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A Letter from the Editor

In this issue, I would like to share the following news items with you concerning the development of this Journal.

Firstly, we are very pleased that several outstanding colleagues Tamás Fleiner (Eötvös Loránd University), Paul Goldberg (Oxford University), Yuichiro Kamada (University of California, Berkeley), Bettina Klaus (University of Lausanne), Jay Sethuraman (Columbia University), and Guoqiang Tian (Texas A&M University) have joined our editorial board over the last year. They have strengthened our editorial team and improved our services.

Secondly, I can report that we continue to make good progress in attracting high quality submissions to the Journal. For the first few issues, regular submissions have been supplemented by invited ones from supporters of the Journal, but the success of the Journal depends on the constant support of professional colleagues as authors and as readers, alongside the work of our editorial board. Our first and last goal is and will always be the same: To publish the best possible papers and to provide the best possible services to the profession for the public interest.

Thirdly, the Society for the Promotion of Mechanism and Institution Design which owns and publishes the Journal has been approved by the UK Charity Commission as a charity. This is an independent learned society and a not-for-profit, unincorporated association. The charity's objects are to advance education for the public benefit in the subject of mechanism and institution design by

1. managing the flagship journal of the Society: Journal of Mechanism and Institution Design;
2. promoting scientific research on designing, improving, analysing and testing economic, financial, political or social mechanisms and institutions;
3. encouraging the development of mechanisms and institutions that improve efficiency, equality, prosperity, stability and sustainability in society;
4. supporting and organising lectures, workshops and conferences on mechanism and institution design;
5. fostering exchanges and discussions among academics, practitioners and policy makers and disseminating the scientific knowledge of the field.

Fourthly, the Society will hold its inaugural conference on Saturday-Sunday, 12th-13th May 2018. It will be hosted by the Business School, Durham University, the third oldest and prestigious university in England. Durham is a beautiful small town with a magnificent Cathedral on the top of a hill, a world heritage site.
Lastly but not least, we wish to thank all editorial members and referees who have provided their timely and valuable reports and all authors who have submitted their papers to the Journal.

Zaifu Yang, York, 7th December, 2017

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ON A SPONTANEOUS DECENTRALIZED MARKET PROCESS

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ABSTRACT

We examine a spontaneous decentralized market process widely observed in real life labor markets. This is a natural random decentralized dynamic competitive process. We show that this process *converges globally and almost surely* to a competitive equilibrium. This result is surprisingly general by assuming only the existence of an equilibrium. Our findings have also meaningful policy implications.

Keywords: Spontaneous market process, decentralized market, competitive equilibrium.

JEL classification: C71, C78, D02, D44, D58.

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> "Every individual endeavors to employ his capital so that its produce may be of greatest value. He generally neither intends to promote the public interest, nor knows how much he is promoting it. He intends only his own security, only his own gain. And he is in this led by an invisible hand to promote an end which was no part of his intention. By pursuing his own interest he frequently promotes that of society more effectually than when he really intends to promote it." Adam Smith, *The Wealth of Nations* (1776).

1. INTRODUCTION

One of the central issues of economic research is to study market processes by which equilibrium prices or wages can be formed. The basic idea of market processes can be traced back at least to Smith (1776), who used his famous metaphor 'the Invisible Hand' to describe the self-regulating nature of a decentralized free market where the spontaneous action of rational economic agents driven by self-interest will produce a socially desirable outcome.

Walras (1874) suggested a price adjustment process known as the tâtonnement process. In it, a fictitious auctioneer announces a price for one good, collecting all the demands for the good, adjusting the price by the law of demand and supply until an equilibrium in this single good market is reached. The same procedure applies to the remaining goods successively one by one. This sequential procedure is, however, very restrictive. A major improvement was made by Samuelson (1941, 1948) who proposed a simultaneous tâtonnement process. Arrow & Hurwicz (1958), Hahn (1958), and Arrow et al. (1959) proved that Samuelson's process converges globally to an equilibrium provided that all goods are perfectly divisible and substitutable. Scarf (1960) showed by examples that this process, however, does not work if the goods are complementary.[1] More recently, efficient market processes such as auctions and job matching have been developed to deal with more realistic markets that permit indivisibilities; see Crawford & Knoer (1981), Kelso & Crawford (1982), Gul & Stacchetti (2000), Milgrom (2000, 2004), Ausubel & Milgrom (2002), Perry & Reny (2005), Ausubel (2004, 2006), and Sun & Yang (2009,

[1] Scarf (1973) proposed a procedure for computing equilibrium prices in markets with perfectly divisible goods.

2014) among others.[2] All these processes are *deterministic and tâtonnement, and are adjusted by an auctioneer in an orderly manner.*

Departing from the deterministic market processes, we will examine a *random decentralized dynamic competitive market process* widely observed in real life labor markets. Although this is a natural *spontaneous market process,* there has been no formal analysis on it in the literature. The first and foremost important question arises here: will this process generate a desirable economic outcome? This is a significant question as it concerns whether decentralized and uncoordinated markets, notably labor markets, can be operated efficiently or not. Our major finding is that starting from an arbitrary market state this process converges with probability one to a competitive equilibrium in finite time, yielding a Pareto optimal outcome. To our surprise, this result holds true by requiring only a minimal assumption that the market has an equilibrium. It does not rely on any particular condition on the underlying market structure and can therefore admit every possible existence condition. It can accommodate indivisibility, complementarity, and uncertainty or randomness, which are prominent features of the market.

The spontaneous decentralized market process under consideration has the following basic features. Firstly, in the market firms and workers are heterogeneous and inherently indivisible, and workers meet directly and randomly in pursuit of higher payoffs over time. Each firm hires as many workers as it wishes, having a revenue value for each group of workers. Each worker has preferences over firms and salaries but works for at most one firm. When employees work for a firm, they generate a joint revenue which is then split among the firm and its employees.

Secondly, all agents (firms or workers) are driven by self-interest and make their own decisions independently and freely and their activities are neither coordinated nor organized. A firm and a group of workers may form a new coalition if they can divide their joint payoff among themselves to make no member of the coalition worse off and at least one member strictly better off. In this process, the firm will probably dismiss some of its own workers and recruit workers from other firms to be called deserted firms, and every deserted firm will at least temporally not change its contracts for its remaining workers. This process is called *a coalition improvement with the status quo maintaining rule*.

This status quo maintaining rule reflects a common practice in real life

[2] See Bajari and Hortaçsu (2004) for a survey on internet auctions.

business. For instance, if a star professor moves from university A to university B, the former will not alter its contracts with the remaining faculty members at least for a short period of time. This process is only assumed to occur with a positive probability conditional on the current state and time. The assumption on such a probability is intended to capture significant uncertainty about market opportunities due to the nature of decentralized decision-making; see e.g., Kelso & Crawford (1982) and Roth & Vande Vate (1990). For example, even a most confident employer, say, a top academic department, cannot be 100% certain that its vacancies would be fully filled. Furthermore, the assumption is a natural requirement that although information about the market is dispersed and incomplete, it should flow freely enough so that all market participants are sufficiently well informed and can therefore have a chance to respond to newly arrived opportunities. The assumption of a positive probability could be viewed as *a degree of market transparency.*

Thirdly, this process is *spontaneous* in the sense that it is the result of human self-interested action but not of conscious human design such as auction or matching design.[3] In Hayek (1988), this kind of market process is called a spontaneous order and is a natural free market process. This process is not, however, a tâtonnement[4] because trading is permitted even if the market is not in equilibrium. In other words, at each time, if a worker is matched with a firm, this worker will provide service for the firm in exchange for a wage from the firm whether it is an equilibrium wage or not. As long as the market has not reached an equilibrium, it will create incentives for some agents to deviate from the current state. This type of deterministic non-tâtonnement process has been studied by Hahn (1962), Hahn & Negishi (1962), Negishi (1961, 1962), Uzawa (1962), Arrow & Hahn (1971, Chapter XIII), and Fisher (1972, 1974, 1989).

Fourthly, this process is not monotonic and permits chaotic, cyclic and myopic behaviors. In the process deserted firms and dismissed workers generally become worse off and thus the total welfare need not be monotonic. A worker may sequentially work for several firms because a latter firm offers a better salary or the worker may have been fired previously; conversely, the same firm hires different workers over time for the same positions as workers who

[3] Besides those auction mechanisms mentioned previously, see also Roth & Sotomayor (1990), Hatfield & Milgrom (2005), Ostrovsky (2008), and Kojima & Pathak (2009) for a variety of matching models.

[4] Recall that a tâtonnement process permits trade to take place only at equilibrium prices.

come later may either work more efficiently or demand lower salaries. It is not uncommon to see that a worker eventually returns to her previous employer but with a different contract. It is also possible that in the process a worker may get a remarkably attractive position at one time but get fired at another time. The process can be chaotic and occasionally cyclical as firms and workers meet directly and randomly, haggling for better deals, and coalitions can be formed hastily and can also dissolve instantly whenever better opportunities arise. Yet, the rules of this spontaneous process are simple, transparent, and detail-free according to the doctrine of Wilson (1987).

Our main result demonstrates that, starting from an arbitrary market state of a matching between firms and workers with a system of salaries, the above random decentralized dynamic process, where every possible coalition improvement with the status quo maintaining rule conditional on the current state and time occurs with a positive probability, almost surely converges in finite time to a competitive equilibrium of the market consisting of an efficient matching between firms and workers and a scheme of supporting salaries (Theorem 1 and Corollary 1), resulting in a Pareto optimal outcome. This result holds true for any market environment as long as there exists a competitive equilibrium with an integral vector of equilibrium salaries or prices.[5] The consideration of such equilibrium prices is very natural and practical, because any transaction in real life business can only happen in integer number of monetary units. A number of sufficient conditions are known to ensure the existence of such an equilibrium.[6] Among them, the Gross Substitutes condition given by Kelso & Crawford (1982) has been widely used and requires every firm to view all workers as substitutes, subsuming the assignment model by Koopmans & Beckmann (1957), Shapley & Shubik (1971), Crawford & Knoer (1981), and Demange et al. (1986) as a special case.[7] A crucial step toward establishing the major result of the paper is to prove that the spontaneous decentralized random process does not get stuck in cycles endlessly. To this end, we develop a novel

5 In contrast, to our best knowledge, in the deterministic settings there is no general existing process that guarantees to find a competitive equilibrium in an economy with indivisibilities if one assumes only the existence of an equilibrium; see those references mentioned in the second paragraph.

6 See Kelso & Crawford (1982), Ma (1998), Sun & Yang (2006), Milgrom & Strulovici (2009), Baldwin & Klemperer (2013), Fujishige & Yang (2015), and Murota (2016).

7 Such models are also called unit-demand models where every consumer demands at most one item (see also Shapley & Scarf (1974)) or every person needs only one opposite sex partner (see Gale & Shapley (1962)).

technique to show the existence of a finite sequence of successive coalition improvements with the status quo maintaining rule from any initial market state to a competitive equilibrium in the market (Theorem 2).

The current study on the spontaneous decentralized market process may help deepen our understanding of Adam Smith's Invisible Hand in complex economic environments and offer a theoretic explanation as to why the spontaneous decentralized market process widely observed in labor markets can perform well and thus justify its very existence. Our findings might also have some interesting policy implications: in general, free markets can work well as Hayek (1944) had passionately advocated, almost surely resulting in socially efficient outcomes in a self-interested and law-abiding but seemingly chaotic and random economic environment, and in particular the price system can marvellously aggregate and communicate information "in a system in which the knowledge of the relevant facts is dispersed among many people" as Hayek (1945) had believed.[8] A caveat is that free markets cannot unconditionally function properly, and in order for them to work well, the government should promote and improve market transparency so as to facilitate the convergence of such market processes.[9] We also discuss (in Section 5) under what circumstance firms can avoid (sometimes controversial) massive dismissals.

We conclude this introductory section by reviewing several related studies. In a pioneering study Kelso & Crawford (1982) introduced a general job matching market where each firm can hire several workers and each worker is employed by at most one firm. They developed a salary adjustment process that converges to an equilibrium provided that every firm treats all workers as substitutes. Although they stress that pervasive uncertainty is an essential feature of the labor market, they do not deal with uncertainty and their process is a deterministic market process. In a seminal article Roth & Vande Vate (1990) reexamined the marriage matching model of Gale & Shapley (1962) in which each man tries to marry his favorite woman and vice versa, and established a decentralized random process for the model; see Ma (1996) for a further study on this process. Kojima & Ünver (2008) generalized the marriage model in a substantial way to allow for instance each college to

[8] See a recent article on the decentralization of stated-owned enterprizes in China by Huang et al. (2017).

[9] Since 1980s, indeed many governments around the world require every employer to publicly announce its job openings at least several weeks or months in advance before the closing date.

admit many students and each student to attend several colleges.[10] They investigated a decentralized random process for a pairwise stable matching outcome and established a probabilistic convergence. Chen et al. (2016) examined a random decentralized process for the assignment market, as a counterpart of the deterministic processes proposed by Crawford & Knoer (1981) and Demange et al. (1986). Nax & Pradelski (2015) discussed a similar issue from the viewpoint of evolutionary dynamics. Ma & Li (2016) studied a decentralized probabilistic double auction process for a financial market.

A crucial and well-recognized difference between the matching models[11] and the competitive market models[12] is that the matching models do not involve (or have flexible) prices nor have a system of competitive prices to support a stable matching outcome, which is the often-used notion of solution to matching models and generally weaker than the concept of competitive equilibrium (see Quinzii (1984)). Feldman (1974) and Green (1974) studied deterministic decentralized processes for certain subclasses of non-transferable utility games. Their approaches do not apply to the labour market or matching models where indivisibility is involved. Chung (2000), Diamantoudi et al. (2004), Klaus & Klijn (2007), and Biró et al. (2014) investigated random processes for roommate and marriage matching models.

The rest of the paper is organized as follows. Section 2 presents the model and basic concepts. Section 3 contains the main results. Section 4 introduces the thought experimental procedure for proving the key result (Theorem 2). Section 5 examines the case of Gross Substitutes which can help firms avoid massive dismissals. Section 6 concludes.

2. THE MODEL

Consider a general labor market with a finite number of heterogeneous firms and workers. Formally, let F be the set of m firms and W the set of n workers, respectively. We assume that each firm can hire as many workers as it wishes

[10] They assume that agents on one side of the market have substitutable preferences and those on the other side have responsive preferences.

[11] Studied by Gale & Shapley (1962), Roth & Sotomayor (1990), Roth & Vande Vate (1990), and Kojima & Ünver (2008) among others.

[12] Studied by Koopmans & Beckmann (1957), Arrow & Hurwicz (1958), Shapley & Shubik (1971), Shapley & Scarf (1974), Kelso & Crawford (1982), Gul & Stacchetti (1999), Ausubel (2006), and Sun & Yang (2006, 2009).

but each worker can work for at most one firm. Each firm $j \in F$ has an integer-valued and weakly increasing revenue function $R^j : 2^W \to \mathbb{Z}$ with $R^j(\emptyset) = 0$. Namely, when firm j hires a group $B \subseteq W$ of workers, it has a revenue of $R^i(B)$ in units of money thus being an integer value. Given a scheme of salaries $s^j = (s^j_i \mid i \in W) \in \mathbf{R}^W_+$ for firm $j \in F$, firm j's net profits are given by $\pi_j(B, s^j) = R^j(B) - \sum_{i \in B} s^j_i$. Each worker $i \in W$ has quasi-linear utility in money and has an integer minimum wage requirement $w^j_i \geq 0$ for being willing to work at firm $j \in F$. Because of the minimum wage requirement, for the same salary worker i may prefer to be hired by firm j rather than by firm k. The integer value assumption of R^j and w^j_i is quite natural and standard, as for example we cannot specify a monetary payoff more closely than to its nearest penny.[13] The information about R^j and w^j_i can be private, as explained in the next section. We use $(F, W, (R^j \mid j \in F), (w^j_i \mid i \in W, j \in F))$ (or (F, W, R^j, w^j_i), in short) to represent this economy. In addition, for any $F' \subseteq F$ and $W' \subseteq W$, let (F', W', R^j, w^j_i) be the economy only consisting of firms in F' and workers in W'. In the sequel a worker or firm may be simply called an agent.

A matching μ in the labor market is a correspondence such that

- for all $i \in W$, either $\mu(i) = i$ or $\mu(i) \in F$,
- for all $j \in F$, $\mu(j) \subseteq W$, and
- for all $i \in W$ and $j \in F$, $\mu(i) = j$ if and only if $i \in \mu(j)$.

At matching μ, for any worker $i \in W$, if $\mu(i) \in F$, then $\mu(i)$ represents the firm to which worker i is assigned. If $\mu(i) = i$, then worker i is not assigned to any firm and we will say that such worker i is *unemployed* or *self-matched*. For any firm $j \in F$, $\mu(j)$ stands for the set of workers hired by firm j. If $\mu(j)$ is empty, then firm j does not employ any worker.

A *salary scheme system* $S = (s^j \mid j \in F)$ consists of salary schemes $s^j \in \mathbf{R}^W_+$ of all firms $j \in F$. A *state or allocation* of the market consists of a salary scheme system $S = (s^j \mid j \in F)$ and a matching μ. At allocation (μ, S), if $\mu(i) = j \in F$ for any worker $i \in W$, then worker i works for firm j and receives salary s^j_i; if $\mu(i) = i$, then worker i does not work for any firm and receives no salary, and firm j hires the group $\mu(j)$ of workers and pays the total amount $s^j(\mu(j)) = \sum_{i \in \mu(j)} s^j_i$ of salary. An allocation (μ, S) induces a *payoff* vector

[13] See e.g., Demange et al. (1986), Roth & Sotomayor (1990), Ausubel (2006), and Sun & Yang (2009).

$u \in \mathbf{R}^{F \cup W}$ such that for every worker $i \in W$, $u_i = s_i^{\mu(i)} - w_i^{\mu(i)}$ when $\mu(i) \in F$, and $u_i = 0$ when $\mu(i) = i$, and for every firm $j \in F$, $u_j = \pi_j(\mu(j), s^j)$. In this way, the state (μ, S) can be alternatively written as (μ, u). Observe that at every state (μ, u) we have $u_j + \sum_{i \in \mu(j)} u_i = R^j(\mu(j)) - \sum_{i \in \mu(j)} w_i^j$ for every firm $j \in F$ and $u_k = 0$ for every worker $k \in W$ with $\mu(k) = k$.

A state (μ, u) is *individually rational* if no agent is worse than she stands alone, i.e., $u_k \geq 0$ for every $k \in F \cup W$. A nonempty group $B \subseteq F \cup W$ of workers and firms is called a coalition. Following Kelso & Crawford (1982, p. 1487), we say that a coalition B is *essential* if it contains either only one worker or only one firm with any number of workers. (Note that we make this definition slightly more general than theirs by including the case of either a single firm or a single worker to cover individual rationality.) In the following any coalition means an essential coalition. Sometimes it is convenient to use (j, B) to express a coalition in order to distinguish the firm and workers, where $j \in F$ or $j = \emptyset$, and $B \subseteq W$.

A state (μ, u) is *weakly blocked by an individually rational coalition* $B \subseteq F \cup W$ (*or simply weakly blocked by a coalition*) (i) if there exists one firm $j \in B$ with a nonnegative payoff vector $r \in \mathbf{R}_+^B$ such that

$$r_k \geq u_k \text{ for every } k \in B, \text{ and} \tag{1}$$

$$\sum_{k \in B} r_k = R^j(B \setminus \{j\}) - \sum_{k \in B \setminus \{j\}} w_k^j \tag{2}$$

with at least one strict inequality for (1), or (ii) if the coalition B contains only one worker i with $r_i = 0 > u_i$. Notice that all members in the coalition B are individually rational, i.e., $r_k \geq 0$ for all $k \in B$, and that in case (i), if B contains only a firm j, then $r_j = 0 > u_j$. The definition says that at least one member in B would be better off and none in B would be worse off if firm j hires only workers $i \in B$ and every worker $i \in B$ works for firm j, or it would be strictly better for the single firm not to hire any worker or for the single worker i not to work at firm $\mu(i)$. Such a B is called an *individually rational weakly blocking coalition* (*or simply a weakly blocking coalition*). An (*individually rational*) *strongly blocking coalition* is defined in the same manner as a weakly blocking coalition, except that (1) is now strengthened as a strict inequality for every member in B. Throughout the paper every (weakly or strongly) blocking coalition means an individually rational blocking coalition unless stated otherwise. With respect to a blocking coalition (j, B) of (μ, u), we say that a firm $k(\neq j)$ is *deserted* if B contains at least one worker from $\mu(k)$, and that a worker $w \in \mu(j)$ is *dismissed* if B does not contain w. A firm $k(\neq j)$ is *unaffected* if B does not contain any worker from $\mu(k)$.

A state is a *core allocation* if it is not strongly blocked by any coalition. Clearly, a state is a core allocation if and only if it is individually rational and is not strongly blocked by any firm with at least one worker. A state is a *strict core allocation* or *a competitive equilibrium* if it is not weakly blocked by any coalition.[14] Observe that unemployment is allowed in equilibrium. It is well known from Kelso & Crawford (1982, p. 1487) that the set of competitive equilibria in this market coincides with the set of strict core allocations.[15]

The following definition describes how the market will transfer from a disequilibrium state to another state through a blocking coalition.

Definition 1 *Let $B \subseteq F \cup W$ be a blocking coalition against the state (μ, u) with the associated payoff vector $r \in \mathbf{R}^B$. A new state (μ', u') is said to be* a coalition improvement of the state (μ, u) through B with every deserted or unaffected firm maintaining the status quo for its remaining employees (*or simply* a coalition improvement with the status quo maintaining rule) *if the new state is constructed as follows:*

(1) *if there exists one firm $j \in B$, let*

(1a) $\mu'(i) = j$ *and* $u'_i = r_i$ *for every worker* $i \in B$,

(1b) $\mu'(j) = B \setminus \{j\}$ *and* $u'_j = r_j$,

(1c) $\mu'(i) = i$ *and* $u'_i = 0$ *for every worker* $i \in \mu(j) \setminus B$,

(1d) $\mu'(i) = \mu(i)$ *and* $u'_i = u_i$ *for every worker* $i \in W \setminus (B \cup \mu(j))$,

(1e) $\mu'(k) = \mu(k) \setminus B$ *and*

$$u'_k = R^k(\mu(k) \setminus B) - \sum_{i \in \mu(k) \setminus B} (w_i^k + u_i)$$

for every firm $k \in F \setminus \{j\}$; or

[14] We point out that in the definition of a blocking coalition we ignore those nonessential blocking coalitions in which some members may have negative payoffs. However, doing so does not lose any generality. See the appendix of this paper for a detailed discussion.

[15] Perry and Reny (1994) provide a noncooperative foundation for this fundamental cooperative solution-the core.

(2) *if the coalition B consists of only one worker i, let*

(2a) $\mu'(i) = i$ *and* $u'_i = 0$,

(2b) $\mu'(k) = \mu(k)$ *and* $u'_k = u_k$ *for every worker* $k \in W \setminus \{i\}$,

(2c) $\mu'(\mu(i)) = \mu(\mu(i)) \setminus \{i\}$ *and*

$$u'_{\mu(i)} = R^{\mu(i)}(\mu(\mu(i)) \setminus \{i\}) - \sum_{k \in \mu(\mu(i)) \setminus \{i\}} (w_k^{\mu(i)} + u_k),$$

(2d) $\mu'(k) = \mu(k)$ *and* $u'_k = u_k$ *for every firm* $k \in F \setminus \{\mu(i)\}$.

By definition, at the new state (μ', u'), if B contains a firm j, then firm j will hire all workers $i \in B \setminus \{j\}$ and share the revenue according to the given specification $r \in \mathbb{R}^B$, whereas workers in $\mu(j)$ not in B will be dismissed by firm j and become unemployed to get a payoff of zero and all other workers outside $B \cup \mu(j)$ will maintain the status quo of (μ, u). Observe that every firm $k \in F \setminus \{j\}$ will continue to hire those workers who are not in the blocking coalition B and who were hired by firm k at (μ, u), and that a deserted firm k's payoff may be negatively affected because firm k will keep the same contract with each of its remaining workers as in the state (μ, u), and that every unaffected firm and its employees will not be affected and thus preserve the status quo of (μ, u). With respect to a coalition improvement with the status quo maintaining rule through the blocking coalition B, we also distinguish *weak coalition improvement* from *strong coalition improvement*, depending on whether the associated blocking coalition B is weak or strong.

The weak coalition improvement with the status quo maintaining rule imitates real life business practices. When a firm and a group of workers find an opportunity to form a weakly blocking coalition, the firm and hired workers are better off but the deserted firms and dismissed workers are usually worse off. Therefore coalition improvements with the status quo maintaining rule are just local not global improvements for the market. In other words, during this process, the overall welfare of the market is not monotonic and can be increasing or decreasing. If an employee leaves a firm for a better offer from another firm, the abandoned firm will usually not change contracts immediately for its remaining employees and needs time to adapt to the new situation. It is fairly common that such a firm will continue to operate for a period of time even if it is in debt. If a worker is fired by a firm, she needs time to find a new job. In such processes, it is not unusual to observe that as in reality, a worker may jump from one position to another and may eventually return to her previous employer, but with a different contract. Weak coalition improvements also allow a firm in debt to be adjudicated bankrupt by firing all its employees, or continue to run by taking no action, or reorganize by hiring and firing, as in real life business.

Observe that we define all concepts such as a blocking coalition and a competitive equilibrium on the basis of real numbers. However, most real life market processes work only on rational or integral salaries or prices. The following lemma shows that, assuming integrality of revenue functions and minimum salary requirements, an integral state (μ, u) is a competitive equilibrium within the domain of real payoffs if (and only if) it is not weakly blocked by any coalition with integral payoffs. It should be noticed that this result holds true without requiring any other extra conditions.

Lemma 1 *Let R^j and w_i^j for all $i \in W$ and $j \in F$ be integral. If a state (μ, u) with $u \in \mathbb{Z}^{F \cup W}$ is not weakly blocked by any coalition B with integral payoffs $v_i \in \mathbb{Z}$ for all $i \in B$, then it cannot be blocked by any coalition T with real payoffs $u'_i \in \mathbf{R}$ for all $i \in T$. Consequently, (μ, u) must be a competitive equilibrium.*

Proof. Suppose to the contrary that an integral state (μ, u) with $u \in \mathbb{Z}^{F \cup W}$ which is not blocked by any group of firm j and workers B with integral payoffs $v_j, v_i \in \mathbb{Z}$ for all $i \in B$, is blocked by a group of firm k and workers T with real payoffs $u'_k, u'_i \in \mathbf{R}$ for all $i \in T$. Because the coalition $T \cup \{k\}$ blocks (μ, u), then we have

$$u'_k + \sum_{i \in T} u'_i = R^k(T) - \sum_{i \in T} w_i^k, \tag{3}$$

$$u'_k \geq u_k \text{ and } u'_i \geq u_i \text{ for all } i \in T \tag{4}$$

with at least one strict inequality. Let $K = \{i \in T \mid u'_i > u_i \text{ and } u'_i \notin \mathbb{Z}\} \cup \{k \mid u'_k > u_k \text{ and } u'_k \notin \mathbb{Z}\}$. If K is empty, we have a contradiction. If K is not empty, it follows from (3) and the integer number $R^k(T) - \sum_{i \in T} w_i^k$ that K contains at least two elements. Take any element $i^* \in K$. Then let $\bar{u}_i = u_i (\in \mathbb{Z})$ for every $i \in K$ with $i \neq i^*$ and $\bar{u}_i = u_i (\in \mathbb{Z})$ for every $i \in (T \cup \{k\}) \setminus K$, and $\bar{u}_{i^*} = R^k(T) - \sum_{i \in T} w_i^k - \sum_{i \in (T \cup \{k\}) \setminus \{i^*\}} u_i (> u_{i^*})$. $\bar{u}_{i^*}$ is an integer. Because u_i for all $i \in F \cup W$, R^j for all $j \in F$, and w_i^j for all $j \in F$ and $i \in W$ are integers, we have the coalition $T \cup \{k\}$ with integer payoffs $\bar{u}_k$ and $\bar{u}_i$ for all $i \in T$ that blocks (μ, u), yielding a contradiction. The case of a singleton coalition is easy to verify. This completes the proof. $\square$

For convenience, a state (μ, u) with an integral payoff vector $u \in \mathbb{Z}^{F \cup W}$ or equivalently integral salaries or prices will be called *an integral state*. We are particularly interested in integral states because transactions in real world business can happen only in integral or rational numbers of monetary units. The above lemma shows that it is sufficient to concentrate on integral states. There are several major sufficient conditions guaranteeing the existence of an integral competitive equilibrium. The most well-known of these conditions is the Gross Substitutes condition of Kelso and Crawford (1982), which will be introduced shortly.

Given a salary scheme $s^j \in \mathbf{R}^W$, let $D^j(s^j)$ be the set of solutions to

$$\max_{T \subseteq W} \pi_j(T, s^j)$$

$D^j(s^j)$ is the collection of those groups of workers which give the firm the highest profit at the offered salaries s^j.

Definition 2 *Firm j satisfies the Gross Substitutes condition if for every pair of salary schemes s^j and t^j with $s^j \leq t^j$ and for every $A \in D^j(s^j)$, there exists $C \in D^j(t^j)$ such that $\{i \mid i \in A \text{ and } s_i^j = t_i^j\} \subseteq C$.*

This condition states that if a firm j hires a group A of workers at salaries s^j and if the salaries are now increased to the new levels t^j, the firm will still want to hire those workers in A whose salaries do not increase.

It is well known that this job matching market admits at least one competitive equilibrium and that the set of strict core allocations coincides with that of competitive equilibria (Kelso & Crawford (1982)). In addition, as all valuations w_i^j and R^j are integers and every firm satisfies the Gross Substitutes condition, the labour market must have at least one strict core allocation with an integral payoff vector $u \in \mathbb{Z}^{F \cup W}$ or an integral salary system $S = (s^1, s^2, \cdots, s^m) \in \mathbb{Z}^{W \times F}$; see Gul & Stacchetti (1999), Ausubel (2006), and Sun & Yang (2009). Notice that the celebrated assignment market of Koopmans & Beckmann (1957) and Shapley & Shubik (1971) automatically satisfies the Gross Substitutes condition and thus has an integral equilibrium. In the rest of the paper in order to avoid repetition, every (market) state is taken to mean an integral (market) state unless stated otherwise.

3. SPONTANEOUS DECENTRALIZED MARKET PROCESSES

In this section we address the central issue whether a spontaneous decentralized random market process can settle the market in a competitive equilibrium or not. Suppose that the market starts at time 0 with an arbitrary state. It is plausible to assume that information about the market is dispersed among all the market participants and no single agent or organization commands complete knowledge of the market. For instance, each firm j possesses private information about its own revenue function R^j and each worker i knows her own minimum wage requirement w_i^j privately. Because firms and workers are self-interested, any individual or group of agents will be willing to grasp any opportunity to improve their wellbeing by forming a new coalition within which the firm may fire some of its workers and hire some workers from other firms, and some workers may abandon their employers. Deserted firms will at least temporarily maintain the status quo for their remaining employees. The formation of the new coalition is a weak coalition improvement against the current state with the status quo maintaining rule. Because the market is totally decentralized and uncoordinated and agents are not assumed to have complete knowledge of the market, such coalition improvements with the status quo maintaining rule cannot be expected to occur with absolute certainty but with a positive probability. Moreover because real life transactions take place only in an integral number of monetary units, it suffices to work with only integral weak coalition improvements with the status quo maintaining rule. Obviously, this spontaneous decentralized random process will continue to move from one disequilibrium state to another until a competitive equilibrium is reached.

A natural and fundamental question arises here: will such a spontaneous decentralized random market process converge to a competitive equilibrium eventually? The following theorem gives an affirmative answer by showing that this general process will almost surely converge to a competitive equilibrium in finite time, provided that at any point in time, every weak coalition improvement with the status quo maintaining rule conditional on the current market state arises with a positive probability bounded away from zero. The assumption of a positive probability for every weak coalition improvement is intended to capture pervasive uncertainty about opportunities in decentralized and uncoordinated markets. For instance, in academic markets, even a top department with attractive offers cannot guarantee to fill its vacancies. This assumption also implies that although information about the market is dispersed among all the agents involved, job-related information flows smoothly enough so that agents can grasp newly arrived opportunities in the market at least with a positive probability.

As it will be clear in the following proof of Theorem 1, the requirement of the positive probability being no less than any fixed small number $\varepsilon > 0$ is really minimal. This probability could be viewed as a measure of market transparency. The magnitude of this positive number ε will not affect the convergence of the decentralized random competitive process but it does have an impact on the convergence speed. In general, the bigger ε is, the faster the process will be.

Theorem 1 *Assume that the labour market (F, W, R^j, w_i^j) has a competitive equilibrium. Then, starting with an arbitrary market state, any random and decentralized process in which every weak coalition improvement with the status quo maintaining rule occurs with a positive probability conditional on the current state and time bounded away from zero, will converge with probability one to a competitive equilibrium in finite time.*

Several remarks are in order. First, the market has an infinite number of (integral) market states amongst which there are only a few equilibrium states and the rest are all disequilibrium states. Each disequilibrium state can be blocked by at least one coalition and is therefore associated with at least one coalition improvement with the status quo maintaining rule. Second, at each disequilibrium state there exists a finite number of coalition improvements with the status quo maintaining rule. Third, as it will be shown in Lemma 2, starting from any given market state, the set of market states that can be generated through all possible successive coalition improvements with the status quo maintaining rule is finite. This means that the assumption of a positive probability in the theorem is actually imposed only upon a finite number of market states which can be historically linked with the initial market state via coalition improvements with the status quo maintaining rule.

Corollary 1 *Assume that every firm in the market (F, W, R^j, w_i^j) satisfies the Gross Substitutes condition. Then, starting with an arbitrary market state, any random and decentralized process in which every weak coalition improvement with the status quo maintaining rule occurs with a positive probability conditional on the current state and time bounded away from zero, converges almost surely to a competitive equilibrium in finite time.*

The proof of the theorem above relies on the following crucial mathematical result, which establishes a link between any initial market state and a competitive equilibrium through only a finite sequence of successive weak coalition improvements with the status quo maintaining rule. The distinguishing feature of finite successive weak coalition improvements with the status quo maintaining rule is essential to capture the decentralized nature of the random market process. Any other path which does not exhibit this feature but may still connect the initial market state with a competitive equilibrium will not achieve the goal. It is also worth pointing out that the proof of Theorem 1 depends critically on the statement of Theorem 2 but not on its proof technique or procedure.

