Article:

Tavakoli, S., Sadeghi, J., Griffin, I. et al. (1 more author) (2018) PI controller tuning for load disturbance rejection using constrained optimization. International Journal of Dynamics and Control, 6 (1). pp. 188-199. ISSN 2195-268X

https://doi.org/10.1007/s40435-016-0290-6

PI controller tuning for load disturbance rejection using constrained optimization

Saeed Tavakoli [a*], Jafar Sadeghi [b], Ian Griffin [c], Peter J. Fleming [c]

[a] Department of Electrical Engineering, The University of Sistan and Baluchestan, Iran

[b] Department of Chemical Engineering, The University of Sistan and Baluchestan, Iran

[c] Department of Automatic Control and Systems Engineering, The University of Sheffield, UK

*Corresponding author; E-mail: tavakoli@ece.usb.ac.ir

Abstract
In this paper, a simple and effective PI controller tuning method is presented. To take both performance requirements and robustness issues into consideration, the design technique is based on optimization of load disturbance rejection with a constraint either on the gain margin or phase margin. In addition, a simplified form of the resulting tuning formulae is obtained for first order plus dead time models. To demonstrate the ability of the proposed tuning technique in dealing with a wide range of plants, simulation results for several examples, including integrating, non-minimum phase and long dead time models, are provided.

Keywords: Constrained optimization, Gain margin, Load disturbance rejection, Phase margin, PI control

1. Introduction

In spite of the recent advances in control theory, PID controller is the most widespread form of feedback compensation. This is mainly due to its noticeable effectiveness and simple structure that is conceptually easy to understand. PID is a simple and useful controller, which gives a powerful solution to the control of a huge number of industrial plants. According to the literature, more than 95% of industrial controllers are PID controllers [1-5]. The key reason for this popularity is that a well-designed PID controller can meet most control requirements [6]. In fact, most of the industrial controllers are PI because the derivative action is very often not used. As a result, good PI tuning methods are extremely desirable due to their widespread use.

Since the 1940s, a large number of analytical and numerical methods, which are usually different in complexity and flexibility, have been proposed for tuning of PID controllers [7-13]. In addition, several well-known control books have chapters on tuning PID controllers [14-17].

Generally, an efficient design method should satisfy the design specifications and be able to deal with a wide range of plants. A satisfactory load disturbance response is often the first goal in control applications. This paper presents a PI tuning method resulting in a set of tuning formulae. To consider performance and robustness requirements, the design objective is the optimization of load disturbance rejection with a constraint either on the gain margin (GM) or the phase margin (PM). As the first order plus dead time (FOPDT) models can approximately model a huge number of industrial plants, the resulting tuning formulae are then applied to these plants to obtain a simple set of tuning formulae.

Nomenclature			
A_m	desired gain margin	$L(s)$	loop transfer function
d	load disturbance signal	PID	proportional-integral-derivative
$G_p(s)$	plant transfer function	PM	phase margin
$G_c(s)$	controller transfer function	ϕ_m	desired phase margin
FOPDT	first order plus dead time	r	reference signal
GM	gain margin	SGM	specified gain margin
IAE	integral of absolute error	SPM	specified phase margin
IE	integral of error	T	time constant of FOPDT model
K_c	proportional gain	τ_d	time delay of FOPDT model
K_i	integral gain	ω	frequency
K_p	gain of FOPDT model	y	output signal

The paper is organised as follows. In section 2, an analytical method to determine the optimal parameters of PI controllers in terms of minimizing an objective function and satisfying a GM or PM constraint is developed. The method is applied to FOPDT plants in section 3. In section 4, the simplified tuning formulae for FOPDT plants are presented, using dimensional analysis and curve-fitting techniques. In Section 5, the resulting tuning formulae are applied to a variety of control examples. Moreover, a comparison between the performance of the proposed tuning formulae and that of one of the most prevalent design methods is given for each example. Finally, the conclusions of the whole study are drawn.

2. Theory

The plant, $G_p(s)$, is controlled by the PI controller in Equation (1).

$$G_c(s) = K_c + \frac{K_i}{s}. \qquad (1)$$

where K_c and K_i are proportional and integral gains, respectively. The aim of control is to reject load disturbance signals, which are the most common and most important disturbances in control that drive systems away from their desired operating points [3]. The output signal of a closed-loop system in the presence of an input load disturbance signal is given by Equation (2).

$$y = \frac{G_p G_c}{1+G_p G_c} r + \frac{G_p}{1+G_p G_c} d. \qquad (2)$$

where r, d and y refer to the reference, load disturbance and output signals, respectively. Step disturbances are applied at the input to the plant. A commonly chosen performance metric is the integral of absolute error (IAE). A significant drawback of this criterion is that it is not suitable for analytical approaches, as the evaluation requires computation of time functions [3]. However, the IAE is equivalent to the integral of error (IE) if the error signal is positive. Moreover, the IE may be a good approximation for the IAE for well-damped closed-loop systems. The reason for using IE is that it is appropriate for analytical approaches as its value is directly related to the integral gain, as shown in Equation (3) [3].

$$IE = \frac{1}{K_i}. \qquad (3)$$

In addition, robustness is a key issue in control systems. It is well known that GM and PM are used as measures of robustness. In order to ensure the robustness of the closed-loop system, the optimization problem is constrained so that a desired GM or a required PM is guaranteed. Moreover, the PM acts as a measure of performance as it is related to the damping of the system [18]. Therefore, the design objective is to maximize K_i subject to satisfying the robustness constraint.

2.1. Tuning formulae for a constraint on GM

Assume that the model of the plant is given by Equation (4).

$$G_p(j\omega) = \alpha(\omega) + j\beta(\omega). \qquad (4)$$

where $\alpha(\omega)$ and $\beta(\omega)$ are real and imaginary parts of the plant. The loop transfer function is then written as shown in Equation (5).

$$L(j\omega) = (\alpha(\omega) + j\beta(\omega))(K_c - j\frac{K_i}{\omega}). \qquad (5)$$

In order to determine the controller parameters that obtain a desired GM, Equations (6) and (7) should be solved.

$$\mathrm{Im}[L(j\omega)] = 0. \qquad (6)$$

$$\mathrm{Re}[L(j\omega)] = \frac{-1}{A_m}. \qquad (7)$$

where A_m is the value of the desired GM. Inserting Equation (5) in Equations (6) and (7) results in the controller parameters given by Equations (8) and (9).

$$K_c = \frac{-\alpha(\omega)}{A_m(\alpha^2(\omega)+\beta^2(\omega))}. \qquad (8)$$

$$K_i = \frac{-\omega\beta(\omega)}{A_m(\alpha^2(\omega)+\beta^2(\omega))}. \qquad (9)$$

The necessary and sufficient conditions for maximizing K_i and satisfying the GM constraint are given by Equations (10) and (11), respectively.

$$\frac{dK_i}{d\omega} = 0. \quad (10)$$

$$\frac{d^2K_i}{d\omega^2} < 0. \quad (11)$$

Equation (9) can be written as shown in Equation (12).

