the following notion of spectrum to study more general transfers. Given k exponents $e_1, \dots, e_k$, we define the *spectrum* of the exponents $(e_1, \dots, e_k)$ as the subset of $\mathbb{R}^k$ described by all k-uples $(e_1(\boldsymbol{\theta}), \dots, e_k(\boldsymbol{\theta}))$ as $\boldsymbol{\theta}$ runs through all points $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ such that $1, \theta_1, \dots, \theta_n$ are $\mathbb{Q}$-linearly independent.

In [8], the author proved the following theorem.

Theorem 1. *For every integer $n \ge 3$, the n uniform exponents $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$ are algebraically independent.*

Using the same construction, it is even possible to show that for every integer $n \ge 3$, the spectrum of $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$ is a subset of $\mathbb{R}^n$ with non empty interior. In this paper, we extend this result as follows.

Theorem 2. *For every integer $n \ge 3$, the $2n$ exponents $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}, \omega_0, \dots, \omega_{n-1}$ are algebraically independent.*

In dimension $n = 2$, the spectrum is fully described in [6]:

Theorem 3 (Laurent, 2009)**.** *In dimension 2, the spectrum of the four exponents $\omega_0, \omega_1, \hat{\omega}_0, \hat{\omega}_1$ is described by the inequalities*

$$\hat{\omega}_0 + 1/\hat{\omega}_1 = 1, \quad 2 \le \hat{\omega}_1 \le +\infty, \quad \frac{\omega_1(\hat{\omega}_1 - 1)}{\omega_1 + \hat{\omega}_1} \le \omega_0 \le \frac{\omega_1 - \hat{\omega}_1 + 1}{\hat{\omega}_1}.$$

When $\hat{\omega}_1 < \omega_1 = +\infty$ we have to understand these relations as $\hat{\omega}_1 - 1 \le \omega_0 \le +\infty$ and when $\hat{\omega}_1 = +\infty$, the set of constraints should be interpreted as $\omega_0 = \omega_1 = +\infty$ and $\hat{\omega}_0 = 1$.

The first equality, relating the two uniform exponents, is known as Jarník's relation [3] and breaks the algebraic independence. Note that this sharpens previously mentioned relations. In dimension $n = 1$ the uniform exponent is always equal to 1.

We refer the reader to [8, §2] for the notation and the presentation of the parametric geometry of numbers, main tool of the proof. We mainly use the notation introduced by D. Roy in [10, 11] which is essentially dual to the one of W. M. Schmidt and L. Summerer [13, 14].

2 Proof of the main Theorem 2

To prove Theorem 2, we place ourselves in the context of parametric geometry of numbers. We fully use Roy's theorem [8, Theorem 5] that reduces the study of spectra of Diophantine approximation to the study of the combinatorial properties of generalized n-systems. We construct explicitly a family of generalized $(n+1)$-systems with $2n$ parameters, which provides the algebraic independence in the spectrum via Roy's theorem.

We fix the dimension $n \geq 3$. Consider any family of positive parameters

$$A_1 = A_2 < A_3 < \cdots < A_{n+1} \ , \ B_2 < B_3 < \cdots < B_n \ , \ C \ , \ D$$

satisfying the following properties for $2 \leq k \leq n$:

$$\begin{aligned} A_1 + A_2 + \cdots + A_{n+1} = 1 \ , \ B_2 < D < CA_2, \\ A_{k+1} < B_k < A_{k+2} \ , \ B_k < CA_k, \end{aligned} \tag{1}$$

where $A_{n+2} = \infty$.

We consider the generalized $(n+1)$-system $\boldsymbol{P}$ on the interval $[1, C]$ depending on the previous parameters whose combined graph is given below by Figure 1, where

$$P_k(1) = A_k \text{ and } P_k(C) = CA_k \text{ for } 1 \leq k \leq n+1.$$

Conditions (1) are consistent with the graph. On each interval between two consecutive division points, there is only one line segment with non zero slope. This line segment has slope 1 on the intervals $[1, \delta_{2,1}]$, $[\delta_{k-1,2}, \delta_{k,1}]$ for $3 \leq k \leq n$, and $[\mu_k, \mu_{k-1}]$ for $n \geq k \geq 1$, and has slope 1/2 on the interval $[\mu_0, C]$ and $[\delta_{k,1}, \delta_{k,2}]$ for $3 \leq k \leq n$, where the two components P_k and P_{k+1} coincide. We have $3n+1$ division points 1, C, $\delta_{k,1}$ and $\delta_{k,2}$ for $2 \leq k \leq n$ and μ_l for $n+1 \geq l \geq 0$. They are all ordinary division points except μ_k for $1 \leq k \leq n$ which are switch points.