Theorem 2 *Assume that the labour market (F,W,R^j,w_i^j) has a competitive equilibrium. Then there exists a finite number of successive weak coalition improvements with the status quo maintaining rule linking an arbitrary market state with a competitive equilibrium.*

We now discuss how to establish Theorem 1 via Theorem 2. As pointed out in the previous section, it is sufficient and also natural to confine ourselves to integral market states. We first demonstrate the following lemma, which provides an important insight into the spontaneous market process under examination, i.e., starting from any initial market state, the number of market states that can be possibly generated by successive coalition improvements with the status quo maintaining rule is finite.

Lemma 2 *For any given initial market state (μ^0,u^0), the set $A(\mu^0,u^0)$ of all market states that can be attained through successive coalition improvements with the status quo maintaining rule is finite.*

Proof. Consider an arbitrary firm $j \in F$. Let $S(j,0)$ be the total salaries that firm j paid to its employees $\mu^0(j)$ at the initial market state (μ^0,u^0). Let $M = \max_{j\in F} R^j(W)$ be the maximum among all firms' revenues. Let (μ^t,u^t), $t = 0,1,2,\cdots$, be any sequence of states that begins with (μ^0,u^0) and can be possibly generated through coalition improvements with the status quo maintaining rule. Observe that this process moves from one disequilibrium (μ^t,u^t) to another state (μ^{t+1},u^{t+1}), because a coalition (j^t,B^t) weakly blocks the state (μ^t,u^t), where $j^t \in F$ or $j^t = \emptyset$ and $B^t \subseteq W$. Now we prove that the payoff of every agent $k \in F\cup W$ is bounded at any time. By coalition improvement with the status quo maintaining rule (Definition 1), as long as any firm j is deserted or unaffected, its net profit cannot be less than $-S(j,0)-1$ ($< R^j(\emptyset) - S(j,0) = -S(j,0)$ which is the worst case for firm j that still pays $S(j,0)$ but receives no revenue). When any firm j emerges in a blocking coalition, its net profit will be weakly increasing and cannot be below zero, and after such adjustments even when it becomes deserted or unaffected or emerges in a blocking coalition again and again, its net profit cannot be less than $-R^j(W)$, because firm j will never pay more than $R^j(W)$ for any group of workers. Firm j's net profit is equal to $R^j(T) - \sum_{i\in T} s_i^{j,t}$, where T is the group of workers hired by firm j at time t and $\sum_{i\in T} s_i^{j,t}$ is the salaries paid by firm j at time t. Obviously, firm j's net profit can never be above $R^j(W)$. Thus firm j's payoff is always between $-\min\{R^j(W), S(j,0)+1\}$ and $R^j(W)$. Every worker i's payoff is clearly between $\min\{0, u_i^0\}$ and M, and therefore is also bounded. This shows that the sequence (μ^t, u^t), $t = 0, 1, 2, \cdots$, is bounded. Observe that the bounded set $A(\mu^0, u^0)$ is finite, because every state is integral and the number of matchings is finite. □

It may be obvious but important to note that starting from any point in the set $A(\mu^0, u^0)$, any sequence of market states that can be generated through coalition improvements with the status quo maintaining rule lies also in the set $A(\mu^0, u^0)$.

We are now ready to prove Theorem 1. Suppose that the market starts with the initial market state (μ^0, u^0) and operates at discrete time $t = 1, 2, \cdots$. Consider a general decentralized random market process in which every time-dependent transition probability from a disequilibrium state in $A(\mu^0, u^0)$ at any time t to another state in $A(\mu^0, u^0)$ at time $t+1$ is no less than a fixed (but sufficiently small) number $\varepsilon \in (0,1)$, namely, every possible weak coalition improvement with the status quo maintaining rule occurs with a positive probability bounded away from zero. With only two classes of states (equilibrium and disequilibrium), it follows that starting from any state (μ, u) in $A(\mu^0, u^0)$, the process either terminates in an equilibrium state and remains in equilibrium afterwards, or continues to move from one disequilibrium state to another disequilibrium state in $A(\mu^0, u^0)$, as the random process by construction always arrives at a state in $A(\mu^0, u^0)$. Suppose that the random process does not converge to an equilibrium state with probability one in the limit. This implies that, starting from a disequilibrium state in $A(\mu^0, u^0)$, the random process almost surely moves around within a (finite) set of disequilibrium states in

$A(\mu^0, u^0)$ forever and therefore some of these disequilibrium states could be visited an infinite number of times. Since each possible weak coalition improvement with the status quo maintaining rule is chosen with a probability no less than ε at each point of time, there is then some state (μ', u') in $A(\mu^0, u^0)$ from which no finite path of weak coalition improvements with the status quo maintaining rule toward equilibrium exists, no matter how the associated weak coalition improvements are chosen, yielding a contradiction to Theorem 2. This completes the proof of Theorem 1.

4. THE EXISTENCE OF A DESIRED PATH TO EQUILIBRIUM

We will prove Theorem 2 in this section. The main difficulty is to show the existence of a finite sequence of successive weak coalition improvements with the status quo maintaining rule linking an arbitrary initial integral market state with a competitive equilibrium. This sequence is *our desired path* and any other path which does not generate successive weak coalition improvements with the status quo maintaining rule but still leads to a competitive equilibrium will not help to establish the theorem.

Because the labor market (F, W, R^j, w_i^j) is assumed to have a competitive equilibrium with integral equilibrium payoffs, we can regard any such competitive equilibrium (μ^*, u^*) as a *reference equilibrium point.* The idea of using a reference point is a conventional thought experiment method and can avoid many practical issues and has been used in theoretical physics. Biró et al. (2014) apply this idea to the unit-demand models such as roommate matching and assignment problems. For our purpose, the real challenge lies in how to *construct a desired path* from any initial state to a competitive equilibrium by using a reference point. It is, however, insufficient to just construct *a path* from any initial state to a competitive equilibrium by using a reference point without requiring that the path should be a path of successive weak coalition improvements with the status quo maintaining rule. It is also worth pointing out that our method of constructing a desired path is very general in the sense that it works for any market as long as the market has an equilibrium. We will use the reference (μ^*, u^*) and an arbitrary initial market state (μ, u) to construct a desired path, denoted by $P((\mu^*, u^*), (\mu, u))$. The path $P((\mu^*, u^*), (\mu, u))$ thus exists and can be seen as a function of the reference point and the initial state. Thus the reference point plays a role as a variable, which means that we do not need to know any specific or precise reference point in order to establish the existence of a desired path from any initial state to a competitive equilibrium. This thought experiment method serves the purpose of proving Theorem 2 but is not a practical economic adjustment process.

Given a state (μ,u), an agent $i \in F \cup W$ is *underpaid* (*overpaid*) at (μ,u) with respect to the reference point (μ^*,u^*) if $u_i \leq u_i^*$ ($u_i > u_i^*$). Given any $U \subseteq W$ and $j \in F$, we define $u(U) = \sum_{i \in U} u_i$ and $w^j(U) = \sum_{i \in U} w_i^j$. For convenience we also use π_j to stand for the payoff u_j of firm $j \in F$ in a state (μ,u).

We will describe a general procedure that, starting from any initial market state (μ,u), generates a finite number of weak coalition improvements with the status quo maintaining rule leading to a competitive equilibrium. We first give a subroutine UPDATE which will be repeatedly used in the procedure.

The subroutine UPDATE tells us how to adjust a state (μ,u) weakly blocked by the coalition (j,U) to a new state (μ',u'). It specifies one type of weak coalition improvement with the status quo maintaining rule.

$$\mathsf{UPDATE}(\mu,u,j,U)$$

(a) Start from an integral market state (μ,u) weakly blocked by (j,U). Go to Step (b).

(b) If $j = \emptyset$, go to Step (e). If $j \in F$, let $F' = \{k \in F \setminus \{j\} \mid \mu(k) \cap U \neq \emptyset\}$ and $F^* = \{k \in F \setminus \{j\} \mid \mu(k) \cap U = \emptyset\}$. Make workers in $\mu(j) \setminus U$ unemployed and their payoffs at zero, i.e., let $\mu'(i) = i$ and $u_i' = 0$ for every $i \in \mu(j) \setminus U$. Firm j hires all workers in U, i.e., let $\mu'(j) = U$. Go to Step (c).

(c) If there exists an overpaid worker $i \in U$, then increase one such u_i' so that
$$u_i' + \sum_{l \in U \setminus \{i\}} u_l = R^j(U) - \pi_j - w^j(U),$$
let $u_l' = u_l$ for every $l \in U \setminus \{i\}$ and $\pi_j' = \pi_j$, and then go to Step (e). Otherwise, go to Step (d).

(d) There exists no overpaid worker in U. Increase all underpaid workers' u_i' as much as possible in such a way that $u_i' \leq u_i^*$ with $u_i' \in \mathbb{Z}$ and $u'(U) \leq R^j(U) - \pi_j - w^j(U)$. If $u'(U) = R^j(U) - \pi_j - w^j(U)$, let $\pi_j' = \pi_j$ and go to Step (e). Otherwise let $\pi_j' = R^j(U) - u'(U) - w^j(U)$ and go to Step (e).

(e) For each $k \in F'$, let $U_k = \mu(k) \setminus U$. Every firm $k \in F'$ continues to hire all remaining workers in U_k under the same contract as in (μ, u), and adjusts its net profit, i.e., let $u'_i = u_i$ for every $i \in U_k$, $\mu'(k) = U_k$ and $\pi'_k = R^k(U_k) - u(U_k) - w^k(U_k)$. For each $k \in F^*$, let $\pi'_k = \pi_k$ and $u'_i = u_i$ for every $i \in \mu(k)$. Go to Step (f).

(f) Set $(\mu, u) = (\mu', u')$ and get a new integral market state (μ, u).

The process comprising Steps (a), (b), (c) and (d) is called **Reshuffle**(μ, u, j, U) while the process consisting of only Step (e) is called **Retention**(μ, u, j, U).

If a state (μ, u) is weakly blocked by a coalition (j, U) with $j \in F$ and $U \subseteq W$, by definition we have

$$R^j(U) - w^j(U) > \pi_j + u(U).$$

Since (μ^*, u^*) is a competitive equilibrium, it follows that

$$\pi^*_j + u^*(U) \geq R^j(U) - w^j(U) > \pi_j + u(U).$$

This implies that UPDATE can be executed.

Lemma 3 *After* UPDATE*, if firm j's payoff π'_j increases, we have $u'_i = u^*_i$ for all $i \in U$ and also $\pi'_j \leq \pi^*_j$.*

Proof. Note that π'_j gets increased only if Step (d) is executed, i.e., $\pi'_j = R^j(U) - u'(U) - w^j(U)$ and $u'_i = u_i$ for all $i \in U$. Since $\pi'_j + u^*(U) = \pi'_j + u'(U) = R^j(U) - w^j(U) \leq \pi^*_j + u^*(U)$, we have $\pi'_j \leq \pi^*_j$. □

Note that if U contains at least one overpaid worker, after the execution of UPDATE firm j's payoff remains the same as in the state (μ, u).

We are now ready to describe the procedure which, starting from an arbitrary integral market state, will generate a finite sequence of successive coalition improvements with the status quo maintaining rule leading to a competitive equilibrium.

The Procedure for Constructing a Desired Path to Equilibrium

Step 0 Let (μ^*, u^*) be a reference equilibrium of the market. Start with an arbitrary integral market state (μ, u). Go to Step 1.

Step 1 If there exists a weakly blocking coalition (j, U) with $j \in F$ and $U \subseteq \mu(j) \cup \mu^*(j)$ against the current state (μ, u), choose one such j and go to Step 2. Otherwise, go to Step 3.

Step 2 Find an optimal solution U^* to the following problem

$$\begin{array}{ll} \max & R^j(X) - u(X) - w^j(X) \\ \text{s.t.} & X \subseteq \mu(j) \cup \mu^*(j). \end{array}$$

Perform $\mathsf{UPDATE}(\mu, u, j, U^*)$ and get a new state (μ, u). Go to Step 1.

Step 3 If the state (μ, u) is weakly blocked by a coalition (j, U), perform $\mathsf{UPDATE}(\mu, u, j, U)$ by giving a new state (μ, u) and go to Step 1. Otherwise stop with the current state (μ, u), which is a competitive equilibrium.

For convenience, the process that the Procedure from **Step 0** goes through **Step 1** and **Step 2** before moving into **Step 3** is called **Phase 1**, while the process that the Procedure from the beginning of **Step 3** goes through **Step 3** before returning to **Step 1** is called **Phase 2**.

The following observation is simple but crucial to the convergence of the Procedure and follows immediately from the construction of the UPDATE process.

Observation 1: In the entire Procedure, every underpaid worker will always remain underpaid. If an overpaid worker becomes underpaid, she will remain underpaid afterwards.

Before proving the convergence of the procedure it is helpful to demonstrate how the procedure works by an example.

An Illustrative Example: Suppose that in a market, there are three firms f_1, f_2, f_3 and four workers w_1, w_2, w_3, and w_4. Every worker i's minimum wage w_i^j equals zero for all firms j. Every firm's value over each group of workers is given in Table 1. Notice that in the table, w_{12} means the group of workers $\{w_1, w_2\}$.

It is easy to verify that although this market does not satisfy any known condition imposed on each individual such as the Gross Substitutes condition of Kelso & Crawford (1982), it does have a unique efficient matching μ^* which

Table 1: Firms' values over each group of workers.

	w_1	w_2	w_3	w_4	w_{12}	w_{13}	w_{14}
f_1	2	2	1	1	7	3	2
f_2	2	2	7	1	3	7	3
f_3	2	2	2	6	5	3	7
	w_{23}	w_{24}	w_{34}	w_{123}	w_{124}	w_{234}	w_{1234}
f_1	4	3	2	7	7	4	9
f_2	7	2	7	7	3	7	10
f_3	3	6	7	6	7	7	8

is supported by many competitive equilibrium price vectors, where $\mu^*(f_1) = \{w_1, w_2\}$, $\mu^*(f_2) = \{w_3\}$, $\mu^*(f_3) = \{w_4\}$, $\mu^*(w_1) = \mu^*(w_2) = f_1$, $\mu^*(w_3) = f_2$, and $\mu^*(w_4) = f_3$. We will take (μ^*, u^*) as a reference equilibrium with $u^*_{f_1} = 1$, $u^*_{f_2} = 4$, and $u^*_{f_3} = u^*_{w_1} = u^*_{w_2} = u^*_{w_3} = u^*_{w_4} = 3$, or $s^j = (3,3,3,3)$ for $j = 1,2,3$.

In Table 2 we show how a desired path is generated by the procedure. The procedure starts with the state (μ^0, u^0) and ends up with an equilibrium state $(\mu^5, u^5) = (\mu^*, u^*)$. Note that in Table 2 'opt' means 'optimal', 'state' means 'market state' and 'coalition' means 'weakly blocking coalition'. For each matching μ^k we only write down the group of workers hired by each firm. For instance, $\mu^1 = (w_{12}, \emptyset, \emptyset)$ means that firm 1 hires workers w_1 and w_2, and firms 2 and 3 hire no worker, and workers 3 and 4 are unemployed. For each payoff vector u^k, its first three components indicate the payoff of the three firms respectively while its last four components specify the payoff of the four workers respectively. In the table (f_1, w_{12}) is a blocking coalition of (μ^0, u^0), and (μ^1, u^1) is the coalition improvement with the status quo maintaining rule of (μ^0, u^0) through (f_1, w_{12}). Initially, at (μ^0, u^0) worker 2 is overpaid but workers 1, 3 and 4 are underpaid. Observe that in the entire process an underpaid worker will always remain underpaid, and an overpaid worker may become underpaid and thereafter will always be underpaid.

Lemma 4 *Let (μ, u) be the current state of the market with which* **Step 3** *of the* Procedure *begins. Then for all $j \in F$, $\mu(j)$ is a maximizer of $R^j(X) - u(X) - w^j(X)$ in $X \subseteq \mu(j) \cup \mu^*(j)$ and there exists no subset X of the set $\mu(j) \cup \mu^*(j)$ such that (j, X) weakly blocks (μ, u).*

Table 2: A desired path generated by the procedure.

state	coalition	opt U^*	matching	payoff
(μ^0,u^0)	(f_1,w_{12})	$\{w_{12}\}$	$\mu^0=(w_{1234},\emptyset,\emptyset)$	$u^0=(0,0,0,1,4,2,2)$
(μ^1,u^1)	(f_2,w_3)	$\{w_3\}$	$\mu^1=(w_{12},\emptyset,\emptyset)$	$u^1=(0,0,0,1,6,0,0)$
(μ^2,u^2)	(f_3,w_4)	$\{w_4\}$	$\mu^2=(w_{12},w_3,\emptyset)$	$u^2=(0,4,0,1,6,3,0)$
(μ^3,u^3)	(f_1,w_1)	$\{w_1\}$	$\mu^3=(w_{12},w_3,w_4)$	$u^3=(0,4,3,1,6,3,3)$
(μ^4,u^4)	(f_1,w_2)	$\{w_{12}\}$	$\mu^4=(w_1,w_3,w_4)$	$u^4=(0,4,3,2,0,3,3)$
(μ^5,u^5)			$\mu^5=(w_{12},w_3,w_4)$	$u^5=(1,4,3,3,3,3,3)$

Proof. By **Step 2** and UPDATE, for every $j\in F$, $\mu(j)$ is a maximizer of $R^j(X)-u(X)-w^j(X)$ in $X\subseteq\mu(j)\cup\mu^*(j)$. Hence,

$$\pi_j=R^j(\mu(j))-u(\mu(j))-w^j(\mu(j))\geq R^j(X)-u(X)-w^j(X),$$

for all $X\subseteq\mu(j)\cup\mu^*(j)$. It follows that there exists no $X\subseteq\mu(j)\cup\mu^*(j)$ such that (j,X) weakly blocks (μ,u). □

Lemma 5 *Every worker $i\in\mu(j)\setminus(U^*\cup\mu^*(j))$ that leaves firm j in* **Step 2** *of the* Procedure *will never return to the firm afterward before the* Procedure *goes to* **Step 3**.

Proof. Each worker $i\in\mu(j)\setminus(U^*\cup\mu^*(j))$ before the UPDATE in **Step 2** becomes self-employed and disappears from $\mu(j)\cup\mu^*(j)$ for the new $\mu(j)$ after the UPDATE in **Step 2**. Since during the repetition of **Step 1** and **Step 2** sets $\mu(j)\cup\mu^*(j)$ for all $j\in F$ do not get enlarged, such a worker i remains unemployed before **Step 3** is invoked. □

Lemma 6 *Every worker $i\in(\cup_{k\in F\setminus\{j\}}\mu(k))\cap U^*$ that moves to firm j in* **Step 2** *of the* Procedure *will never return to her previous firm afterward (but possibly becomes unemployed by being fired by firm j) before the* Procedure *goes to* **Step 3**.

Proof. Similarly to the proof of the previous lemma, it follows from the definition of the UPDATE process. □

Observe that every worker $i \in (\mu(j) \cap \mu^*(j)) \setminus U^*$ gets $u_i = 0$ and this will remain the same afterward before the Procedure goes to **Step 3**. Also note that if the set $\mu(j) \cup \mu^*(j)$ remains the same after UPDATE, i.e., only some workers in $\mu(j) \cap \mu^*(j)$ leave firm j, there is no weakly blocking coalition (j, U) with $U \subseteq \mu(j) \cup \mu^*(j)$ after the UPDATE, which can be shown by argument as similar to the proof of Lemma 4.

Lemma 7 *Let (μ, u) be the current state of the market with which* **Step 3** *of the* Procedure *begins. Then it holds*

$$\pi_j^* + u^*(\mu^*(j)) = \pi_j^* + u^*(\mu(j)) = \pi_j + u(\mu^*(j)) = \pi_j + u(\mu(j)) \tag{5}$$

for all $j \in F$.

Proof. It follows from Lemma 4 that for every firm $j \in F$

$$\begin{aligned} \pi_j^* + u^*(\mu^*(j)) &= R^j(\mu^*(j)) - w^j(\mu^*(j)) \\ &\leq R^j(\mu(j)) - u(\mu(j)) - w^j(\mu(j)) + u(\mu^*(j)) \\ &= \pi_j + u(\mu^*(j)). \end{aligned} \tag{6}$$

On the other hand, since (μ^*, u^*) is a competitive equilibrium, we have for all $j \in F$

$$\begin{aligned} \pi_j^* + u^*(\mu(j)) &\geq R^j(\mu(j)) - w^j(\mu(j)) \\ &= \pi_j + u(\mu(j)). \end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\begin{aligned} \sum_{j \in F} (\pi_j + u(\mu(j))) &\leq \sum_{j \in F} (\pi_j^* + u^*(\mu(j))) \\ &\leq \sum_{j \in F} (\pi_j^* + u^*(\mu^*(j))) \\ &\leq \sum_{j \in F} (\pi_j + u(\mu^*(j))) \\ &\leq \sum_{j \in F} (\pi_j + u(\mu(j))). \end{aligned} \tag{8}$$

Hence every inequality in (6)–(8) hold with equality. This leads to

$\pi_j^* + u^*(\mu^*(j)) = \pi_j^* + u^*(\mu(j)) = \pi_j + u(\mu^*(j)) = \pi_j + u(\mu(j))$ for all $j \in F$.

□

The above proof and lemma also imply the following result.

Lemma 8 *Let (μ, u) be the current state of the market with which* **Step 3** *of the* Procedure *begins. Then it holds that*

$$u_i = 0 \text{ for all } i \in W \setminus \bigcup_{j \in F} \mu^*(j) \text{ and } u_i^* = 0 \text{ for all } i \in W \setminus \bigcup_{j \in F} \mu(j).$$

Now, we examine the behavior of **Step 3** of the Procedure. Take any weakly blocking coalition (j, U) and define $F' = \{k \in F \mid U \cap \mu(k) \neq \emptyset\}$. Because of the definitions of (j, U) and (μ^*, u^*) we have

$$\pi_j + u(U) < R^j(U) - w^j(U) \leq \pi_j^* + u^*(U). \tag{9}$$

If $j \notin F'$, let $F' = F' \cup \{j\}$. Then from (5) we have

$$\begin{aligned} &\sum_{k \in F'} (\pi_k + u(\mu(k))) \\ &= \sum_{k \in F' \setminus \{j\}} (\pi_k + u(\mu(k) \setminus U)) + u(\mu(j) \setminus U) + (\pi_j + u(U)) \\ &= \sum_{k \in F'} (\pi_k^* + u^*(\mu(k))). \end{aligned} \tag{10}$$

It follows from (9) and (10) that for some firm $k \in F' \setminus \{j\}$

$$\pi_k + u(\mu(k) \setminus U) > \pi_k^* + u^*(\mu(k) \setminus U) \tag{11}$$

or

$$u(\mu(j) \setminus U) > u^*(\mu(j) \setminus U). \tag{12}$$

Case (I): (11) holds. It follows from (11) and the equilibrium (μ^*, u^*) that we have

$$\pi_k + u(\mu(k) \setminus U) > \pi_k^* + u^*(\mu(k) \setminus U) \geq R^k(\mu(k) \setminus U) - w^k(\mu(k) \setminus U). \tag{13}$$

Then by Step (e) of UPDATE we have

$$\pi_k' = R^k(\mu(k) \setminus U) - w^k(\mu(k) \setminus U) - u(\mu(k) \setminus U).$$

Hence π_k', the new π_k, strictly decreases compared with previous π_k in (13).

Case (II): (12) holds. Then there exists at least one overpaid worker in $\mu(j)\setminus U$. By Step (b) of UPDATE all workers $i\in\mu(j)\setminus U$ become unemployed and at least one overpaid worker in $\mu(j)\setminus U$ becomes underpaid.

We can now establish the following major convergence result of this section and thus prove Theorem 2.

Theorem 3 *The* Procedure *generates a finite sequence of weak coalition improvements with the status quo maintaining rule leading to a competitive equilibrium.*

Proof. Recall that the number of integral feasible market states in $A(\mu^0,u^0)$ with the initial allocation (μ^0,u^0) is finite, due to Lemma 2. Furthermore, every market state generated by the Procedure is an integral feasible state satisfying $(**)$ because of Step 2 and Step 3 of the Procedure. If the Procedure does not produce a finite sequence of weak coalition improvements with the status quo maintaining rule leading to a competitive equilibrium, it must yield a finite cycle. The Procedure would repeat the cycle forever. Without loss of generality we assume that at least one **Phase 1** is executed before the Procedure reaches the cycle.

Notice that Case (II) in **Phase 2** never occurs along the cycle, because there are at most n overpaid workers and if any overpaid worker becomes unemployed, it will remain underpaid forever by Observation 1. Thus only Case (I) may occur in **Phase 2** along the cycle. Then, since in Case (I) π_k strictly decreases, π_k should be increased to recover the loss along the cycle, which can only be done by Reshuffle(μ,u,k,U) in **Phase 1** when $U(=\mu(k)$ later) does not contain any overpaid workers. Note that each Retention in **Phase 1** makes the value of π_k less than or equal to that of π_k given at the end of the previous **Phase 1**, since at the end of **Phase 1** then obtained $\mu(k)$ for every $k\in F$ is a maximizer obtained in **Step 2**, so that removing some workers from $\mu(k)$ results in a lower revenue than that given at the end of the previous **Phase 1** for firm k, while Retention in **Phase 2** may only reduce π_k because of the same reason. Also note that after the Reshuffle(μ,u,k,U) we have $\pi_k\le\pi_k^*$ due to Lemma 3 and comments right after that and then keep $\pi_k\le\pi_k^*$ thereafter. On the other hand, if $\pi_k\le\pi_k^*$, then because of (11) in Case (I) in **Phase 2** there must be at least one overpaid worker in $\mu(k)\setminus U$, where we update $\mu(k)$ by setting $\mu(k)=\mu(k)\setminus U$. Hence along the cycle there exists at least one overpaid worker i who becomes unemployed and thus remains underpaid thereafter forever by Observation 1. But this is impossible along the cycle, because there are only at most n overpaid workers. In other words, there will be no overpaid worker along the cycle, yielding a contradiction. Hence the Procedure terminates (in a finite number of successive integral weak coalition improvements with the status quo maintaining rule) with a final integral state (μ, u) that has no integral weak blocking, which is a competitive equilibrium due to Lemma 1. □

It should be noted that the above proof implies that if there exists a competitive equilibrium, then for every initial state (μ^0, u^0) there also exists a competitive equilibrium within $A(\mu^0, u^0)$.

5. THE BENCHMARK CASE OF GROSS SUBSTITUTES

Weak coalition improvements with the status quo maintaining rule cover all kinds of hiring and firing procedures and some of these procedures could be too general and too complicated for firms to handle, for instance, in practice sometimes it can be very difficult and even controversial for a firm to dismiss a large number of employees at one time. However, under the Gross Substitutes condition it is possible to obtain the following much easier, more intuitive and more well-behaved form of hiring and firing procedure.

A weakly blocking coalition (j, B) against a state (μ, u) is called a *basic weakly blocking coalition* if $j = \emptyset$ or if $j \in F$ and one of the following holds:

(1) $B = \mu(j) \cup \{k\}$ for $k \in W \setminus \mu(j)$;

(2) $B = (\mu(j) \cup \{l\}) \setminus \{k\}$ for some worker $k \in \mu(j)$ and some worker $l \in W \setminus \mu(j)$;

(3) $B = \mu(j) \setminus \{k\}$ for some worker $k \in \mu(j)$.

A weak coalition improvement with the status quo maintaining rule (μ', u') of (μ, u) through (j, B) is called a *basic weak coalition improvement with the status quo maintaining rule* if (j, B) is a basic weakly blocking coalition. With respect to (j, B), in Case (1), firm j hires a new worker, in Case (2), firm j simultaneously dismisses a worker and hires a new worker, and in Case (3), firm j fires a worker.

It is immediately clear that a basic weakly blocking coalition (j, B) against a state (μ, u) occurs if and only if $j = \emptyset$ or if $j \in F$ and one of the following occurs:

(1) For a firm $j \in F$ and a worker $k \in W \setminus \mu(j)$,

$$u_j + \sum_{i \in \mu(j)} (u_i + w_i^j) + u(k) + w_k^j < R^j(\mu(j) \cup \{k\}).$$

(2) For a firm $j \in F$, a worker $k \in W \setminus \mu(j)$, and a worker $l \in \mu(j)$,

$$u_j + \sum_{i \in \mu(j) \setminus \{l\}} (u_i + w_i^j) + u(k) + w_k^j < R^j((\mu(j) \cup \{k\}) \setminus \{l\}).$$

(3) For a firm $j \in F$ and a worker $k \in \mu(j)$,

$$u_j + \sum_{i \in \mu(j) \setminus \{k\}} (u_i + w_i^j) < R^j(\mu(j) \setminus \{k\}).$$

The following important characterization is called the *Single Improvement* (SI) property and shown by Gul & Stacchetti (1999, 2000) to be equivalent to the Gross Substitutes condition of Kelso & Crawford (1982).

Definition 3 *Firm j satisfies the Single Improvement property if for every salary scheme s^j and $A \notin D^j(s^j)$, there exists $B \in D^j(s^j)$ such that $R^j(B) - \sum_{i \in B} s_i^j > R^j(A) - \sum_{i \in A} s_i^j$, $|A \setminus B| \leq 1$ and $|B \setminus A| \leq 1$.*

Next we show that under the Gross Substitutes condition it is indeed sufficient to consider only basic weakly blocking coalitions.

Theorem 4 *Under the Gross Substitutes condition, if there is a weakly blocking coalition, there must be a basic weakly blocking coalition.*

Proof. It suffices to consider blocking coalitions (j, B) with $j \in F$. Suppose we are given a weakly blocking coalition (j, B) for some firm $j \in F$ and worker group $B \subseteq W$. By definition, there exists a salary scheme t^j such that $t_i^j - w_i^j \geq s_i^{\mu(i)} - w_i^{\mu(i)}$ for every $i \in B$ and $\pi_j(B, t^j) \geq \pi_j(\mu(j), s^j)$ with at least one strict inequality.

Now define a new salary scheme $\tilde{t}^j \in \mathbf{R}^W$ by

$$\tilde{t}_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} + w_i^j, \qquad \forall i \in W. \tag{14}$$

Observe that $\tilde{t}_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} + w_i^j = s_i^j$ for every $i \in \mu(j)$, and $\tilde{t}_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} + w_i^j \leq t_i^j - w_i^j + w_i^j = t_i^j$ for every $i \in B$. Then we have

$$\pi_j(B, \tilde{t}^j) \geq \pi_j(B, t^j) \geq \pi_j(\mu(j), s^j) = \pi_j(\mu(j), \tilde{t}^j). \tag{15}$$

It follows from the definition of (j, B) that at least one of the two inequalities in (15) is strict, i.e.,

$$\pi_j(B, \tilde{t}^j) > \pi_j(\mu(j), \tilde{t}^j). \tag{16}$$

Hence from (16) we have $\mu(j) \notin D^j(\tilde{t}^j)$. Now because of the GS condition it follows from the SI property one of the three cases must occur:

(4) there is a worker $k \in W \setminus \mu(j)$ such that

$$R^j(\mu(j) \cup \{k\}) - \sum_{i \in \mu(j) \cup \{k\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i \in \mu(j)} \tilde{t}_i^j.$$

(5) there are a worker $k \in W \setminus \mu(j)$ and a worker $l \in \mu(j)$ such that

$$R^j((\mu(j) \cup \{k\}) \setminus \{l\}) - \sum_{i \in (\mu(j) \cup \{k\}) \setminus \{l\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i \in \mu(j)} \tilde{t}_i^j.$$

(6) there is a worker $k \in \mu(j)$ such that

$$R^j(\mu(j) \setminus \{k\}) - \sum_{i \in \mu(j) \setminus \{k\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i \in \mu(j)} \tilde{t}_i^j.$$

Note that $\tilde{t}_i^j - w_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} = u_i$ for all $i \in W$ due to (14). We first consider case (4). Since $u_i = \tilde{t}_i^j - w_i^j = s_i^j - w_i^j$ and $\tilde{t}_i^j = s_i^j$ for every $i \in \mu(j)$, and $u_j = R^j(\mu(j)) - \sum_{h \in \mu(j)} \tilde{t}_i^j = R^j(\mu(j)) - \sum_{h \in \mu(j)} s_i^j$, we have

$$R^j(\mu(j) \cup \{k\}) - \sum_{i \in \mu(j)} \tilde{t}_i^j - \tilde{t}_k^j > u_j$$

which can be written as

$$u_j + \sum_{i \in \mu(i)} (u_i + w_i^j) + u(k) + w_k^j < R^j(\mu(j) \cup \{k\})$$

This corresponds to case (1).

Similarly, using $u_j = R^j(\mu(j)) - \sum_{h\in\mu(j)} \tilde{t}^j_i = R^j(\mu(j)) - \sum_{h\in\mu(j)} s^j_i$, one can show that case (5)

$$R^j((\mu(j)\cup\{k\})\setminus\{l\}) - \sum_{i\in(\mu(j)\cup\{k\})\setminus\{l\}} \tilde{t}^j_i > R^j(\mu(j)) - \sum_{i\in\mu(j)} \tilde{t}^j_i,$$

implies case (2)

$$u_j + \sum_{i\in\mu(j)\setminus\{l\}} (u_i + w^j_i) + u(k) + w^j_k < R^j((\mu(j)\cup\{k\})\setminus\{l\}),$$

and that case (6) implies case (3). □

Under the Gross Substitutes condition we can establish the following major refinement of Theorem 2.

Theorem 5 *For the labour market (F, W, R^j, w^j_i) under the Gross Substitutes condition, there exists a finite number of basic weak coalition improvements with the status quo maintaining rule from an arbitrary market state to a competitive equilibrium.*

From this result one can easily write down the corresponding refinement of Theorem 1 under the Gross Substitutes condition. The proof of the above theorem follows from the Procedure in the previous section, Theorem 3 and the next lemma.