$$K_i = \omega\, f(\omega). \quad (12)$$

where $f(\omega)$ is given by Equation (13).

$$f(\omega) = \frac{-\beta(\omega)}{A_m(\alpha^2(\omega) + \beta^2(\omega))}. \quad (13)$$

Inserting Equation (12) into Equation (10) gives the necessary condition shown in Equation (14).

$$\frac{dK_i}{d\omega} = f(\omega) + \omega f'(\omega) = 0. \quad (14)$$

where $f'(\omega)$ is the derivative of $f(\omega)$ with respect to ω. ω can be determined by inserting $f(\omega)$ from Equation (13) and $f'(\omega)$ into Equation (14), resulting in Equation (15).

$$\omega = \frac{1}{2\dfrac{\alpha(\omega)\alpha'(\omega) + \beta(\omega)\beta'(\omega)}{\alpha^2(\omega) + \beta^2(\omega)} - \dfrac{\beta'(\omega)}{\beta(\omega)}}. \quad (15)$$

where $\alpha'(\omega)$ and $\beta'(\omega)$ are the derivatives of $\alpha(\omega)$ and $\beta(\omega)$ with respect to ω, respectively. Inserting Equation (14) into Equation (11), the sufficient condition is obtained as shown in Equation (16).

$$\frac{d^2K_i}{d\omega^2} = -2\, f(\omega) + \omega^2 f''(\omega) < 0. \quad (16)$$

The maximizing ω is given by Equation (15) subject to satisfying Equation (16). The optimal controller parameters are given by inserting the maximizing ω into Equations (8) and (9). This analytical tuning method is referred to as specified gain margin (SGM) because the closed-loop system satisfies a desired GM. An iterative technique, such as the Newton-Raphson method is required to solve Equation (15).

2.2. Tuning formulae for a constraint on PM

Assuming the loop transfer function in Equation (5), Equations (17) and (18) should be solved to determine the controller parameters that obtain a desired PM.

$$|L(j\omega)| = 1. \quad (17)$$

$$\pi + \angle L(j\omega) = \phi_m. \quad (18)$$

where ϕ_m is the value of the desired PM. Inserting Equation (5) into Equations (17) and (18) results in Equations (19) and (20).

$$K_c^2 + \frac{K_i^2}{\omega^2} = \frac{1}{\alpha^2(\omega) + \beta^2(\omega)}. \quad (19)$$

$$\pi + \tan^{-1}\left(\frac{\beta(\omega)}{\alpha(\omega)}\right) - \tan^{-1}\left(\frac{K_i}{\omega K_c}\right) = \phi_m. \quad (20)$$

Equation (20) can be written as shown in Equation (21).

$$\omega T_i = \frac{\alpha(\omega)\cos(\phi_m) + \beta(\omega)\sin(\phi_m)}{-\alpha(\omega)\sin(\phi_m) + \beta(\omega)\cos(\phi_m)}. \quad (21)$$

where T_i is given by Equation (22).

$$T_i = \frac{K_c}{K_i}. \quad (22)$$

Considering Equations (19), (21) and (22), PI parameters can be written as shown in Equations (23) and (24).

$$K_c = -\frac{\alpha(\omega)\cos(\phi_m) + \beta(\omega)\sin(\phi_m)}{\alpha^2(\omega) + \beta^2(\omega)}. \qquad (23)$$

$$K_i = \omega \frac{\alpha(\omega)\sin(\phi_m) - \beta(\omega)\cos(\phi_m)}{\alpha^2(\omega) + \beta^2(\omega)}. \qquad (24)$$

Writing Equation (24) in the form of Equation (12) with $f(\omega)$ shown in Equation (25)

$$f(\omega) = \frac{\alpha(\omega)\sin(\phi_m) - \beta(\omega)\cos(\phi_m)}{\alpha^2(\omega) + \beta^2(\omega)}. \qquad (25)$$

and applying the necessary condition for maximizing K_i, represented in Equation (14), to Equation (25) results in Equation (26).

$$\omega = \frac{1}{2\frac{\alpha(\omega)\alpha'(\omega) + \beta(\omega)\beta'(\omega)}{\alpha^2(\omega) + \beta^2(\omega)} - \frac{\alpha'(\omega)\sin(\phi_m) - \beta'(\omega)\cos(\phi_m)}{\alpha(\omega)\sin(\phi_m) - \beta(\omega)\cos(\phi_m)}}. \qquad (26)$$

whereas the sufficient condition is again given by Equation (16). If the maximizing ω given by Equation (26) satisfies Equation (16), the optimal PI parameters are given by Equations (23) and (24). This tuning method is referred to as specified phase margin (SPM).

3. Tuning formulae for FOPDT plants

In this section, the SGM and SPM methods are applied to an important category of industrial plants and simplified versions of Equations (8), (9), (15), (23), (24) and (26) are presented.

3.1. SGM tuning formulae for FOPDT plants

A huge number of industrial plants can be modelled by a FOPDT model, shown in Equation (27).

$$G_p(s) = \frac{K_p e^{-\tau_d s}}{Ts + 1}. \qquad (27)$$

To design PI controllers for this class of plants, the SGM design method is applied to the FOPDT models. The real and imaginary parts of the plant are given by Equations (28) and (29).

$$\alpha(\omega) = \frac{K_p}{1 + \omega^2 T^2}(\cos(\omega\tau_d) - \omega T \sin(\omega\tau_d)). \qquad (28)$$

$$\beta(\omega) = \frac{-K_p}{1 + \omega^2 T^2}(\sin(\omega\tau_d) + \omega T \cos(\omega\tau_d)). \qquad (29)$$

Inserting Equations (28) and (29) into Equations (8), (9) and (13) results in Equations (30)-(32).

$$K_c = \frac{-\cos(\omega\tau_d) + \omega T \sin(\omega\tau_d)}{A_m K_p}. \qquad (30)$$

$$K_i = \frac{\omega(\sin(\omega\tau_d) + \omega T \cos(\omega\tau_d))}{A_m K_p}. \qquad (31)$$

$$f(\omega) = \frac{\sin(\omega\tau_d) + \omega T \cos(\omega\tau_d)}{A_m K_p}. \qquad (32)$$

Maximizing ω shown in Equation (33) is given by inserting $f(\omega)$ from Equation (32) and $f'(\omega)$ into Equation (14).

$$\omega = \frac{\sin(\omega\tau_d) + \omega T \cos(\omega\tau_d)}{-(T + \tau_d)\cos(\omega\tau_d) + \omega T \tau_d \sin(\omega\tau_d)}. \qquad (33)$$

The sufficient condition for maximizing K_i, shown in Equation (34), is determined by inserting $f(\omega)$ and $f''(\omega)$ into Equation (16).

$$A\sin(\omega\tau_d) + B\cos(\omega\tau_d) > 0. \qquad (34)$$

where A and B are given by Equations (35) and (36).

$$A = 2 + \omega^2 \tau_d (2T + \tau_d). \qquad (35)$$

$$B = \omega T(2 + \omega^2 \tau_d^2). \qquad (36)$$

Finding $\cos(\omega \tau_d)$ from Equation (33) and substituting it into Equation (34), the sufficient condition is given by Equation (37).

$$C \sin(\omega \tau_d) > 0. \qquad (37)$$

where C is given by Equation (38).

$$C = (2 + \omega^2 \tau_d^2) \frac{\tau_d (1 + \omega^2 T^2) + T}{2T + \tau_d} + 2\omega^2 T \tau_d. \qquad (38)$$

$C > 0$ and it can easily be investigated that $\omega \tau_d < \pi$ holds for the SGM method. As a result, the sufficient condition is always satisfied.