The points which will be most relevant for the proof are labeled with black dots. Note that from 1 to $\delta_{n,2}$, the combined graph is the same as in [8, §5].

We extend $\boldsymbol{P}$ to the interval $[1, \infty)$ by self-similarity. This means, $\boldsymbol{P}(q) = C^m \boldsymbol{P}(C^{-m} q)$ for all integers m. In view of the value of $\boldsymbol{P}$ and its derivative at 1 and C, one sees that the extension provides a generalized $(n+1)$-system on $[1, \infty)$.

The relation between exponents and n-systems [8, Proposition 1] suggests to define $2n$

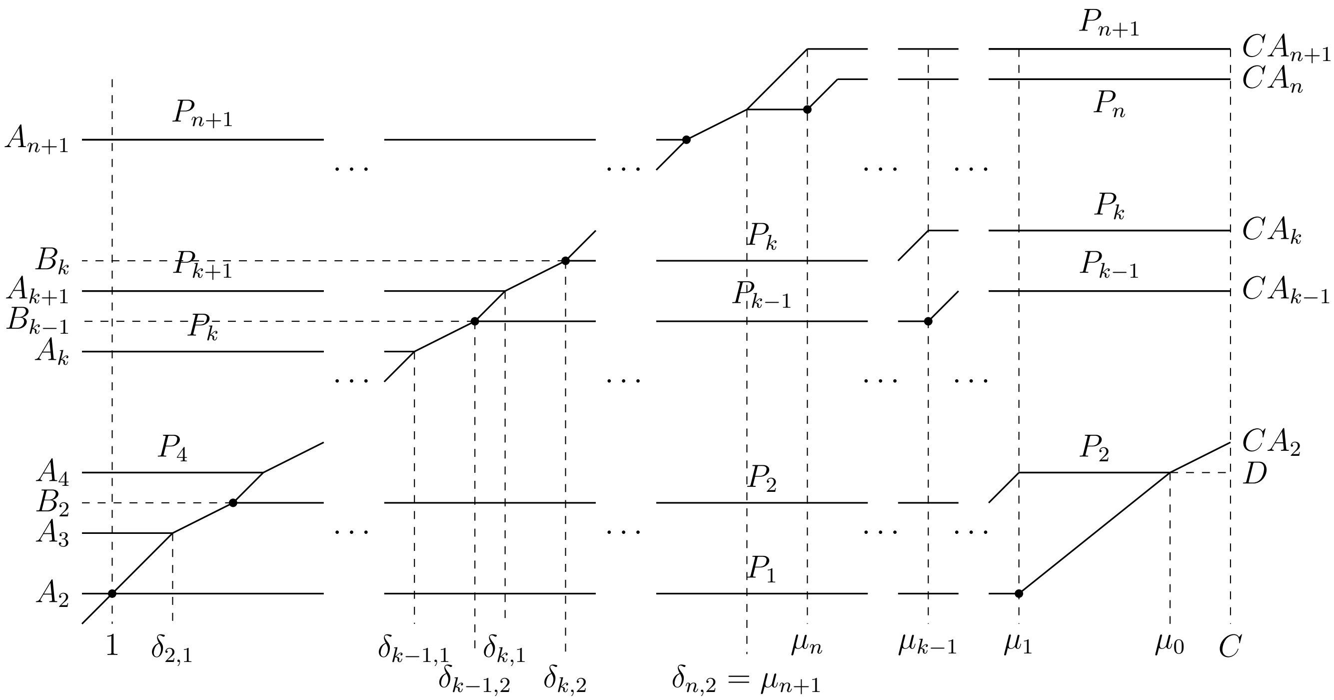

Figure 1: Pattern of the combined graph of $\boldsymbol{P}$ on the fundamental interval $[1, C]$