Lemma 9 *In* **Step 2** *of the* Procedure*, there exists a finite sequence of basic weak coalition improvements with the status quo maintaining rule from the current $\mu(j)$ to a maximizer U^* within $\mu(j)\cup\mu^*(j)$.*

Proof. Let $p_k = u_k + w^j_k$ for every $k \in \mu(j)\cup\mu^*(j)$. Consider the following problem

$$\begin{array}{ll} \max & R^j(X) - u(X) - w^j(X) = R^j(X) - \sum_{k\in X} p_k \\ \text{s.t.} & X \subseteq \mu(j)\cup\mu^*(j) \end{array}$$

Let $D^j(p)$ be the collection of optimal solutions to the problem. Since $D^j(p)$ satisfies the Gross Substitutes condition, it satisfies the Single Improvement property. Consequently, there exists a finite sequence of basic weak coalition improvements with the status quo maintaining rule from the current $\mu(j)$ to a maximizer U^* within $\mu(j)\cup\mu^*(j)$. □

It is also possible to obtain a refined version of Theorems 1 and 2 under the Gross Substitutes and Complements condition of Sun & Yang (2006, 2009).

6. CONCLUSION

Economic processes are fundamental instruments by which markets are operated and equilibrium prices or salaries are generated. Such processes can be roughly classified into two major groups. One group comes out from deliberate human design, such as auctions, which have been widely used to sell mobile-phone licenses, electricity, treasure bills, mineral rights, keywords, pollution permits, and many other commodities and services involving a staggering value of hundreds of billions of dollars (see Krishna (2002), Klemperer (2004), and Milgrom (2004)). In some sense, a conscious human design market process like auction, matching and school choice design can be regarded as a visible hand. The other are spontaneous market processes which arise naturally from human economic action but are not designed by human beings. They are perhaps literally the “true” invisible hand as conceived by Adam Smith. Uncoordinated decentralized markets, notably labor markets, are of this nature. While many important results have been obtained for the first type of market processes, we have far less understanding of the second. This paper shed light on the second type of processes—spontaneous market processes—for a large class of decentralized and uncoordinated markets.

In the paper we have analyzed a general decentralized and uncoordinated labour market where heterogeneous self-interested firms and workers meet directly and randomly in pursuit of higher payoffs over time. Each firm hires as many workers as it wishes. Each worker has preferences over firms and salaries but takes at most one position. Each economic agent makes her own decision independently and freely. The information of the market is dispersed among all separate market participants. In other words, information is incomplete to every individual. At any time any firm and any group of workers can form a new coalition if all members in the coalition divide their joint payoff in such a way that makes no member of the coalition worse off and at least one member strictly better off. In the process, the firm may fire some of its own workers and hire workers from other firms and each deserted firm will at least shortly maintain the status quo for its remaining workers. An important feature is that in the process the total welfare need not be monotonic, because every abandoned firm and dismissed worker could be worse off. As information is incomplete and decision-making is decentralized, it is natural to assume that this coalition improvement with the status quo maintaining rule occurs only with a positive probability conditional on the current state and time. This spontaneous random decentralized competitive dynamic process exhibits several salient features that are widely observed in real life decentralized markets. We have shown that starting with any initial market state, the spontaneous market process converges almost surely in finite time to a competitive equilibrium, thus resulting in an efficient allocation of resources. The result holds true for any competitive market as long as there exists an equilibrium with an integral vector of equilibrium salaries or prices. An important example for equilibrium existence is the well-known Gross Substitutes condition of Kelso & Crawford (1982).

Our study provides a theoretical foundation for affirming Adam Smith's Invisible Hand in complex economic environments involving uncertainty, indivisibility and incomplete information and sheds new light on a large class of spontaneous decentralized, random and dynamic market processes. This study also has interesting and meaningful policy implications: Free markets can in general achieve an efficient distribution of resources even in a chaotic, random and imperfect information environment. More specifically, the price system can play a vital role in efficiently communicating information "in a system in which the knowledge of the relevant facts is dispersed among many people, prices can act to coordinate the separate actions of different people in the same way as subjective values help the individual to coordinate the parts of his plan" as Hayek (1945, pp. 525-527) had believed. A word of caution is that in order for free markets to perform well, the government should improve market transparency and offer some coordination among market participants.

Our model is very general and natural in almost all respects but its zero search cost assumption. An important direction for future research is to relax this assumption. For instance, firms and workers do not and cannot always make contact with one another immediately. Firms are trying to find workers and workers are looking for jobs. This search process usually requires resources and time, thus creating frictions in the market; see e.g., Diamond (1971, 1981). How will such search frictions affect efficiency of the market and convergence of the spontaneous process? Another important question is to address the computational complexity issue of the random decentralized dynamic process studied in the current paper.

We hope that the current study will prove to be useful in understanding fundamental issues concerning spontaneous random decentralized market processes in the complex world.

Appendix

A state (μ,u) is *weakly blocked by a general coalition* $B \subseteq F \cup W$ (i) if there exists one firm $j \in B$ with a payoff vector $r \in \mathbf{R}^B$ such that

$$r_k \geq u_k \text{ for every } k \in B, \text{ and} \tag{17}$$

$$\sum_{k \in B} r_k = R^j(B \setminus \{j\}) - \sum_{k \in B \setminus \{j\}} w_k^j$$

with at least one strict inequality for (17), or (ii) if the coalition B contains only one worker i with $r_i = 0 > u_i$. B is called *a general weakly blocking coalition*. Notice that this definition does not require every member of the coalition B to be individually rational, i.e., $r_k \geq 0$ for all $k \in B$.

A state is a *strict core allocation* or *a competitive equilibrium* if it is not weakly blocked by any general coalition. See Kelso & Crawford (1982, pp. 1487-1488) for this conventional definition. Below we show that using individually rational weakly blocking coalitions can achieve the same competitive equilibrium.

Lemma 10 *If a state (μ,u) is weakly blocked by a general coalition B, it must be blocked by an individually rational coalition.*

Proof. Because B weakly blocks (μ,u), there exists $r \in \mathbf{R}^B$ such that $r_k \geq u_k$ for every $k \in B$ with at least one strict inequality. If $r_k \geq 0$ for all $k \in B$, we are done. If there is some $r_k < 0$, then the state (μ,u) is weakly blocked by the individually rational coalition $\{k\}$ with the payoff $\bar{r}_k = 0$. □

Lemma 11 *A state is a competitive equilibrium if it is not weakly blocked by any individually rational coalition.*

Proof. Let (μ,u) be a market state that is not weakly blocked by any individually rational coalition. Suppose that (μ,u) is weakly blocked by a general coalition B. There exists some $k \in B$ such that $u_k \leq r_k < 0$. Then the state (μ,u) must be weakly blocked by the individually rational coalition $\{k\}$, yielding a contradiction. We have shown that (μ,u) is not weakly blocked by any general coalition and thus must be a competitive equilibrium. □

The above results demonstrate that it is sufficient to focus on individually rational weakly blocking coalitions.

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TIMING AND PRESENTATION EFFECTS IN SEQUENTIAL AUCTIONS

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ABSTRACT

This paper investigates two hitherto unexplored dimensions inherent in online sequential auctions, namely, how the time elapsed between the end of an auction and the end of the next one and the order of presentation on the website affect prices. Using a state-of-the-art-dataset on train-ticket auctions with a particular institutional design feature that enables a causal interpretation of these dimensions, it is demonstrated that both dimensions have a significant impact on price formation in sequential auctions.

Keywords: Sequential auctions, presentation order, timing afternoon effect.

JEL Classification Numbers: D02, D44.

1. INTRODUCTION

Sequential auction formats are commonly adopted when selling multiple units of a good within a predetermined time frame. Examples include the selling of flowers, highway paving contracts, school milk contracts, timber, and wine Jofre-Bonet & Pesendorfer (2003). There are several reasons for the popularity of the format. Firstly, it requires relatively little information

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exchange among the buyers and the auctioneer. Secondly, it easily accommodates scenarios in which buyers enter and leave the market. Thirdly, it allows the auctioneer to allocate items incrementally (Bae et al., 2009). A formal theoretical analysis of the auction format is, however, not straightforward. Consequently, much of the attention in the literature has been restricted to the case when items are identical and buyers have unit-demand. In this setting, Milgrom & Weber (1982) show that expected prices in a sequence of second-price auctions (Vickrey, 1961) with identical items should be equal. The empirical literature has, however, refuted this prediction by showing that prices are decreasing in rank order, i.e., the order in which the auctions in the sequence are terminated (see, e.g., Ashenfelter, 1989; McAfee & Vincent, 1993; van den Berg et al., 2001).[1] This decreasing price anomaly has been coined "the afternoon effect" because auctions later in a sequence typically took place in the afternoon whereas early auctions typically took place in the morning. Evidence of increasing price sequences has been observed by Raviv (2006) and Gandal (1997).[2] Hence, the phenomenon does not seem universal.

Studying the rank order effect on prices of identical items in sequential auctions is natural as this is the most easily observed variation in happenstance auction data; see e.g., van den Berg et al. (2001). This paper takes one step further and utilizes a state-of-the-art data set on online train-ticket auctions in Sweden to investigate two hitherto unexplored dimensions inherent in sequential auctions. The first concerns the order in which the auctioned items are presented on the website and the second concerns the time elapsed between auction terminations. Using a particular institutional design, random variation in these dimensions are exploited to make causal inference of their effect on prices. In sum, it is found that both these dimensions have a significant impact on prices.

It has previously been demonstrated that the time elapsed between auctions in a sequence may affect prices. For example, Andersson et al. (2012) find that when auction ends are identical, considerable price heterogeneity is observed

1 Ashenfelter (1989), Beggs & Graddy (1997) and Hong et al. (2015) found that this observation also holds for heterogeneous items. Decreasing price sequences have also been found in auction formats where auction winners have the option to buy items in subsequent auctions (Février et al., 2007) or when they have the option to place the first bid in subsequent auctions (Lambson & Thurston, 2006).

2 Ginsburgh (1998) and Deltas & Kosmopoulou (2004) find both increasing and decreasing price sequences. They argue that these observations could be due to non-strategic absent bidders who submit their bids previous to the auction start.

between auctions, and Zeithammer (2006) shows that bids are lower when similar items are being auctioned off in the near future. On the other hand, Anwar et al. (2006) observe that bidders are less likely to bid on identical items if their respective auction end times are far apart. Hence, the time elapsed between auctions may be an important factor for price formation in sequential auctions, which calls for further studies. In sequential auctions the time elapsed between the auctions is, typically, held constant, but the difference between auction formats could be considerable ranging from just a few minutes up to several days apart. Typically, however, the exact timing of auctions has seldom been recorded in previous studies which makes it hard to study the timing effect. To the contrary, the data investigated in this paper include exact recordings of auction ends and, moreover, exhibit a random variation between auction ends (exact details to be described below). This gives a unique possibility to study the timing effect.

Regarding the presentation order, note that choices in Milgrom & Weber (1982) are modeled from the perspective of a set of items being auctioned off. Yet, bidders, in real life, typically face lists of items being auctioned off and the presentation order on the list may influence their decisions. This may create a "primacy effect", i.e., items with a high position on a list are more prominent than items with a low position. A reversed "recency effect" means that items placed low are more prominent than items placed high(see, e.g., Rubinstein & Salant, 2006, for a discussion).[3] Deltas & Kosmopoulou (2004) study sequential auctions of rare books. The books are described in an auction catalogue and placed in alphabetical order so, although being heterogenous items, they are seemingly randomly placed. They find an overall increasing price pattern and, furthermore, that books described further down in the catalogue receive fewer bids from e-mail bidders.[4] This suggests a primacy effect. However, in their data, the rank order is identical to the presentation order and, as a consequence, their result may also be due to a rank-order effect effect or a combination of the two. This paper seeks to separate the effect of rank order to that of the timing and presentation order.

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[3] In the literature on sponsored search (e.g., Agarwal et al., 2011) and position auctions (e.g., Edelman et al., 2007), evidence of a primacy effect, from the perspective of the seller, is found. Recently, a primacy effect has been found in terms of downloads of new working papers from the NBER website (Feenberg et al., 2015).

[4] Bids could be submitted by e-mail before the auction start or placed "on the floor" during the auction.

This paper uses data from more than 42,000 online train ticket auctions grouped into 7,200 sequential auctions.[5] From the perspective of the bidders, each ticket in a sequential auction is completely homogenous in terms of its characteristics (route, class, and time of departure), its seller (the state-owned train company), its reserve price (1 SEK), and its transaction cost (literally zero as tickets are delivered by an SMS text message to the buyer). This feature is quite rare in the literature on sequential auctions. In fact, the only variables that vary within an group of sequential auctions are the termination time of the auctions, i.e., their *rank order* (the first auction has rank order one, the second two, and so on), and the order in which they are presented on the auction website, i.e., *presentation order* (the auction presented at the very top of the website has presentation order one, and so forth). In every sequential auction sequence, each ticket is assigned an auction number which determines the presentation order on the auction website. Moreover, each ticket auction in a sequence is set to end within a given hour. However, the exact ending minute of each auction is determined by a random draw. Consequently, the rank order is different from the presentation order and the time elapsed between two consecutive auction ends is random. These novel design features are used to identify and separate the effect of rank order to that of the timing and presentation order (i.e., it is a natural field experiment in the terminology of Harrison & List, 2004). To the best of our knowledge, this article is the first to make this separation and causally estimate how these dimensions affect prices in sequential auctions.[6]

The analysis reveals that the presentation order has a non-linear impact on prices, so that it pays off to be first or last in the list. Moreover, if ending times of two consecutive auctions are close, this has a negative impact on prices in both auctions. However, the relationship shows substantial non-linearities so being placed far apart also depresses prices. From an auction-designer's perspective, these insights are important when trying to curb the afternoon effect to secure higher revenues. Finally, in line with the bulk of the previous empirical literature, a negative rank order effect (i.e., an afternoon effect) is found. We also note that the rank order effect is essentially unchanged when controlling for presentation order and timing and, hence, a substantial part of the anomaly persists indicating that further research is needed to fully

[5] See Bajari & Hortacsu (2004) for an excellent summary of internet-auction studies.

[6] However, using field experiments to study behavior in auctions is not new. See, for example, Lucking-Reiley (1999), List & Lucking-Reiley (2000), or Grether et al. (2015).

understand the afternoon effect.

The remaining part of the paper is outlined as follows. Section 2 describes the sequential online auction format. The data are described and analyzed in Sections 3 and 4, respectively. Section 5 concludes that paper. The Appendix contains some additional estimations.

2. THE SEQUENTIAL TRAIN TICKET AUCTION

Statens Järnvägar (SJ, henceforth) is a publicly owned company that mainly runs passenger trains in Sweden. In 2007, they began to auction train tickets on the eBay-owned website Tradera which is the leading auction site in Sweden.[7] This section describes the sequential auction mechanism that was adopted by SJ on November 15, 2010 and for the duration of the collected data.

Before introducing the notion of a sequence of auctions, it is noted that each train ticket is auctioned off in a separate auction. More specifically, each ticket is displayed on the website of Tradera exactly at 9pm at date t, and the following information is available for potential bidders; (i) date of departure, (ii) the "time block" when the train departures (05:00am–09:59am, 10:00am–02:59pm, or 03:00pm–08:59pm), (iii) departure station, and (iv) final destination. The reserve price of a ticket is always set to 1 SEK (approximately 0.11 USD). Bidders place bids by entering a maximum amount that they are willing to pay for the ticket. An automatic bidder (a proxy bidding agent) places bids on behalf of the agent using an automatic bid increment amount which depends on the current standing bid. The proxy bidder will only bid as much as necessary to make sure that the bidder remains the highest bidder up to the bidder's maximum amount (note that a bidder's maximum willingness to pay is kept confidential until it is exceeded by another bidder). The winner is the bidder with the highest bid when the auction ends, and the winner pays one bid increment above the highest losing bid. In this sense, each train ticket auction resembles a second-price auction (Vickrey, 1961) even if it can be argued that it is a hybrid between a second-price and a first-price auction due to the role of the bid increment (see Hickman, 2010).[8] Upon winning an auction,

7 Because the train ticket auctions are organized by SJ, the tickets are not resales. See Yoshimoto & Nakabayashi (2016) for a recent paper on resales of train tickets in Japan.

8 It has also been noted by Hickman (2010), Hickman et al. (2011) and Tukiainen (2013) that bid shading may be optimal when the price is set to one increment above highest loosing bid. However, we conjecture that this issue is orthogonal to what we study here as the bid shading

the winning bidder is directly asked to fill in his name and cell phone number, and the ticket is subsequently sent as a personal ticket by an SMS text message. This final feature makes it hard to resell a previously-won ticket, if the name and phone number of that buyer were not available to the auction winner at the time of winning the auction.

Two tickets are considered to be *identical* if conditions (i)–(iv) from the above are identical as there is no way of distinguishing between the tickets. A set of auctions with identical tickets is called a *sequential auctions* (SA, henceforth). Each ticket auction in a SA is given an auction number which determines the *presentation order* on the website, i.e., the auction with the lowest number is displayed highest up on the website and has presentation order 1, the auction that is displayed second highest up has presentation order 2, and so on.

The *rank order* of an auction within a given SA is defined as the order in which the auction is terminated, i.e., the auction that terminates first in a given SA has rank order 1, the auction that terminates second in the same SA has rank order 2, and so forth. Each auction in a SA ends on date $t+2$ between 9pm and 10pm, but more importantly, the exact ending minute of the auction is determined by a random draw from a uniform distribution on [0,59] that is made before the auction is announced on the website. As a consequence, the rank order and the presentation order for a ticket in a given SA may differ, and, moreover, the time span between two consecutive auctions in a SA is random. Hence, this allows us to disentangle the effect of timing and presentation order from that of the rank order effect. Yet, we note that the presentation order is based on the default ordering on the website and a bidder may alter this ordering by using different filters (e.g., auction ending time) which may put a downward bias on how our definition of presentation order affects prices.

3. DATA DESCRIPTION

The data set consists of all train tickets sold by SJ on Tradera during the period 2010–11–15 to 2011–06–14. In total, $42,007$ tickets were sold for 42 different departure–destination routes during this period. These tickets can be partitioned into 7,202 sequential auctions with a total of more than 15,000 participating bidders.

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will be constant within an auction sequence.

Table 1 summarizes the main characteristics of interest by route. As can be seen from Table 1, some routes, usually from and/or to small communities, contain very few observations. These may be routes where auctions failed to generate enough bidding or “trial routes” that were discontinued. In addition, some particular routes, typically only the most popular ones (e.g., Göteborg C – Stockholm C), contain a few sequential auctions with very long sequences. In the empirical analysis, we exclude the 10 routes with fewer than 100 tickets, and all sequential auctions containing more than 15 tickets. We also exclude auctions with only one bidder. Given these restrictions, the reduced dataset still contains 5,999 sequential auctions with a varying number of bidders and tickets.

Table 1: Summary statistics for the considered departure–destination routes.

Departure – Destination	#A	#B/A (mean)	#B/A (sd)	#SA	Price (mean)	Price (sd)	Min price	Max price
Arvika stn – Stockholm C	13	2.2	1.3	5	51.8	114.2	1.0	330.0
Duved stn – Stockholm C	166	2.5	1.2	41	72.4	87.9	1.0	350.0
Falun C – Stockholm C	2393	3.5	1.5	433	70.5	51.9	1.0	263.0
Göteborg C – Kalmar C	5	1.2	0.4	5	1.2	0.4	1.0	2.0
Göteborg C – Koebenhavn H	470	4.0	1.8	119	63.2	51.4	1.0	235.0
Göteborg C – Malmö C	885	3.6	1.7	184	68.5	55.1	1.0	265.0
Göteborg C – Oesterport	559	3.8	1.8	103	56.8	50.4	1.0	280.0
Göteborg C – Stockholm C	5385	5.5	2.6	480	188.2	141.2	1.0	900.0
Göteborg C – Sundsvall C	2	1.0	0.0	1	1.0	0.0	1.0	1.0
Halmstad C – Stockholm C	111	5.0	2.4	27	213.0	147.2	1.0	500.0
Hudiksvall stn – Stockholm C	3	1.7	1.2	1	2.3	2.3	1.0	5.0
Kalmar C – Göteborg C	2	1.0	0.0	2	1.0	0.0	1.0	1.0
Karlstad C – Stockholm C	2548	3.3	1.7	564	81.7	82.9	1.0	460.0
Kiruna C – Luleå C	416	2.2	1.1	177	50.0	61.5	1.0	245.0
Koebenhavn H – Göteborg C	281	4.1	1.6	70	75.1	51.0	1.0	250.0
Koebenhavn H – Stockholm C	147	5.9	2.2	44	193.9	117.0	1.0	560.0
Luleå C – Kiruna C	400	2.3	1.2	169	60.3	67.7	1.0	280.0
Luleå C – Narvik	379	2.2	1.1	158	51.5	62.8	1.0	389.0
Malmö C – Göteborg C	951	3.6	1.7	217	67.7	59.6	1.0	305.0
Malmö C – Stockholm C	3219	5.6	2.3	437	205.9	124.9	1.0	710.0
Mora stn – Stockholm C	3	1.7	0.6	1	6.7	8.1	1.0	16.0
Narvik – Luleå C	370	2.1	1.2	161	45.8	59.6	1.0	327.0
Odense – Stockholm C	23	5.3	2.3	5	144.9	114.6	7.0	370.0
Oesterport – Göteborg C	925	3.5	1.7	152	54.1	48.1	1.0	231.0
Oesterport – Stockholm C	219	6.4	2.3	53	264.5	141.8	4.0	630.0
Stockholm C – Borlänge C	392	3.0	1.5	105	50.8	47.3	1.0	260.0
Stockholm C – Duved stn	178	2.8	2.0	50	85.0	121.9	1.0	491.0
Stockholm C – Falun C	2098	3.3	1.7	447	63.1	56.5	1.0	295.0
Stockholm C – Göteborg C	5411	5.0	2.8	469	169.1	157.2	1.0	1075.0
Stockholm C – Halmstad C	45	3.3	2.5	14	122.0	153.1	1.0	598.0
Stockholm C – Karlstad C	2582	3.1	1.6	487	70.6	79.2	1.0	510.0
Stockholm C – Koebenhavn H	224	5.7	2.3	73	240.6	130.8	1.0	667.0
Stockholm C – Malmö C	1931	5.8	2.2	347	234.7	136.8	1.0	1225.0
Stockholm C – Odense	4	4.3	0.5	1	169.8	65.6	105.0	261.0
Stockholm C – Oesterport	786	4.9	2.3	166	193.9	147.8	1.0	1125.0
Stockholm C – Sundsvall C	3176	3.8	1.8	451	104.5	96.5	1.0	550.0
Stockholm C – Åre stn	508	3.3	1.8	106	117.4	121.1	1.0	550.0
Stockholm C – Östersund C	597	3.5	1.4	158	129.6	120.6	1.0	760.0
Sundsvall C – Stockholm C	3095	4.2	1.9	449	119.2	95.8	1.0	600.0
Ånge stn – Stockholm C	1	1.0	–	1	1.0	–	1.0	1.0
Åre stn – Stockholm C	379	3.7	2.0	84	126.0	115.3	1.0	580.0
Östersund C – Stockholm C	725	4.0	1.9	185	150.7	132.4	1.0	810.0
Total	42007	4.3	2.3	7202	129.6	125.8	1.0	1225.0

Notes: A set of auctions with identical tickets is called a sequential auctions or SA for short. #A = number of auctions, #B/A = number of bidders per auction, #SA = number of SA, and sd = standard deviation.

Table 2: The effects of rank order, presentation order, and timing on prices.

Dependent variable: Price	Model (1)	Model (2)	Model (3)
Rank order	-3.863***	-3.837***	-4.133***
	(0.850)	(0.850)	(0.840)
Rank order^2	0.140*	0.138*	0.159**
	(0.0739)	(0.0740)	(0.0732)
Presentation order		-1.260***	-0.988**
		(0.461)	(0.460)
Presentation order^2		0.0815**	0.0644*
		(0.0383)	(0.0380)
Minutes to previous			2.802***
			(0.224)
Minutes to previous^2			-0.0964***
			(0.0108)
Minutes to next			1.615***
			(0.215)
Minutes to next^2			-0.0568***
			(0.0104)
Constant	154.0***	157.2***	141.8***
	(2.079)	(2.329)	(2.513)
Observations	18,718	18,718	18,718
R-squared	0.011	0.012	0.031
Number of SA	5,036	5,036	5,036

Notes: Sample restricted by excluding first and last auction to make samples in the three models identical. Robust standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$.

4. ANALYSIS

A linear fixed effects model where each SA is treated as an unbalanced panel is estimated in the sample. The chosen specifications are quite flexible as we allow for quadratic terms in all variables of interest. Table 2 shows the main estimation results. Model (1) is the baseline model in which no controls for presentation order and timing are included. Model (2) includes a control for presentation order, and Model (3) also includes controls for timing by measuring the distance, in minutes, to the previous ("Minutes to previous") and next ("Minutes to next") auction in a SA.[9] Model (1) confirms the existence

[9] Note that the sample is kept constant in Models (1)–(3), i.e., since the first and last auctions in a SA must be excluded in Model (3), to include the timing variables, these auctions are also absent in Models (1) and (2). Table 4 in the Appendix reports the results where Models

of an afternoon effect when not controlling for timing and presentation order. This is in line with previous findings in the literature, e.g., Ashenfelter (1989), McAfee & Vincent (1993), and van den Berg et al. (2001).

The left panel in Figure 1 shows the estimated average marginal effect of rank order on prices for Model (3) in Table 2, and it illustrates that the negative price trend continues to hold also for high rank orders even though the effect is diminishing as the rank order increases. The right panel in Figure 1 shows the estimated marginal effect of presentation order on prices for Model (3) in Table 2. From the figure, it can be seen that the optimal placement, from the perspective of the seller, is at the top of the website. This indicates a primacy effect and corroborates previous findings from sponsored search literature (see, e.g., Agarwal et al., 2011). However, in contrast to the literature on sponsored search, where the items are heterogenous and their position usually is determined by a position auction (see, e.g., Edelman et al., 2007; Varian, 2007), the train tickets in a given SA are completely homogenous. Note also that even though there is a negative effect associated with presentation order, the effective size is less than half the effect of the rank order of the auction. As noted earlier, this may be partly explained by bidders using filters to re-order the presentation of tickets.

The left panel (right panel) in Figure 2 shows the distribution in minutes to the next (previous) auction, and the right panel (left panel) in Figure 3 shows the estimated average effect of time from subsequent (previous) auction on prices for for Model (3) in Table 2. As can be seen in the figure, there is an upward sloping trend for small time differences. This trend can, at least partly, be explained by the results in Andersson et al. (2012) and the theoretical predictions in Zeithammer (2006). The former suggests that if auction ends are identical, bidders may have problems cross bidding, i.e., it may be difficult for bidders to coordinate bids between auctions with similar termination times. The latter, on the other hand, suggests that bidders are more likely to be forward looking when subsequent auctions are closer which makes bidders less willing to place high bids in the current auction.

In general, the timing effect shows an inverted U-shape to the previous (subsequent) auction. One potential explanation for this price hike is that the more time that has passed since the previous auction ended, the more

(1) and (2) include the first and last auction in a SA. The results are qualitatively similar but coefficients on rank order is smaller. The model is also estimated using the full sample in Table 5 in the Appendix.

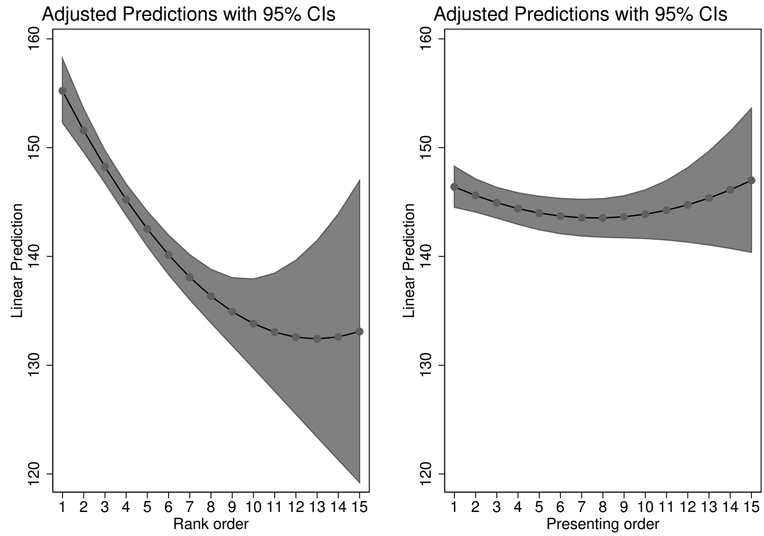

Figure 1: Marginal effect of rank order.

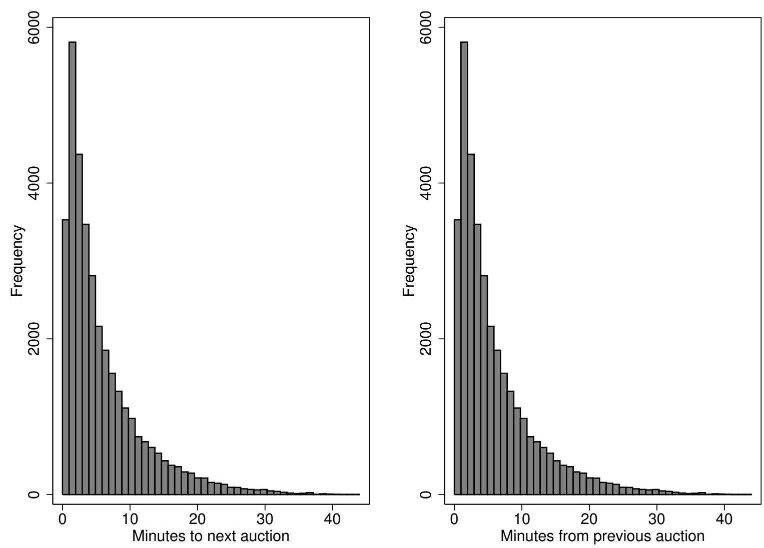

Figure 2: Minutes to previous and next auction.

bidders have been able to enter and place bids in the current auction. The fact that timing seems to matter for the evolution of price sequences is interesting and we have not found any such result in the literature. According to the estimations, the optimal timing is to have auction ends approximately 15 minutes apart, which may counteract the rank order effect.

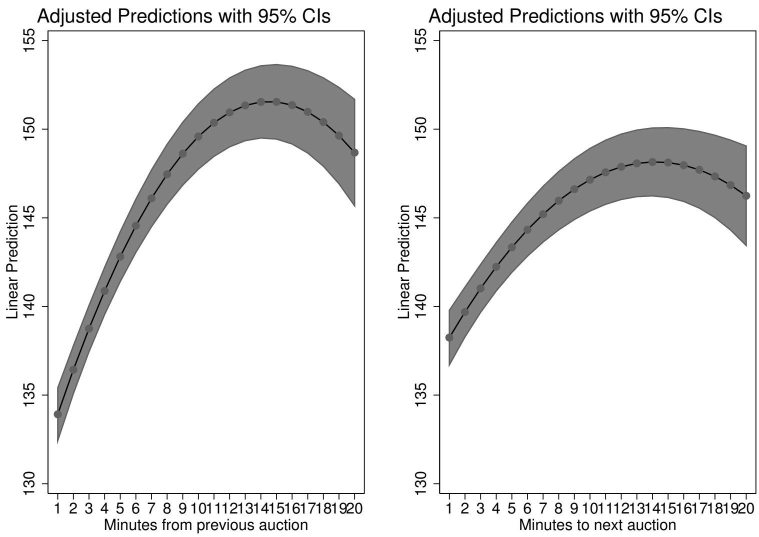

Figure 3: Marginal effect of timing.

4.1. Unit-demand Bidders

The previous analysis shows that both the time elapsed between auction ends and the presentation order are important for price formation in sequential auctions, but that they cannot fully explain the negative rank order effect (i.e., the afternoon effect). One possible explanation of the latter finding is that the analysis in Milgrom & Weber (1982) explicitly considers unit-demand bidders. Unfortunately, as in many real-world applications and all empirical studies on sequential auctions, it is not possible to observe whether bidders have unit-demand or not. What is possible to observe in the current study, however, is the number of tickets within a specific SA that a particular bidder has won.[10]

Table 3: The effects of rank order, presentation order, and timing on prices (unit demand).

Dependent variable: Price	Model (1)	Model (2)	Model (3)
Rank order	-1.551*	-1.529*	-4.720***
	(0.897)	(0.893)	(1.522)
Rank order^2	0.162	0.161	0.270*
	(0.0995)	(0.0991)	(0.143)
Presentation order		-1.590**	-1.020
		(0.792)	(0.993)
Presentation order^2		0.130	0.0825
		(0.0833)	(0.0988)
Minutes to previous			3.151***
			(0.343)
Minutes to previous^2			-0.104***
			(0.0148)
Minutes to next			1.978***
			(0.360)
Minutes to next^2			-0.0618***
			(0.0165)
Constant	165.4***	168.5***	149.5***
	(1.676)	(2.231)	(4.559)
Observations	14,035	14,035	7,463
R-squared	0.001	0.001	0.031
Number of SA	3,496	3,496	2,661

Notes: Sample restricted to SA's where each bidder won at most one ticket. Robust standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$.

In Table 3, the same set of regressions as in Table 4 are run but restricting the sample to situations where each bidder in the SA only won at most one auction. As can be seen from the table, the rank order effect is still negative but smaller and now only significant at the ten percent level for Models (1) and (2). Model (3) is more inline with its full-sample counterpart, displaying a sizable negative rank order effect. Results from the first two specifications

[10] In the investigated auction, it was possible to buy multiple train tickets. However, it should be noticed that each ticket was personal and a ticket holder may have had to show identification when validating the ticket with the conductor. It is, however, unknown to the authors whether this institution was strictly enforced or not. Another possibility to buy multiple tickets was to place bids using different accounts on the auction platform.

suggest that multi-unit demand may partly help to explain the decreasing price anomaly. Yet, the third specification suggests not. What is special with this last specification, except for controlling for timing effects, is that it excludes the first and the last auction in a SA. If the price drop is most notable in the very first auction, then this may explain the effect but there is no way to tell if this is true or not given the limitations of the data. Finally, note that the coefficients on timing are intact to the sample restriction. The size of the effect of presentation order are similar albeit insignificant under the third specification, which may be explained by the smaller sample.

5. CONCLUSIONS

This paper has investigated how prices are affected by the time elapsed between auction ends and the presentation order on the website. It has been demonstrated that the presentation order on the auction website has a significant impact on prices. Yet, the effect follows a U-shape. Hence, it pays off to be one of the first or last auctions presented. When it comes to the effect on prices of the time elapsed between two consecutive auctions, it is found that having closeness in the time dimension has a negative impact on prices in both auctions. Consequently, the policy implication for the auction designer is that one should be careful in posting two auctions "too close" to each other as it may result in lower prices for both items. Moreover, by randomizing the presentation order, the primacy effect may be down-played, but as it is hard to inherently control this order, its effect on prices may be small.