3.2. SPM tuning formulae for FOPDT plants

Substituting Equations (28) and (29) into Equations (23), (24) and (25) results in Equation (39)-(41).

$$K_c = \frac{-\cos(\omega \tau_d + \phi_m) + \omega T \sin(\omega \tau_d + \phi_m)}{K_p}. \qquad (39)$$

$$K_i = \frac{\omega(\sin(\omega \tau_d + \phi_m) + \omega T \cos(\omega \tau_d + \phi_m))}{K_p}. \qquad (40)$$

$$f(\omega) = \frac{\sin(\omega \tau_d + \phi_m) + \omega T \cos(\omega \tau_d + \phi_m)}{K_p}. \qquad (41)$$

Maximizing ω shown in Equation (42) is given by inserting $f(\omega)$ from Equation (41) and $f'(\omega)$ into Equation (14).

$$\omega = \frac{\sin(\omega \tau_d + \phi_m) + \omega T \cos(\omega \tau_d + \phi_m)}{-(T + \tau_d)\cos(\omega \tau_d + \phi_m) + \omega T \tau_d \sin(\omega \tau_d + \phi_m)}. \qquad (42)$$

Inserting $f(\omega)$ and $f''(\omega)$ into Equation (16), results in the sufficient condition shown in Equation (43).

$$C \sin(\omega \tau_d + \phi_m) > 0. \qquad (43)$$

$\omega \tau_d + \phi_m < \pi$ holds for the SPM method, therefore, the sufficient condition is always satisfied.

4. Simplified tuning formulae for FOPDT models

Although simpler versions of Equations (15) and (26) for FOPDT plants are presented in Equations (33) and (42), an iterative method is still required to solve these nonlinear equations. Using dimensional analysis and curve-fitting methods, simple PI tuning formulae are presented in this section.

The PI controller in Equation (1) can be written as shown in Equation (44).

$$G_c(s) = K_c(1 + \frac{1}{T_i s}). \qquad (44)$$

To obtain the optimal PI tuning formulae for a FOPDT model given in Equation (27), the PI parameters can be defined based on the model parameters, as shown in Equations (45) and (46).

$$K_c = f_1(K_p, \tau_d, T). \qquad (45)$$

$$T_i = f_2(K_p, \tau_d, T). \qquad (46)$$

Functions f_1 and f_2 should be determined to optimize the objective function and satisfy the GM or PM constraint. Obviously, it is a challenging task to obtain these functions as each controller parameter is a function of three model parameters. To cope with this issue, we use dimensional analysis to simplify the procedure for determining f_1 and f_2 [19].

To simplify a problem through reducing the number of its variables to the smallest number of essential variables, dimensional analysis can be employed [20]. Without any change in a given physical system behaviour, relations between variables in the system are defined as relations between dimensionless numbers, using dimensional analysis. A dimensionless number has no physical unit and is formed as a product or ratio of quantities that have units.

Consider a system expressed by Equation (47)

$$x_1 = f(x_2, x_3, \ldots, x_n). \qquad (47)$$

with non-zero $x_1, x_2, \ldots, x_n$. According to Buckingham's pi-theorem [20], this equation can be substituted with Equation (48)

$$\pi_1 = g(\pi_2, \pi_3, \ldots, \pi_{n-m}). \qquad (48)$$

where $\pi_2, \ldots, \pi_{n-m}$ are independent dimensionless numbers and m is the minimum number of $x_2, x_3, \ldots, x_n$, which includes all the units in $x_1, x_2, \ldots, x_n$

For the model given by Equation (27), the unit of the dead time (τ_d) and the time constant (T) is the second. The unit of the plant gain (K_p) depends on the nature of the system. As the plant gain along with either the dead time or the time constant cover all the units in Equations (45) and (46), m is equal to 2. Hence, $\frac{\tau_d}{T}$, named dimensionless dead time, is the only dimensionless number in the model. In the PI controller shown in Equation (44), the unit of integral time (T_i) is the second. The unit of controller gain is the inverse of the unit of plant gain. Therefore, the remaining dimensionless numbers are dimensionless gain ($K_p K_c$) and dimensionless integral time ($\frac{T_i}{\tau_d}$ or $\frac{T_i}{T}$). According to Buckingham's pi-theorem, these dimensionless numbers are functions of the dimensionless dead time, as shown in Equations (49) and (50) [19].

$$K_p K_c = g_1\left(\frac{\tau_d}{T}\right). \qquad (49)$$

$$\frac{T_i}{\tau_d} = g_2\left(\frac{\tau_d}{T}\right). \qquad (50)$$

4.1. Simplified SGM tuning formulae for FOPDT models

Having a constraint on GM, the following procedure is proposed for generating formulae for PI controller tuning.

Step 1. A range of values of $\frac{\tau_d}{T}$ is selected.

Step 2. Using Equation (33), ω is determined for each selected value of $\frac{\tau_d}{T}$.

Step 3. For each value of $\frac{\tau_d}{T}$, the optimal values of K_c and T_i are obtained by inserting the resulting ω from step 2 into Equations (30), (31) and (22).

Step 4. The optimal values of $K_p K_c A_m$ and $\frac{T_i}{\tau_d}$ versus $\frac{\tau_d}{T}$ are drawn.

Step 5. Functions g_1 and g_2 in Equations (49) and (50) are determined by using curve-fitting methods.

Assuming the values of $\frac{\tau_d}{T}$ range from 0.1 to 2, Figures 1 and 2 show the optimal values of $K_p K_c A_m$ and $\frac{T_i}{\tau_d}$ across the selected values of $\frac{\tau_d}{T}$, respectively. It can be seen from Figure 1 that $K_p K_c A_m$ is a function of $\frac{\tau_d}{T}$, as shown in Equation (51).

$$K_p K_c A_m = A_1 + \frac{B_1}{\frac{\tau_d}{T}}. \qquad (51)$$

Similarly, the values of $\frac{T_i}{\tau_d}$ are determined from the values of $\frac{\tau_d}{T}$, using Equation (52).

$$\frac{T_i}{\tau_d} = \frac{A_2}{\frac{\tau_d}{T} + B_2}. \qquad (52)$$

Using the least-squares minimization approach, A_1 , B_1 , A_2 and B_2 are determined for the best match with Figures 1 and 2. As a result, the optimal values of A_1 , B_1 , A_2 and B_2 are 0.4331, 1.1191, 1.8095 and 0.8344, respectively.

