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Does $2 + 3 = 5$? In defence of a near absurdity¹

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James Robert Brown asks, “Is anyone really agnostic about $2 + 3 = 5$, and willing only to give assent to $PA \rightarrow 2 + 3 = 5$?” (where PA stands for the Peano axioms for arithmetic). In fact Brown should qualify his ‘anyone’ with ‘anyone not already hopelessly corrupted by philosophy’, since, as he knows full well, there are plenty of so-called *nominalist* philosophers – myself included – who, wishing to avoid commitment to abstract (that is, non-spatiotemporal, acausal, mind- and language-independent) objects, take precisely this attitude to mathematical claims.

Why on earth might one question such a basic claim as ‘ $2 + 3 = 5$ ’? First of all, we should be clear about what is not being questioned. That two apples plus three more apples makes five apples is not something that is in question, and neither is the generalized version of this, ‘For any F, if there are two Fs, and three more Fs, then there are five Fs’. Of course this generalisation may fail for some Fs (think of rabbits or raindrops), but suitably qualified so that we only plug in the right kind of predicates as replacements for ‘F’, this generalization will not worry nominalist philosophers of mathematics – indeed, each of its instances are straightforward logical truths expressible and derivable in first-order predicate logic, without any mention of numbers at all.

But isn’t this what ‘ $2 + 3 = 5$ ’ really says? That any two things combined with any three more (combined in the right kind of way so that no things are created or destroyed in the process) will make five things? If we *only* understood ‘ $2 + 3 = 5$ ’ as a quick way of writing a general claim of the latter sort, then again nominalist philosophers of mathematics would not worry.² But ‘ $2 + 3 = 5$ ’ as a

¹ I am grateful to James Robert Brown, Mark Colyvan, and an anonymous referee for this journal for helpful comments.

² Indeed, it is because of the relation of provable-in-PA claims such as ‘ $(1+1) + ((1 + 1) + 1) = (((1 + 1) + 1) + 1) + 1$ ’ (abbreviated, once suitable definitions are in place, to ‘ $2 + 3 = 5$ ’), to true logically true generalizations such as ‘any two things combined with any three other things make five things’ that we are interested in the Peano axioms in the first place. A mathematical Platonist – i.e., a defender of the view that mathematics consists in a body of truths about abstract mathematical objects – might say that, far from believing that ‘ $2 + 3 = 5$ ’ on the basis of its following from the Peano axioms, we come to see that the Peano axioms correctly characterize the natural numbers on account of their implying claims, such as the claim that ‘ $2 + 3 = 5$ ’, that we already know to be true of the natural numbers (something like this line of thinking is suggested by Russell (1924, p. 325), who points out that “When pure mathematics is organized as a deductive system. ... Some of the premises are much less obvious than some of their consequences, and are believed chiefly because of their consequences”). The contention of this article is that this line of reasoning is incorrect, since we do not know ‘ $2 + 3 = 5$ ’ to be true of numbers considered as mathematical objects (since we do not know that there are any such objects).

mathematical claim is more than a mere abbreviation of a generalization about counting. This can be seen in the fact that it has logical consequences that are not consequences of the generalisation to which it relates. It follows logically from ' $2 + 3 = 5$ ' that there is an object (namely, 2), that, added to three makes 5. And this is not a logical consequence of the general claim 'For any F, if there are two Fs, and three more Fs, then there are five Fs'. For this general claim can be true in finite domains consisting entirely of physical objects, with no numbers in them at all. Since nominalist philosophers question whether there are any numbers (on the grounds that, were there to be such things they would have to be abstract – nonspatiotemporal, acausal, mind- and language-independent – to serve as appropriate truthmakers for the claims of standard mathematics), they see fit to question claims such as ' $2 + 3 = 5$ ' precisely because they logically imply the existence of objects such as the number 2, which, they take it, may fail to exist (as in our finite domain example) even though the general claim 'any two things added to any three further things make five things' is true.

Some philosophers – inspired by the philosopher/logician Gottlob Frege – try to rule out such finite domains by arguing that the existence of the natural numbers is a consequence of an analytic (or conceptual) truth, this truth being the claim that, effectively, if the members of two collections can be paired off with one another exactly, then they share the same number:

'for any F and G, the number of Fs = the number of Gs if and only if $F \approx G$ '

(where ' $F \approx G$ ' is short for 'the Fs and Gs can be put into one-one correspondence'). This claim (which has become known as 'Hume's principle', since Frege first presented the principle citing its occurrence in Book 1 of David Hume's *Treatise of Human Nature* (1738)), is argued to be analytic of our concept of number since anyone who grasps the concept of number will grasp the truth of this claim.