Received theory predicts that prices in a sequence of auctions with identical items should be equal. Yet, even when controlling for presentation order, the timing of auctions, and by restricting the dataset to unit-demand bidders, it is found that average auction prices are declining in a sequence. This further strengthens the anomalous findings (i.e., the afternoon effect) in the existing empirical auction literature. Given this insight, it is logical to search for a theoretical explanation of this phenomenon that can be empirically tested. Unfortunately, none of the theoretical explanations that we are aware of (e.g., Ashenfelter, 1989; McAfee & Vincent, 1993; von der Fehr, 1994; Mezzetti, 2011; Rosato, 2014) have predictions that can be tested empirically unless estimates of the risk preferences, loss aversion or participation costs are available. The problem of finding alternative theoretical explanations to the afternoon effect is left for future research.

6. APPENDIX: ADDITIONAL ESTIMATIONS

Table 4 describes the re-estimation of the models in Table 2 but allowing the sample to vary over the three models. Table 5 replicates the estimation in Table 2 but using the full sample.

Table 4: The effects of rank order, presentation order, and timing on prices.

Dependent variable: Price	Model (1)	Model (2)	Model (3)
Rank order	-2.788***	-2.767***	-4.133***
	(0.566)	(0.567)	(0.840)
Rank order^2	0.148***	0.148***	0.159**
	(0.0542)	(0.0543)	(0.0732)
Presentation order		-0.966**	-0.988**
		(0.404)	(0.460)
Presentation order^2		0.0531	0.0644*
		(0.0347)	(0.0380)
Minutes to previous			2.802***
			(0.224)
Minutes to previous^2			-0.0964***
			(0.0108)
Minutes to next			1.615***
			(0.215)
Minutes to next^2			-0.0568***
			(0.0104)
Constant	154.5***	157.0***	141.8***
	(1.199)	(1.470)	(2.513)
Observations	29,927	29,927	18,718
Number of SA	5,999	5,999	5,036

Notes: Robust standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$.

Table 5: The effects of rank order, presentation order, and timing on prices. Full sample.

Dependent variable: Price	Model (1)	Model (2)	Model (3)
Rank order	-3.235***	-3.209***	-4.094***
	(0.237)	(0.237)	(0.258)
Rank order^2	0.0335***	0.0329***	0.0473***
	(0.00554)	(0.00556)	(0.00679)
Presentation order		-0.527***	-0.318***
		(0.117)	(0.121)
Presentation order^2		0.00204	-0.00126
		(0.00313)	(0.00315)
Minutes to previous			3.173***
			(0.190)
Minutes to previous^2			-0.113***
			(0.00933)
Minutes to next			1.729***
			(0.182)
Minutes to next^2			-0.0620***
			(0.00884)
Constant	144.4***	146.9***	132.9***
	(1.084)	(1.186)	(1.620)
Observations	42,007	42,007	27,778
R-squared	0.033	0.034	0.076
Number of SA	7,202	7,202	6,145

Notes: Full sample. In Model (3) first and last auction in SA is excluded due to timing variables. Robust standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$.

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EFFICIENCY AND FAIR ACCESS IN KINDERGARTEN ALLOCATION POLICY DESIGN

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ABSTRACT

We examine Kindergarten allocation practices in an Estonian municipality, Harku. Based on our recommendations, the allocation mechanism in Harku was redesigned in 2016. The new mechanism produces a child-optimal stable matching, with priorities primarily based on siblings and distance. We evaluate seven policy designs based on 2016 admission data in order to understand efficiency and fairness trade-offs. In addition to the descriptive data analysis, we conduct a counter-factual policy comparison and sensitivity analysis using computational experiments with generated preferences. We fix the allocation mechanism to be the child-oriented Deferred-Acceptance algorithm, but we vary how the priorities are created by altering sibling and distance factors. Different lotteries are included for breaking ties. We find that different ways of considering the same priority factors can have a significant aggregate effect on the allocation. Additionally, we survey a dozen special features that can create significant challenges (both theoretical and practical) in redesigning the

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allocation mechanism in Estonian Kindergartens, and potentially elsewhere as well.

Keywords: Kindergarten allocation; policy design; mechanism design

JEL Classification Numbers: C78, D47, D50, H75

1. INTRODUCTION

Families have become a much-debated issue in all developed countries and they form the focal point of debates about "new risks" and the much needed "new policies" for Western welfare states. The questions of who should care for children, to what extent and for how long, lie at the centre of conflicts about the values that shape not only policies and struggles around policies, but also individual and family choices (Saraceno, 2011). Moreover, in Eastern Europe, the Soviet legacy has paved the way for the dominance of publicly provided care, but in many countries, including the case examined here, there is a shortage of early childhood care places for children aged 18 months to three years. This shortage of places has forced municipalities, who are the main providers, to set priorities for the allocation of these places. Priorities are aimed not only at solving the problem of oversubscription, but also at implementing social goals. Thus, we conceptualise the process of implementing priorities accompanied with allocation principles (matching design) as policy design.

Policy design entails taking the approach of a matching mechanism design in order to propose a good way to allocate children to Kindergartens. There are process descriptions about the (re-)design of school choice mechanisms, e.g. in various cities in the US (Pathak & Sönmez, 2013; Pathak & Shi, 2013; Ergin & Sönmez, 2006) and in Amsterdam (de Haan et al., 2015). Nevertheless, to the best of our knowledge, our paper is the first to report such a redesign of a Kindergarten place allocation mechanism. However, our theoretical foundation relies on the mechanism design literature motivated by related applications, such as school choice (Abdulkadiroğlu & Sönmez, 2003; Abdulkadiroğlu et al., 2005a), college admissions (Biró et al., 2010b; S.-H. Chen et al., 2012) and job assignments (Roth, 2008). Mechanism design provides methods for allocation under given welfare criteria and selection priorities, but it does not prescribe the way in which these priorities should be applied. The general policy considerations for school choice are the allocation of siblings to the same school and the proximity of the school. Some countries also use some affirmative action measures, e.g. prioritising children of low socio-economic status. Similar principles are applicable to our Kindergarten policy design case study while aiming for the clear-cut implementation and operationalisation of policies. The latter not only concerns a clear definition of proximity as a priority (i.e. defined as a walk-zone (Shi, 2015) or a continuous cardinal measure (West et al., 2004)) or the ordering of priority classes but also allows for the implementation of welfare considerations in policy evaluation.

Our welfare considerations aim at two social goals: efficiency and fairness. We define efficiency as the ability of a policy to meet predefined goals, the utility of families (high rank in their preferences and siblings in the same Kindergarten) accompanied with social goals such as minimising the travel distance or time to Kindergartens. Defining fairness is more problematic and entails more uncertainty. Our definition of fairness is based on the idea of equal access. It is operationalised by the probability that the child is assigned to her first preference.

Our study considers a local municipality Harku, in Estonia. Instead of implementing certain social goals by policy design, the most commonly used priority in Estonian municipalities is the date of application, while in limited cases, catchment areas are applied to ensure proximity. Children are ordered on the basis of the application date in a manner similar to a serial dictatorship mechanism, thus forcing one-sided matchings without enabling the implementation of affirmative action policies or social goals, such as fairness. In addition, parental preferences are either not considered or have been limited. In the Harku case, the number of preferences was bounded by three until 2015. The latter restriction implies that preferences are not revealed truthfully and moreover, the matching has been done manually.

Between 2014 and 2016, as part of an Estonian project we collaborated with representatives of the Harku municipality. We monitored their 2015 allocation practice and suggested a revision which led to a transitory system in 2016. In the 2016 allocation, the standard student-proposing Deferred-Acceptance mechanism was used under a special priority setting which is described in detail in Section 2.2. This mechanism is known to be strategy-proof, and the parents were encouraged to submit full preference lists, so we can expect the submitted applications to be truthful. We made a comparative assessment of policies using the 2016 data. As an input we used preference data collected from 152 families who have the right to a Kindergarten place.

In the assessment, we proposed seven different policies which consist of different metrics of indicating distance (as absolute, relative or binary measures), siblings, quotas; and their priority order. Ties are broken by assigning random numbers either with a single or with multiple lotteries. Our research methods are partially inspired by Shi (2015), but we investigated some novel policies as well. Perhaps the most interesting aspect of these policies is the way that distance is used in the priorities.

The classic way of creating proximity priorities is the *catchment area* system, where the city is partitioned into areas and the students living in an area have the highest priority in all schools in that area. This simple method can be seen as unfair, as one student can have a higher priority than another student, even though the actual distance of her location to the school is greater than for the other child. Therefore instead of catchment areas, most applications have switched to absolute or relative distance based priorities. The simplest absolute distance based policy is the walk-zone priority scheme, used in many US cities (e.g. New York (Abdulkadiroğlu et al., 2005a)), where the children living within a well-defined walking distance are in the high distance priority group for that school and the ties are broken by lottery. Strict priorities based on absolute distances are used in Sweden as well (Andersson, 2017). However, there were also discussions and court cases about the fairness of such absolute distance based priorities[1].

The absolute distance based priority schemes can be unfair for those living far from all (or most) of the (good) schools, therefore the so-called *relative distance* based methods are also commonly used in many applications (e.g. Calsamiglia & Güell (2014); Shi (2015)). The relative distance priority means that we give the highest priority to all children for their closest Kindergarten, no matter how far that is, and the children will be in the second priority group in their second closest Kindergarten, and so on. A rough version of this rule is to give high priority for all children in a given number of closest schools.

Barcelona changed its catchment area systems to a relative distance system in 2007. After the change, students have priority in at least six of their closest

[1] In the city of Lund parents have challenged allocation decisions in court based on an alternative option distance argument. The city used the absolute distance priority in their allocation, but some parents have found this policy unfair, as they would have to travel 1000m more to their second choice school than to their first choice school, whilst there was another student who would only need to travel 650m more if allocated to their second choice school rather than their first choice school. The court accepted this argument and gave a seat to the appealing student in their first choice school.

schools (Calsamiglia & Güell, 2014, Section 5.1)[2]. In Boston, another relative distance policy was proposed recently by Shi (2015), mainly in order to reach the city's aim to cut down busing costs.

Note that there are also applications where the distance based priorities are considered unfair, as they can limit equal access to good schools. The Amsterdam school choice system (de Haan et al., 2015) does not use any distance based priority, only a pure lottery. In the Harku case, where Kindergartens are of more or less the same quality, the authority was in favour of using the distance based priorities in order to decrease the overall commuting costs and also to satisfy the preferences of the parents (typically for nearby Kindergartens). Based on the unfairness of the catchment area system described above, we only considered absolute and relative distance based priority approaches. We explain the distance-based priorities that we studied in more detail in Section 3 with examples.

Besides the distance, we also investigated different ways of taking the sibling priorities into account and also the way the lotteries are conducted in case of ties. The way the distance and sibling factors are considered has already been studied in the literature (Dur et al., 2013). The particular solution chosen for the 2016 transitory system is an interesting rotation priority scheme, which can lead to a well-balanced solution with respect to the two factors. Regarding the lotteries, we analysed the effects of using a single lottery for all Kindergartens compared to using multiple lotteries (one at each Kindergarten), and we have seen results similar to other research papers (Ashlagi & Nikzad, 2015).

As the second main contribution of our paper, we present a sensitivity analysis of various metrics of fairness and efficiency of policy designs based on counter-factual preference profiles. The policies that provide the best solutions for the current Harku data may not be ideal for other applications or robust for Harku, where the preferences of the parents are different. This can be the case in cities, or in other countries with different Kindergarten/school qualities, or for applications at different education levels (e.g. primary and

[2] "Before 2007, the city was divided into fixed neighbourhoods. The neighbourhoods varied in size for semi-public and public schools, but were conceptually the same. For semi-public schools, the neighbourhood coincided with the administrative district. For public schools, the neighbourhoods were smaller areas within the administrative district. The new neighbourhoods are based on distance between schools and family residences. An area (specifically, a minimum convex polygon) around every block of houses in the city was established to include at least the six closest schools (three public and three semi-public)."

secondary schools). Therefore, we found it important to investigate the effects of changes in priorities in the performance of different policies (i.e. different priority structures for the student-optimal Deferred-Acceptance mechanism). As a novel approach, we studied the fairness (or equal access) of the allocations measured by the probabilities of getting placed in the first choice schools.

In general our results indicate that preference structures, more precisely their endogeneity with respect to proximity, influence policy design. However, we advocate a relatively simple policy that prioritises siblings first and relative distance second. Relative distance gives all children priority in the closest Kindergarten independently of absolute distance from it. This policy is superior to others by our fairness criteria, especially when preferences of the families are aligned with policy priorities.

We structure our paper as follows. In Section 2 we review the practices and processes of Kindergarten choice of an Estonian municipality, Harku, before the process was redesigned on the basis of our recommendations in 2016. In Section 3 we define seven alternative policies and the descriptive statistics of our data, including our results from computational experiments. Finally, we discuss additional mechanism design challenges with some policy recommendations in Section 4 and give conclusions in Section 5.

2. MATCHING MECHANISM DESIGN

The design of an allocation mechanism is usually based on a two-sided matching market model, in this case between 1) families and 2) Kindergartens. Participants on both sides have linear orderings over the participants on the other side. Families have preferences over Kindergartens and they seek to get places at their most preferred Kindergartens. Kindergartens have a priority ranking over children. Priorities become important if there are fewer places available in a particular Kindergarten than the number of families who would like to be allocated to that Kindergarten. In those circumstances, Kindergartens accept children who are higher on their priority list, which in practice usually means children who live closer and/or who have a sibling in the Kindergarten. Kindergartens do not seek to admit higher priority children. This practice is different from some applications of two-sided markets. In college admissions for example (Gale & Shapley, 1962), both students and colleges seek to get more preferred matches, therefore they might act strategically in the allocation mechanism.

There are two prominent strategy-proof mechanisms for solving matching problems, the Deferred-Acceptance (DA) and the Top-Trading Cycles (TTC) mechanisms (Abdulkadiroğlu & Sönmez, 2003). The DA mechanism guarantees that no preferences and priorities (policies in our case) are violated, and there is no child who could get a place in a more preferred Kindergarten by priority, so there are no blocking pairs. A matching with no blocking pairs is called *stable*. A blocking pair can also be seen as a child having justified envy, since there is a family that would prefer a Kindergarten that either has free places or has accepted a child with lower priority. These kinds of justified envy situations are not tolerated in most applications (Pathak & Sönmez, 2013), and are sometimes even prohibited by law. Thus, stability is a crucial property for most applications.

While there are potentially a number of stable allocations (Knuth, 1997), the child-proposing DA mechanism that is usually implemented results in the best possible preference for all families among the stable solutions, and this option also makes it safe for the families to reveal their true preferences.

The theoretical properties and disadvantages of DA were studied by Haeringer & Klijn (2009), backed by evidence from laboratory experiments (Calsamiglia et al., 2010) and by practical applications across the world (Pathak & Sönmez, 2013). In addition to advocating DA, the main policy implications of these studies indicate that for an efficiency gain, it is advised to increase the bounds on the number of collected preferences or to abolish the limit on the number of submitted preferences.

Before its redesign, the application process of the Harku municipality had many design features, but it was not a transparent system. Families could submit up to three ordered choices. The application date and the home address were also collected. The application date was relevant for the allocation, as families with an earlier application date had higher priority. Therefore, families tended to submit their applications as early as possible, usually a few weeks after the birth of the child. The application data typically remained unchanged until the actual allocation occurred, which could make the originally true preferences out of date (e.g. it was possible that the family moved to a different place or their older sibling has received a place in a different Kindergarten during the waiting period). The address could be a factor, as some heads of Kindergartens considered it when assigning places. Secondly, a qualifying condition for a Kindergarten place is that the parents have to be registered residents in Harku, and residency is based on where local taxes are collected.

Moreover, the matching was done manually using the following procedural rules. First, the number of vacant places was settled by January of each year, when the allocation process started. Place offers were made to families by the heads of Kindergartens if their Kindergarten was the first choice of the family. Second, if there were more families than places, then priority was given to the applications with earlier registration dates, although proximity or siblings could also be occasionally relevant. Third, if an offer was accepted, the child became assigned to the Kindergarten, otherwise that place was offered to the subsequent family on the waiting list.

In the case of unassigned children, the procedural rules where complicated and discretionary. Generally the heads of the Kindergarten communicated with each other to find a place for the children who remained unassigned. In the case of families who ordered popular Kindergarten at the top of their list and remained unassigned in the first round, their second or third choice was considered, although these could already be full. If that was the case, the families with an earlier application date would be rejected from their second choice because the children already assigned there had listed that Kindergarten as their first preference, irrespective of their application dates. Thus, some children were allocated to a less preferred Kindergarten, simply because of how the family ordered their preferences. This is a well-known property of the Immediate-Acceptance mechanism (e.g. Abdulkadiroğlu & Sönmez, 2003) and the procedure that had been used in Harku until 2015 was very similar to this.

2.1. Building a mechanism for Kindergarten seat allocation

Our redesign of the Harku Kindergarten allocation mechanism inspired by the literature has four main areas as described in Table 1. The application procedure before 2016 which was initiated by collecting preferences had several drawbacks. First, since parents could get higher priority if they applied earlier, they tended to apply soon after the birth of their child. However, during the subsequent three years, the preferences of the families could change. This situation was usually not reflected in the application data, thus resulting in a high number of cancellations. Second, families could only list their top three choices. Limited preference not only created a large number of unassigned children, but also manipulation with the revelation of preferences.

Our design changed the data collection procedure and the number of

Table 1: Redesign of Harku mechanism

2015	2016
Application procedure	
Applications are collected after the birth of the child due to prioritising according to application dates	Applications are collected from 1 January until 1 February for allocating places from 1 September of the same year
Limited preference lists	
Limited to three Kindergartens	List all Kindergartens they are willing to attend (no limit)
Priorities (policies)	
Not clearly defined	See Section 3.2 for policy design alternatives
Matching mechanism	
Decentralised mechanism which has some properties of Serial dictatorship and Boston (Immediate-Acceptance)	Deferred-Acceptance

preferences collected. Families make application in the matching platform[3] during monthly period six months before the service delivery (1. September) and list all their preferences. Giving up application date as a priority will be a necessary result of the procedural amendments.

Finally, the central allocation mechanism applied until 2016 was not transparent, the priorities were not clearly defined or adhered to by the heads of the Kindergartens. The first priority of the application date was sometimes violated. Children with siblings were usually considered to have higher priority, but not always. Our design introduces clearly defined priority metrics and a centralised allocation system that ensures that the criteria are always followed. Moreover, instead of an unstable and manipulable Immediate-Acceptance mechanism we propose the child-proposing DA. This is a standard method for school choice (Abdulkadiroğlu & Sönmez, 2003), which eliminates justified envy, and gives incentive for the families to state their true preferences.

[3] `https://www.haldo.ee/`

2.2. Particularities of the 2016 system

Before the final implementation of our platform-based matching design, there was a transitory system in place in Harku in 2016 that partially applied our design recommendations, but experimented with priorities. Families were asked to rank all seven Kindergartens. Additionally, the home address, application date, status of siblings and the child's birth date were collected. The allocation process was designed on the basis of the DA mechanism with slots (Dur et al., 2013) while policy transformation regarding fixing priorities was more complex. There were four types of priorities that are defined per position as follows, in the order of precedence:

1. siblings, distance, age, application date
2. distance, age, application date, siblings
3. application date, siblings, distance, age
4. age, application date, siblings, distance
5. siblings, distance, age, application date
6. distance, age, application date, siblings

...

The positions are considered in order, with families first applying to the first position, then the second position, etc. This can also be thought of as each Kindergarten being split into a number of seats, with each seat potentially having a unique priority criteria. Then, the preferences of the families are modified so that within each Kindergarten, they rank the position with the higher precedence more highly. If the number of available places is not exactly divisible by four, then some type of priorities might have more positions available than others.

The main reason for the complicated policy design or for considering the four types of priorities rotationally was backed by the argument of equal treatment. Granting equal opportunity to all ”types of families” (the ones that have siblings; those living nearby; early applicants; and families with an older child) was the preference of the local municipality. In future allocations, the application date will not be used. It was used here as some families still had the expectation of being allocated by the application date.

The precedence order of priority classes matters in the allocation procedure, as shown by Dur et al. (2013) by demonstrating that a simple priority scheme might be discriminating for some groups. For instance, let us assume there are five seats with siblings and distance priority and a further five seats with only distance priority. There are more than five children with a sibling and in total more than ten children. If for the first five positions we would consider children with siblings and then by distance, this would be disadvantageous for children with siblings compared to first only considering distance and then siblings as well as distance. In the latter case, some children with siblings might already be allocated by distance alone, so other children with siblings have lower competition and a better chance of getting a desired place. On the other hand, it might occur that some children living closer have an unfair disadvantage. The aim of the rotating scheme is to balance these two effects. That leads us to the equal treatment issues related to policy design.

3. POLICY DESIGN

3.1. Efficiency and fairness

In mechanism design the goals are usually related to designing an allocation method that maximises a form of efficiency, while not violating some constraint(s). In the matching domain, the usual criterion is selecting a Pareto optimal matching among a set of stable matchings. In a public resource two-sided matching setting, e.g. school seats, usually in fact two selections are made: first, the priorities of applicants and second, the mechanism. In a school choice setting, the priorities are often based on siblings and distance, although there are other alternatives (MatchinginPractice, 2016). However, in designing the allocation mechanism these priorities are usually treated as a given.

When evaluating the allocation methods we concentrate on two main criteria: efficiency and fairness. Efficiency characterises the level at which we, as designers, can satisfy the preferences of the applicants. Thus, we look at the average allocated preference. We also include the percentage of applicants receiving their first preference as this is often the case and the average might not always be a good indicator.

In addition to efficiency and stability (lack of envy), our policy design is driven by equality concerns. In the literature on distributive justice, discussion on fairness (fair access in our case) is often accompanied by discussion on the principles of affirmative action, i.e. the Rawlsian difference principle (Rawls, 1971). In our case, fair access is defined as the chance for the family to access their most preferred Kindergarten. Moreover, we include in our design some positive discrimination, or controlled choice, through policies such as prioritising siblings.

Fair access is essentially different from the efficiency metrics for the priorities of local municipalities and the preferences of families. The goal of fair access is to provide an opportunity for every child to get a place at their most preferred Kindergarten. As some families might live far away from all Kindergartens (see Appendix C), they would always be low on the priority list for any Kindergarten. We measure fair access as the proportion of families placed in their most preferred Kindergarten on two levels, at least 10% chance and 50% chance. This is similar to access to quality in (Shi, 2015) where quality, in addition to being ranked high, contains an objective quality metric. Since there is no quality ranking for a Kindergarten in our case and only a small number of Kindergartens we look at the probability of being allocated to the first choice. Since not all policy designs use lotteries, some will be inherently unfair in terms of fair access.

The mechanism also allows the local authorities to have social objectives, which are usually, but not always aligned with the preferences of the parents. The two most prominent goals are

- having siblings in the same Kindergarten, and
- placing children in a Kindergarten near their home.

Prioritisation of proximity and siblings is also recommended by the regulations responsible for the allocation of Kindergarten places ("Preschool Child Care Institutions Act", 2014). While prioritising proximity and siblings is common practice in the case of school and Kindergarten choice design, being favoured as the means to sustain community cohesion and avoid unreasonable transportation costs (see Shi, 2015, for instance), this practice may cause various concerns. The proximity principle may lead to problems in segregated areas, where it may result in the concentration of children from a similar socio-economic background into the same Kindergartens. Further social objectives could be the prioritisation of disadvantaged families or children with special needs, but there was no access to this kind of information in the data, so those goals were disregarded in this study. However, the main goal is still to provide families with a place at their most preferred Kindergartens.

3.2. Operationalisation of policy designs

A short list of social objectives indicated in the previous section does not mean that policy designs are limited to two alternatives, as the priority structures for siblings and proximity have many variants. Children with siblings might always have priority over others, or might only be prioritised over families living further away. Proximity can also be considered in many different ways, such as a walk-zone or a catchment area or a geographical distance.

A simple way to consider geographic aspects is to define catchment areas for each Kindergarten, and prioritise the children living in the catchment area where the Kindergarten is located. The drawback of this method is that these priorities may not reflect the personalised distances, as a Kindergarten might be relatively far from an address in the same area, whilst another Kindergarten in a different area can actually be nearby. Therefore, it may be more appropriate to use personalised distances. We can use continuous (real) distances or discretise them somehow, for instance giving priority to a Kindergarten within a 10-minute walking distance, or giving priority to the closest, or several closest Kindergartens. Another option is to give high priority to a child in a number of nearby Kindergartens. A special version of the latter so-called *menu* system has been evaluated and used in Boston school choice (Shi, 2015). Below we specify the distance-based priorities that we used in our policies.

- Absolute: Strict priorities based on the personalised absolute distances between the child's location and the school, measured in walk time or kilometres.

- Walk-zone: Coarse priorities based on the above-described absolute distance. A child is in the high priority group for a school if she lives within a 10-minute walking distance to this school.

- Relative: Every child is in the highest distance-based priority group in her closest school, she is in the second highest priority group in the second closest school, and so on.

- 3 closest: A binary variant of the above-defined relative distance policy, where every child is in the high priority group of a school, if this school is among the three closest schools for this child.

When we consider the children in walk-zones to have a higher priority, followed by children with siblings, the following priority groups are obtained:

1. siblings in walk-zones, 2. children in walk-zones, 3. siblings, 4. the rest. Siblings could also be considered to have a higher priority, which would result in the priority groups: 1. siblings in walk-zones, 2. siblings, 3. children in walk-zones, 4. the rest. This simple classification is used in many US cities, such as New York (Abdulkadiroğlu et al., 2005a) and Boston (Abdulkadiroğlu et al., 2005b), together with a randomised lottery for breaking ties. The lottery can also be conducted in two ways, either as a single lottery which is used in all Kindergartens, or as multiple lotteries, one for each Kindergarten. The typical choice, used in most US school choice programmes and also in Irish higher education admissions (L. Chen, 2012), is the single lottery. We will investigate both in our computational experiments. This question is discussed further by Ashlagi & Nikzad (2015) and Pathak & Sethuraman (2011).

If it is considered undesirable that a high proportion of children get admitted by sibling priority, then one option is to set a quota for siblings, for example 50% of the places. In this case, there is high priority for siblings for only some proportion of the places available, and the remaining places are prioritised by distance only. In such a setting, how the allocation is implemented is crucial. It can be done by allocating the places for siblings first and then the remaining seats or in reverse. Dur et al. (2013) showed that the reverse approach can benefit children with siblings, and Hafalir et al. (2013) showed that reserving places for a certain minority results in a better allocation for the minority than limiting the quota for the majority does. Under the latter policy, both groups (minority and majority) could be worse off. We evaluate policy design by the reservation of places for siblings or for families living nearby. In Harku, only about 20% of children have a sibling, so 20% of the places were set to have a sibling priority.

The Deferred-Acceptance algorithm can be slightly modified to accommodate for reserves and quotas. The priority quotas can be considered as separate Kindergartens. In this variant, the child is first placed in a quota group high in the precedence order and, if rejected, the child is then placed lower, etc. Thus, each child will be placed in the highest possible precedence quota group.

In this study, in order to explore the described aspects, we settled on seven priority policies (summarised in Table 2) for evaluation:

DA1. Children with siblings always have the highest priority and children living closer have higher priority. Priority classes would be considered in the order: 1) siblings; 2) walking distance.

Table 2: Summary of policies (priority order in parentheses)

Policy	**Distance** (D)	**Siblings** (S)	**Lottery**	**Quotas (Precedence)**
DA1	absolute (2)	(1)	no	no
DA2	walk-zone (2)	(1)	(3)	no
DA3	walk-zone (1)	(2)	(3)	no
DA4	3 closest (2)	(1)	(3)	no
DA5	absolute (2)	(1)	no	[80%, 20%] ([D, S+D])
DA6	absolute (2)	(1)	no	[20%, 80%] ([S+D, D])
DA7	relative (2)	(1)	(3)	no

DA2. Children with siblings always have the highest priority, then children in the walk-zone have higher priority. The walk-zone is defined as a 10-minute walking distance from home. Additional ties are ordered by a random lottery for all Kindergartens. The order of priority classes is: 1) siblings + walk-zone; 2) siblings; 3) walk-zone; 4) the remainder.

DA3. Children in the walk-zone always have the highest priority, then children with siblings have higher priority. Additional ties are ordered by a random lottery for all Kindergartens. The order of priority classes is: 1) siblings + walk-zone; 2) walk-zone; 3) siblings; 4) the remainder.

DA4. Children with siblings always have the highest priority, and children have higher priority for the three closest Kindergartens. Additional ties are ordered by a random lottery for all Kindergartens. Priority precedence order: 1) siblings + one-of-three-closest; 2) siblings; 3) one-of-three-closest; 4) the remainder.

DA5. Children with siblings have the highest priority for the reserved 20% of places, otherwise priority is by distance. Precedence order: 1) by distance up to 80%; 2) children with siblings + distance up to 20%; 3) remaining places, if any, by distance.

DA6. Children with siblings have the highest priority for the reserved 20% of places, otherwise priority is by distance. Precedence order: 1) children with siblings + distance up to 20%; 2) remaining places, if any, by distance.

DA7. Children with siblings always have the highest priority, and children have higher priority in the closest Kindergarten, second highest in the second-closest, etc. Additional ties are ordered by a random lottery for all Kindergartens. Priority precedence order: 1) siblings; 2) closest-number.

To demonstrate the effect of policies we construct a simple example. Let us assume we have four children $C = \{c_1, c_2, c_3, c_4\}$ and four Kindergartens $K = \{k_1, k_2, k_3, k_4\}$. In Table 3 we show the distances between homes and Kindergartens. We have no children with siblings in this example.

Table 3: Distances between homes and Kindergartens (km-s)

km	k_1	k_2	k_3	k_4
c_1	.7	1.2	1.0	1.7
c_2	.4	.6	.3	.7
c_3	.9	.5	.4	.3
c_4	.8	.3	.9	1.0

Assuming that walk-zone distance is $\leq .6$ km, the resulting priorities are in Table 4. We can observe that with absolute distance or walk-zone the child c_1 would not have a high priority in any Kindergarten. However with the 3-closest policy, there is at least some chance of having the highest priority in some Kindergarten, and with relative distance, each child has the highest priority in at least one Kindergarten. While this is not always guaranteed with relative distance, the lottery has lower impact compared to the 3-closest policy.

3.3. Data and initial policy design comparison

From a total of 152 families, 151 ranked all seven Kindergartens and only one family submitted a single Kindergarten as their preference. Table 5 shows the number of available places in each Kindergarten. Also 37 (about 24% of) children have a sibling in one of the Kindergartens.

Table 6 compares the allocations over all the policies with the submitted preferences. The listed Harku allocation does not exclude those few families who declined their assigned place. However, many (115, i.e. 76%) of the families were allocated to their most preferred Kindergarten. Since most

Table 4: Distance priorities

	absolute (DA1)	walk-zone (DA2, DA3)
k_1	$c_2 \prec c_1 \prec c_4 \prec c_3$	$c_2 \prec \{c_1, c_3, c_4\}$
k_2	$c_4 \prec c_3 \prec c_2 \prec c_1$	$\{c_2, c_3, c_4\} \prec c_1$
k_3	$c_2 \prec c_3 \prec c_4 \prec c_1$	$\{c_2, c_3\} \prec \{c_1, c_4\}$
k_4	$c_3 \prec c_2 \prec c_4 \prec c_1$	$c_3 \prec \{c_1, c_2, c_3\}$

	3-closest (DA4)	relative (DA7)
k_1	$\{c_1, c_2, c_4\} \prec c_3$	$c_1 \prec \{c_2, c_4\} \prec c_3$
k_2	$\{c_1, c_2, c_3, c_4\}$	$c_4 \prec \{c_1, c_2, c_3\}$
k_3	$\{c_1, c_2, c_3, c_4\}$	$c_2 \prec \{c_1, c_3\} \prec c_4$
k_4	$c_3 \prec \{c_1, c_2, c_4\}$	$c_3 \prec \{c_1, c_2, c_4\}$

families ranked all Kindergartens and there are more places than children, no children remained unassigned.

For policies that included lotteries, we computed averages over 20 lotteries. In the parentheses we show the standard error over the lotteries. In addition, we compared policies using a single (S) lottery for all Kindergartens or multiple (M) lotteries, one for each Kindergarten.

By using a simpler policy such as the DA1, we saw that there are fewer families receiving a place at their first choice Kindergarten[4] than with the transitory Harku priority system. Moreover, two children (about 5%) are not allocated to the same Kindergarten as their siblings with the transitory rule, but with most other policies all siblings end up in the same Kindergarten. The only exception to this is DA3, which has siblings as a second priority over walk-zone, and on average also allocated 95% of siblings in the same Kindergarten, but fewer children to their first preferences.

It seems that the transitory policy of Harku invoked the so-called *vacancy chains* (Blum et al., 1997), where at the expense of one child with a sibling several others could obtain better places along an augmenting path. In particular, by denying places for two children in the same Kindergarten as their sibling, around seven more families could obtain their first choices. This leads

[4] A more detailed allocated preference data is available in appendix B

Table 5: Harku allocation

Kindergarten	**Number of places**
A	20
B	20
C	34
D	18
E	20
F	38
G	5
Total	155

to an interesting trade-off between the goals of satisfying the sibling priority or granting the first choice of slightly more parents.

In 2016, the allocations based on policies DA5 and DA6 were exactly the same. This indicates that the gain in allocating more children to their first preference with Harku's policy is not due to allocating children to a closer Kindergarten, but due to application date and age priorities. Therefore, if these two criteria are not to be used in future policies, we expect that the rotation scheme based only on siblings and proximity will provide allocations similar to DA1, DA5 and DA6, assuming that the proportion of children and seats is similar.