After simplification, the PI parameters can explicitly be determined using Equations (53) and (54).

$$K_pK_c = \frac{1}{A_m}\left(\frac{10T}{9\tau_d} + \frac{3}{7}\right). \tag{53}$$

$$\frac{T_i}{\tau_d} = \frac{\frac{9}{5}}{\frac{\tau_d}{T} + \frac{5}{6}}. \tag{54}$$

4.2. Simplified SPM tuning formulae for FOPDT models

Having a constraint on PM, the procedure mentioned in section 4.1 is used when Equation (33) in step 2 and Equations (30) and (31) in step (3) are substituted by Equations (42), (39) and (40), respectively. Also, $K_pK_cA_m$ in step 4 should be replaced by K_pK_c. Having obtained the optimal values of K_pK_c and $\frac{T_i}{\tau_d}$ for each value of $\frac{\tau_d}{T}$, the dimensionless gain and dimensionless integral time are given by Equations (55) and (56), using cure-fitting techniques.

$$K_pK_c = A_1(\phi_m) + \frac{B_1(\phi_m)}{\frac{\tau_d}{T}}, \quad \frac{\pi}{6} \le \phi_m \le \frac{\pi}{3}. \tag{55}$$

$$\frac{T_i}{\tau_d} = \frac{A_2(\phi_m)\frac{\tau_d}{T} + B_2(\phi_m)}{\frac{\tau_d}{T} + C_2(\phi_m)}, \quad \frac{\pi}{6} \le \phi_m \le \frac{\pi}{3}. \tag{56}$$

where

$$A_1(\phi_m) = \frac{2}{5}\phi_m + \frac{1}{7}. \tag{57}$$

$$B_1(\phi_m) = -\frac{4}{7}\phi_m + \frac{22}{23}. \tag{58}$$

$$A_2(\phi_m) = \frac{5}{6}\phi_m^2 - \frac{8}{11}\phi_m + \frac{3}{7}. \tag{59}$$

$$B_2(\phi_m) = -\frac{2}{7}\phi_m^2 + \frac{8}{11}\phi_m + \frac{3}{5}. \tag{60}$$

$$C_2(\phi_m) = -\frac{3}{10}\phi_m + \frac{4}{11}. \tag{61}$$

Figures 3 and 4 show values of K_pK_c and $\frac{T_i}{\tau_d}$ across $\frac{\tau_d}{T}$.

5. Simulation results

Tuning is a trade-off between conflicting design objectives. Both robustness and setpoint regulation are design objectives in conflict with load disturbance rejection [8]. In this section, the SGM and SPM controllers are compared with the Astrom-Panagopoulos-Hagglund (APH) controller [7]. Like the proposed method, the APH technique aims at optimal load disturbance rejection. Similarly, this is done by minimizing the IE criterion. Robustness is guaranteed by requiring that the maximum sensitivity is less than a specified value.

Example 1:

$$G_1(s)=\frac{1}{(s+1)^3}.$$

Inserting $s=j\omega$ into $G_1(s)$ results in

$$G_1(j\omega)=\alpha(\omega)+j\beta(\omega)$$

where

$$\alpha(\omega)=\frac{1-3\omega^2}{(1+\omega^2)^3}$$

$$\beta(\omega)=\frac{\omega(\omega^2-3)}{(1+\omega^2)^3}$$

For a constraint on GM, optimal PI parameters are determined by solving Equation (15) and inserting the resulting ω into Equations (8), (9) and (22). Solving Equation (15) by a trial and error method results in $\omega=1.225\,\frac{r}{s}$. Applying Equation (62)

$$f''(\omega)=\lim_{\Delta\to 0}\frac{f(\omega+2\Delta)-2f(\omega+\Delta)+f(\omega)}{\Delta^2}. \qquad (62)$$

to $f(\omega)$ in Equation (13), Equation (16) gives $\frac{d^2K_i}{d\omega^2}=-14.71$. Hence, the sufficient condition is satisfied. PI parameters are given by $K_c=\frac{3.5}{A_m}$ and $T_i=\frac{14}{9}$. Closed-loop step responses for different values of GM are shown in Figure 5. The comparison results are shown in Table 1.

An interesting property of the SGM tuning formulae is that the value of GM can be indicated as a parameter to compromise between performance and robustness. Figure 5 clearly shows that a higher value of GM results in an inferior load disturbance rejection but a better setpoint regulation. It should be noted that higher values of $\frac{IE}{IAE}$ are associated with less oscillatory systems.

For a constraint on PM, optimal PI parameters are determined by solving Equation (26) and inserting the resulting ω into Equations (23), (24) and (22). Considering $f(\omega)$ in Equation (25) and for $PM=40^\circ$, the SPM method results in $\omega=0.697$, $K_c=1.476$ and $T_i=2.02$. The sufficient condition in Equation (16) is also satisfied as $\frac{d^2K_i}{d\omega^2}=-5.527$. Table 2 summarises the comparison results for different values of PM.

It can be seen from Table 2 that the sufficient condition is satisfied for the selected values of PM. Closed-loop step responses for different values of PM are shown in Figure 6. Clearly, a better setpoint regulation but an inferior load disturbance rejection is provided by a higher value of PM.

To compare the performance of the SGM, SPM and APH methods, closed-loop step responses are drawn in Figure 7. A slightly better setpoint regulation is given by the SPM due to a higher value of PM. The setpoint response given by the APH controller is improved using a two-degree of freedom structure. Table 3 shows the comparison results.

An advantage of the SGM and SPM methods is that as soon as ω is determined and subject to satisfying the sufficient condition, the controller parameters are directly given. However, the APH controller parameters cannot be resulted from an explicit set of tuning formulae. They should be computed using a procedure, which may lead to complicated situations [7].

Example 2:

In this example, the SGM method is applied to a non-minimum phase plant, a pure time delay unit, a long dead time plant and a plant with complex poles.

$G_2(s)=\frac{1-2s}{(s+1)^3}$. $\qquad G_3(s)=e^{-s}$.

$$G_4(s)=\frac{e^{-15s}}{(s+1)^3}. \qquad G_5(s)=\frac{9}{(s+1)(s^2+as+9)}.$$

$G_2(s)$ and $G_5(s)$ are not common in control, however, they are included to demonstrate the wide applicability of the design procedure. Closed-loop step responses for different values of GM are shown in Figure 8. The comparison results are shown in Table 4. Figure 9 show the fairly similar closed-loop step responses provided by the SGM and APH methods.

Results of comparison of the SGM and APH methods are summarised in Table 5. Results of applying the SPM methods to $G_2(s)$, $G_3(s)$ and $G_4(s)$ are shown in Table 6. Comparing to each SGM controller, the corresponding SPM controller has a too high gain, resulting in a low gain margin and a high maximum sensitivity.