Since the existence of numbers falls directly out of Hume's principle, if Hume's principle is part of our concept of number then it follows from this that anyone who grasps that concept thereby grasps that numbers exist. This derivation of the existence of numbers from our concept of number is reminiscent of St Anselm's ontological argument, deriving the existence of God from our concept of God as a being 'than which no greater can be conceived'. (Such a being couldn't exist *merely* in the

Nevertheless, we can mirror this reasoning from an anti-Platonist perspective to provide a justification for PA over other candidate axiom systems: we choose to work on *this* system, and are interested in what follows from *its* axioms, in no small part because of the relation of its quantifier-free theorems to logical truths such as the truth that 'For any F, if there are two Fs, and three more Fs, then there are five Fs'.

imagination, Anselm argues, because if we can conceive of God at all then we can also conceive of Him existing *in reality*. And since existing in reality is greater than existing merely in the imagination, if God existed *only* in the imagination, we could conceive of something even greater – a really existing God – contradicting our definition of God as a being ‘than which no greater can be conceived’.) For nominalist philosophers, the Fregeans’ derivation of the existence of numbers from our concept of number is at least as fishy as this supposed derivation of the existence of God from our concept of God. Since nominalist philosophers take themselves to have a concept of number without believing in the existence of numbers, they will reject Hume’s principle as a conceptual truth, believing only that Hume’s Principle characterises our concept of number in the sense that, *in order for any objects to count as satisfying that concept*, then Hume’s Principle would have to be true of them, while remaining agnostic on the question of whether there are in fact any numbers.

But *why* remain agnostic about whether there are numbers? And what even hinges on this? Mathematicians talk about mathematical objects and mathematical truths all the time, and indeed are able to *prove* that their mathematical theorems are true. Isn’t it absurd in the face of accepted mathematical practice to say, ‘I know you *think* you’ve proved Fermat’s Last Theorem, Prof Wiles, but actually since we have no reason to believe there are any numbers, we have no reason to believe FLT’? (Actually, the situation is even worse than that: if there are no numbers then FLT is trivially true since, it follow *a fortiori* that there are no numbers $n > 2$ such that $x^n + y^n = z^n$, so Wiles’ efforts were truly wasted.) The philosopher David Lewis certainly thought it would be absurd for philosophers to question the truth of mathematical claims. As he puts it,

Mathematics is an established, going concern. Philosophy is as shaky as can be. To reject mathematics for philosophical reasons would be absurd. ...

That’s not an argument, I know. Rather, I am moved to laughter at the thought of how *presumptuous* it would be to reject mathematics for philosophical reasons. How would *you* like the job of telling the mathematicians that they must change their ways, and abjure countless errors, now that *philosophy* has discovered that there are no classes? Can you tell them, with a straight face, to follow philosophical argument wherever it may lead? If they challenge your credentials, will you boast of philosophy’s other great discoveries: that motion is impossible, that a Being than which no greater can be conceived cannot be conceived not to exist, that it is unthinkable that there is anything outside the mind, that time is unreal, that no theory has ever been made at all probable by the evidence (but on the other hand that an empirically ideal theory cannot possibly be false), that it is a wide-

open scientific question whether anyone has ever believed anything, and so on, and on, *ad nauseum*?

Not me! (Lewis 1990: 58-9)

Just to put this in some perspective, David Lewis is the philosopher best known for believing that, for every true claim about what's possible (such as, 'possibly, Trump will win'), there's a world *just like our own* in respect of its reality (i.e., physical, concrete, though spatiotemporally inaccessible to us) at which that claim is actual (i.e., at that world, there is a counterpart to our own Donald Trump, who becomes President of that world's counterpart to our USA).³ If a philosophical view is so absurd that *even David Lewis can't stomach it*, then maybe it's time to rethink.