3.4. Policy sensitivity to preferences

When comparing policies, one may wonder how sensitive the results are to changes in the preferences of parents. This can also be important when applying our policy recommendations in other applications. In Kindergarten allocation, and sometimes also in school choice, when the Kindergartens are more or less of the same quality, the most important factor influencing the preferences of parents is the location. Therefore, we conducted a comparative study wherein the intensities of this factor in the preferences of parents is varied. We evaluated the efficiency and fairness of the alternative policies accordingly. For the generation of preferences, we use the locations and the information on the siblings from the 2016 preference data. The detailed description of how

Table 6: Year 2016 comparison of policies using reported preferences

Policy	**Mean preference**	**First**	**Mean distance (km)**	**With siblings**
Harku	1.68	115	4.24	95 %
DA 1	1.76	110	4.26	100 %
DA 2 (M)[a]	1.85 (0.01)	98.75 (0.61)	4.59 (0.02)	100 % (0.0 %)
DA 2 (S)	1.72 (0.01)	108.05 (0.61)	4.44 (0.01)	100 % (0.0 %)
DA 3 (M)	1.83 (0.01)	98.30 (0.79)	4.51 (0.02)	95 % (0.25 %)
DA 3 (S)	1.72 (0.01)	107.75 (0.38)	4.45 (0.02)	96 % (0.3 %)
DA 4 (M)	1.91 (0.01)	89.25 (1.06)	4.53 (0.02)	100 % (0.0 %)
DA 4 (S)	1.75 (0.01)	104.85 (0.7)	4.49 (0.01)	100 % (0.0 %)
DA 5	1.76	110	4.26	100 %
DA 6	1.76	110	4.26	100 %
DA 7 (M)	1.78 (0.01)	107.60 (0.47)	4.30 (0.01)	100 % (0.0 %)
DA 7 (S)	1.76 (0.01)	107.75 (0.47)	4.31 (0.01)	100 % (0.0 %)

[a] For policies with lotteries, (M) indicates multiple tie-breaking lotteries and (S) single. The standard errors over lotteries are in parentheses.

we generate the preferences of parents can be found in the Appendix A.

We characterise preference profiles by the conditional probability of a family ranking a closer Kindergarten higher ($\Pr(r_i \succ r_j \mid d_i < d_j), i \neq j$) and ranking a Kindergarten with a sibling higher ($\Pr(r_i \succ r_j \mid s_i > s_j), i \neq j$). Where r_i is rank of Kindergarten i, d_i is distance to Kindergarten i and s_i is one when there is a sibling and zero otherwise. In the collected 2016 preference data, the $\Pr(r_i \succ r_j \mid d_i < d_j) = 0.81$ and the $\Pr(r_i \succ r_j \mid s_i > s_j) = 1.0, i \neq j$.

The main dimensions of the evaluation are the preference rank achieved in an allocation as well as the effect of the average distance from Kindergartens and the share of siblings in the same Kindergarten.

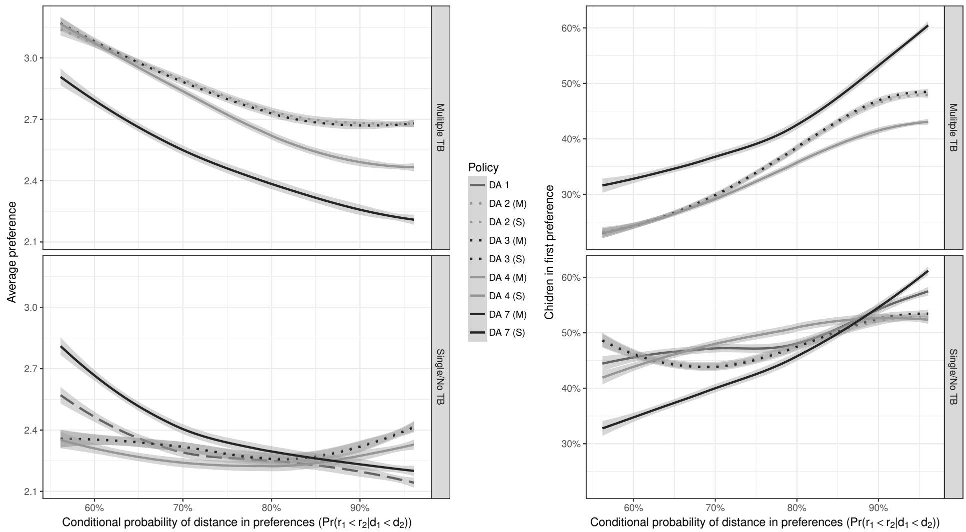
For statistical comparison, we generated twenty preference profiles of each of the parameter values. A total of 200 preference profiles were generated. For each policy that has a lottery, we ran twenty different randomised lotteries for each instance. As we saw in Table 6 the standard errors over the twenty lotteries are small. All the figures of the results show the smoothed[5] results of the ten allocations over policies with a 95% confidence bound. For policies with lotteries, there are results with a single (S) and multiple (M) lotteries over Kindergartens.

Each year the number of available Kindergarten positions varies. However, on average about 20 places should be available in each Kindergarten each year, as one group of children leaves for school. Occasionally, there might be more or fewer places. In our experiments, we set the number of available places at 20 in each Kindergarten. However, this creates additional competition and the resulting matched ranks will be lower (see Ashlagi et al., 2013a,b) in these experiments than in the actual data in Table 6. Additionally, in our interpretations we implicitly assume the effect of the competition will be similar for all the policies. We discuss here only the Deferred-Acceptance based results[6]. In addition we removed policies DA5 and DA6 from the chart, as these matchings were usually almost the same as DA1.

Figures 1a and 1b demonstrate the average preferences obtained and the proportion of families getting their first choices for all policies. Policy DA7 is the most sensitive to changes in the preferences of families. When preferences are strictly based on distance with conditional probability of $\Pr(r_i \succ r_j \mid d_i < d_j) \to 1.0$, one of the highest average rank scores is produced, one similar to

[5] smoothed with local polynomial regression

[6] In Appendix D we also provide for comparison results based on Top Trading Cycles algorithms, as defined by Abdulkadiroğlu & Sönmez (2003).



(a) Average preference (b) Proportion with first preference

Figure 1: Conditional probability of distance

other policies such as DA1, DA5 and DA6. Surprisingly, when the preferences of families are close to random, with conditional probability of $\Pr(r_i \succ r_j \mid d_i < d_j) \to 0.5$, then DA7 (S) is the policy that has one of the lowest average ranks and the lowest number of families with a first preference. Policies that do worse are the ones using multiple lotteries, one per Kindergarten. In addition, the difference of having a single or multiple lotteries for Kindergartens is not very significant for DA7, most likely due to lower usage of tie-breaking in this policy compared to others with a lottery.

At face value, DA7 seems to be the most egalitarian policy as every family has the highest priority in at least one of the Kindergartens. However, it seems that families that do not prefer to be in the closest Kindergarten tend to be rejected more often from their preferred Kindergartens further away where they have a lower priority. As the matched rank drops more in DA7 than other policies, when $\Pr(r_i \succ r_j \mid d_i < d_j) \to 0.5$. Since the preferences and priorities are not aligned, the probability of the family being rejected in some round of the process is higher. The probability of being rejected at a certain point seems to be smaller for other policies.

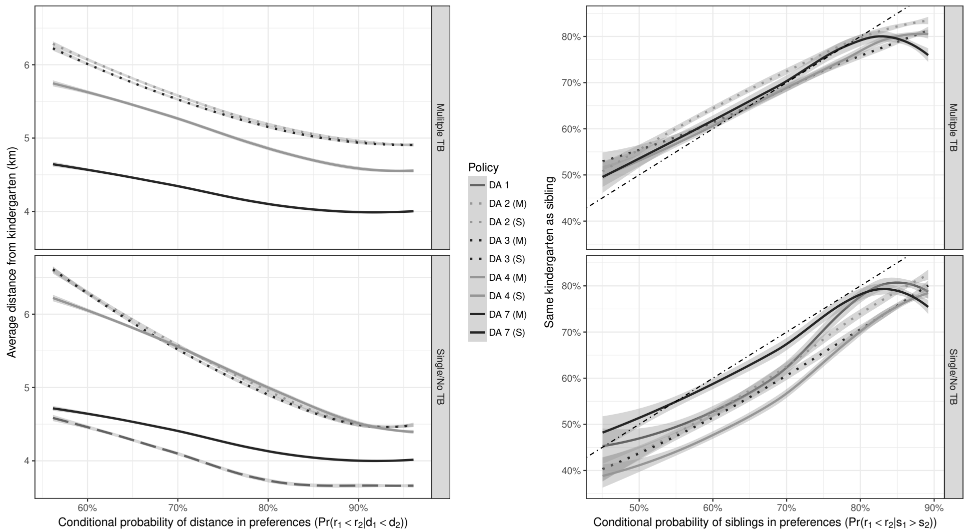
In terms of average matched preference rank, the policies DA2 and DA3 are almost indistinguishable from each other, most likely because there are too few siblings in this data. Nevertheless, it is always better to use a single rather than multiple tie-breaking lotteries for both of these policies. The average preference achieved is always better with a single lottery and also there are more families with their first preference (Figure 1b). Policies with a single lottery, such as DA2 (S), DA3 (S) and DA4 (S) – with the exception of DA7 (S) – are significantly better for families in most situations. Only when $\Pr(r_i \succ r_j \mid d_i < d_j) > 0.9$, did policies DA1, DA5 and DA6, which use absolute distance, turn out to be better than the single lottery policies.

The policies DA1 and DA6 always produce exactly the same matching, DA5 is occasionally slightly different (for about 2-6 children), but the aggregate results are still very similar. This is most likely because the selected reserve of 20 % is close to the percentage of siblings in the data.

Interestingly, most policies, with the exception of DA7, are quite robust to changes in preferences. The same proportion of families almost always receive their first preferences, about 50% to 60% with DA2, DA3 and DA4 and 60% to 70% with DA1, DA5 and DA6. There is a slight increase in the average preference when preferences become determined by distance. With DA7, the proportion varies widely between 40% and 70%, and families fare better when preferences are aligned with distance.

Figure 2a shows the average distance between families and Kindergartens. The average distance is smaller for all policies when the preferences of families are determined more by distance. As might be expected, the smallest average distance is always with DA1 (including DA5 and DA6), as these policies are aimed to minimise distance. The average distance is the largest with DA2 and DA3, policies based on walk-zones, probably caused by the randomness in the priorities of Kindergartens. Furthermore, these policies have a slightly lower average distance with a single tie-breaking lottery, when preferences are correlated with distance. On the other hand, it is usually the case that if preferences are random, the multiple tie-breaking lotteries have a lower average distance than single lotteries. A small improvement in average distance in policies with lotteries is obtained by not using discretisation by walk-zones and instead having a higher priority for a fixed number of Kindergartens, as in DA4.

With random preferences, there is a trade-off between achieved preference and average distance in the results obtained by DA7 (M) and, DA2 (M) and DA3 (M), where DA4 (M) is at the middle point among these policies in this



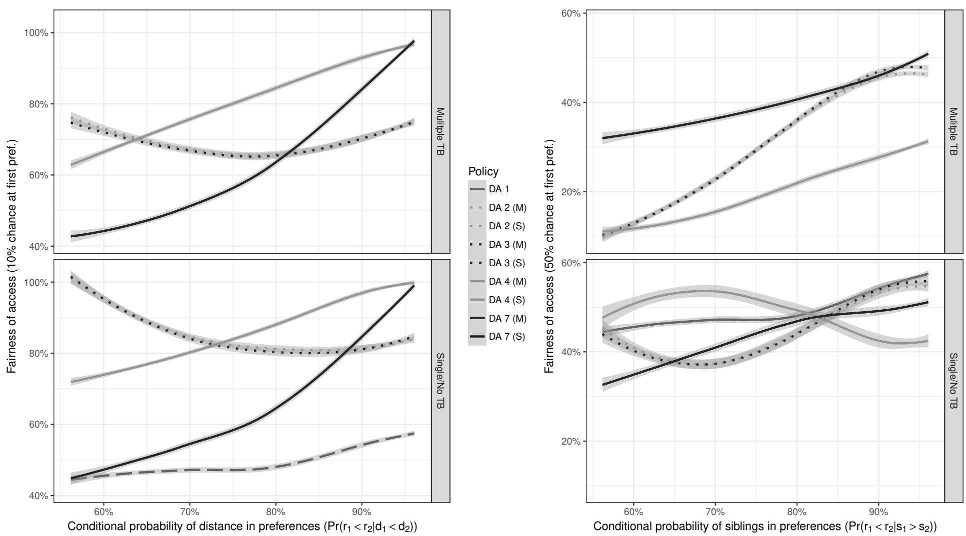
(a) (b)

Figure 2: Average distance and conditional probability of distance in preferences

aspect. Policy DA7 always achieves the lowest average distance among the lottery policies, others produce better matched ranking. When preferences are more correlated with distance, DA7 performs better according to both average preference and distance.

Figure 2b depicts the probability of children being in the same Kindergarten as their siblings. When the preferences of families are random with respect to siblings, most policies place about 40% to 60% of children in the same Kindergarten as their siblings. When families prefer closer Kindergartens, then more siblings end up in the same place. This higher percentage is most likely due to siblings already being in a nearby Kindergarten. We have also added a 45 degree line, indicating that policies that are below this level have some children, who would prefer a Kindergarten with a sibling, result in children being assigned to a different Kindergarten. Multiple lottery policies seem to be better at placing children in the same Kindergarten with siblings.

In Figures 3a and 3b, the probability of a child being matched to the family's first preference in at least one lottery is measured. This is a measure of fairness, or fair (equal) access to Kindergartens, which is similar to the



(a) (b)

Figure 3: Fairness of access

measure of access to quality used by Shi (2015). We have plotted the fairness of access for policies DA1, DA5 and DA6, even though there is no sensible interpretation, since there are no lotteries. However, these policies are still useful for comparison.

With the lottery policies DA2, DA3 and DA4, with both single and multiple lotteries, about 60% to 95% of families have about a 10% chance of being granted a place in a Kindergarten that is their first preference. The DA4 (S) is the best performer when preferences are aligned with distance and DA2 (S) and DA3 (S) when preferences only have a Kindergarten effect. Policy DA7 (S) comes close to DA4 (S) only when preferences are almost perfectly aligned with distance.

However, when we make our fairness notion slightly stronger, i.e. when there has to be at least a 50% chance of a place in the family's first choice Kindergarten, the proportion of families achieving this drops to only about 40%. This is even lower than the case with deterministic policies like DA1. Therefore, it seems that with lotteries we can give some families a small 10%, chance of getting their first preference, but as a result, some families lose their first preferences. With a larger chance, 50%, there are more families losing their first preference than those gaining.

In terms of trade-offs, the policy DA4 (S) is better on fairness and average matched preference, but worse on average matched distance. DA1 and similar policies do better on average matched rank and distance, however they fare worse on fairness, i.e. families living far away from all Kindergartens have a smaller chance of a preferred match. When preferences are not entirely determined by distance, then these two (DA1 and DA4) are the best options to choose from. However, with distance-based preferences, DA7 can prove to be an improvement. In the case of DA7, fairness is almost as good as with DA4, average distance was a significant improvement over DA4 and average allocated rank very close to DA1.

4. FURTHER ISSUES

We identify a dozen additional special features that should be further considered in the (re-)design of the mechanism in Harku. However, many of these features may pose significant challenges and require additional research. We describe these issues and give recommendations for possible adjustments in the allocation mechanism.

Children with special needs. In larger cities, there are schools for children with special needs, but in smaller municipalities these pupils are mixed with others. The standard practice is for Kindergartens to reserve places for children with special needs who require more attention and are thus considered to take up the space of three children. Usually, it is not known beforehand if there will be any such cases and the special needs may only become evident later. However, in most cases the extra places remain free and can be subsequently allocated to other children. Obviously, this has some effect on the fairness of the allocation.

A possible solution would be to have this data available before allocation and to take it into account in the allocation process. However, evaluating all of the applicants in advance could be very costly compared to the extra efforts needed for the reallocation process and the potential issues arising from the extended solution. It would be helpful if the parents of children that are likely to need special treatment were to register for evaluation. It should then be guaranteed that their chances of admission to their preferred Kindergartens would not be worsened, perhaps by giving priority for a number of places in each Kindergarten to such children.

Allocation in multiple rounds. Harku currently allocates students in multiple rounds, since two extra places could arise in each Kindergarten to which no student with special needs is admitted. The proportion of disadvantaged families is about 10% (PTVRR, 2015) and children with special needs make up about 3% (Paat et al., 2011). This question is similar to the question of the design of two-stage allocation mechanisms (Dur & Kesten, 2014) and also to the design of appeal processes (Dur & Kesten, 2015). The first option is to allocate the extra places exclusively among the unmatched children. This is a simple method with no reallocation of children, but it can be seen as unfair to families who are allocated seats in the main round and would prefer an extra place in a Kindergarten where they have higher priority than the unallocated children who get those extra places in the second round. The final solution could cause justified envy for the families. In addition, the parents might also act strategically in the main round, perhaps by not accepting an offer from the Kindergarten listed second, especially if they have information that they are first in the waiting list and the creation of extra places is very likely. Therefore, it appears reasonable to let everyone apply for the extra places, as is currently done in Harku. However, if the process is not centralised, then those who were assigned a place in the first round but now get a better match, would consequently create new available places. Even if this decentralised process could be continued until a stable solution was reached, this proposal-rejection chain would result in a stable matching that is the worst possible stable matching for the reallocated children, as proved by Blum et al. (1997). Therefore, this process would not be strategy-proof for the parents either. Hence, the only possible solution that is strategy-proof for the parents and avoids justified envy is a centralised second round, where parents can re-apply to all Kindergartens with the option of keeping their assignment if they wish to do so (technically this is achieved by putting the children already assigned to the Kindergarten at the top of the Kindergartens' rankings). Yet, this solution may affect a significant number of children, and in theory possibly all of them, which could result in high reallocation costs. These costs would be accepted by the parents, since they would always have the option of not changing their assignment, but could be seen as undesired by the local council and the Kindergartens.

Children with existing places. The parents of some children may request a transfer. This is especially relevant for children attending a class for 2-3-year olds who would like to go to a different Kindergarten for the 3-6-year period, since the classes for children aged 2-3 may not be available in the

Kindergartens preferred by the families. It is therefore a question of whether the reallocation of these children should be conducted as part of the yearly matching round. If so, these children should be guaranteed to get at least as good a seat in the reallocation, i.e. they should have the highest priority in their current Kindergarten. This question has been studied in the context of Danish daycare allocation (Kennes et al., 2014), and also for the reallocation of French teachers (Combe et al., 2015).

Overlapping admission processes. Some parents may be registered in more than one municipality, so they are able to apply for a place for their child in two systems, for example in Harku and in neighbouring Tallinn. This can lead to inefficiencies due to cancellations. Similar problems arise in some US cities where state schools and charter schools hold their admissions separately. Furthermore, the same phenomenon has also appeared in European college admission programmes, where an increasing number of students are applying for programmes in several countries disturbing the national matching schemes.

Outside options with subsidies. Somewhat related to the previous issue is the fact that private Kindergartens operate in Estonia, and some parents also consider the option of home schooling. However, if a municipality cannot provide enough Kindergarten places for its resident population, in some cases it may subsidise parents who choose an alternative option. In Harku, the local council financially supports parents who do not receive a place in a Kindergarten, but the council may withdraw their support if the parents do not accept a place that is offered. This conditional support can lead to strategic considerations, since some parents may find an alternative home or private option preferable to a local school, if and only if they receive the financial support, but this cannot be stated in the application. This special case can be modelled with the matching with contracts framework. A similar special feature is found in the Hungarian higher education matching scheme, where students can study on the same course under two different contracts, either free of charge or with a tuition fee. Furthermore, US cadets (Sönmez & Switzer, 2013) also face such a situation when they decide whether or not they are willing to take on some extra years of service in order to increase their chances of admission. The recommended solution is to let the parents list the option of not having a place in the Kindergarten but receiving financial support instead, when they give their applications to Kindergartens. Thus, all the listed options are considered preferable to the outside option with no financial support. In such a case, it is crucial that the parents-optimal stable solution is implemented so as to make the parents reveal their true preferences for these outside options.

Lower quotas, opening of new groups. Sometimes Kindergartens are able to cancel groups or open new ones to fit with the applications. In particular, there are regulations determining the minimum number of children needed to start a new group. This feature is similar to the lower quotas used in the Hungarian higher education matching scheme (Biró et al., 2010a), where programmes may be cancelled if there is a lack of students. This is a natural requirement that makes the education service economical, but the theoretical model for college admissions with lower quotas is not always solvable. This means that a fair solution does not always exist and the problem of finding a fair solution is NP-hard. The problem becomes even more complicated if new groups can be created, since both the closures and the openings in a Kindergarten affect the number of students admitted elsewhere. However, clever heuristics and robust optimisation techniques, such as integer programming (Biró et al., 2014) can be used to tackle these generalised problems.

Homogeneous age groups and mixed groups. In Estonia, there are both homogeneous age groups and mixed groups. Having only same age groups can vary the number of groups opened in a Kindergarten, as a Kindergarten with five groups could open only one group every three years. This would be unsatisfactory for the local children in the years when no groups are opened. When mixed groups are created, the number of children admitted can be relatively stable if the available places are always filled. However, if there are some free places left in a year, then the age distribution of the children can be distorted.

Sharing places. In some Kindergartens, it is possible that some children only attend part of the week and the rest of the time is taken up by other children. This possibility again makes the underlying problem challenging to solve. Specifically, when there is a large number of part-time students then one might face the same problem as when allocating doctors and couples to hospitals, which is an NP-hard problem (McDermid & Manlove, 2010).

Historic dependence of preferences. In Harku, the applications of registered parents are listed on a public website. In Tallinn, the number of applications already submitted to the Kindergartens is also published. If the registration date is a criterion for priority and the parents can see the applications or the number of applications made before their turn, then this can affect their true as well as their submitted preferences. Potentially, if there are more applications than places, then parents will find it risky to apply. This can depend on the birth date of the child, because if a child was born soon after 1 October, then the parents could have a good chance of obtaining a place everywhere, and so be more truthful. We did not find much evidence of significant changes in the preferences over time in the Harku data. However, in a similar study for Tallinn or other places where the registration date is important, attention should be paid to the potentially biased preferences caused by the published information about past applications.

Smooth transition to a new system. When designing the new mechanism, it may be important to consider how to engineer a smooth transition between the old and the final systems. This process is especially challenging in Harku, since the old priorities were based on registration date, and those parents who registered early may see it as unfair if this priority that they earned in the past is suddenly neglected. Therefore, in the 2016 transitory system, the priority of those who have already registered in the old regime is partly kept, as described above. Regarding the future years, how long these priorities should be kept, or whether they should be replaced with some age priority which is in correlation with the registration dates, is still to be debated.

The role of the heads of the Kindergartens. The heads of the Kindergartens were actively involved in the allocation system until 2015. The discussions among the heads and the personal communication with the parents were crucial in eliciting the true preferences of the parents and finding relatively good solutions through informal negotiations. In the centrally coordinated system, the head may fear losing their chance to influence the allocations, and the same could be true for the employees of the local municipality. It should be considered whether the heads of the Kindergartens could still have some power to adjust the priorities, or to make other decisions about their Kindergartens, for instance whether to open a new group or to create mixed groups.

The fairness of using proximity as a priority. Whether the use of proximity is fair may depend on the ease and/or cost of registering. Specially circumstances may vary, e.g., it is almost costless (as in Hungary); there may be some significant costs such as renting or having a flat in the area; or the family truly has to live there (for example in Barcelona, where somebody who is proved not to live at the stated address can lose their place). When it is easy to register at an address, then the parents may play a strategic game in which the first stage is to choose an address. When ownership and actual residency are required, and the priorities are important for the parents, this can affect the housing choices of the families, and influence house prices as well as the socio-economic distribution of the population.

Restricting the choice of the parents. A simple restriction is to allow families to only apply to nearby Kindergartens. A more sophisticated method is to provide personalised choice menus, such as the system proposed in the Boston school choice mechanism (Shi, 2015). This would potentially provide parents with a choice of schools close to them where a child's siblings may have attended, with a limited number of further options. The advantage of this method over restricting the number of applications is that the mechanism remains strategy-proof, and the parents have a simpler task of ranking the available options. However, the disadvantage is the difficulty of estimating the preferences of the parents and therefore, there is a risk that some highly preferred Kindergartens could be missed out from some menus. In general, this type of restrictive policy can improve the overall quality of the allocation from the point of view of the municipality, perhaps by reducing the total travel distance. That was the main motivation in the Boston school choice redesign, as the bus costs had to be limited. However, the overall welfare of the children could be badly affected. We do not recommend this policy for Harku, due to the small size of the municipality, but it is suggested for consideration in larger cities, like Tallinn.

5. CONCLUSION AND DISCUSSION

We have reviewed the Kindergarten matching practices in one Estonian municipality, Harku. Until 2015, the collected preferences were unlikely to reflect the true preferences of the parents, since the data were out-of-date by the time of the allocation, the number of applications were limited and the allocation mechanism was not incentive-proof either. Therefore, the resulting allocation could create justified envy and it was also lacking transparency. In 2016, the municipality changed its allocation system mostly based on our recommendations.

In our study, we first listed well-known practices from matching mechanism design that present solutions to some of the problems and also provide policy tools for the local municipalities. These practices consist of:

- getting complete rather than limited preferences from families,
- using child-proposing stable matching for allocating places,

- defining clear policies for the local municipality based on a transparent priority system.

In assisting in the redesign of the allocation mechanism, it emerged that although the policy goals might be clear, the choice of exactly which implementation method to use can create significant differences in the results. In most cases, the goals of the local municipalities are to have siblings in the same Kindergarten and to provide a place in a Kindergarten close to home, in addition to the main consideration of providing a place in the most preferred Kindergartens of the families. We evaluated seven different policies for implementing the policy goals, first based on data from 2016, and then based on generated data. The 2016 transitory system that follows our main recommendations provides a child-optimal stable allocation under a rotational priority structure based on four factors, such as location, siblings, registration and birth dates. The limit on the number of applications was also removed, so the preferences of the families can be considered truthful. Our main findings regarding the seven policies evaluated on the real data and in the computational experiments are summarised below.

The simplest policy is to give higher priority to children with siblings and to families living nearby, which is policy DA1. This was also demonstrated to be one of the most effective policies. The resulting allocation had, on average, matched a lot of families with their most preferred Kindergarten, while also having one of the smallest average distances. This remained true when the preferences of families were agnostic about distance.

Policy DA1 might occasionally seem unfair, as small differences in distance might affect whether families are placed in their first preference or a lower one. Policies DA2, DA3 and DA4 group Kindergartens by distance within equal priority classes, DA2 and DA3 by defining a walk-zone and DA4 by having high priority in the three closest Kindergartens. Families in the walk-zone are treated equally and priorities are defined by lottery. It appears that the multiple tie-breaking rule might create a more egalitarian access to Kindergartens, however it is not without its cost. The average number of children who are placed in their most preferred Kindergarten is usually significantly lower and the average distance is greater. However, with a single tie-breaker over Kindergartens, families are on average allocated to their more preferred Kindergarten, even when compared to deterministic policies like DA1. Nevertheless, an allocation based on randomness might prove hard to justify to families. If having more egalitarian access is important, policy DA4 with a single tie-breaker would be the best of the three. The level of fair access is the same, satisfaction with average preferences is the best, and distance is the lowest.

Siblings always being given higher priority might prove another source of seemingly unfair treatment. If a family already has a child in a particular Kindergarten, they are almost guaranteed to get a place in the same Kindergarten for a sibling, even when there is another family living closer than them. We considered two policies, DA5 and DA6, which limit the number of places in a Kindergarten that consider having a sibling a priority at up to 20%. Even though the number of places reserved for siblings was low, most families still received a place in that Kindergarten if they preferred it. There is almost no difference from policy DA1 on any measure, nor between DA5 and DA6, although theoretically DA6 should provide more opportunity to nearby families, and DA5 to children with siblings.

A clear oddity is policy DA7, which was initially designed to deliver more equal access to Kindergartens for families who live far away from all Kindergartens. While policy DA1 would give such families low priority everywhere, DA7 would give them the highest priority in their closest Kindergarten. When most families have a high preference for nearby Kindergartens and for those where their siblings are, DA7 results in one of the best policy designs in all aspects. DA7 gives many families their first preference, it has the shortest average distance and one of the best results for equality of access. However, the result is radically different when family preferences are mostly idiosyncratic and are almost independent from distance. In this case, DA7 is the worst policy of all for families. On average less than 40% of children get matched to their first preferences, but the average distance is the one of lowest. Thus the lesson from policy DA7 seems to be that the policy designer needs to predict the preferences of the society fairly accurately to select a good trade-off. When preferences and priorities are aligned, both of the main goals can be met. A downside of this policy is that it is vulnerable when preferences and priorities are misaligned, and then the price paid is significant in terms of efficiency and fairness. If a local municipality aims to minimise the distances between homes and Kindergartens, then DA1 is the best option. The latter objective recently turned out to be crucial in Boston, where the local authority became concerned about the busing costs (Shi, 2015).

Finally, there remain several unsolved issues that we have not tried to address in the redesign. A dozen issues were listed along with a discussion about possible solutions. For example, it would be reasonable to coordinate the allocation between neighbouring municipalities, but cooperation is usually hard to achieve. Similarly, it would be best to know about children with special needs before the allocation, but this is often not feasible.

A potential way to manage the shortage of Kindergarten places is to provide monetary incentives for parents to stay at home with their children or to seek a place in private childcare. The question of how to set this monetary compensation in an optimal manner is also interesting in terms of future research. Here, optimality could mean minimising the total cost of providing childcare services in the municipality.

There remain a few interesting aspects related to designing a more flexible mechanism which might improve the allocation outcome for families. Making decisions on the size and the age composition of the groups in Kindergartens and determining this in an optimal way based on the application data could give an additional boost to the number of families receiving a place at their most preferred Kindergarten. Some of this research has been done in terms of lower quotas for opening groups (Biró et al., 2010a).

A. GENERATING COUNTER-FACTUAL PREFERENCES

We use the 2016 data for counter-factual policy evaluation. To generate the counter-factual preferences only we use the distance between homes and Kindergartens and sibling status in a Kindergarten. The collected preference data is used to understand which features to use in the ranking function, the functional form of the utility function and the fixed effects of Kindergartens.

For each family and Kindergarten we know the geographical location from address lookup from google maps[7] and Estonian Land Board (Maa-amet[8]) and distance calculations taken from Google maps distance[9]. We have a rich dataset for distance, as for each family-Kindergarten pair we know the driving and walking distances in kilometres and minutes. We also have the direct distance between the two points calculated with the haversine formula. The features are described in Table 7.

We fit a multinomial rank-ordered logit model (Croissant, 2011), which is similar to the model used by Shi (2015). The model assumes that families

[7] https://developers.google.com/maps/documentation/geocoding/intro
[8] http://inaadress.maaamet.ee/geocoder/bulk
[9] https://developers.google.com/maps/documentation/distance-matrix/intro

Table 7: Family's Kindergarten features

Feature	Description
preference rank	Families rank of the Kindergarten, between 1-7
walking_distance_sec	walking time between family's home and Kindergarten, based on The Google Maps Distance Matrix API (2015)
walking_distance_m	walking distance between family's home and Kindergarten, based on The Google Maps Distance Matrix API (2015)
driving_distance_sec	driving time between family's home and Kindergarten, based on The Google Maps Distance Matrix API (2015)
driving_distance_m	driving distance between family's home and Kindergarten, based on The Google Maps Distance Matrix API (2015)
haversine_distance_m	direct distance between family's home and Kindergarten
walking_distance_rank	Kindergarten rank by walking distance
driving_distance_rank	Kindergarten rank by driving distance
haversine_distance_rank	Kindergarten rank by haversine distance
sibling	1 if Kindergarten has a sibling already attending, 0 otherwise
log_walking_distance_sec	$\log(walking\_distance\_sec)$
sqrt_walking_distance_sec	$\sqrt{walking\_distance\_sec}$
log_walking_distance_m	$\log(walking\_distance\_m)$
sqrt_walking_distance_m	$\sqrt{walking\_distance\_m}$
log_driving_distance_sec	$\log(driving_{d}istance_{s}ec)$
sqrt_driving_distance_sec	$\sqrt{driving\_distance\_sec}$
log_driving_distance_m	$\log(driving\_distance\_m)$
sqrt_driving_distance_m	$\sqrt{driving\_distance\_m}$
log_haversine_distance_m	$\log(haversine\_distance\_m)$
sqrt_haversine_distance_m	$\sqrt{haversine\_distance\_m}$

have an utility function of the form,

$$u_{ij} = \alpha_j + \sum_k \beta_k \cdot x_{kij} + \varepsilon_{ij} \tag{1}$$

where α_j are fixed effect of Kindergartens, β_k is the coefficient for feature k and ε_{ij} is the family's personal unexplained preference. We further use the utilities to find a probability if a ranking. In a ranked-order logit model the probability of a ranking is a multiple of a Kindergarten begin is a particular position, which in our case is $Pr(ranking 1,2,...,7) = Pr(ranking = 1) \cdot Pr(ranking = 2) \cdot ... \cdot Pr(ranking = 7)$. The probability of family i ranking Kindergarten j at some position are,

$$\begin{cases} Pr_{ij}(ranking = 1) = \frac{e^{u_{ij}}}{\sum_{r=1}^{7} e^{u_{ir}}} \\ Pr_{ij}(ranking = 2) = \frac{e^{u_{ij}}}{\sum_{r=2}^{7} e^{u_{ir}}} \\ ... \\ Pr_{ij}(ranking = 6) = \frac{e^{u_{ij}}}{\sum_{r=6}^{7} e^{u_{ir}}} \end{cases} \tag{2}$$

First our aim is to select one of the distance metrics from Table 7 to include in the utility model (1). For this we do 100 bootstrap runs with each metric. In Figure 4 we plot the resulting log-likelihood with its standard error. We see that the $\sqrt{driving\_distance\_sec}$ provides the best prediction on average. We also see that including the sibling status would improve the prediction accuracy, however the statistical significance of the coefficient is low (Table 8) in any combination of features. So we select the model (1) from Table 8 as our final model.

For policy comparison we generate the ranking over all Kindergartens. We do not model the cut-off levels for outside options, when the family would rather keep the child at home. We assume they would always rather have a place in any of Harku's Kindergartens.