quantities $W_{n-1}, \dots, W_0, \hat{W}_{n-1}, \dots, \hat{W}_0$ by

$$\frac{1}{1+\hat{W}_{n-k}} := \limsup_{q \to +\infty} \frac{P_1(q) + \cdots + P_k(q)}{q} \text{ for } 1 \leq k \leq n,$$
$$\frac{1}{1+W_{n-k}} := \liminf_{q \to +\infty} \frac{P_1(q) + \cdots + P_k(q)}{q} \text{ for } 1 \leq k \leq n.$$

Indeed with this setting, Roy's Theorem provides the existence of a point $\boldsymbol{\theta}$ in $\mathbb{R}^n$ such that $\hat{\omega}_k(\boldsymbol{\theta}) = \hat{W}_k$ and $\omega_k(\boldsymbol{\theta}) = W_k$ for every $0 \leq k \leq n-1$.

Here, self-similarity ensures that the $\limsup$ (resp. $\liminf$) is in fact the maximum (resp. the minimum) on the interval $[1, C[$. Note that for $1 \leq k \leq n$, the function $P_1 + \cdots + P_k$ has slope 1 on the intervals $[1, \delta_{k,1}]$ and $[\mu_k, C[$, slope $1/2$ on the interval $[\delta_{k,1}, \delta_{k,2}]$ and is constant on the interval $[\delta_{k,2}, \mu_k]$. Therefore the minimum of the function $q \mapsto q^{-1}(P_1(q) + \cdots + P_k(q))$ is reached at μ_k and its maximum is reached either at $\delta_{k,1}$ or at $\delta_{k,2}$, when slope changes from 1 to $1/2$ or from $1/2$ to 0. Namely, the maximum is reached at $\delta_{k,1}$ if

$$\frac{P_1(\delta_{k,1}) + \cdots + P_k(\delta_{k,1})}{\delta_{k,1}} \geq \frac{1}{2} \tag{2}$$

and at $\delta_{k,2}$ if the lefthand side is $\leq 1/2$. We deduce that for $1 \leq k \leq n$,

$$\begin{aligned} \hat{W}_{n-k} &= \frac{P_{k+1}(q) + \cdots + P_{n+1}(q)}{P_1(q) + \cdots + P_k(q)} \text{ where } q = \begin{cases} \delta_{k,1} & \text{if (2) is satisfied} \\ \delta_{k,2} & \text{otherwise} \end{cases}, \\ W_{n-k} &= \frac{P_{k+1}(\mu_k) + \cdots + P_{n+1}(\mu_k)}{P_1(\mu_k) + \cdots + P_k(\mu_k)}. \end{aligned}$$

It is easy to check that the parameters

$$\begin{gathered} C = 3,\ A_1 = A_2 = 2^{-n}, A_k = 2^{-n+k-2} \text{ for } 3 \leq k \leq n+1 \\ D = \frac{11}{8} 2^{-n+1}, B_k = \frac{5}{4} 2^{-n+k-1} \text{ for } 2 \leq k \leq n \end{gathered} \tag{3}$$

satisfy the conditions (1). For this choice of parameters, the lefthand side of inequality (2) is $> 1/2$ for $1 \leq k \leq n-1$ and $< 1/2$ for $k = n$. This property remains true for $(C, A_2, \ldots, A_n, D, B_2, B_3, \ldots, B_n)$ in an open neighborhood of the point

$$(3, 2^{-n}, \ldots, 2^{-2}, \frac{11}{8} 2^{-n+1}, \frac{5}{2} 2^{-n}, \ldots, \frac{5}{2} 2^{-2})$$

provided that we set $A_1 = A_2$ and $A_{n+1} = 1 - (A_1 + \cdots + A_n)$. In this neighborhood, the quantities $W_0, \ldots, W_{n-1}, \hat{W}_0, \ldots, \hat{W}_{n-1}$ are given by the following rational fractions in $\mathbb{Q}(C, A_2, \ldots, A_n, D, B_2, B_3, \ldots, B_n)$:

$$\begin{aligned} \hat{W}_{n-1} &= \frac{1}{A_2} - 1, & \hat{W}_0 &= \frac{1 - (2A_2 + A_3 + A_4 + \cdots + A_n)}{A_2 + (B_2 + \cdots + B_{n-1})}, \\ \hat{W}_{n-k} &= \frac{1 - (2A_2 + A_3 + A_4 + \cdots + A_{k+1}) + B_k}{A_2 + (B_2 + \cdots + B_k)} & &\text{for } 2 \leq k \leq n-1, \\ W_{n-k} &= \frac{C(1 - (2A_2 + A_3 + A_4 + \cdots + A_k))}{A_2 + B_2 + \cdots + B_k} & &\text{for } 2 \leq k \leq n, \\ W_{n-1} &= \frac{D + C(1 - 2A_2)}{A_2}. \end{aligned}$$

Since $W_0, \ldots, W_{n-1}, \hat{W}_0, \ldots, \hat{W}_{n-1}$ come from a generalized $(n+1)$-system $\boldsymbol{P}$, Roy's Theorem provides the existence of a point $\boldsymbol{\theta}$ in $\mathbb{R}^n$ such that $\hat{\omega}_k(\boldsymbol{\theta}) = \hat{W}_k$ and $\omega_k(\boldsymbol{\theta}) = W_k$ for every $0 \leq k \leq n-1$. Therefore, to prove Theorem 2 it is sufficient to show that the rational fractions $W_0, \ldots, W_{n-1}, \hat{W}_0, \ldots, \hat{W}_{n-1} \in \mathbb{Q}(C, A_2, A_3, \ldots, A_n, D, B_2, B_3, \ldots, B_n)$ are algebraically independent.

First, note that only W_{n-1} depends on D and $\hat{W}_{n-1}$ only depends on A_2. Therefore, it is enough to prove that the $2n-2$ other rational fractions are algebraically independent over

$\mathbb{Q}(A_2)$. For the calculation, it is convenient to successively make the following two changes of variables. First, we set

$$\begin{aligned} M_k &:= 1-\sum_{i=1}^{k} A_i \text{ for } 2 \le k \le n+1, \\ N_k &:= A_1+\sum_{i=2}^{k} B_i \text{ for } 1 \le k \le n. \end{aligned}$$

Note that $M_{n+1} = 0$ and $N_1 = A_1$. We get the formulae

$$\begin{aligned} \hat{W}_0 &= \frac{M_n}{N_{n-1}}, \\ W_{n-k} &= \frac{CM_k}{N_k} \text{ for } 2 \le k \le n, \\ \hat{W}_{n-k} &= 1+\frac{M_{k+1}-N_{k-1}}{N_k} \text{ for } 2 \le k \le n-1. \end{aligned}$$

Then, we set

$$U_k := \frac{M_k}{N_k} \text{ and } V_k := \frac{M_{k+1}}{N_k} \text{ for } 2 \le k \le n,$$

and $V_1 = \frac{1-2A_2}{A_2}$ getting the formulae

$$\begin{aligned} \hat{W}_0 &= V_{n-1}, \\ W_{n-k} &= CU_k \text{ for } 2 \le k \le n, \\ \hat{W}_{n-k} &= 1+V_k-\frac{U_k}{V_{k-1}} \text{ for } 2 \le k \le n-1. \end{aligned}$$

Hence, the $2n-2$ independent parameters $C, A_3, \cdots, A_n, B_2, \cdots, B_n$ provide the $2n-2$ independent parameters $C, U_2, \ldots, U_n, V_2, \ldots, V_{n-1}$. Thus, it is sufficient to show that the rational fractions $W_0, \ldots, W_{n-2}, \hat{W}_0, \ldots, \hat{W}_{n-2} \in \mathbb{Q}(A_2)(C, U_2, U_3, \ldots, U_n, V_2, V_3, \ldots, V_{n-1})$ are algebraically independent over $\mathbb{Q}(A_2)$.