Well if nominalist philosophers are going to find mathematics wanting in the way Lewis suggests (calling on mathematicians to renounce their errors and change their practices), and indeed if as suggested above they're going to have to dismiss important results such as Wiles's proof, then they probably do deserve to be laughed out of town. But contemporary nominalists typically wish to leave mathematics just as it is. Indeed, nominalists can even preserve mathematicians' judgements concerning the truth of their theories and the existence of mathematical objects, in at least this sense: there is a notion of truth internal to mathematics according to which to be true *mathematically* just is to be an axiom or a logical consequence of accepted (minimally, logically possible – or *coherent*) mathematical axioms, and to exist *mathematically* just is to be said to exist in an accepted (minimally, logically possible) mathematical theory. Thus in expressing his puzzlement over Frege's account of axioms in mathematics as truths that are true of an intuitively grasped subject matter, David Hilbert writes in response to a letter from Frege:

You write: '...From the truth of the axioms it follows that they do not contradict one another.' I found it very interesting to read this very sentence in your letter. For as long as I have been thinking and writing on these things, I have been saying the exact reverse: if the arbitrarily given axioms do not contradict each other in all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence. (Hilbert, letter to Frege, 1899 (reprinted in Frege (1980))

³ Typically, in presenting Lewis's account, people appeal to one of his own colourful examples (from Lewis 1986) of a merely possible but nevertheless improbable and in fact non-actual world, for example, a world in which there are talking donkeys. At time of writing (March 2016), I thought I would pick an alternative possible but surely similarly improbable (and thus pretty likely to be non-actual) scenario, one in which Donald Trump becomes President. Lewis's account has as a consequence that whatever the actual outcome of the 2016 election, there would be some possible world in which Trump won it. What I wasn't banking on was that that world would turn out to have been our own.

If by truth in mathematics we just mean ‘axiom or logical consequence of a coherent axiom system’, and *if* by existence in mathematics we just mean ‘existence claim that follows logically from the assumption of a coherent axiom system’, then again the nominalist philosopher will not balk at the *mathematical* truth of the theorems of standard mathematics, or the *mathematical* existence claims that follow from these theorems. Mathematicians are welcome to the truth of their theorems, and the existence of mathematical objects, in this sense.⁴

But then what is it that nominalist philosophers *do* balk at. In what sense of truth and existence do they wish to say that we have no reason to believe that the claims of standard mathematics are true, or that their objects exist? If we agree that $2 + 3 = 5$ is true in this Hilbertian sense (of being a consequence of coherent axioms), and also true in a practical applied sense (when understood as shorthand for a generalization about what you get when you combine some things and some other things), then what is the nominalist worrying about when she worries whether this sentence is *really* true, or whether its objects *really* exist? The issue arises because being true ‘in the Hilbertian sense’ is not always enough. At least outside of pure mathematics, the mere internal coherence of a framework of beliefs is not enough to count those beliefs as true. Perhaps the notion of an omniscient, omnipotent, omnibenevolent being is *coherent*, in the sense that the existence of such a being is at least logically possible, but most would think that there remains a further question as to whether there really is a being satisfying that description. And, in more down to earth matters, Newtonian gravitational theory is internally coherent, but we now no longer believe it to be a true account of reality. Granted this general distinction between the mere internal coherence of a theory and its truth, the question arises as to whether we ever have to take our mathematical theories as more than merely coherent – as getting things right about an independently given subject matter. To answer this, we need to understand how we do mathematics – how mathematical theories are

⁴ Those familiar with Hilbert’s work will know that, in his correspondence with Frege, Hilbert would have assumed a syntactic notion of logical consequence, so strictly speaking his criterion of truth and existence was *deductive consistency* (so that an axiomatic theory would be true, mathematically speaking, if no contradiction could be derived from those axioms). In light of Gödel’s incompleteness theorems, we now know that if we take the second-order Peano axioms (with the full second-order induction axiom, rather than a first-order axiom scheme), and conjoin with this the negation of the Gödel sentence for this theory (defined in relation to a particular derivation system for it), no contradiction will be derivable from this theory, but nevertheless the theory has no model (in the standard second-order semantics). The syntactic notion of deductive consistency thus comes apart (in second-order logic) from the semantic notion of *logically possibly true*. I have used Stewart Shapiro’s (1997) terminology of ‘coherence’ as opposed to ‘consistency’ to indicate this semantic notion of logically possible truth. This notion is adequately modelled in mathematics by the model theoretic notion of satisfiability, though I take the lesson of Georg Kreisel (1967) to be that the intuitive notion of logically possible truth is neither model theoretic nor proof theoretic (though adequately modelled by the model theoretic notion).