To obtain a full ranking of Kindergartens we use the probabilities from (2). For counter-factual preferences we vary the coefficient for distance. The parameter values are in (3). For each combination of parameters we generate several (7) different preference profiles and evaluate the policies on the average over all the preference profiles.

$$\beta_1 \in \{0.0, 0.05, 0.1, 0.23, 0.25, 0.5, 1, 2, 4, 10\} \tag{3}$$

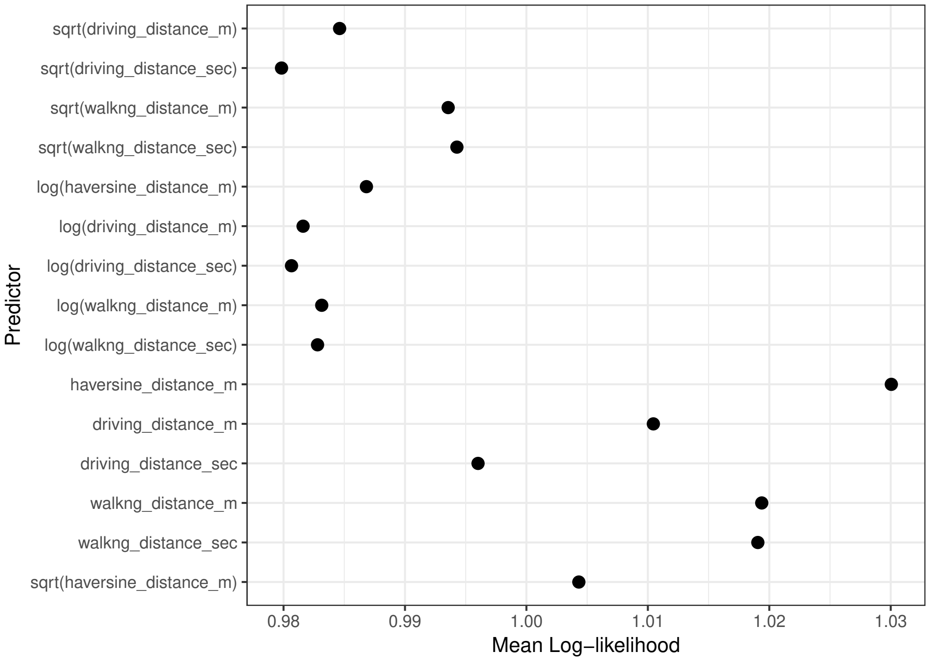


Figure 4: Predictive features

Table 8: Rank-ordered logit coefficients

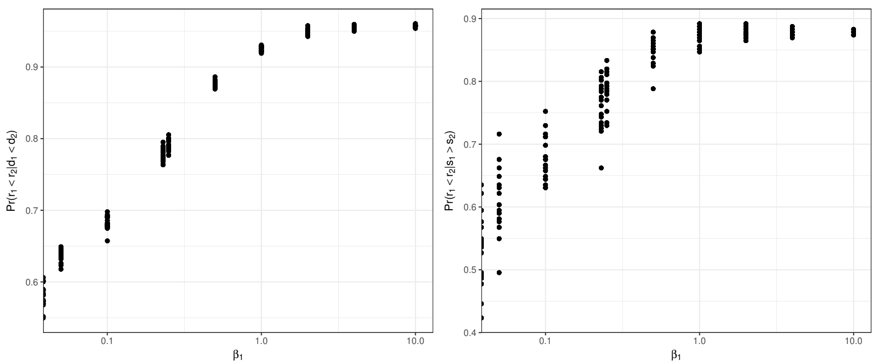
	preference rank		
	(1)	(2)	(3)
α_B	-0.690^{***}	-0.685^{***}	-0.560^{***}
	(0.150)	(0.150)	(0.143)
α_C	-0.565^{***}	-0.540^{***}	0.471^{***}
	(0.173)	(0.176)	(0.145)
α_D	0.157	0.185	1.479^{***}
	(0.176)	(0.182)	(0.154)
α_E	0.476^{***}	0.500^{***}	1.187^{***}
	(0.156)	(0.159)	(0.146)
α_F	0.275	0.351^{*}	1.608^{***}
	(0.176)	(0.181)	(0.153)
α_G	-1.769^{***}	-1.789^{***}	-1.580^{***}
	(0.193)	(0.195)	(0.179)
β_1	-0.229^{***}	-0.220^{***}	
$\sqrt{driving\_distance\_sec}$	(0.015)	(0.015)	
β_2		20.750	20.812
sibling		(2,676.852)	(1,651.629)
Observations	906	906	906
Log Likelihood	−882.862	−840.256	−958.955
Note:			*p<0.1; **p<0.05; ***p<0.01

To better interpret the results we look at the results by conditional probabilities of a parameter set. We look at two conditional effects: (a) probability of ranking Kindergarten higher given it is closer and (b)probability of ranking a Kindergarten higher given a Kindergarten has a sibling. Formally the conditional probability are defined in (4) and (5).

$$\Pr(r_1 < r_2 \mid d_1 < d_2) = \frac{\Pr(d_1 < d_2, r_1 < r_2)}{\Pr(d_1 < d_2)} \tag{4}$$

$$\Pr(r_1 < r_2 \mid s_1 > s_2) = \frac{\Pr(s_1 > s_2, r_1 < r_2)}{\Pr(s_1 > s_2)} \tag{5}$$

The mean conditional probability with fitted regression parameter, $\beta = 0.25$, is $\Pr(r_1 < r_2 \mid d_1 < d_2) \approx 0.79 \pm 0.02$[10]. This is similar to what we observe in the 2016 data, where $\Pr(r_i \succ r_j \mid d_i < d_j) = 0.81$, $i \neq j$. Figure 5a shows the relationship between the logistic parameters and the conditional probabilities.



(a) Conditional probability on distance (b) Conditional probability on siblings

Figure 5: Coefficients and conditional probabilities

[10] 1.96 standard deviations, 95% probability

B. ALLOCATED PREFERENCES

Table 9: Year 2016 allocated preference comparison DA

Policy	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**
Harku	115	14	6	3	4	8	2
IA 1	122	8	2	5	8	3	4
DA 1	110	17	3	6	6	9	1
DA 2 (M)[a]	98.75	19.95	9.65	11.05	7.20	4.85	1.22
	(0.61)	(0.61)	(0.49)	(0.78)	(0.43)	(0.36)	(0.15)
DA 2 (S)	108.05	19.90	4.65	4.95	7.70	5.85	1.20
	(0.61)	(0.56)	(0.41)	(0.39)	(0.37)	(0.36)	(0.11)
DA 3 (M)	98.30	21.95	8.85	9.9	8.35	3.75	1.5
	(0.79)	(1.09)	(0.6)	(0.49)	(0.3)	(0.24)	(0.29)
DA 3 (S)	107.75	20.65	4.79	4.95	6.60	6.2	1.86
	(0.38)	(0.38)	(0.50)	(0.29)	(0.39)	(0.3)	(0.25)
DA 4 (M)	89.25	27.2	13.1	9.85	7.7	4.35	1.38
	(1.06)	(0.84)	(0.88)	(0.76)	(0.53)	(0.43)	(0.26)
DA 4 (S)	104.85	19.15	7.05	8.15	8.05	4.10	1.63
	(0.70)	(0.70)	(0.63)	(0.6)	(0.46)	(0.28)	(0.26)
DA 5	110	17	3	6	6	9	1
DA 6	110	17	3	6	6	9	1
DA 7 (M)	107.6	15.85	5.7	8.9	6.95	5.9	1.47
	(0.47)	(0.52)	(0.36)	(0.42)	(0.33)	(0.32)	(0.17)
DA 7 (S)	107.75	16.90	5.20	7.40	8.40	5.15	1.33
	(0.47)	(0.44)	(0.26)	(0.36)	(0.37)	(0.33)	(0.14)

[a] For policies with lotteries in parenthesis are the standard errors

Table 10: Year 2016 allocated preference comparison TTC

Policy	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**
Harku	115	14	6	3	4	8	2
IA 1	122	8	2	5	8	3	4
TTC 1	112	16	1	5	9	8	1
TTC 2 (M)	110.25 (0.52)	17.85 (0.49)	4.00 (0.32)	6.55 (0.41)	7.75 (0.45)	4.70 (0.52)	1.38 (0.21)
TTC 2 (S)	110.10 (0.45)	17.50 (0.56)	3.85 (0.33)	6.40 (0.56)	7.95 (0.36)	4.80 (0.35)	2.00 (0.23)
TTC 3 (M)	109.50 (0.54)	18.95 (0.72)	4.55 (0.34)	5.75 (0.37)	7.00 (0.40)	5.20 (0.28)	1.50 (0.25)
TTC 3 (S)	110.55 (0.61)	17.05 (0.51)	4.65 (0.36)	6.20 (0.52)	7.05 (0.39)	5.75 (0.35)	1.25 (0.13)
TTC 4 (M)	109.25 (0.56)	18.65 (0.74)	3.78 (0.33)	7.40 (0.62)	8.80 (0.57)	3.60 (0.36)	1.80 (0.29)
TTC 4 (S)	109.35 (0.38)	18.75 (0.48)	4.47 (0.42)	7.30 (0.37)	7.45 (0.45)	4.05 (0.25)	1.31 (0.17)
TTC 7 (M)	109.40 (0.39)	15.45 (0.37)	3.75 (0.28)	7.45 (0.39)	8.55 (0.36)	5.70 (0.25)	1.89 (0.23)
TTC 7 (S)	109.25 (0.41)	16.00 (0.45)	3.63 (0.34)	7.25 (0.37)	9.15 (0.33)	5.60 (0.29)	1.53 (0.12)

C. MAP OF THE MUNICIPALITY

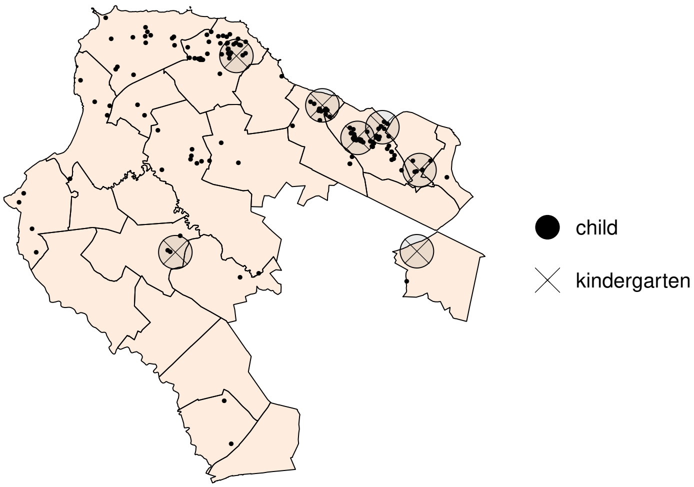


Figure 6: Locations of children and Kindergartens (with walk-zones) in 2016

D. RESULTS WITH TOP TRADING CYCLES (TTC)

Table 11: Year 2016 comparison of policies using reported preferences

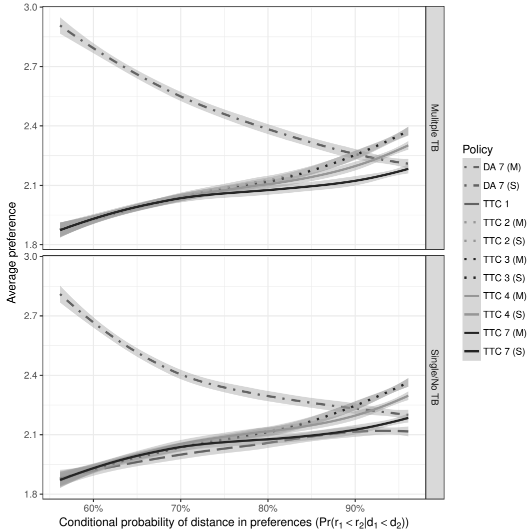
Policy	Mean preference	First	Mean distance (km)	With siblings	Frac. children with JE[a]	BP per child with JE
DA 1	1.76	110	4.26	100 %		
DA 2 (M)[b]	1.85 (0.01)	98.75 (0.61)	4.59 (0.02)	100 % (0.0 %)		
DA 2 (S)	1.72 (0.01)	108.05 (0.61)	4.44 (0.01)	100 % (0.0 %)		
DA 3 (M)	1.83 (0.01)	98.30 (0.79)	4.51 (0.02)	95 % (0.25 %)		
DA 3 (S)	1.72 (0.01)	107.75 (0.38)	4.45 (0.02)	96 % (0.3 %)		
DA 4 (M)	1.91 (0.01)	89.25 (1.06)	4.53 (0.02)	100 % (0.0 %)		
DA 4 (S)	1.75 (0.01)	104.85 (0.7)	4.49 (0.01)	100 % (0.0 %)		
DA 7 (M)	1.78 (0.01)	107.60 (0.47)	4.30 (0.01)	100 % (0.0 %)		
DA 7 (S)	1.76 (0.01)	107.75 (0.47)	4.31 (0.01)	100 % (0.0 %)		

[a] JE - Justified Envy, BP - Blocking Pairs
[b] For policies with lotteries, (M) indicates multiple tie-breaking lotteries and (S) single. The standard errors over lotteries are in parentheses.

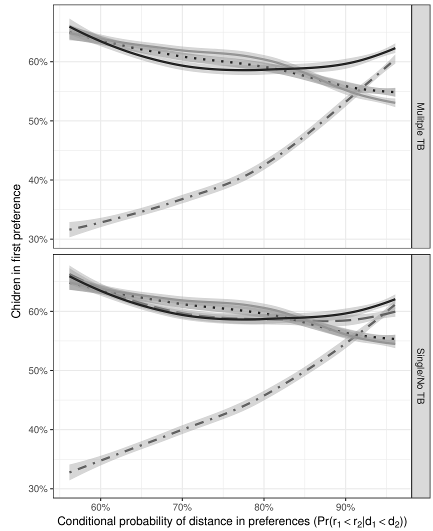
Table 12: Year 2016 comparison of policies using reported preferences

Policy	Mean preference	First	Mean distance (km)	With siblings	Frac. children with JE[a]	BP per child with JE
TTC 1	1.76	112	4.39	100 %	11 %	1.06
TTC 2 (S)	1.71 (0.01)	110.1 (0.45)	4.49 (0.02)	100 % (0.0 %)	9 % (1.28 %)	1.08 (0.03)
TTC 2 (M)	1.69 (0.01)	110.25 (0.52)	4.51 (0.03)	100 % (0.0 %)	21 % (0.82 %)	2.16 (0.03)
TTC 3 (S)	1.7 (0.01)	110.55 (0.61)	4.47 (0.02)	96 % (0.37 %)	14 % (1.24 %)	1.27 (0.06)
TTC 3 (M)	1.69 (0.01)	109.5 (0.54)	4.46 (0.02)	95 % (0.33 %)	22 % (0.71 %)	2.16 (0.04)
TTC 4 (S)	1.69 (0.01)	109.35 (0.38)	4.6 (0.02)	100 % (0.0 %)	20 % (1.08 %)	1.86 (0.05)
TTC 4 (M)	1.7 (0.01)	109.25 (0.56)	4.58 (0.02)	100 % (0.0 %)	26 % (0.52 %)	2.26 (0.04)
TTC 7 (S)	1.77 (0.01)	109.25 (0.41)	4.48 (0.01)	100 % (0.0 %)	22 % (0.82 %)	1.70 (0.06)
TTC 7 (M)	1.78 (0.01)	109.4 (0.39)	4.51 (0.02)	100 % (0.0 %)	23 % (0.83 %)	1.90 (0.07)

[a] JE - Justified Envy, BP - Blocking Pairs

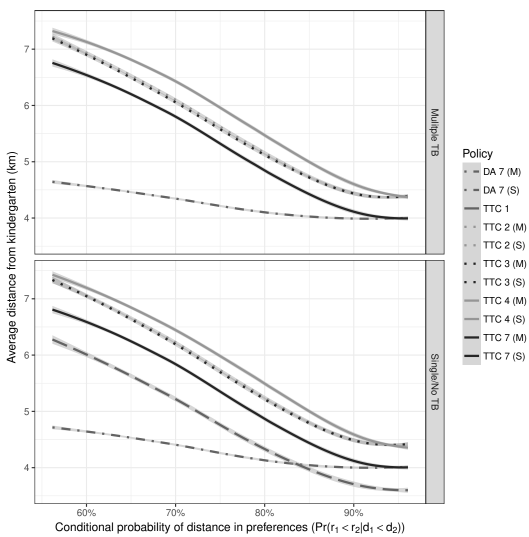


(a) Average preference

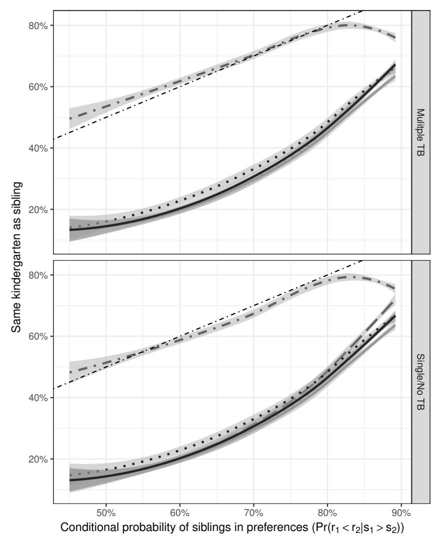


(b) Proportion with first preference

Figure 7: Conditional probability of distance (TTC)

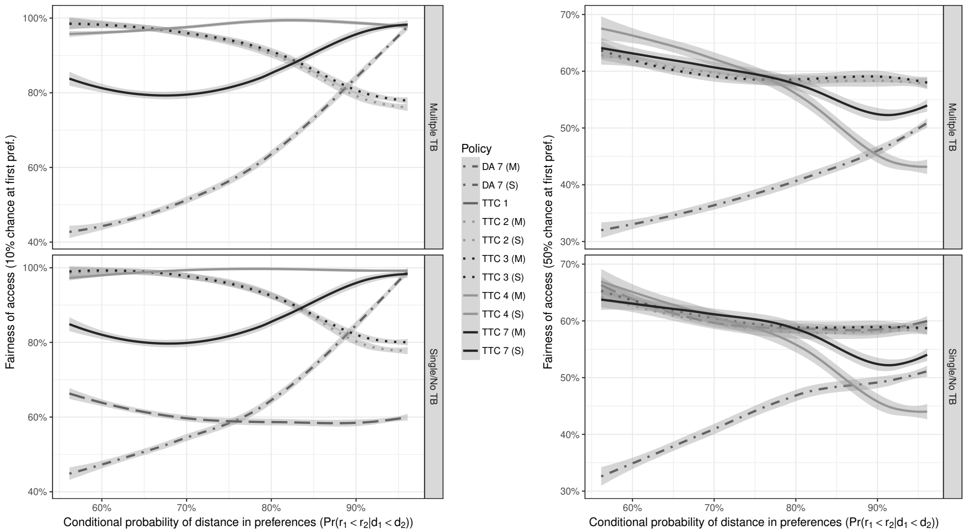


(a)



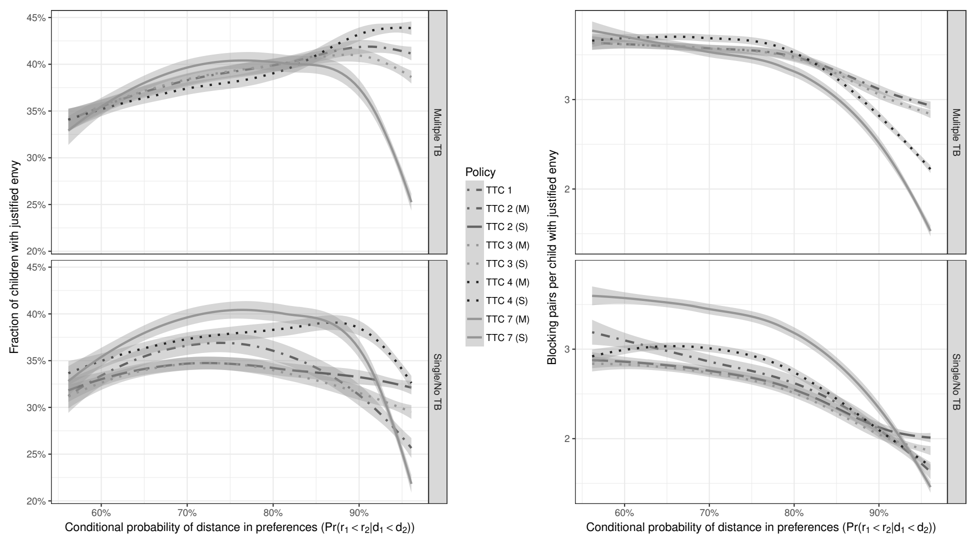
(b)

Figure 8: Average distance and conditional probability of distance in preferences (TTC)



(a) (b)

Figure 9: Fairness of access



(a) (b)

Figure 10: Justified envy and blocking pairs (TTC)

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ON THE PARTNERSHIP FORMATION PROBLEM

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ABSTRACT

In this paper we re-examine the partnership formation problem, which is a generalization of the classical assignment game. We show that the former problem can be transformed into the latter one in some sense; more precisely, we demonstrate that using an equilibrium in an associated assignment game, we can find an equilibrium in the partnership formation problem if it exists. Based on this, we devise an algorithm to compute an equilibrium of the partnership formation problem, and show that the proposed algorithm can be seen as a variant of the one by Andersson et al. (2014b).

Keywords: Partnership formation, equilibrium, assignment game, adjustment process.

JEL Classification Numbers: C71, D44.

1. INTRODUCTION

In this paper we reconsider the partnership formation problem as studied by Talman & Yang (2011). In such a problem, there is a group of agents, and each agent works alone or works together with another agent. If an agent works alone, then the agent generates a value for himself, and if an agent works with a partner, then the agent and his partner generate a joint value, which is shared by them in an appropriate way. The goal of the partnership formation problem is to find an equilibrium, where no agent has incentive to change his partner, to break up an existing partnership to become alone,

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or to form a new partnership. Typical instances of the partnership formation problem can be found in the professional tennis tournament, pair programming in software development, etc. (see, e.g., Andersson et al. (2014b); Eriksson & Karlander (2001); Talman & Yang (2011)). Similar but different models of the partnership formation are also discussed in Chiappori et al. (2014); Alkan & Tuncay (2013). The partnership formation problem has been also formulated as the roommate problem with transferable utility, the matching game, or the one-sided matching problem (Eriksson & Karlander, 2001; Biró et al., 2012; Klaus & Nichifor, 2010).

The partnership formation problem is closely related to the classical assignment game (Koopmans & Beckmann, 1957; Shapley & Shubik, 1971). The assignment game can be regarded as a special case of the partnership formation problem, where the set of agents is partitioned into two groups, the one corresponding to buyers (or firms) and the other to sellers (or workers), and any two agents in the same group cannot be a pair. An equilibrium always exists in the assignment game (Koopmans & Beckmann, 1957; Shapley & Shubik, 1971), and can be found by price adjustment processes (see, e.g., Crawford & Knoer (1981), Demange et al. (1986)).

In contrast, the partnership formation problem may not have an equilibrium (Talman & Yang, 2011). Some sufficient (and necessary) conditions for the existence of equilibrium are provided by Eriksson & Karlander (2001) and Talman & Yang (2011). An adjustment process is proposed for the partnership formation problem by Andersson et al. (2014b), which can always either find an equilibrium or disprove the existence of an equilibrium. The adjustment process of Andersson et al. (2014b) computes a certain payoff vector in a similar way as in the adjustment process by Demange et al. (1986), and the obtained payoff vector is used to find an equilibrium of the partnership formation problem.

The main aim of this paper is to clarify the relationship between the partnership formation problem and the assignment game. As mentioned above, the assignment game is a very special case of the partnership formation problem. We show in this paper that the converse is also true in some sense. That is, we prove that if we obtain an equilibrium in a certain assignment game, then we can find an equilibrium in the partnership formation problem or disprove the existence of an equilibrium.

For this, we associate an assignment game with a given partnership formation problem in Section 3.1, and show in Sections 3.2 and 3.3 that the associated assignment game has various properties that are useful in finding an equilibrium of the partnership formation problem. In particular, we show that an equilibrium in the partnership formation problem corresponds to an equilibrium in the associated assignment game satisfying a certain condition (Theorems 3.3 and 3.4). Then, it is shown that using an equilibrium payoff in the associated assignment game, the problem of finding an equilibrium in the partnership formation problem can be reduced to the problem of finding a matching among the agents such that each agent is matched to one of his favorite agents (Theorem 3.5). Base on this property, we devise an algorithm for computing an equilibrium of the partnership formation problem.

It is observed that our algorithm is similar to the algorithm by Andersson et al. (2014b). Indeed, the starting point of our current research is to better understand the behavior of their algorithm. In Section 4 we discuss the connection between our algorithm and the one by Andersson et al. (2014b), and show that our algorithm can be seen as a variant of the algorithm by Andersson et al. (2014b) with more flexibility.

We finally note that our approach by using an associated assignment game is not totally new, and similar approach is already used in Biró et al. (2012) and Chiappori et al. (2014). Indeed, Biró et al. (2012) and Chiappori et al. (2014) associate certain assignment games with a given partnership formation problem, and use the maximum weight of a matching in the associated assignment games to characterize the existence of an equilibrium in the partnership formation problem. Based on the characterization, Biró et al. (2012) and Chiappori et al. (2014) propose algorithms for checking the existence of an equilibrium in partnership formation problem. A drawback of their algorithms is that the information about the joint values of agents pairs are needed to compute the maximum weight of a matching.

In contrast, the associated assignment game used in this paper is similar to but different from the ones in Biró et al. (2012) and Chiappori et al. (2014) (see Section 3.1 for more discussion on the difference of the associated assignment games). This difference makes it possible to obtain a characterization for the existence of an equilibrium in terms of demand correspondences of agents. This characterization is useful in designing an adjustment process for checking the existence of an equilibrium, which does not require the information about the joint values of agents pairs.

2. PRELIMINARIES

We review definitions and fundamental properties for the partnership formation problem and the assignment game. In the following, we denote by $\mathbb{Z}_+$ and $\mathbb{R}_+$ the sets of non-negative integers and non-negative real numbers, respectively.

2.1. Partnership Formation Problem

We explain the model of the partnership formation problem in Biró et al. (2012), which is (slightly) more general than the original one in Talman & Yang (2011).

An instance of the partnership formation problem is given by a tuple (N,E,v), where (N,E) is an undirected graph and $v : E \to \mathbb{R}_+$ is an edge weight function taking non-negative real numbers. The vertex set N represents a set of agents, where it is assumed that there are n agents and $N = \{1,2,\dots,n\}$. Each agent works alone or works together with another agent. We consider the setting where possible partners of agents are restricted for some reasons such as their skills and/or human relationship, and possible pairs are represented by the edge set $E \subseteq \{(i,j) \mid i,j \in N,\ i \neq j\}$. That is, two agents i and j can work together if and only if $(i,j) \in E$. The original model of Talman & Yang (2011) corresponds to the case where any two agents can be a pair, i.e., $E = \{(i,j) \mid i,j \in N,\ i \neq j\}$. We note that $(i,j) \in E$ if and only if $(j,i) \in E$ for every $i,j \in N$. For $(i,j) \in E$, the edge weight $v(i,j)$ represents the (joint) value generated by the two agents i and j. We assume, without loss of generality, that the value generated by a single agent $i \in N$ is equal to zero.

A vector $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^N$ is called a *payoff*. A *matching* is a function $\mu : N \to N$ such that for $i,j \in N$, we have $\mu(i) = j$ if and only if $\mu(j) = i$. For $i \in N$, if $\mu(i) \neq i$ then agent $\mu(i)$ is the partner of agent i in the matching μ, while $\mu(i) = i$ means that agent i has no partner in the matching μ. Therefore, a matching corresponds to a partition of N into pairs of agents and single agents.

A pair of a matching μ and a payoff p is called an *equilibrium* in the partnership formation problem (N,E,v) if the following conditions hold:[1]

$$p_i + p_j \geq v(i,j) \qquad (\forall (i,j) \in E), \tag{2.1}$$
$$p_{\mu(i)} + p_i = v(\mu(i),i) \qquad (\forall i \in N \text{ with } \mu(i) \neq i), \tag{2.2}$$
$$p_i \geq 0 \qquad (\forall i \in N), \tag{2.3}$$
$$p_i = 0 \qquad (\forall i \in N \text{ with } \mu(i) = i). \tag{2.4}$$

An equilibrium in the partnership formation problem may not exist (see, e.g., Chiappori et al. (2014); Talman & Yang (2011)). For example, it is easy to see that the partnership formation problem with $N = \{1,2,3\}$, $E = \{(1,2),(2,3),(1,3)\}$, and $v(i,j) = 1$ for every $(i,j) \in E$ has no equilibrium.

A matching (resp., a payoff) in an equilibrium is called an *equilibrium matching* (resp., an *equilibrium payoff*). The partnership formation problem is called the *matching game* in Biró et al. (2012), where the set of equilibrium payoffs in the partnership formation problem is called the *core*, while an equilibrium payoff is called a *core allocation*.

The conditions (2.1)–(2.4) for an equilibrium can be simply rewritten as follows:

$$p_i + p_j \geq v(i,j) \qquad (\forall (i,j) \in \overline{E}), \tag{2.5}$$
$$p_{\mu(i)} + p_i = v(\mu(i),i) \qquad (\forall i \in N), \tag{2.6}$$

where

$$v(i,i) = 0 \quad (i \in N), \qquad \overline{E} = E \cup \{(i,i) \mid i \in N\}.$$

Note that the pair $(N,\overline{E})$ can be seen as an undirected graphs with a self-loop at each vertex.

The conditions (2.5) and (2.6) can be further rewritten in terms of demand correspondences. For an agent $i \in N$ and a payoff $p \in \mathbb{R}^N$, we define the *demand correspondence* $D_i(p) \subseteq N$ as the set of agents $j \in N$ that maximize $v(i,j) - p_j$, i.e.,

$$D_i(p) = \arg\max\{v(i,j) - p_j \mid j \in N,\ (i,j) \in \overline{E}\}.$$

We consider the following two conditions, which are shown to be equivalent to (2.5) and (2.6); the first condition means that the partner of each agent is one of his favorite agents, and the second condition means that the payoff of

[1] The symbol "$\forall$" reads "for all" or "for every."

each agent $i \in N$ is equal to the maximum of the value $v(i,j) - p_j$ among all possible partners $j \in N$.

$$\mu(i) \in D_i(p) \qquad (\forall i \in N), \tag{2.7}$$
$$p_i = \max\{v(i,j) - p_j \mid j \in N,\ (i,j) \in \overline{E}\} \quad (\forall i \in N). \tag{2.8}$$

Proposition 2.1. *Let $\mu : N \to N$ and $p \in \mathbb{R}^N$ be a matching and a payoff, respectively, in the partnership formation problem (N,E,v). Then, the pair (μ, p) is an equilibrium in (N,E,v) if and only if it satisfies the conditions (2.7) and (2.8).*

Proof. The conditions (2.5) and (2.6) hold if and only if

$$v(\mu(i), i) - p_{\mu(i)} = p_i \geq v(i,j) - p_j \quad (\forall (i,j) \in \overline{E}),$$

which is equivalent to the combination of (2.7) and (2.8). □

So far we consider characterizations of equilibria in the partnership formation problem. We then present necessary and sufficient conditions for the existence of an equilibrium by using the following dual pair of linear programming problems:

$$\begin{array}{lll}
\text{(P)} & \text{Maximize} & \displaystyle\sum_{(i,j)\in E} v(i,j)x_{ij} \\
& \text{subject to} & \displaystyle\sum_{j\in N\setminus\{i\},\,(i,j)\in E} x_{ij} \leq 1 \qquad (\forall i \in N), \\
& & x_{ij} \geq 0 \qquad (\forall (i,j) \in E), \\
\text{(D)} & \text{Minimize} & \displaystyle\sum_{i\in N} p_i \\
& \text{subject to} & p_i + p_j \geq v(i,j) \qquad (\forall (i,j) \in E), \\
& & p_i \geq 0 \qquad (\forall i \in N).
\end{array}$$

Recall that E is a set of distinct (unordered) pairs of agents in N. Problem (P) is a linear programming relaxation for the problem of finding a maximum-weight matching in (N,E,v), where the weight of a matching $\mu : N \to N$ is given by $\sum_{i\in N} v(\mu(i), i)$. By the duality theorem for linear programming problems, the optimal value of (P) is equal to the optimal value of (D).

Proposition 2.2 (cf. Talman & Yang (2011); Biró et al. (2012))**.** *For the partnership formation problem (N,E,v), the following three conditions are*

equivalent.
(a) *There exists an equilibrium in* (N,E,v).
(b) *Problem* (P) *has an integral optimal solution.*
(c) *The maximum weight of a matching in* (N,E,v) *is equal to the optimal value of* (P) *(or the optimal value of* (D)*).*

From Proposition 2.2, we can obtain the following properties; the last property (iii) means that a matching and a payoff in an equilibrium can be chosen independently of each other.

Proposition 2.3 (cf. Talman & Yang (2011))**.** *Suppose that there exists an equilibrium in the partnership formation problem* (N,E,v). *Let* $\mu : N \to N$ *be a matching and* $p \in \mathbb{R}^N$ *be a payoff.*
(i) μ *is an equilibrium matching if and only if it is a maximum-weight matching in* (N,E,v).
(ii) p *is an equilibrium payoff if and only if it is an optimal solution of the linear programming problem* (D).
(iii) *If* μ *is an equilibrium matching and* p *is an equilibrium payoff, then* (μ, p) *is an equilibrium.*

Example 1. To illustrate our results shown in this paper, we will use the following example of the partnership formation problem (N,E,v) such that

$$N = \{1,2,3,4,5\}, \qquad E = \{(i,j) \mid i,j \in N,\ i \neq j\},$$

and the values $v(i,j)$ for $(i,j) \in E$ are given by the following table:

$i \setminus j$	1	2	3	4	5
1	0	5	4	1	2
2	5	0	2	4	1
3	4	2	0	3	1
4	1	4	3	0	1
5	2	1	1	1	0

This problem has an equilibrium; indeed, the two matchings $\mu^a, \mu^b : N \to N$ given by

$$\mu^a(1) = 2,\ \mu^a(2) = 1,\ \mu^a(3) = 4,\ \mu^a(4) = 3,\ \mu^a(5) = 5, \tag{2.9}$$

$$\mu^b(1) = 3,\ \mu^b(2) = 4,\ \mu^b(3) = 1,\ \mu^b(4) = 2,\ \mu^b(5) = 5 \tag{2.10}$$

are equilibrium matchings. By Proposition 2.3 (i), they are maximum-weight matchings in (N,E,v) with the weight equal to 8. The set of equilibrium payoffs is given as

$$P = \{(p_1,p_2,p_3,p_4,p_5) \in \mathbb{R}^N \mid 2 \leq p_1 \leq 3,\ p_2 = 5 - p_1,\\ p_3 = 4 - p_1,\ p_4 = -1 + p_1,\ p_5 = 0\}.$$

By Proposition 2.3 (iii), a pair (μ, p) of a matching $\mu : N \to N$ and a payoff $p \in \mathbb{R}^N$ is an equilibrium in (N,E,v) if and only if $\mu \in \{\mu^a, \mu^b\}$ and $p \in P$. □

Remark 2.4. Problem (P) is known as the *fractional matching problem* in the literature of combinatorial optimization (see, e.g., Schrijver (2003, Chapter 30)). The optimal value of problem (P) as well as an optimal solution can be computed by solving the maximum-weight matching problem on a bipartite graph (see, e.g., Nemhauser & Trotter (1975), Pulleyblank (1987), Schrijver (2003, Chapter 30); see also Biró et al. (2012)).