Example 3:

In this example, the SPM method is applied to the following integrating plants.

$$G_6(s)=\frac{1}{s(s+1)^2}. \qquad G_7(s)=\frac{e^{-s}}{s}.$$

Closed-loop step responses for different values of PM are shown in Figure 10. The comparison results are shown in Table 6. Figure 11 shows the closed-loop step responses resulting from the SPM and APH methods. As shown in Figure 11, the setpoint response of the SPM controller can easily be improved using the setpoint weight. For these methods, the comparison results are summarised in Table 7.

The SPM controller for a FOPDT plant is given by solving Equation (42) and inserting the resulting ω into Equations (39), (40) and (22). A plant with dead time and a single pole at origin is a special case of a FOPDT plant when the time constant becomes infinite. Such a plant can be described by Equation (63).

$$G_{int}(s)=\lim_{T\to\infty}\frac{K_p e^{-\tau_d s}}{Ts+1}=\frac{K_p^{'} e^{-\tau_d s}}{s}. \tag{63}$$

where $K_p^{'}$ is given by Equation (64).

$$K_p^{'}=\frac{K_p}{T}. \tag{64}$$

For the plant in Equation (63), Equation (42) is simplified to Equation (65).

$$\omega=\frac{2}{\tau_d}\cot(\omega\tau_d+\phi_m). \tag{65}$$

Controller parameters are given by inserting the resulting ω into Equations (66) and (67).

$$K_c=\frac{\omega\sin(\omega\tau_d+\phi_m)}{K_p^{'}}. \tag{66}$$

$$T_i=\frac{\tan(\omega\tau_d+\phi_m)}{\omega}. \tag{67}$$

Using Equations (65)-(67), results shown in Table 6 for $G_7(s)$ are obtained in a simpler manner.

Results of applying the SGM method to $G_6(s)$ and $G_7(s)$ are shown in Table 4. Comparing to the corresponding SPM controller, the SGM controller does not have a large enough integral time, resulting in a low phase margin and a high maximum sensitivity.

6. Conclusions

To consider both performance and robustness requirements, this paper presented a PI tuning method for the optimization of load disturbance rejection with a constraint either on the GM or on the PM. The design method resulted in the SGM and SPM tuning formulae that could be adapted for the type of system required. Using dimensional analysis and curve-fitting techniques, a simplified form of tuning formulae for FOPDT models was also determined. Simulation results for a variety of examples including integrating, non-minimum phase and long dead time plants showed that the proposed tuning method was effective in dealing with a wide range of plants.

For industrial applications, it is often required that GM and PM specifications fall into desirable ranges. Future research will attempt to minimize the IE criterion subject to simultaneously satisfying predefined constraints on gain and phase margins.

7. References

[1] Deshpande, PB., (1989) Multivariable Process Control. ISA, RTP, North Carolina.

[2] Koivo, HN, Tanttu, JT., (1991) Tuning of PID controllers: survey of SISO and MIMO techniques. Proc. Intelligent Tuning and Adaptive Control, Singapore.

[3] Astrom, KJ, Hagglund, T., (1995) PID Controllers: Theory, Design and Tuning. Instrument Society of America.

[4] Bialkowski, WL., (1993) Dreams versus reality, a view from both sides of the gap. Pulp and Paper Canada, 94, (11), 19-27.

[5] Desbourough, L, Miller, R., (2002) Increasing customer value of industrial control performance monitoring-Honeywell's experience. Proc. 6th International Conference on Chemical Process Control, AIChE Symposium, 98, (326).

[6] Fruehauf, PS, Chien, IL, Lauritsen, MD., (1994) Simplified IMC-PID tuning rules. ISA Transactions, 33, 43-59.

[7] Åstrom, KJ, Panagopoulos, H, Hagglund T., (1998) Design of PI controllers based on non-convex optimization. Automatica, 34, (5), 585-601.

[8] Åström K, Hägglund T., (2004) Revisiting the Ziegler–Nichols step response method for PID control. Journal of process control, 14, 635-50.

[9] Skogestad, S., (2003) Simple analytic rules for model reduction and PID controller tuning. Journal of Process Control, 13, 291-309.

[10] Åström KJ, Hägglund T., (2006) Advanced PID Control. ISA.

[11] Cominos P, Munro N., (2002) PID controllers: recent tuning methods and design to specification. IEE Proceedings-Control Theory and Applications, 46-53.

[12] Tavakoli, S., Fleming, PJ., (2003) Optimal tuning of PI controllers for first order plus dead time/long dead time models using dimensional analysis. Proceedings of the European control conference, Cambridge, UK.

[13] Tavakoli S, Griffin I, Fleming PJ., (2007) Multi-objective optimization approach to the PI tuning problem. IEEE Congress on Evolutionary Computation, 3165-3171.

[14] McMillan GK., (2014) Tuning and Control Loop Performance. Momentum Press.

[15] Marlin TE., (2000) Process Control: Designing Processes and Control Systems for Dynamic Performance. McGraw-Hill.

[16] Seborg DE, Mellichamp DA, Edgar TF, Doyle FJ., (2010) Process Dynamics and Control. John Wiley & Sons.

[17] Smith CA, Corripio AB., (2006) Principles and Practice of Automatic Process Control. Wiley, Hoboken, NJ.

[18] Ho, WK, Gan, OP, Tay, EB, Ang, EL., (1996) Performance and gain and phase margins of well-known PID tuning formulae. IEEE Transactions on Control Systems Technology, 4, (4), 473-477.

[19] Tavakoli, S, Griffin I, Fleming PJ., (2005) Robust PI controller for load disturbance rejection and setpoint regulation. Proc. IEEE International Conference on Control Applications, Canada.

[20] Zlokarnik, M., (1991) Dimensional Analysis and Scale-up in Chemical Engineering. Springer-Verlag, Berlin.

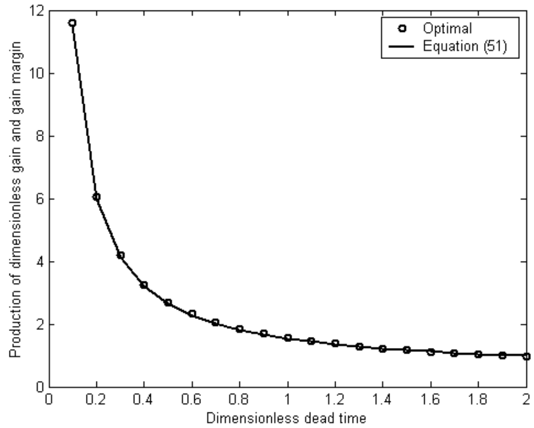


Fig. 1. Optimal values of $K_p K_c A_m$ and the values of $K_p K_c A_m$ given by Equation (51) versus $\frac{\tau_d}{T}$