Suppose that there exists an irreducible polynomial $R \in \mathbb{Q}(A_2)[X_1, \ldots, X_{2n-2}]$ such that

$$R\left(\hat{W}_0, \ldots, \hat{W}_{n-2}, W_0, \ldots, W_{n-2}\right) = 0.$$

Specializing C in 1, we obtain

$$R\left(V_{n-1}, V_{n-1}+1-\frac{U_{n-1}}{V_{n-2}}, \ldots, V_2+1-\frac{U_2}{V_1}, U_n, \ldots, U_2\right) = 0 \tag{4}$$

where the $2n-3$ last rational fractions generate the field $\mathbb{Q}(A_2)(U_2,\dots,U_n,V_2,\dots,V_{n-1})$ over $\mathbb{Q}(A_2)$. Therefore, they are algebraically independent. We investigate their relation with the first coordinate, that will provide information on R. Observe that for $2 \le k \le n-1$,

$$\hat{W}_{n-k} = 1 + V_k - \frac{U_k}{V_{k-1}}$$

provide the relation

$$V_k = \hat{W}_{n-k} - 1 + \frac{W_{n-k}}{V_{k-1}}.$$

Since $\hat{W}_0 = V_{n-1}$, we can compute by finite induction

$$\hat{W}_0 = V_{n-1} = (\hat{W}_1 - 1) + \frac{W_1}{V_{n-2}} = f_0 + \mathop{\mathrm{K}}_{k=1}^{n-2} \frac{e_k}{f_k}$$

where

$$\begin{cases} e_k &= W_k \qquad\qquad \text{for } 1 \le k \le n-2 \\ f_k &= \hat{W}_{k+1} - 1 \qquad \text{for } 0 \le k \le n-3 \\ f_{n-2} &= V_1 = \dfrac{1-2A_2}{A_2} \end{cases}$$

and

$$f_0 + \mathop{\mathrm{K}}_{k=1}^{n-2} \frac{e_k}{f_k} = f_0 + \cfrac{e_1}{f_1 + \cfrac{e_2}{f_2 + \cfrac{\ddots}{f_{n-2}}}}$$

is Gauss' notation for a (finite) generalized continued fraction. Denote by $\left(\dfrac{E_k}{F_k}\right)_{k=0}^{n-2}$ the finite sequence of its convergents.

We set

$$\tilde{R} = F_{n-2}\hat{W}_0 - E_{n-2}$$

where F_{n-2} and E_{n-2} are seen as polynomials in $\mathbb{Q}(A_2)[W_0,\dots,W_{n-2},\hat{W}_0,\dots,\hat{W}_{n-2}]$. Note that F_{n-2} and E_{n-2} do not depend on $\hat{W}_0$ since none of the $(e_k)_{1\le k\le n-2}$ and $(f_k)_{0\le k\le n-2}$ do. Hence, $\tilde{R}$ is a polynomial of degree 1 with respect to $\hat{W}_0$. Writing the Euclidean division of R by $\tilde{R}$ in $\mathbb{Q}(A_2,\hat{W}_1,\dots,\hat{W}_{n-2},W_0,\dots,W_{n-2})[\hat{W}_0]$ we get

$$R = \tilde{R}Q + P$$

with $\deg_{\hat{W}_0}(P) = 0$. Hence P can be seen as a polynomial in the $2n-3$ variables $\hat{W}_1,\dots,\hat{W}_{n-2},W_0,\dots,W_{n-2}$ over $\mathbb{Q}(A_2)$. The latter are algebraically independent over $\mathbb{Q}(A_2)$

because their specializations at $C = 1$ are. We deduce that $P = 0$, and by irreducibility of R, the polynomial Q is a constant:

$$R = \alpha \left(F_{n-2}\hat{W}_0 - E_{n-2} \right)$$

with $\alpha \in \mathbb{Q}(A_2)$.

Specializing C in 0, we obtain

$$R\left(V_{n-1}, V_{n-1} + 1 - \frac{U_{n-1}}{V_{n-2}}, \ldots, V_2 + 1 - \frac{U_2}{V_1}, 0, \ldots, 0\right) = 0$$

where the $n-1$ non zero rational fractions generate the field $\mathbb{Q}(V_1)(U_3, \ldots, U_{n-1})(V_{n-1}, V_{n-2}, \ldots, V_2, U_2)$ over $\mathbb{Q}(V_1)(U_3, \ldots, U_{n-1})$. Therefore, they are algebraically independent over $\mathbb{Q}(A_2) = \mathbb{Q}(V_1)$. We deduce that the constant monomial of R seen in $\mathbb{Q}(A_2, \hat{W}_0, \ldots, \hat{W}_{n-2})[W_0, \ldots, W_{n-2}]$ should be zero.