The bipartite graph used for solving the problem (P) is given as follows. Let N' be a copy of the set N, and denote by $i' \in N'$ the copy of $i \in N$, i.e., $N' = \{i' \mid i \in N\}$. Let us consider the bipartite graph $(N,N';F_0)$ with vertex set $N \cup N'$ and edge set F_0 given by

$$F_0 = \{(i,j') \in N \times N' \mid (i,j) \in E\}. \tag{2.11}$$

Therefore, for each $(i,j) \in E$ with $i \neq j$, set F_0 contains two edges (i,j') and (j,i'), and for each $i \in N$, there is no edge between the vertices $i \in N$ and $i' \in N'$. For $(i,j') \in F_0$, we define the edge weight $w(i,j')$ by $w(i,j') = v(i,j)$.

Then, the half of the weight of a maximum-weight matching in the bipartite graph $(N,N';F_0)$ is equal to the optimal value of the problem (P). Moreover, for a maximum-weight matching $M \subseteq F_0$ in $(N,N';F_0)$, the vector $x^* = (x^*_{ij} \mid (i,j) \in E)$ given by

$$x^*_{ij} = \begin{cases} 1 & \text{(if } M \text{ contains both of } (i,j') \text{ and } (j,i')\text{)}, \\ 1/2 & \text{(if } M \text{ contains exactly one of } (i,j') \text{ and } (j,i')\text{)}, \\ 0 & \text{(otherwise, i.e., } M \text{ contains neither of } (i,j') \text{ and } (j,i')\text{)} \end{cases}$$

is an optimal solution of (P).

In the following sections we will use a bipartite graph similar to $(N,N';F_0)$ to reveal the connection between the partnership formation problem and the assignment game. □

2.2. Assignment Game

The *assignment game* is a special case of the partnership formation problem, where the set of agents N is partitioned into two disjoint sets A and B corresponding to *sellers* and *buyers*, respectively, and any two distinct sellers (buyers) cannot be a pair.

In the following, we represent an instance of the assignment game by a tuple (A,B,F,w), where $(A,B;F)$ is a bipartite graph and $w : F \to \mathbb{R}_+$ is an edge weight function taking non-negative real numbers. The vertex sets A and B represent the sets of sellers and buyers, respectively, and $F \subseteq A \times B$ is the set of possible pairs of sellers and buyers, i.e., buyer i and seller j can be a pair if and only if $(i,j) \in F$. For $(i,j) \in F$, $w(i,j)$ represents the joint values generated by pairs of agents $i \in A$ and $j \in B$. Note that the original model of the assignment game coincides with the case with $F = A \times B = \{(i,j) \mid i \in A,\ j \in B\}$, i.e., any seller and buyer can be a pair.

For notational convenience, we consider a dummy seller, denote by 0, and regard each buyer $j \in B$ with no partner as a pair $(0,j)$ with the dummy seller. We assume that the joint value generated by a pair with the dummy seller is zero (i.e., $w(0,j) = 0$ for all $j \in B$), and the dummy seller can be a pair with an arbitrary number of buyers. A *matching* in the assignment game (A,B,F,w) is a function $\eta : B \to A \cup \{0\}$ satisfying the following conditions:

$$\begin{aligned}
&(\eta(j), j) \in F \text{ for every } j \in B \text{ with } \eta(j) \neq 0 \\
&\qquad\qquad\qquad\qquad\qquad \text{(i.e., every pair } (\eta(j), j) \text{ is "feasible")}, \\
&\text{for each } i \in A \text{ there exists at most one } j \in B \text{ with } \eta(j) = i \\
&\qquad\qquad\qquad \text{(i.e., every seller can be a pair with at most one buyer).}
\end{aligned}$$

Vectors $q = (q_i \mid i \in A) \in \mathbb{R}^A$ and $r = (r_j \mid j \in B) \in \mathbb{R}^B$ are called *sellers' payoff* and *buyers' payoff*, respectively. A seller's payoff is sometimes called a *price vector*. The pair (q,r) is simply called a *payoff*.

The concept of equilibrium in the partnership formation problem is specialized to the assignment game as follows. The tuple (η, q, r) of a matching η and a payoff (q,r) is called an *equilibrium* in the assignment game (A,B,F,w)

if the following conditions hold with $q_0 = 0$:

$$q_i + r_j \geq w(i,j) \qquad (\forall (i,j) \in F), \tag{2.12}$$
$$q_{\eta(j)} + r_j = w(\eta(j),j) \qquad (\forall j \in B), \tag{2.13}$$
$$q_i \geq 0 \quad (\forall i \in A), \quad r_j \geq 0 \quad (\forall j \in B), \tag{2.14}$$
$$q_i = 0 \qquad (\forall i \in A \setminus \{\eta(j) \mid j \in B\}). \tag{2.15}$$

Every assignment game has an equilibrium (Shapley & Shubik, 1971).

The conditions (2.12) and (2.13) can be rewritten in terms of demand correspondences. For a vector of sellers' payoff $q \in \mathbb{R}^A$ and a buyer $j \in B$, we define the *demand correspondence* $\widetilde{D}_j(q) \subseteq A \cup \{0\}$ of buyer j by

$$\widetilde{D}_j(q) = \arg\max\{w(i,j) - q_i \mid i \in A \cup \{0\},\ (i,j) \in F \cup \{(0,j)\}\}. \tag{2.16}$$

We consider the following conditions:

$$\eta(j) \in \widetilde{D}_j(q) \qquad (\forall j \in B), \tag{2.17}$$
$$r_j = \max\{w(i,j) - q_i \mid i \in A \cup \{0\},\ (i,j) \in F \cup \{(0,j)\}\} \quad (\forall j \in B), \tag{2.18}$$

where $q_0 = 0$.

Proposition 2.5. *Let $\eta : B \to A \cup \{0\}$ be a matching in the assignment game (A,B,F,w), and $(q,r) \in \mathbb{R}^A \times \mathbb{R}^B$ be a payoff. Then, the tuple (η,q,r) is an equilibrium if and only if it satisfies the conditions* (2.14), (2.15), (2.17), *and* (2.18).

A matching η and a payoff (q,r) in an equilibrium (η,q,r) are called an *equilibrium matching* and an *equilibrium payoff*, respectively. Equilibrium matching and payoff can be characterized as optimal solutions of certain optimization problems (see, e.g., Shapley & Shubik (1971); Roth & Sotomayor (1990)). In addition, any combination of an equilibrium matching and an equilibrium payoff gives an equilibrium.

Proposition 2.6. *Let $\eta : B \to A \cup \{0\}$ be a matching in the assignment game (A,B,F,w), and $(q,r) \in \mathbb{R}^A \times \mathbb{R}^B$ be a payoff.*
(i) *η is an equilibrium matching if and only if it maximizes the weight $\sum_{j \in B} w(\eta(j),j)$ among all matchings.*

(ii) (q,r) *is an equilibrium payoff if and only if it is an optimal solution of the following linear programming problem:*

$$\begin{array}{ll} \text{Minimize} & \sum_{i\in A} q_i + \sum_{j\in B} r_j \\ \text{subject to} & q_i + r_j \geq w(i,j) \quad (\forall (i,j) \in F), \\ & q_i \geq 0 \quad (\forall i \in A), \qquad r_j \geq 0 \quad (\forall j \in B). \end{array}$$

(iii) *If* η *is an equilibrium matching and* (q,r) *is an equilibrium payoff, then* (η,q,r) *is an equilibrium.*

Define

$$\mathscr{H}_0 = \{q \in \mathbb{R}^A \mid (q,r) \text{ is an equilibrium payoff in } (A,B,F,w) \text{ for some } r \in \mathbb{R}^B\}. \tag{2.19}$$

If we regard $q \in \mathbb{R}^A$ as a price vector of sellers' goods, then $\mathscr{H}_0$ can be seen as the set of equilibrium price vectors in the assignment game. Hence, a minimal vector in the set $\mathscr{H}_0$ is uniquely determined (see, e.g., Shapley & Shubik (1971)).

It is known that if an equilibrium price vector $q \in \mathscr{H}_0$ is fixed, then the corresponding buyers' payoff r in an equilibrium payoff (q,r) is uniquely determined by (2.18).

Proposition 2.7. *Let* $(q,r) \in \mathbb{R}^A \times \mathbb{R}^B$ *be a payoff in the assignment game* (A,B,F,w)*. Then,* (q,r) *is an equilibrium payoff if and only if* $q \in \mathscr{H}_0$ *and* r *is given by* (2.18).

An equilibrium matching can be characterized by using demand correspondences.

Proposition 2.8. *Let* $\eta : B \to A \cup \{0\}$ *be a matching in the assignment game* (A,B,F,w)*, and* $q \in \mathbb{R}^A$ *be a vector with* $q \in \mathscr{H}_0$*. Then,* η *is an equilibrium matching if and only if* η *and* q *satisfy* (2.15) *and* (2.17).

Proof. We define a vector $r \in \mathbb{R}^B$ by (2.18). By Proposition 2.7, (q,r) is an equilibrium payoff, and therefore it satisfies the condition (2.14). This fact, together with Proposition 2.6 (iii), implies that η is an equilibrium matching if and only if (η,q,r) is an equilibrium. By Proposition 2.5, (η,q,r) is an equilibrium if and only if η is a matching satisfying (2.15) and (2.17). □

Finally, we present a price adjustment process (also known as a dynamic auction) for computation of a vector in $\mathscr{H}_0$. It is known that the unique minimal vector in $\mathscr{H}_0$ can be obtained by an ascending-type price adjustment process, which is explained below.

For a vector $q \in \mathbb{R}^A_+$ and a set $Y \subseteq A$, we define

$$\begin{aligned} \widetilde{O}(Y,q) &= \{j \in B \mid \widetilde{D}_j(q) \subseteq Y\}, \\ \widetilde{U}(Y,q) &= \{j \in B \mid \widetilde{D}_j(q) \cap Y \neq \emptyset\}. \end{aligned} \tag{2.20}$$

The set $\widetilde{O}(Y,q)$ consists of buyers who only demand sellers in Y at price q, while $\widetilde{U}(Y,q)$ is the set of buyers who demand some seller in Y at price q. Obviously, $\widetilde{O}(Y,q) \subseteq \widetilde{U}(Y,q)$ holds. A set $Y \subseteq A$ is said to be *overdemanded* if $|\widetilde{O}(Y,q)| > |Y|$. A set $X \subseteq A$ is said to be *in excess demand* at price q if it satisfies

$$|\widetilde{U}(Y,q) \cap \widetilde{O}(X,q)| > |Y| \qquad (\emptyset \neq \forall Y \subseteq X).$$

It is known that a maximal set in excess demand is uniquely determined (Mo et al. (1988); Sankaran (1994)). Moreover, the maximal set X in excess demand is an overdemanded set that maximizes the value $|\widetilde{O}(X,q)| - |X|$ (Murota et al. (2016)).

An ascending-type price adjustment process due to Mo et al. (1988) and Sankaran (1994), which is a variant of the one in Demange et al. (1986), is described as follows. We here assume that the values $w(i,j)$ $((i,j) \in F)$ are non-negative integers; this assumption implies that the minimal vector in $\mathscr{H}_0$ is integral.

Algorithm VICKREYENGLISH
Step 0: Set $q \in \mathbb{Z}^A$ by $q := (0,0,\ldots,0)$.
Step 1: Collect the demand correspondences $\widetilde{D}_j(q)$ for $j \in B$.
Step 2: If $|\widetilde{O}(S,q)| \leq |S|$ holds for every $S \subseteq A$, then output q and stop.
Step 3: Find the unique maximal set $\widetilde{S}^* \subseteq A$ in excess demand at payoff q, update q by $q_i := q_i + 1$ $(i \in \widetilde{S}^*)$, and go to Step 1. □

Proposition 2.9. *For the assignment game* (A,B,F,w)*, assume that the values* $w(i,j)$ $((i,j) \in F)$ *are non-negative integers. Then, the algorithm* VICKREYENGLISH *outputs the unique minimal vector in* $\mathscr{H}_0$ *(i.e., the unique minimal equilibrium price vector).*

For more price adjustment processes for computing an equilibrium price vector, see Appendix and also Andersson et al. (2010), Andersson & Erlanson (2013) and Mishra & Parkes (2009).

3. REDUCTION OF PARTNERSHIP FORMATION PROBLEM TO ASSIGNMENT GAME

As mentioned in Section 2.2, the assignment game is a special case of the partnership formation problem. Hence, if we can compute an equilibrium in the partnership formation problem, then we can also compute an equilibrium in the assignment game.

In this section, we show that the converse is also true in some sense, i.e., if we can compute an equilibrium in the assignment game, then we can also compute an equilibrium in the partnership formation problem if it exists. For this, we associate an assignment game with the partnership formation problem in Section 3.1, and present its fundamental properties in Section 3.2. In Section 3.3 we present the main theorems on the relationship between the partnership formation problem and the associated assignment game. Based on these theorems, we propose an algorithm for computing an equilibrium in the partnership formation problem. Proofs are given in Section 3.4.

3.1. Assignment Game Associated with Partnership Formation Problem

For the partnership formation problem (N,E,v), we associate the assignment game (N,N',F,w) as follows, where F_0 is given by (2.11).

$$\begin{aligned}
N' &= \{i' \mid i \in N\}, \text{ where } i' \text{ is a copy of } i, \\
F &= F_0 \cup \{(i,i') \mid i \in N\} \\
&= \{(i,j') \in N \times N' \mid (i,j) \in E\} \cup \{(i,i') \mid i \in N\}, \\
w(i,j') &= \begin{cases} v(i,j) & (\text{if } i \neq j), \\ 0 & (\text{if } i = j). \end{cases} \qquad ((i,j') \in F).
\end{aligned}$$

We see that the associated assignment game (N,N',F,w) is similar to the bipartite graph $(N,N';F_0)$ in Remark 2.4.

Example 2. To illustrate the associated assignment game defined above, we consider the partnership formation problem (N,E,v) in Example 1. The associated assignment game (N,N',F,w) is given by

$$N=\{1,2,3,4,5\},\quad N'=\{1',2',3',4',5'\},$$
$$F=\{(i,j')\mid i\in N,\ j'\in N'\},$$

and values $w(i,j')$ for $(i,j')\in F$ are given by the following table:

$i\backslash j'$	$1'$	$2'$	$3'$	$4'$	$5'$
1	0	5	4	1	2
2	5	0	2	4	1
3	4	2	0	3	1
4	1	4	3	0	1
5	2	1	1	1	0

.

The assignment game (N,N',F,w) has two equilibrium matchings $\eta^a,\eta^b : N'\to N$ given as

$$\eta^a(1')=2,\ \eta^a(2')=1,\ \eta^a(3')=4,\ \eta^a(4')=3,\ \eta^a(5')=5, \tag{3.1}$$
$$\eta^b(1')=3,\ \eta^b(2')=4,\ \eta^b(3')=1,\ \eta^b(4')=2,\ \eta^b(5')=5. \tag{3.2}$$

The set of equilibrium payoffs is given as

$$\begin{aligned}\widetilde{P}=\{(q,r)\in\mathbb{R}^N\times\mathbb{R}^{N'}\mid\ & 2\le q_1\le 3,\ 2\le q_2\le 3,\\ & q_3=q_2-1,\ q_4=q_1-1,\ q_5=0,\\ & r_{1'}=5-q_2,\ r_{2'}=5-q_1,\ r_{3'}=3-q_4,\\ & r_{4'}=3-q_3,\ r_{5'}=0\}.\end{aligned} \tag{3.3}$$

By Proposition 2.6 (iii), a tuple (η,q,r) is an equilibrium in the assignment game (N,N',F,w) if and only if $\eta\in\{\eta^a,\eta^b\}$ and $(q,r)\in\widetilde{P}$. □

Assignment games associated with the partnership formation problem are also considered in Biró et al. (2012) and Chiappori et al. (2014), which are similar to ours, but different in the use of pairs $\{(i,i')\mid i\in N\}$. In the following, we discuss the difference among the three assignment games. While the difference is subtle, it is crucial to our main results.

The associated assignment game in Biró et al. (2012) is obtained from ours by deleting the pairs $\{(i,i')\mid i\in N\}$ from F, i.e., the set F is replaced with F_0 given by (2.11). Hence, the assignment game in Biró et al. (2012)

is given by the bipartite graph $(N,N';F_0)$ in Remark 2.4. Biró et al. (2012) use the maximum weight of a matching in the associated assignment game to characterize the existence of an equilibrium in the partnership formation problem (N,E,v) (see Proposition 2.2 and Remark 2.4).

The associated assignment game in Chiappori et al. (2014) uses the pairs $\{(i,i') \mid i \in N\}$ as in ours, but for a different purpose. Chiappori et al. (2014) consider a more general model of the partnership formation problem, where each $i \in N$ corresponds to a type of agents and there are multiple agents of the same type. Each pair (i,i') in the assignment game of Chiappori et al. (2014) is used to represent a pair of agents of the type $i \in N$. In contrast, in our assignment game, each pair (i,i') corresponds to a single agent $i \in N$ with no partner. Chiappori et al. (2014) use the associated assignment game to provide a necessary and sufficient condition for the existence of an equilibrium in their general model, in a similar way as in Biró et al. (2012).

3.2. Properties of Equilibria in the Associated Assignment Game

The associated assignment game (N,N',F,w) defined in Section 3.1 is "symmetric" in the sense that $(i,j') \in F$ holds if and only if $(j,i') \in F$, and the pairs (i,j') and (j,i') have the same weight. Due to the symmetric structure, various nice properties of equilibria in (N,N',F,w) can be obtained. In the following discussion, we often identify the set N' with N through the natural one-to-one correspondence, and regard a vector in $\mathbb{R}^{N'}$ (resp., $\mathbb{R}^{N}$) as a vector in $\mathbb{R}^{N}$ (resp., $\mathbb{R}^{N'}$).

We consider a matching in (N,N',F,w) such that every agent in $N' \cup N$ has a partner. We call such a matching a *perfect matching*, following the graph theory terminology. Note that a perfect matching indeed exists in (N,N',F,w) since $\{(i,i') \mid i \in N\} \subseteq F$.

The next proposition shows that the assignment game (N,N',F,w) has an equilibrium matching that is a perfect matching. It should be noted that this property does not hold for general assignment games.

Proposition 3.1. *There exists an equilibrium matching $\eta : N' \to N \cup \{0\}$ in the assignment game (N,N',F,w) such that η is a perfect matching.*

Proof. Proof is given in Section 3.4.1. □

In the following discussion, we consider a perfect matching whenever we refer to a matching in (N,N',F,w). Use of a perfect matching is a key to obtain the main results of this paper.

For the assignment game (N,N',F,w) and a sellers' payoff $q \in \mathbb{R}^N$, we have defined the demand correspondence $\widetilde{D}_{j'}(q) \subseteq N \cup \{0\}$ of buyer $j' \in N'$ by (2.16), i.e.,

$$\widetilde{D}_{j'}(q) = \arg\max\{w(i,j') - q_i \mid i \in N \cup \{0\},\ (i,j') \in F \cup \{(0,j')\}\}. \quad (3.4)$$

By the definition of the weight $w(i,j')$, we have

$$\widetilde{D}_{j'}(q) = \arg\max\{v(i,j) - q_i \mid i \in N \cup \{0\},\ (i,j) \in E \cup \{(0,j)\}\}$$

with $v(0,j) = 0$, and therefore $\widetilde{D}_{j'}(q) \setminus \{0\} = D_j(q)$ holds if $\widetilde{D}_{j'}(q) \setminus \{0\} \neq \emptyset$ (i.e., $\max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} \geq 0$).

For the assignment game (N,N',F,w), we have defined the set $\mathscr{H}_0 \subseteq \mathbb{R}^N$ of equilibrium price vectors by (2.19), i.e., $\mathscr{H}_0$ is given as

$$\mathscr{H}_0 = \{q \in \mathbb{R}^N \mid (q,r) \text{ is an equilibrium payoff in } (N,N',F,w) \text{ for some } r \in \mathbb{R}^{N'}\}. \quad (3.5)$$

We show that the definition and characterizations (Propositions 2.7 and 2.8) of an equilibrium (η,q,r) in the associated assignment game (N,N',F,w) can be simplified if we restrict η to a perfect matching. Recall that an equilibrium in the assignment game is defined by the four conditions (2.12)–(2.15).

Lemma 3.2. *Let $\eta : N' \to N$ and $(q,r) \in \mathbb{R}^N \times \mathbb{R}^{N'}$ be a perfect matching and a payoff in the assignment game (N,N',F,w), respectively.*
(i) *(η,q,r) is an equilibrium in (N,N',F,w) if and only if the following conditions hold:*

$$q_i + r_{j'} \geq w(i,j') \qquad (\forall (i,j') \in F), \quad (3.6)$$

$$q_{\eta(j')} + r_{j'} = w(\eta(j'),j') \qquad (\forall j' \in N'), \quad (3.7)$$

$$q_i \geq 0 \quad (\forall i \in N), \qquad r_{j'} \geq 0 \quad (\forall j' \in N'). \quad (3.8)$$

(ii) *(q,r) is an equilibrium payoff in (N,N',F,w) if and only if $q \in \mathscr{H}_0$ and r is given by*

$$r_{j'} = \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} \quad (j' \in N'). \quad (3.9)$$

Hence, the formula (3.5) *can be rewritten as*

$$\mathscr{H}_0 = \{q \in \mathbb{R}^N \mid \text{ the pair } (q,r) \text{ of } q \text{ and } r \in \mathbb{R}^{N'} \text{ given by (3.9)} \\ \text{is an equilibrium payoff in } (N,N',F,w)\}. \quad (3.10)$$

(iii) *Suppose that* $q \in \mathscr{H}_0$ *holds. Then, we have* $\widetilde{D}_j(q) \setminus \{0\} \neq \emptyset$ $(\forall j \in N)$. *Moreover,* η *is an equilibrium matching in* (N,N',F,w) *if and only if*

$$\eta(j') \in \widetilde{D}_{j'}(q) \setminus \{0\} \qquad (\forall j' \in N').$$

Proof. Proof is given in Section 3.4.2. □

3.3. Theorem and Algorithm for Partnership Formation Problem

We first present a characterization of equilibria in the partnership formation problem (N,E,v) in terms of the associated assignment game (N,N',F,w). For a matching $\mu : N \to N$ in (N,E,v), we define a matching $\eta_\mu : N' \to N$ in (N,N',F,w) associated with μ by

$$\eta_\mu(j') = \mu(j) \qquad (j' \in N'). \quad (3.11)$$

Note that η_μ is a perfect matching in (N,N',F,w).

Theorem 3.3. *Let* $\mu : N \to N$ *and* $p \in \mathbb{R}^N$ *be a matching and a payoff in the partnership formation problem* (N,E,v)*, respectively. Also, let* $\eta_\mu : N' \to N$ *be the (perfect) matching in the associated assignment game* (N,N',F,w) *given by* (3.11)*. Then, the following three statements are equivalent:*
(a) (μ, p) *is an equilibrium in* (N,E,v).
(b) (η_μ, p, p) *is an equilibrium in* (N,N',F,w).
(c) (η_μ, q, r) *is an equilibrium in* (N,N',F,w) *for some payoff* $(q,r) \in \mathbb{R}^N \times \mathbb{R}^{N'}$ *with* $(1/2)(q+r) = p$.

Proof. Proof is given in Section 3.4.3. □

The next theorem clarifies the relationship between equilibrium matchings (resp., payoffs) in the partnership formation problem (N,E,v) and equilibrium matchings (resp., payoffs) in the associated assignment game (N,N',F,w).

Theorem 3.4.
(i) *For a matching $\mu : N \to N$ in the partnership formation problem (N,E,v), let $\eta_\mu : N' \to N$ be the (perfect) matching given by (3.11) in the associated assignment game (N,N',F,w). Then, μ is an equilibrium matching in (N,E,v) if and only if η_μ is an equilibrium matching in (N,N',F,w).*
(ii) *Suppose that there exists an equilibrium in (N,E,v). Then, a payoff $p \in \mathbb{R}^N$ is an equilibrium payoff in (N,E,v) if and only if there exists an equilibrium payoff $(q,r) \in \mathbb{R}^N \times \mathbb{R}^{N'}$ in (N,N',F,w) such that $(1/2)(q+r) = p$.*

Proof. Proof is given in Section 3.4.4. □

We finally show that an equilibrium in the partnership formation problem (N,E,v) can be obtained by using an equilibrium payoff in the associated assignment game (N,N',F,w), provided that an equilibrium exists in (N,E,v).

Theorem 3.5. *Let $\mu : N \to N$ be a matching in the partnership formation problem (N,E,v), and $q \in \mathbb{R}^N$ be a payoff with $q \in \mathscr{H}_0$.*
(i) *μ is an equilibrium matching in (N,E,v) if and only if*

$$\mu(i) \in D_i(q) \qquad (\forall i \in N). \tag{3.12}$$

(ii) *Suppose that μ is an equilibrium matching in (N,E,v). Define $p \in \mathbb{R}^N$ by*

$$p_i = (1/2)(v(\mu(i),i) + q_i - q_{\mu(i)}) \qquad (i \in N). \tag{3.13}$$

Then, p is an equilibrium payoff in (N,E,v).

Proof. Proof is given in Section 3.4.5. □

Example 3. To illustrate the statements of Theorem 3.5, we consider the partnership formation problem (N,E,v) in Example 1. We have

$$\begin{aligned}\mathscr{H}_0 = \{q \in \mathbb{R}^N \mid 2 \le q_1 \le 3,\ 2 \le q_2 \le 3,&\\ q_3 = q_2 - 1,\ q_4 = q_1 - 1,\ q_5 = 0\}.&\end{aligned} \tag{3.14}$$

For every $q \in \mathscr{H}_0$, it holds that

$$\begin{aligned}&\{2,3\} \subseteq D_1(q) = D_4(q) \subseteq \{2,3,5\},\\ &D_2(q) = \{1,4\},\\ &\{1,4\} \subseteq D_3(q) \subseteq \{1,4,5\},\\ &\{5\} \subseteq D_5(q) \subseteq \{1,3,4,5\}.\end{aligned}$$

It is observed that the (equilibrium) matchings $\mu^a, \mu^b : N \to N$ given by (2.9) and (2.10), respectively, satisfy the condition $\mu^a(j), \mu^b(j) \in D_j(q)$ $(\forall j \in N)$ by Theorem 3.5 (i).

For the matching $\mu = \mu_a$ and the payoff $q = (2,3,2,1,0) \in \mathscr{H}_0$, the vector $p \in \mathbb{R}^N$ given by (3.13) is equal to $(2,3,2,1,0)$, which is an equilibrium payoff in (N,E,v) by Theorem 3.5 (ii). □

Theorem 3.5 implies that an equilibrium in the partnership formation problem (N,E,v) can be found by the following algorithm. The idea of the algorithm is as follows. Let $q \in \mathbb{R}^N$ be a vector in $\mathscr{H}_0$, which can be computed by finding an equilibrium in the associated assignment game (N,N',F,w) (see Remark 3.6). By Theorem 3.5 (i), there exists an equilibrium matching in (N,E,v) if and only if there exists a matching $\mu : N \to N$ in (N,E,v) satisfying the condition (3.12). Therefore, it suffices to check the existence of a matching satisfying the condition (3.12), which can be done by finding a maximum-cardinality matching on a certain undirected graph (see Remark 3.7).

Algorithm COMPUTEEQUILIBRIUM
Step 1: Find a vector $q \in \mathscr{H}_0$.
Step 2: If there exists no matching $\mu : N \to N$ in (N,E,v) satisfying (3.12), then assert that "no equilibrium exists in (N,E,v)" and stop.
Step 3: Find a matching $\mu : N \to N$ in (N,E,v) satisfying (3.12), and let $p \in \mathbb{R}^N$ be a vector given by (3.13).
Output (μ, p) as an equilibrium of (N,E,v). □

Remark 3.6. We discuss the computation of a vector $q \in \mathscr{H}_0$ in Step 1. If the information of the demand correspondences $\widetilde{D}_{j'}(q)$ $(j' \in N')$ is available, then a vector $q \in \mathscr{H}_0$ can be computed by the algorithm VICKREYENGLISH in Section 2.2 or other price adjustment processes (see Appendix). In the case where the information of the sets $D_j(q)$ $(j \in N)$ is available, a vector $q \in \mathscr{H}_0$ can be computed by an adjustment process by Andersson et al. (2014b); see Section 4 for details. □

Remark 3.7. Computation of a matching μ in (N,E,v) satisfying (3.12) can be reduced to finding a maximum-cardinality matching in an undirected graph with vertex set N and the edge set given by $\{(i,j) \mid i,j \in N,\ i \in D_j(p)\}$ (see Andersson et al. (2014b)). A maximum-cardinality matching in an undirected graph can be found in strongly polynomial time, i.e., the time required to find a maximum-cardinality matching can be bounded by a polynomial in $n = |N|$ (see, e.g., Schrijver (2003)). □

Remark 3.8. The algorithm COMPUTEEQUILIBRIUM requires the information about the joint values of agents pairs only in the computation of an equilibrium payoff p in Step 3, and other steps can be performed by using demand correspondences only. This means that the information about the joint values of agents pairs is not needed when we just check the existence of an equilibrium and find an equilibrium matching (if it exists). □

3.4. Proofs

3.4.1. Proof of Proposition 3.1

We prove the claim by a graph-theoretic argument. Let us consider a bipartite graph $G = (N, N'; F)$ with vertex set $N \cup N'$, edge set F, and edge weight $w(i, j')$ for $(i, j') \in F$. Recall that a matching in the graph G is a set $M \subseteq F$ of edges such that for each vertex $i \in N \cup N'$ there exists at most one edge in M incident to i. It is easy to see that matchings in the assignment game (N, N', F, w) have a natural one-to-one correspondence with matchings in G; moreover, Proposition 2.6 (i) implies that a matching in (N, N', F, w) is an equilibrium matching if and only if its corresponding matching in G is a maximum-weight matching. Hence, to prove Proposition 3.1 it suffices to show that there exists a maximum-weight matching in G that is a perfect matching.

Let $M \subseteq F$ be a maximum-weight matching in G. We assume that M is not a perfect matching. Due to the symmetry of the graph G,

$$M' = \{(j, i') \in F \mid (i, j') \in M\}$$

is also a maximum-weight matching in G.

Using the two matchings M and M', we define an edge set X as follows. For a vertex i in the graph G, we denote by $\deg_X(i)$ the number of edges in X incident to i. We initially set $X = M \cup M'$. Then, we have $\deg_X(i) = \deg_X(i') \leq 2$ for $i \in N$. For each $i \in N$, if $\deg_X(i) = \deg_X(i') = 1$ then we add to X (one copy of) the edge (i, i'), and if $\deg_X(i) = \deg_X(i') = 0$ then we add to X two copies of the edge (i, i'). Then, the resulting edge set X satisfies $\deg_X(i) = \deg_X(i') = 2$ for each $i \in N$. This implies that X can be decomposed into two perfect matchings, which are denoted as X_1 and X_2. Moreover, the total weight of the edge set X is twice the weight of a maximum-weight matching since X contains all edges in $M \cup M'$ and each edge (i, i') has zero weight. Hence, both of the matchings X_1 and X_2 are maximum-weight matchings that are perfect matchings.

3.4.2. *Proof of Lemma 3.2*

[Proof of (i)] By definition, the tuple (η, q, r) is an equilibrium if and only if it satisfies the conditions (3.6)–(3.8) and an additional condition that $q_i = 0$ holds for all $i \in N \setminus \{\eta(j') \mid j' \in N'\}$. Since η is a perfect matching, the set $N \setminus \{\eta(j') \mid j' \in N'\}$ is empty, and therefore the additional condition holds immediately. Hence, the claim (i) holds.

[Proof of (ii)] By Proposition 2.7, (q, r) is an equilibrium payoff in (N, N', F, w) if and only if $q \in \mathscr{H}_0$ and r is given by

$$r_{j'} = \max\{w(i, j') - q_i \mid i \in N \cup \{0\},\ (i, j') \in F \cup \{(0, j')\}\} \qquad (j' \in N').$$

To derive the equation (3.9), it suffices to show that for $q \in \mathscr{H}_0$ we have

$$\max\{w(i, j') - q_i \mid i \in N,\ (i, j') \in F\} \geq w(0, j') - q_0 = 0 \qquad (\forall j' \in N'). \tag{3.15}$$

Assume that η is an equilibrium matching in (N, N', F, w) that is a perfect matching; such η exists by Proposition 3.1. By Proposition 2.6 (iii), (η, q, r) is an equilibrium in (N, N', F, w), and therefore we have (3.7) and (3.8), implying that

$$w(\eta(j'), j') - q_{\eta(j')} = r_{j'} \geq 0 \qquad (\forall j' \in N').$$

Therefore, (3.15) holds.

[Proof of (iii)] By (3.15), the set $\widetilde{D}_{j'}(q) \setminus \{0\}$ is nonempty for all $j' \in N'$. It follows from the proof of (i) and Proposition 2.8 that η is an equilibrium matching if and only if $\eta(j') \in \widetilde{D}_{j'}(q)$ holds for all $j' \in N'$. Since $\eta(j') \neq 0$, we have $\eta(j') \in \widetilde{D}_{j'}(q)$ if and only if $\eta(j') \in \widetilde{D}_{j'}(q) \setminus \{0\}$.

3.4.3. *Proof of Theorem 3.3*

We first prove the equivalence of (a) and (b), and then the equivalence of (b) and (c).

[Proof of (a)⇔(b)] Recall that (μ, p) is an equilibrium in (N, E, v) if and only if it satisfies the conditions (2.5) and (2.6). By the definitions of F,

w, and η_μ, the conditions can be rewritten in terms of the assignment game (N, N', F, w) as

$$p_i + p_{j'} \geq w(i, j') \qquad (\forall (i, j') \in F), \tag{3.16}$$

$$p_{\eta_\mu(j')} + p_{j'} = w(\eta_\mu(j'), j') \qquad (\forall j' \in N'). \tag{3.17}$$

The condition (3.16) implies $p_i \geq 0$ for $i \in N$ since $2p_i = p_i + p_{i'} \geq w(i, i') = 0$ for $i \in N$. By Lemma 3.2 (i), the conditions (3.16) and (3.17) hold if and only if (η_μ, p, p) is an equilibrium in (N, N', F, w).