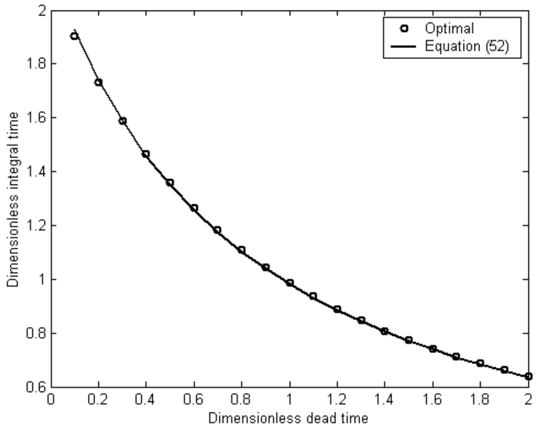


Fig. 2. Optimal values of $\frac{T_i}{\tau_d}$ and the values of $\frac{T_i}{\tau_d}$ given by Equation (52) versus $\frac{\tau_d}{T}$

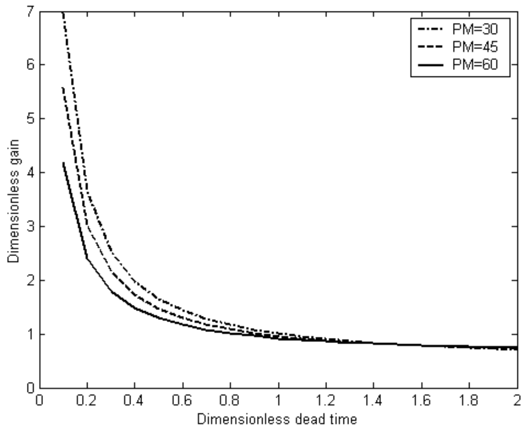


Fig. 3. Values of $K_p K_c$ given by Equation (55) versus $\frac{\tau_d}{T}$

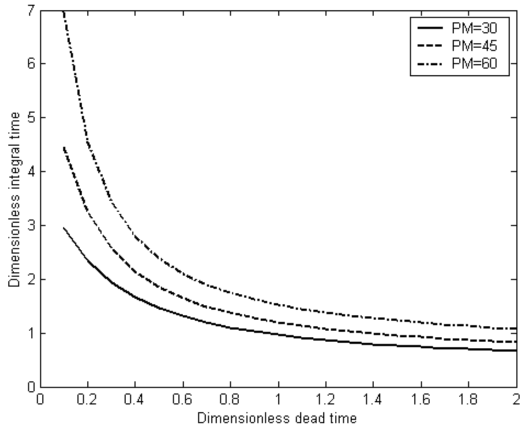


Fig. 4. Values of $\frac{T_i}{\tau_d}$ given by Equation (56) versus $\frac{\tau_d}{T}$

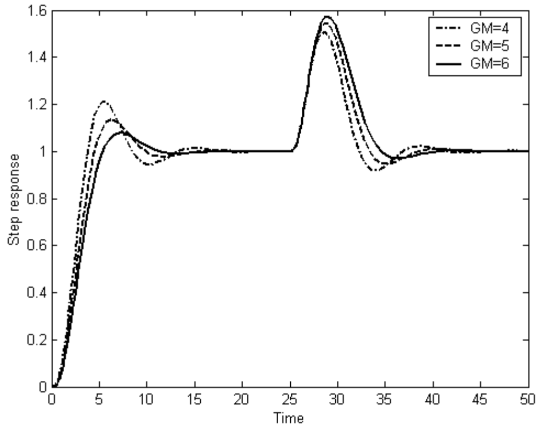


Fig. 5. Closed-loop step responses for different values of GM

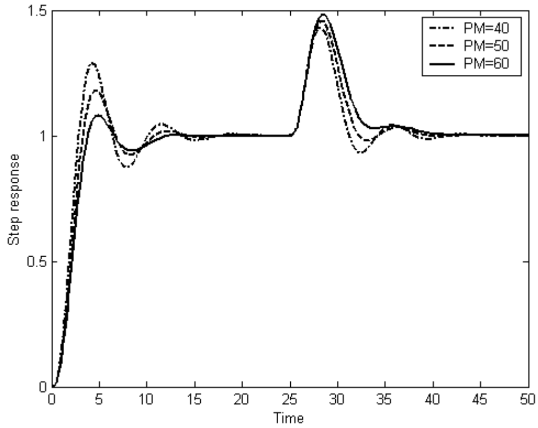


Fig. 6. Closed-loop step responses for different values of PM

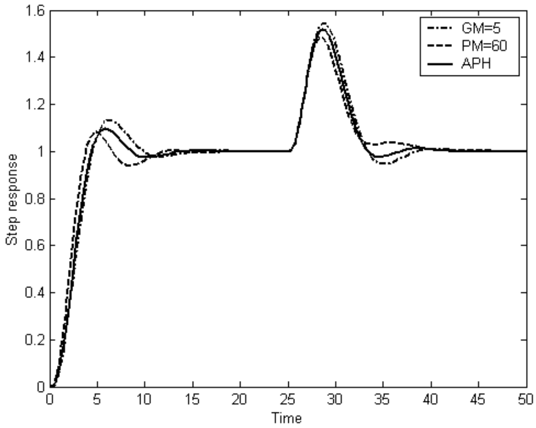
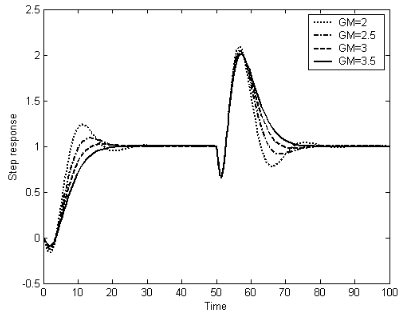
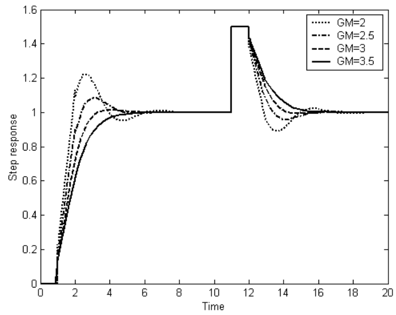


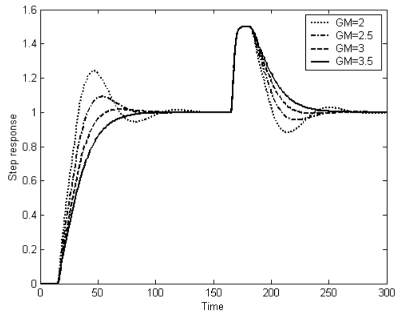
Fig. 7. Closed-loop step responses resulting from the SGM, SPM and APH design methods