We now compute the constant monomial of $F_{n-2}\hat{W}_0 - E_{n-2}$ seen in $\mathbb{Q}(A_2, \hat{W}_0, \ldots, \hat{W}_{n-2})[W_0, \ldots, W_{n-2}]$. We use the classical recurrence formulae for the convergents

$$E_{k+1} = e_{k+1}E_k + f_{k+1}E_{k-1} \text{ and } F_{k+1} = e_{k+1}F_k + f_{k+1}F_{k-1}$$

to compute the constant term of E_{n-2} and F_{n-2} to be

$$\prod_{k=0}^{n-2} f_k \text{ and } \prod_{k=1}^{n-2} f_k$$

respectively. Thus the constant monomial of $F_{n-2}\hat{W}_0 - E_{n-2}$ seen in $\mathbb{Q}(A_2, \hat{W}_0, \ldots, \hat{W}_{n-2})[W_0, \ldots, W_{n-2}]$ is

$$\left(\prod_{k=1}^{n-2} f_k\right) \hat{W}_0 - \prod_{k=0}^{n-2} f_k = (\hat{W}_0 - \hat{W}_1 + 1)\frac{1 - 2A_2}{A_2} \prod_{k=1}^{n-3} (\hat{W}_{k+1} - 1).$$

The fact that $\hat{W}_{k+1} \neq 1$ and $\hat{W}_0 + 1 \neq \hat{W}_1$ induces that this constant monomial is non zero. Hence α and R are zero.

This proves the algebraic independence of the $2n$ exponents. ☐

References

[1] Yann Bugeaud and Michel Laurent. On exponents of homogeneous and inhomogeneous diophantine approximation. *Moscow Math. J.*, 5:747–766, 2005.

[2] Yann Bugeaud and Michel Laurent. Exponents of diophantine approximation. *Diophantine Geometry Proceedings*, 4:101–121, 2007.

[3] Vojtěch Jarník. Zum khintchineschen "Übertragungssatz". *Trav. Inst. Math. Tbilissi*, 3:193–212, 1938.

[4] Alexander Ya. Khinchin. Über eine klasse linearer diophantischer approximationen. *Rend. Circ. Mat. Palermo 50*, pages 170–195, 1926.

[5] Alexander Ya. Khinchin. Zur metrischen theorie der diophantischen approximationen. *Math.Z.*, 24:706–714, 1926.

[6] Michel Laurent. Exponents of diophantine approximation in dimension two. *Canad. J. Math.*, 61:165–189, 2009.

[7] Michel Laurent. On transfer inequalities in Diophantine approximation. In *Analytic number theory*, pages 306–314. Cambridge Univ. Press, Cambridge, 2009.

[8] Antoine Marnat. About Jarník's type relation in higher dimension. *Annales de l'Institut Fourier*, to appear.

[9] Hermann Minkowski. *Geometrie der Zahlen*. Bibliotheca Mathematica Teubneriana, Band 40. Johnson Reprint Corp., New York-London, 1968.

[10] Damien Roy. On Schmidt and Summerer parametric geometry of numbers. *Ann. of Math.*, 182:739–786, 2015.

[11] Damien Roy. Spectrum of the exponents of best rational approximation. *Math. Z.*, 283:143–155, 2016.

[12] Wolfgang M. Schmidt. On heights of algebraic subspaces and diophantine approximations. *Ann. of Math. (2)*, 85:430–472, 1967.

[13] Wolfgang M. Schmidt and Leonhard Summerer. Parametric geometry of numbers and applications. *Acta Arithmetica*, 140(1):67–91, 2009.

[14] Wolfgang M. Schmidt and Leonhard Summerer. Diophantine approximation and parametric geometry of numbers. *Monatsch. Math.*, 169(1):51–104, 2013.