developed and applied – and ask whether anything in those practices requires us to say that mathematics is true in anything more than what I have been calling the Hilbertian sense.⁵

It is here where recent debate in the philosophy of mathematics has turned its attention to the role of mathematics in empirical scientific theorizing. Of course even in unapplied mathematics, *mere* coherence isn't enough.⁶ Mathematicians are concerned with developing mathematically interesting theories, axiom systems that are not *merely* coherent but which capture intuitive concepts, or have mathematically fruitful consequences. But accounting for the role of these further desiderata does not seem to require that we think of our mathematical theories in the way the Platonist does as answerable to how things really are with a realm of mathematical objects (even if there were such objects, what grounds would we have for thinking that the truths about them should be intuitive, interesting, fruitful...?). When we turn to the role of mathematics in science we have at least a *prima facie* case for taking more than the mere logical possibility of our applied mathematical theories to be confirmed. In particular, close attention has been played to the alleged explanatory role played by mathematical entities in science. We believe in unobservable theoretical objects such as electrons in part because they feature in the best explanations of observed phenomena: if we explain the appearance of a track in a cloud chamber as having been caused by an electron, but go on to add 'but I don't believe in electrons', we seem to undermine our claim to have explained the phenomenon of the track. The same, say many Platonist philosophers of mathematics, goes for mathematical objects such as numbers. If we explain the length of cicada periods (Baker 2005, see also Mark Colyvan's recent paper in this journal) as the optimal adaptive choice 'because 13 and 17 are prime numbers', but then go on to add, 'but I don't believe in numbers', don't we similarly undermine the explanation we have tried to give? On behalf of the nominalist side in this debate, I have argued elsewhere that while mathematics is playing an explanatory role in such cases, it is not mathematical objects that are doing the explanatory work. Rather, such explanations, properly understood, are structural explanations: they explain by showing

⁵ It is worth noting that Hilbert did not stick with his position that non-contradictoriness is all that is required for truth in mathematics, choosing in his later work to interpret the claims of finitary arithmetic as literal truths about finite strings of strokes (thus straying from his original position which saw axioms as implicit definitions of mathematical concepts, potentially applicable to multiple systems of objects). This later, also Hilbertian, sense of truth (truth when interpreted as claims about syntactic objects), is not the one I wish to advocate in this discussion.

⁶ It should be noted that, in speaking of 'mere' coherence, I do not mean to suggest that establishing the logical possibility of an axiom system is a trivial matter. Substantial work goes into providing relative consistency proofs, and of course the consistency – and so, *a fortiori* coherence – of base theories such as ZFC is something about which there is active debate.

(a) what would be true in any system of objects satisfying our structure-characterizing mathematical axioms, and (b) that a given physical system satisfies (or approximately satisfies) those axioms. It is because the (axiomatically characterised) natural number structure is instantiated in the succession of summers starting from some first summer at which cicadas appear that the theorem about the optimum period lengths to avoid overlapping with other periods being prime applies. But making use of this explanation does not require any abstract mathematical objects satisfying the Peano axioms, but only that they are true (at least approximately – idealizing somewhat to paper over the fact of the eventual destruction of the Earth) when interpreted as about the succession of summers.

The debate over whether the truth of mathematics, and the existence of mathematical objects (over and above the *Hilbert-truth* and *Hilbert-existence* that comes with mere coherence) is confirmed by the role of mathematics in empirical science rumbles on. But note that whatever philosophers of science conclude about this issue, it does not impinge on mathematicians continuing to do mathematics as they like, and indeed continuing to make assertions about the (Hilbert)-truth of their theorems and the (Hilbert)-existence of their objects. Nominalists will claim that Hilbert-truth and Hilbert-existence is all that matters when it comes to mathematics, and in this sense it is perfectly fine to agree that $2 + 3 = 5$ (since this is a logical consequence of the Peano Axioms). And they will agree that this particular axiom system is of particular interest to us because of the relation of its formally provable claims to logically true generalizations ('If you have two things and three more things, then you have five things'). But to the extent that it is the more-than-mere-coherence *literal* truth of mathematics as a body of claims about a domain of abstract objects that philosophers are concerned about, while nominalists may worry whether we have any reason to believe that mathematical claims are true in that sense, perhaps mathematicians can be happy with Russell's definition of mathematics as "the subject in which we never know what we are talking about, nor whether what we are saying is true" (Russell (1910), 58).

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