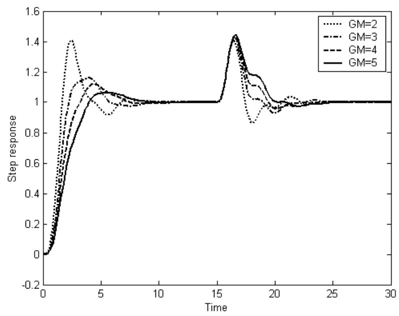
$G_2(s)$



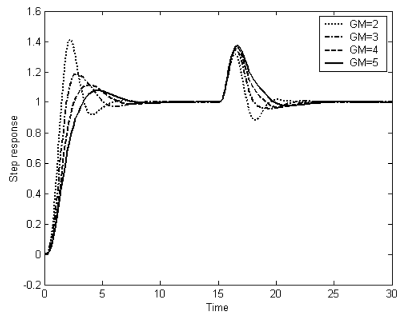
$G_3(s)$



$G_4(s)$



$G_5(s), a = 1$



$G_5(s), a = 2$

Fig. 8. Closed-loop step responses for different values of GM

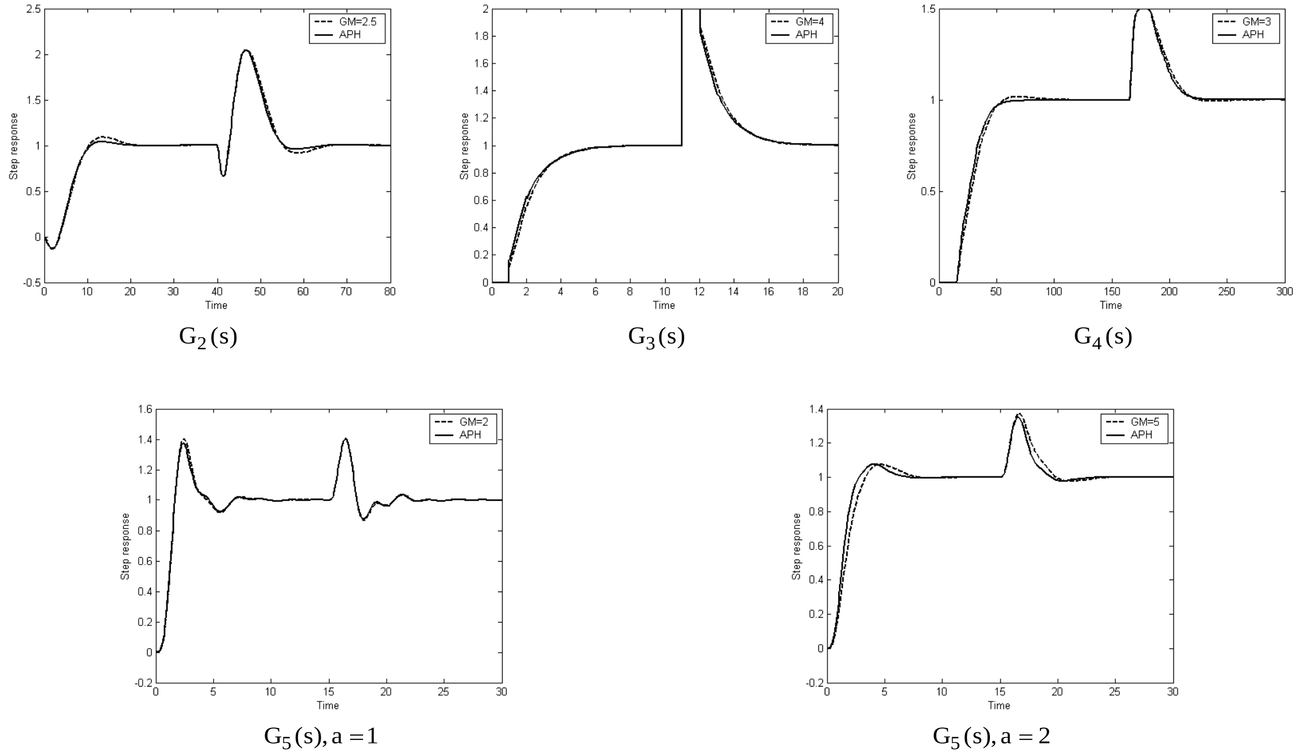


Fig. 9. Closed-loop step responses resulting from the SGM and APH methods

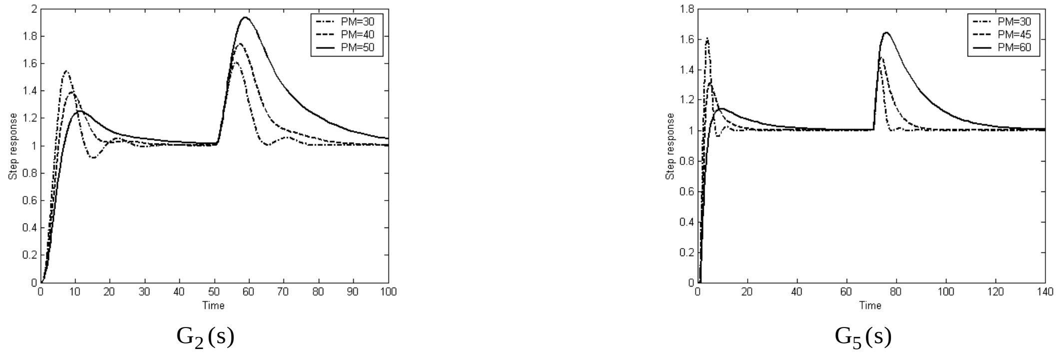


Fig. 10. Closed-loop step responses for different values of PM

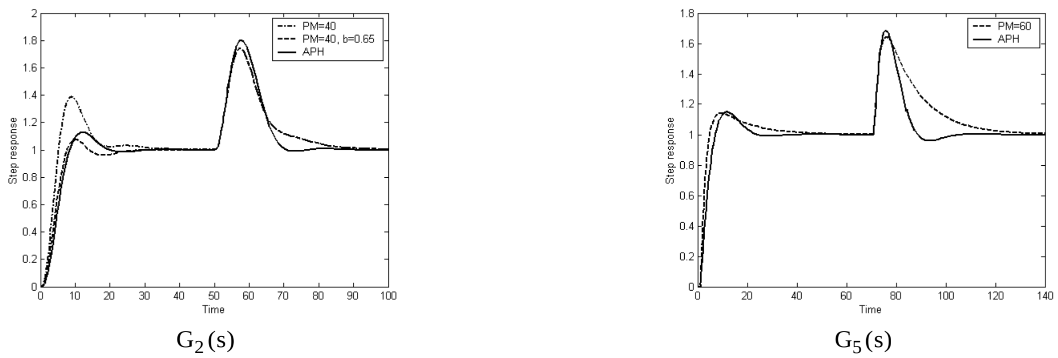


Fig. 11. Closed-loop step responses resulting from the SPM and APH methods

Table 1. Comparison results of the SGM controllers to control $G_1(s)$

GM	3.000	4.000	5.000	6.000
K_c	1.167	0.875	0.700	0.583
T_i	1.556			
M_s	2.153	1.783	1.599	1.486
PM	37.45	47.52	54.72	60.01
$\frac{IE}{IAE}$	0.658	0.783	0.870	0.928

Table 2. Comparison results of the SPM controllers to control $G_1(s)$

PM	40	45	50	55	60
ω	0.697	0.650	0.606	0.565	0.523
K_c	1.476	1.374	1.287	1.215	1.154
T_i	2.020	2.123	2.241	2.380	2.541
M_s	2.112	1.947	1.818	1.715	1.633
GM	2.963	3.296	3.646	4.006	4.374
$\frac{d^2K_i}{d\omega^2}$	-5.527	-4.949	-4.442	-3.994	-3.599
$\frac{IE}{IAE}$	0.812	0.894	0.965	1.000	1.000

Table 3. Comparison results of the SGM, SPM and APH methods to control $G_1(s)$

Method	SGM	SPM	APH
K_c	0.700	1.154	0.862
T_i	1.556	2.541	1.870
b	1.000	1.000	0.930
M_s	1.599	1.633	1.600
GM	5.000	4.374	4.789
PM	54.72	60.00	56.90
$\frac{IE}{IAE}$	0.870	1.000	0.952

Table 4. Comparison results of the SGM controllers

Plant	ω	K_c	T_i	M_s	GM	PM	$\frac{d^2K_i}{d\omega^2}$	$\frac{IE}{IAE}$
$G_2(s)$	0.491	0.268	1.319	2.225	2.000	46.18	-3.368	0.571
		0.214		1.825	2.500	55.19		0.737
		0.179		1.624	3.000	61.12		0.841
		0.153		1.502	3.500	65.31		0.888
$G_3(s)$	2.029	0.177	0.243	1.772	2.500	57.84	-5.482	0.856
		0.147		1.584	3.000	63.35		0.974
		0.126		1.470	3.500	67.23		1.000
		0.111		1.394	4.000	70.12		1.000
$G_4(s)$	0.114	0.231	4.486	2.156	2.000	48.94	-4.933	0.641
		0.185		1.778	2.500	57.48		0.845
		0.154		1.589	3.000	63.05		0.972
		0.132		1.474	3.500	66.98		1.000
$G_5(s)$, $a=1$	2.236	0.056	0.040	2.090	2.000	37.55	-3.192	0.493
		0.037		1.649	3.000	48.52		0.721
		0.028		1.479	4.000	55.70		0.792
		0.022		1.384	5.000	60.86		0.884
$G_5(s)$, $a=2$	2.345	0.417	0.248	2.221	2.000	37.04	-14.37	0.521
		0.278		1.671	3.000	47.77		0.705
		0.208		1.489	4.000	54.58		0.799
		0.167		1.391	5.000	59.54		0.858
$G_6(s)$	0.707	0.500	4.000	5.115	2.000	11.81	-2.83	0.324
		0.400		4.203	2.500	14.24		0.352
		0.333		3.788	3.000	15.64		0.357
$G_7(s)$	1.077	0.474	1.726	5.235	2.000	11.19	-3.656	0.223
		0.374		4.832	2.500	12.00		0.221
		0.316		4.744	3.000	12.15		0.214

Table 5. Comparison results of the SGM and APH controllers

Plant	Method	SGM	APH
$G_2(s)$	K_c	0.214	0.265
	T_i	1.319	1.640
	b	1.000	0.870
	M_s	1.825	1.797
	GM	2.500	2.476
	PM	55.19	57.93
	$\frac{IE}{IAE}$	0.737	0.798
$G_3(s)$	K_c	0.111	0.158
	T_i	0.243	0.335
	b	1.000	1.000
	M_s	1.394	1.400
	GM	4.000	3.846
	PM	70.12	71.71
	$\frac{IE}{IAE}$	1.000	1.000
$G_4(s)$	K_c	0.154	0.208
	T_i	4.486	5.870
	b	1.000	1.000
	M_s	1.589	1.599
	GM	3.000	2.888
	PM	63.05	64.70
	$\frac{IE}{IAE}$	0.972	1.000
$G_5(s)$, $a=1$	K_c	0.056	0.090
	T_i	0.040	0.065
	b	1.000	1.000
	M_s	2.090	2.002
	GM	2.000	2.005
	PM	37.55	39.24
	$\frac{IE}{IAE}$	0.493	0.510
$G_5(s)$, $a=2$	K_c	0.167	0.313
	T_i	0.248	0.373
	b	1.000	0.880
	M_s	1.391	1.400
	GM	5.000	3.843
	PM	59.54	59.16
	$\frac{IE}{IAE}$	0.858	0.867

Table 6. Comparison results of the SPM controllers

$G_2(s)$	PM	30	45	60
	ω	0.360	0.306	0.258
	K_c	0.560	0.594	0.639
	T_i	1.951	2.506	3.337
	M_s	4.280	3.347	3.090
	GM	1.339	1.457	1.500
	$\frac{d^2K_i}{d\omega^2}$	-2.797	-2.501	-2.202
	$\frac{IE}{IAE}$	0.319	0.519	0.722
$G_3(s)$	PM	30	45	60
	ω	1.605	1.404	1.213
	K_c	0.529	0.580	0.636
	T_i	0.388	0.507	0.680
	M_s	5.042	3.945	3.702
	GM	1.288	1.373	1.392
	$\frac{d^2K_i}{d\omega^2}$	-3.880	-3.234	-2.678
	$\frac{IE}{IAE}$	0.260	0.400	0.541
$G_4(s)$	PM	30	45	60
	ω	0.090	0.078	0.068
	K_c	0.543	0.591	0.644
	T_i	7.086	9.211	12.30
	M_s	4.904	3.800	3.512
	GM	1.284	1.379	1.412
	$\frac{d^2K_i}{d\omega^2}$	-4.727	-4.262	-3.624
	$\frac{IE}{IAE}$	0.283	0.425	0.589
$G_6(s)$	PM	30	40	50
	ω	0.396	0.317	0.246
	K_c	0.439	0.338	0.255
	T_i	8.372	11.93	18.54
	M_s	2.258	1.782	1.505
	GM	3.467	4.925	7.005
	$\frac{d^2K_i}{d\omega^2}$	-0.902	-0.584	-0.354
	$\frac{IE}{IAE}$	1.000	1.000	1.000
$G_7(s)$	PM	30	45	60
	ω	0.707	0.528	0.350
	K_c	0.667	0.510	0.345
	T_i	3.998	7.187	16.30
	M_s	2.429	1.742	1.395
	GM	2.068	2.899	4.464
	$\frac{d^2K_i}{d\omega^2}$	-1.533	-0.845	-0.367
	$\frac{IE}{IAE}$	0.998	1.000	1.000

Table 7. Comparison results of the SPM and APH controllers

Plant	Method	SPM	APH
$G_6(s)$	K_c	0.338	0.286
	T_i	11.934	9.000
	b	1.000	0.570
	M_s	1.782	1.801
	GM	4.925	5.436
	PM	40.00	36.92
	$\frac{IE}{IAE}$	1.000	0.989
$G_7(s)$	K_c	0.345	0.282
	T_i	16.30	6.746
	b	1.000	0.660
	M_s	1.395	1.400
	GM	4.464	5.218
	PM	60.00	46.71
	$\frac{IE}{IAE}$	1.000	0.897