[Proof of (b)⇔(c)] The implication "(b)⇒(c)" is easy to see by setting $q = r = p$. The converse can be shown as follows.

By assumption, η_μ and (q, r) are an equilibrium matching and an equilibrium payoff in (N, N', F, w), respectively. Due to the symmetric structure of (N, N', F, w), we can easily observe that the tuple $(\widehat{\eta}, r, q)$ with the matching $\widehat{\eta} : N' \to N$ given by

$$\widehat{\eta}(j') = i \text{ if } \eta_\mu(i') = j \qquad (\forall j' \in N')$$

is an equilibrium in (N, N', F, w). Hence, (r, q) is also an equilibrium payoff in (N, N', F, w). By Proposition 2.6 (ii), the set of an equilibrium payoffs in (N, N', F, w) is given as a convex polyhedron. Hence, the payoff $(p, p) = (1/2)[(q, r) + (r, q)]$ is also an equilibrium payoff in (N, N', F, w). By Proposition 2.6 (iii), (η_μ, p, p) is an equilibrium in (N, N', F, w).

3.4.4. Proof of Theorem 3.4

The claim (i) can be shown by the following chain of equivalence:

μ is an equilibrium matching in (N, E, v)

$\iff$ (μ, p) is an equilibrium in (N, E, v) for some payoff $p \in \mathbb{R}^N$

(by the definition of equilibrium matchings)

$\iff$ (η_μ, q, r) is an equilibrium in (N, N', F, w)

for some payoff $(q, r) \in \mathbb{R}^N \times \mathbb{R}^{N'}$ (by Theorem 3.3)

$\iff$ η_μ is an equilibrium matching in (N, N', F, w)

(by the definition of equilibrium matchings).

The claim (ii) can be also shown in a similar way. Let $\mu : N \to N$ be an equilibrium matching in (N, E, v); the existence follows from the assumption of the claim (ii). Then, the claim (i) shown above implies that the matching $\eta_\mu : N' \to N$ in (N,N',F,w) given by (3.11) is an equilibrium matching in (N,N',F,w). Therefore, we obtain the following chain of equivalence:

$$
\begin{aligned}
& p \text{ is an equilibrium payoff in } (N,E,v) \\
& \iff (\mu,p) \text{ is an equilibrium in } (N,E,v) \\
& \qquad \text{(by the definition of equilibrium payoffs and Proposition 2.3 (iii))} \\
& \iff (\eta_\mu,q,r) \text{ is an equilibrium in } (N,N',F,w) \text{ for some payoff} \\
& \qquad (q,r) \in \mathbb{R}^N \times \mathbb{R}^{N'} \text{ with } (1/2)(q+r) = p \qquad \text{(by Theorem 3.3)} \\
& \iff \text{there exists an equilibrium payoff } (q,r) \in \mathbb{R}^N \times \mathbb{R}^{N'} \\
& \qquad \text{in } (N,N',F,w) \text{ such that } (1/2)(q+r) = p \\
& \qquad \text{(by the definition of equilibrium payoffs and Proposition 2.6 (iii)).}
\end{aligned}
$$

3.4.5. *Proof of Theorem 3.5*

Let $\eta_\mu : N' \to N$ be the matching in (N,N',F,w) given by (3.11). By Theorem 3.4 (i), μ is an equilibrium matching in (N,E,v) if and only if η_μ is an equilibrium matching in (N,N',F,w), which is in turn equivalent to the condition

$$\eta_\mu(i') \in \widetilde{D}_{i'}(q) \setminus \{0\} \qquad (\forall i' \in N') \tag{3.18}$$

by Lemma 3.2 (iii). Since the set $\widetilde{D}_{i'}(q) \setminus \{0\}$ is nonempty by Lemma 3.2 (iii), we have $\widetilde{D}_{i'}(q) \setminus \{0\} = D_i(q)$ for $i \in N$. We also have $\mu(i) = \eta_\mu(i') \neq 0$ for $i \in N$, which implies that the condition (3.18) can be rewritten as $\mu(i) \in D_i(q)$ $(\forall i \in N)$. This concludes the proof of (i).

We then prove the claim (ii). Define a vector $r \in \mathbb{R}^N$ by

$$r(i) = v(\mu(i),i) - q_{\mu(i)} \qquad (i \in N). \tag{3.19}$$

We will show below that (q,r) is an equilibrium payoff in (N,N',F,w). Since $p = (1/2)(q+r)$ holds, it follows from Theorem 3.4 (ii) that p is an equilibrium payoff in (N,E,v).

We now show that (q,r) is an equilibrium payoff in (N,N',F,w). Since $\eta_\mu(i') = \mu(i)$ for $i \in N$ and $v(i,j) = w(i,j')$ for $(i,j) \in E$, the equation (3.19) can be rewritten as

$$r(i') = w(\eta_\mu(i'),i') - q_{\eta_\mu(i')} \qquad (\forall i' \in N').$$

Since η_μ is an equilibrium matching in (N,N',F,w), the condition (3.18) follows from the proof of the claim (i). Hence, we have

$$\begin{aligned} r(i') &= w(\eta_\mu(i'),i') - q_{\eta_\mu(i')} \\ &= \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} \quad (\forall i' \in N'). \end{aligned}$$

Since $q \in \mathscr{H}_0$, this equation and Lemma 3.2 (ii) imply that (q,r) is an equilibrium payoff in (N,N',F,w).

4. CONNECTION TO THE ALGORITHM OF ANDERSSON ET AL.

4.1. Theorems

We consider the algorithm by Andersson et al. (2014b) for finding an equilibrium in the partnership formation problem, and discuss the connection with our algorithm COMPUTEEQUILIBRIUM. The difference between the two algorithms is in the choice of a vector q; while q is selected from the set $\mathscr{H}_0$ in our algorithm, it is computed by a certain price adjustment process in the algorithm by Andersson et al. (2014b). In the following, we show that the vector q used in Andersson et al. (2014b) is the unique minimal vector in $\mathscr{H}_0$. This implies that the algorithm in Andersson et al. (2014b) can be viewed as a specific implementation of our algorithm. We also discuss the relationship between the price adjustment process used in the algorithm of Andersson et al. (2014b) and the algorithm VICKREYENGLISH for the assignment game in Section 2.2.

We first explain the algorithm by Andersson et al. (2014b). In this section, we assume that the values $v(i,j)$ are integers, as in Andersson et al. (2014b). For a payoff $p \in \mathbb{R}^N$ and a set $S \subseteq N$ of agents, we denote by $O(S,p)$ the set of agents $j \in N$ such that all of j's best (possible) partners under the payoff p are contained in the set S, and by $U(S,p)$ the set of agents $j \in N$ such that at least one of j's best (possible) partners of j under the payoff p is contained in S. That is,

$$\begin{aligned} O(S,p) &= \{j \in N \mid D_j(p) \subseteq S\}, \\ U(S,p) &= \{j \in N \mid D_j(p) \cap S \neq \emptyset\}. \end{aligned}$$

A set $S \subseteq N$ is said to be *overdemanded* if $|O(S,p)| > |S|$, and *underdemanded* if $|U(S,p)| < |S|$. If some set $S \subseteq N$ is overdemanded, then it is impossible to assign distinct partners in S to all agents in $O(S,p)$, implying that there exists no matching $\mu : N \to N$ such that $\mu(i) \in D_i(p)$ $(\forall i \in N)$. Similarly, if an underdemanded set exists, then there exists no matching μ with $\mu(i) \in D_i(p)$ $(\forall i \in N)$.

We say that S is *in excess demand* at payoff p if the following condition holds:

$$|U(T,p) \cap O(S,p)| > |T| \qquad (\emptyset \neq \forall T \subsetneq S).$$

It is known (see Andersson et al. (2010, 2013); Mo et al. (1988)) that a set in excess demand exists if an overdemanded set exists (i.e., $|O(S,p)| > |S|$ holds for some $S \subseteq N$), and a maximal set in excess demand is uniquely determined.

We describe below the algorithm by Andersson et al. (2014b) in its variant given in Andersson et al. (2014a). The difference between the algorithm PartnershipFormation and our algorithm is only in the choice of a vector q in Step 1.

Algorithm PartnershipFormation

Step 1: Compute a vector $q \in \mathbb{R}^N$ by the algorithm ProcessAEGHK given below.

Step 2: If there exists no matching $\mu : N \to N$ in (N,E,v) satisfying (3.12), then assert that "no equilibrium exists in (N,E,v)" and stop.

Step 3: Find a matching $\mu : N \to N$ in (N,E,v) satisfying (3.12), and let $p \in \mathbb{R}^N$ be a vector given by (3.13).
Output (μ, p) as an equilibrium of (N,E,v). □

Algorithm ProcessAEGHK

Step 0: Set $q \in \mathbb{Z}^N$ by $q := (0,0,\ldots,0)$.

Step 1: Collect the demand correspondences $D_j(q)$ for $j \in N$.

Step 2: If $|O(S,q)| \leq |S|$ holds for every $S \subseteq N$, then output q and stop.

Step 3: Find the unique maximal set $S^* \subseteq N$ in excess demand at payoff q, update q by $q_i := q_i + 1$ $(i \in S^*)$, and go to Step 1. □

We denote by q^* the output of the algorithm ProcessAEGHK, i.e., q^* is the vector q used in PartnershipFormation. We also denote by $\widehat{q}$ the unique minimal vector in $\mathcal{H}_0$. We will show that $q^* = \widehat{q} \in \mathcal{H}_0$ holds. Hence, the algorithm ProcessAEGHK can be seen as a specific implementation of Step 1 in our algorithm.

Theorem 4.1. *The vector q^* found in the algorithm* ProcessAEGHK *is equal to the unique minimal vector $\widehat{q}$ in $\mathcal{H}_0$.*

Proof. Proof is given in Section 4.2.1. □

We then show that the trajectory of the vector q in the algorithm PROCESSAEGHK is the same as that of the algorithm VICKREYENGLISH applied to the associated assignment game (N,N',F,w). It should be noted that the algorithms PROCESSAEGHK and VICKREYENGLISH use different kind of demand correspondences; the former uses $D_i(q) \subseteq N$, while the latter uses $\widetilde{D}_{j'}(q) \subseteq N \cup \{0\}$.

Theorem 4.2. *Suppose that the algorithm* PROCESSAEGHK *is applied to the partnership formation problem (N,E,v). Then, the trajectory of the vector q is the same as that of the algorithm* VICKREYENGLISH *applied to the associated assignment game (N,N',F,w).*

Proof. Proof is given in Section 4.2.2. □

4.2. Proofs

4.2.1. Proof of Theorem 4.1

Two lemmas are given before we start the proof of Theorem 4.1. We define

$$\mathscr{H} = \{q \in \mathbb{R}^N \mid q_i \geq 0 \ (\forall i \in N),\ |O(S,q)| \leq |S| \ (\forall S \subseteq N)\}. \tag{4.1}$$

This set is used in Andersson et al. (2014b) to prove the validity of their algorithm. In particular, the following fact is proved.

Lemma 4.3 (Andersson et al. (2014b, Theorem 2))**.** *A minimal vector in $\mathscr{H}$ is uniquely determined and equal to the vector q^*.*

We will prove that a vector q is in the set $\mathscr{H}_0$ if and only if q is a vector in $\mathscr{H}$ satisfying the additional condition that

$$\max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} \geq 0 \qquad (\forall j' \in N'). \tag{4.2}$$

This additional condition means that the payoff of every buyer $j' \in N'$ in the associated assignment game (N,N',F,w) is non-negative under the prices given by q.

Lemma 4.4. *The set $\mathscr{H}_0 \subseteq \mathbb{R}^N$ given by (3.5) can be rewritten as*

$$\mathscr{H}_0 = \{q \in \mathscr{H} \mid q \text{ satisfies } (4.2)\}. \tag{4.3}$$

Proof. For $q \in \mathbb{R}^N$, we define a vector $r \in \mathbb{R}^{N'}$ by

$$r_{j'} = \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} \qquad (j' \in N'). \tag{4.4}$$

To prove the equation (4.3), we will show the following equivalence:

$$q \in \mathscr{H}_0 \qquad \text{if and only if} \qquad q \in \mathscr{H} \text{ and } r_{j'} \geq 0 \text{ for all } j' \in N'.$$

We first prove the "only if" part of this equivalence, and then the "if" part.

[Proof of "only if" part] Let $q \in \mathscr{H}_0$. Then, the pair (q,r) is an equilibrium payoff in the associated assignment game (N,N',F,w) by Lemma 3.2 (ii), and therefore we have

$$q_i \geq 0\ (\forall i \in N), \qquad r_{j'} \geq 0\ (\forall j' \in N').$$

To prove $q \in \mathscr{H}$, we need to show that $|O(S,q)| \leq |S|$ holds for every $S \subseteq N$ since $q_i \geq 0$ holds for all $i \in N$. Let $\eta : N' \to N$ be an equilibrium matching in the associated assignment game (N,N',F,w) such that η is a perfect matching. Since $q \in \mathscr{H}_0$, we have $\eta(j') \in \widetilde{D}_{j'}(q) \setminus \{0\} = D_j(q)$ for $j' \in N'$ by Lemma 3.2 (iii). Therefore, it holds that

$$|O(S,q)| = |\{\eta(j') \mid j \in O(S,q)\}| \leq |\bigcup_{j \in O(S,q)} D_j(q)| \leq |S|,$$

where the last inequality is by $D_j(q) \subseteq S$ for $j \in O(S,q)$.

[Proof of "if" part] Suppose that $q \in \mathscr{H}$ and $r_{j'} \geq 0$ for all $j' \in N'$. To prove $q \in \mathscr{H}_0$, it suffices to show that there exists a perfect matching $\eta : N' \to N$ in the associated assignment game (N,N',F,w) such that (η,q,r) is an equilibrium in (N,N',F,w).

For every $j' \in N'$, we have $\widetilde{D}_{j'}(q) \setminus \{0\} = D_j(q)$ since

$$\max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} = r_{j'} \geq 0.$$

It follows that for every $S \subseteq N$,

$$\widetilde{U}(S,q) = \{j' \in N' \mid \widetilde{D}_{j'}(q) \cap S \neq \emptyset\} = \{j \in N \mid D_j(q) \cap S \neq \emptyset\} = U(S,q).$$

We also have $U(S,q) = N \setminus O(N \setminus S,q)$. Hence, it holds that

$$|\widetilde{U}(S,q)| = |U(S,q)| = |N| - |O(N \setminus S,q)| \geq |N| - |N \setminus S| = |S|,$$

where the inequality holds by $q \in \mathscr{H}$. By the well-known Hall's theorem in graph theory (see, e.g., Schrijver (2003)), the condition $|\widetilde{U}(S,q)| \geq |S|$ $(\forall S \subseteq N)$ implies the existence of a perfect matching $\eta : N' \to N$ in (N, N', F, w) such that $\eta(j') \in \widetilde{D}_{j'}(q) \setminus \{0\}$ $(\forall j' \in N')$. Then, it is not difficult to see that (η, q, r) satisfies the conditions (3.6), (3.7), and (3.8), i.e., (η, q, r) is an equilibrium by Lemma 3.2 (i). Thus, we have $q \in \mathscr{H}_0$ by the definition of $\mathscr{H}_0$. □

We now prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.4, it holds that $\widehat{q} \in \mathscr{H}_0 \subseteq \mathscr{H}$. This implies that $q^* \leq \widehat{q}$ since q^* is the unique minimal vector in $\mathscr{H}$. Therefore, for every $j' \in N'$ it holds that

$$
\begin{aligned}
&\max\{w(i,j') - q_i^* \mid i \in N,\ (i,j') \in F\} \\
&\geq \max\{w(i,j') - \widehat{q}_i \mid i \in N,\ (i,j') \in F\} \geq 0,
\end{aligned}
$$

where the last inequality is by $\widehat{q} \in \mathscr{H}$. Hence, we have $q^* \in \mathscr{H}_0$ by Lemma 4.4. From the minimality of $\widehat{q}$ in $\mathscr{H}_0$, we have $q^* \geq \widehat{q}$, implying that $q^* = \widehat{q}$. □

4.2.2. *Proof of Theorem 4.2*

We assume that the trajectory of the vector q is the same until the k-th iteration of the two algorithms, and show that q remains the same in the $(k+1)$-st iteration. For this, we will prove that the set S^* in Step 3 of the k-th iteration in ProcessAEGHK is the same as the set $\widetilde{S}^*$ in Step 2 of the k-th iteration in VickreyEnglish. Recall that S^* and $\widetilde{S}^*$ are defined by using demand correspondences of different kinds.

We consider the vector q at the beginning of the k-th iteration in ProcessAEGHK (and in VickreyEnglish). Since q^* and $\widehat{q}$ are outputs of ProcessAEGHK and VickreyEnglish, respectively, we have $q \leq q^* = \widehat{q}$, where the equality is by Theorem 4.1.

To prove $\widetilde{S}^* = S^*$, we show that the following conditions hold:

$$\widetilde{D}_{j'}(q) \setminus \{0\} = D_j(q) \qquad (\forall j \in N), \tag{4.5}$$

$$\widetilde{U}(Y,q) = U(Y,q) \qquad (\forall Y \subseteq N). \tag{4.6}$$

$$\widetilde{O}(\widetilde{S}^*,q) = O(\widetilde{S}^*,q), \tag{4.7}$$

$$\widetilde{O}(S^*,q) = O(S^*,q). \tag{4.8}$$

The proofs of (4.5)–(4.8) are given later.

We first show that $\widetilde{S}^* \subseteq S^*$ holds. Since S^* is the maximal set in excess demand with respect to the demand correspondences $D_j(q)$, it suffices to show that $\widetilde{S}^*$ is a set in excess demand with respect to the demand correspondences $D_j(q)$. For each Y with $\emptyset \neq Y \subseteq \widetilde{S}^*$, we have

$$|U(Y,q) \cap O(\widetilde{S}^*, q)| = |\widetilde{U}(Y,q) \cap \widetilde{O}(\widetilde{S}^*, q)| > |Y|,$$

where the equality is by (4.6) and (4.7), and the inequality is by the definition of $\widetilde{S}^*$. This shows that $\widetilde{S}^*$ is a set in excess demand with respect to the demand correspondences $D_j(q)$. By the maximality of S^*, we have $\widetilde{S}^* \subseteq S^*$.

Since $\widetilde{S}^*$ is the maximal set in excess demand with respect to the demand correspondences $\widetilde{D}_j(q)$, we can show the inclusion $\widetilde{S}^* \supseteq S^*$ in the same way as above by using (4.6) and (4.8). Hence, we have $\widetilde{S}^* = S^*$.

To conclude the proof, we show that the conditions (4.5)–(4.8) hold.

We first prove (4.5). Since $q \leq \widehat{q}$, we have

$$\begin{aligned}&\max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\}\\&\geq \max\{w(i,j') - \widehat{q}_i \mid i \in N,\ (i,j') \in F\} \geq 0 \quad (\forall j' \in N'),\end{aligned} \tag{4.9}$$

where the last inequality is by $\widehat{q} \in \mathscr{H}_0$ and Lemma 4.4. Therefore, we have

$$\begin{aligned}\widetilde{D}_{j'}(q) &= \arg\max\{w(i,j') - q_i \mid i \in N \cup \{0\},\ (i,j') \in F \cup \{(0,j')\}\}\\&\supseteq \arg\max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\}\\&= \arg\max\{v(i,j) - q_i \mid i \in N,\ (i,j) \in \overline{E}\} = D_j(q)\end{aligned}$$

for $j \in N$, from which (4.5) follows.

The equation (4.6) can be obtained from (4.5) as follows:

$$\widetilde{U}(Y,q) = \{j' \in N' \mid \widetilde{D}_{j'}(q) \cap Y \neq \emptyset\} = \{j \in N \mid D_j(q) \cap Y \neq \emptyset\} = U(Y,q).$$

We finally prove (4.7) and (4.8). By (4.5), we have $D_j(q) \subseteq \widetilde{D}_{j'}(q)$ for all $j \in N$, from which follows that

$$\widetilde{O}(Y,q) = \{j' \in N' \mid \widetilde{D}_{j'}(q) \subseteq Y\} \subseteq \{j \in N \mid D_j(q) \subseteq Y\} = O(Y,q).$$

for every $Y \subseteq N$. Hence, it suffices to show that

$$O(\widetilde{S}^*, q) \subseteq \widetilde{O}(\widetilde{S}^*, q), \qquad O(S^*, q) \subseteq \widetilde{O}(S^*, q).$$

In the following, we prove the latter only; the former can be proven similarly.

To prove the inclusion $O(S^*,q) \subseteq \widetilde{O}(S^*,q)$, we show that $j' \in \widetilde{O}(S^*,q)$ holds for every $j \in O(S^*,q)$.

Let $j \in O(S^*,q)$. Then, $D_j(q) \subseteq S^*$ holds by the definition of $O(S^*,q)$. If $0 \notin \widetilde{D}_{j'}(q)$, then (4.5) implies that $\widetilde{D}_{j'}(q) = D_j(q) \subseteq S^*$, i.e., $j' \in \widetilde{O}(S^*,q)$ holds. We assume, to the contrary, that $0 \in \widetilde{D}_{j'}(q)$ and derive a contradiction.

Since $0 \in \widetilde{D}_{j'}(q)$ and $\widetilde{D}_{j'}(q) \setminus \{0\} = D_j(q) \neq \emptyset$, we have

$$\max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} = 0. \tag{4.10}$$

Let $\bar{q} \in \mathbb{Z}^N$ be a vector given by

$$\bar{q}_i = \begin{cases} q_i + 1 & (\text{if } i \in S^*), \\ q_i & (\text{otherwise}) \end{cases} \qquad (i \in N).$$

That is, $\bar{q}$ is the vector q in the $(k+1)$-st iteration in PROCESSAEGHK. Since $q \leq \bar{q} \leq q^* = \widehat{q}$, we have

$$\begin{aligned} 0 &= \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} \\ &\geq \max\{w(i,j') - \bar{q}_i \mid i \in N,\ (i,j') \in F\} \\ &\geq \max\{w(i,j') - \widehat{q}_i \mid i \in N,\ (i,j') \in F\} \geq 0, \end{aligned}$$

where the equality is by (4.10) and the last inequality is by (4.9). Hence, we have

$$\max\{w(i,j') - \bar{q}_i \mid i \in N,\ (i,j') \in F\} = \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\}. \tag{4.11}$$

It holds that

$$\begin{aligned} &\max\{w(i,j') - \bar{q}_i \mid i \in S^*,\ (i,j') \in F\} \\ &= \max\{w(i,j') - (q_i + 1) \mid i \in S^*,\ (i,j') \in F\} \\ &= \max\{w(i,j') - q_i \mid i \in S^*,\ (i,j') \in F\} - 1 \\ &\leq \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} - 1. \end{aligned} \tag{4.12}$$

We also have

$$\begin{aligned} &\max\{w(i,j') - \bar{q}_i \mid i \in N \setminus S^*,\ (i,j') \in F\} \\ &= \max\{w(i,j') - q_i \mid i \in N \setminus S^*,\ (i,j') \in F\} \\ &\leq \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} - 1, \end{aligned} \tag{4.13}$$

where the inequality is by the facts that $\widetilde{D}_{j'}(q) \setminus \{0\} \subseteq S^*$ and $w(i,j')$ and q_i are integer-valued. It follows from (4.12) and (4.13) that

$$\begin{aligned}
&\max\{w(i,j') - \bar{q}_i \mid i \in N,\ (i,j') \in F\} \\
&= \max\big[\max\{w(i,j') - \bar{q}_i \mid i \in S^*,\ (i,j') \in F\}, \\
&\qquad\qquad \max\{w(i,j') - \bar{q}_i \mid i \in N \setminus S^*,\ (i,j') \in F\}\big] \\
&\le \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\} - 1 \\
&< \max\{w(i,j') - q_i \mid i \in N,\ (i,j') \in F\},
\end{aligned}$$

a contradiction to (4.11). This concludes the proof of (4.8).

A. APPENDIX: REVIEW OF PRICE ADJUSTMENT PROCESSES FOR ASSIGNMENT GAMES

We review price adjustment processes for the assignment game (A,B,F,w). A price adjustment process is an algorithm (a mechanism, more precisely) for finding an equilibrium price vector of the assignment game (i.e., a vector in $\mathscr{H}_0$) by iteratively updating the price vector, where the information of buyers' demand correspondences are used. We have already presented the algorithm VICKREYENGLISH in Section 2.2 as an example of a price adjustment process. We present some other price adjustment processes based on Lyapunov function minimization.

For buyer $j \in B$, we define the indirect utility function $V_j : \mathbb{R}^A_+ \to \mathbb{R}$ by

$$V_j(q) = \max\{w(i,j) - q_i \mid i \in A \cup \{0\}\} \qquad (q \in \mathbb{R}^A_+).$$

We also define the Lyapunov function $L : \mathbb{R}^A \to \mathbb{R}$ by

$$L(q) = \sum_{j \in B} V_j(q) + \sum_{i \in A} q_i \qquad (q \in \mathbb{R}^A_+).$$

Throughout this section, we assume that the values $w(i,j)$ $((i,j) \in F)$ are non-negative integers. It is known (Ausubel (2006), Murota et al. (2016)) that the set of minimizers of the Lyapunov function L coincides with the set of equilibrium price vectors in the assignment game. Moreover, the integrality of values $w(i,j)$ implies the existence of an integral minimizer of the Lyapunov function L; in particular, the unique minimal and maximal minimizers of the Lyapunov function L are integral vectors. Based on this fact, we can compute an equilibrium price vector by finding an integral minimizer of the Lyapunov function L.

The following ascending-type price adjustment process finds a minimizer of the Lyapunov function (Ausubel (2006)). For a set $S \subseteq A$, we denote by $\chi_S \in \{0,1\}^A$ the characteristic vector of S, i.e.,

$$(\chi_S)_i = \begin{cases} 1 & (\text{if } i \in S), \\ 0 & (\text{otherwise}). \end{cases}$$

Algorithm ASCENDMINIMAL

Step 0: Set $q := q^\circ$, where $q^\circ \in \mathbb{Z}^A_+$ is an arbitrary vector that is a lower bound of the minimal vector in $\mathscr{H}_0$ (e.g., $q^\circ = (0,0,\dots,0)$).

Step 1: Collect the demand correspondences $\widetilde{D}_j(q)$ for $j \in B$.

Step 2: Find the unique minimal set $S \subseteq A$ minimizing $L(q+\chi_S)$.

Step 3: If $L(q+\chi_S) = L(q)$, then output q and stop.

Step 4: Update q by $q := q+\chi_S$, and go to Step 1. □

It should be noted that computation of a set $S \subseteq A$ minimizing $L(q+\chi_S)$ in Step 2 can be done by using the demand correspondences $\widetilde{D}_j(q)$ $(j \in B)$ since it holds that

$$L(q+\chi_S) - L(q) = |S| - |\widetilde{O}(S,q)| \qquad (S \subseteq A).$$

It is shown in Murota et al. (2016) that the behavior of the algorithm ASCEND-MINIMAL is exactly the same as the algorithm VICKREYENGLISH in Section 2.2. In particular, the set S computed in Step 2 of each iteration is equal to the the maximal set in excess demand at payoff q.

We consider a variant of the algorithm ASCENDMINIMAL, called ASCENDMAXIMAL, where the initial vector q° is chosen to be a lower bound of the unique maximal vector in $\mathscr{H}_0$ in Step 0, and the unique maximal set $S \subseteq A$ minimizing $L(q+\chi_S)$ is used in Step 2.

Each of the algorithms ASCENDMINIMAL and ASCENDMAXIMAL finds an equilibrium price vector of the assignment game. Moreover, ASCENDMINIMAL (resp., ASCENDMAXIMAL) finds the unique minimal (resp., maximal) equilibrium price vector.

Theorem A.1 (Ausubel (2006); Murota et al. (2016))**.** *For the assignment game (A,B,F,w), the algorithm* ASCENDMINIMAL *(resp.,* ASCENDMAXIMAL*) outputs the unique minimal (resp., maximal) price vector in the set $\mathscr{H}_0$ given by* (2.19).

We then present a descending-type price adjustment process, where the price vector decreases monotonically in each iteration.

Algorithm DESCENDMAXIMAL

Step 0: Set $q := q^\circ$, where $q^\circ \in \mathbb{Z}_+^A$ is an arbitrary vector that is an upper bound of the maximal vector in $\mathscr{H}_0$.

Step 1: Collect the demand correspondences $\widetilde{D}_j(q)$ for $j \in B$.

Step 2: Find the unique minimal set $S \subseteq A$ minimizing $L(q - \chi_S)$.

Step 3: If $L(q - \chi_S) = L(q)$, then output q and stop.

Step 4: Update q by $q := q - \chi_S$, and go to Step 1. □

We also consider a variant, called DESCENDMINIMAL, where the initial vector q° is chosen to be an upper bound of the unique minimal vector in $\mathscr{H}_0$ in Step 0, and the unique maximal set $S \subseteq A$ minimizing $L(q - \chi_S)$ is used in Step 2. It is shown in Murota et al. (2016) that the behavior of DESCENDMINIMAL coincides with that of the descending-type price adjustment process due to Mishra & Parkes (2009).

Theorem A.2 (Ausubel (2006); Murota et al. (2016))**.** *For the assignment game* (A, B, F, w)*, the algorithm* DESCENDMINIMAL *(resp.,* DESCENDMAXIMAL*) outputs the unique minimal (resp., maximal) price vector in the set* $\mathscr{H}_0$ *given by* (2.19)*.*

We finally present the so-called two-phase algorithms for finding an equilibrium (see Murota et al. (2016); see also Murota (2016)). A two-phase algorithm consists of the ascending phase and the descending phase, and the price vector is increasing in the ascending phase and then decreasing in the descending phase. That is, a two-phase algorithm is a combination of two price adjustment processes of ascending-type and descending-type. An important merit of two-phase algorithms is that any price vector can be used as an initial vector, which is impossible in ascending- or descending-type adjustment processes. This flexibility enables us to reduce the number of iterations by selecting an initial vector that is close to an equilibrium price vector.

By the combination of ASCENDMINIMAL/ASCENDMAXIMAL and DESCENDMINIMAL/DESCENDMAXIMAL, we can obtain four variants of two-phase algorithms. For example, the combination of ASCENDMINIMAL and DESCENDMAXIMAL yields the following algorithm (Murota (2016); Murota et al. (2016)).

Algorithm TWOPHASEMINMAX

Step 0: Set $q := q^\circ$, where $q^\circ \in \mathbb{Z}_+^A$ is an arbitrary vector.

Go to Ascending Phase.
Ascending Phase:
Step A1: Collect the demand correspondences $\widetilde{D}_j(q)$ for $j \in B$.
Step A2: Find the unique minimal set $S \subseteq A$ minimizing $L(q+\chi_S)$.
Step A3: If $L(q+\chi_S) = L(q)$, then go to Descending Phase.
Step A4: Update q by $q := q+\chi_S$, and go to Step A1.
Descending Phase:
Step D1: Collect the demand correspondences $\widetilde{D}_j(q)$ for $j \in B$.
Step D2: Find the unique minimal set $S \subseteq A$ minimizing $L(q-\chi_S)$.
Step D3: If $L(q-\chi_S) = L(q)$, then output q and stop.
Step D4: Update q by $q := q-\chi_S$, and go to Step D1. □

The ascending phase stops at the vector $\hat{q}$ that is the unique minimal minimizer of the Lyapunov function in the region $\{q \mid q \geq q^\circ\}$, and the descending phase stops at the vector $\check{q}$ that is the unique maximal minimizer of the Lyapunov function in the region $\{q \mid q \leq \hat{q}\}$, which turns out to be a global minimizer of the Lyapunov function, i.e., an equilibrium price vector (Murota (2016); Murota et al. (2016)).

Theorem A.3 (Murota (2016); Murota et al. (2016)). *For the assignment game* (A,B,F,w)*, the algorithm* TwoPhaseMinMax *outputs a vector in the set* $\mathscr{H}_0$ *given by* (2.19).

The algorithm TwoPhaseMinMax has the following characteristics:

- TwoPhaseMinMax terminates in a smaller number of iterations than the other three variants of the two-phase algorithms.
- The output of TwoPhaseMinMax is neither of minimal and maximal equilibrium price vectors in general.

We finally discuss bounds for the number of iterations required by the algorithms explained above. It can be shown that the worst case bound for the number of iterations required by the algorithms explained above is proportional to $\max\{w(i,j) \mid i \in A,\ j \in B\}$ (see, e.g., Andersson & Erlanson (2013); Murota (2016); Murota et al. (2016)). This means that the number of iterations required by the algorithms is *pseudo-polynomial* in the terminology of algorithm theory.

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HETEROGENEITY IN PREFERENCES AND BEHAVIOR IN THRESHOLD MODELS

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ABSTRACT

A coordination game is repeatedly played on a graph by players (vertices) who have heterogeneous cardinal preferences and whose strategy choice is governed by the individualistic asynchronous logit dynamic. The idea of potential driven autonomy of sets of players is used to derive results on the possibility of heterogeneous preferences leading to heterogeneous behavior. In particular, a class of graphs is identified such that for large enough graphs in this class, diversity in ordinal preferences will nearly always lead to heterogeneity in behavior, regardless of the cardinal strength of the preferences. These results have implications for network design problems, such as when a social planner wishes to induce homogeneous/heterogeneous behavior in a population.

Keywords: Heterogeneity, potential, networks.

JEL Classification Numbers: C72, C73, D02.

1. INTRODUCTION

Ever since the classic treatment of Lewis (1969), game theory has concerned itself with the behavior of individuals and groups within societies when interactions between individuals take the form of a coordination game. One area of this literature has studied perturbed adaptive dynamics (Freidlin & Wentzell,

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1984; Foster & Young, 1991) and looked at long run behavior (Young, 1993; Kandori et al., 1993; Blume, 1996; Peski, 2010; Neary, 2012; Staudigl, 2012) and the speed of convergence of behavior (Young, 2011; Ellison, 2000; Montanari & Saberi, 2010; Newton & Angus, 2015) in binary-choice coordination games under different interaction structures, which can be represented by graphs, with players represented by vertices and interactions between players represented by edges.

A set of players is said to be *autonomous* if predictions can be made about the behavior of players within the set without considering the behavior of players outside of the set. Young (2011) shows how one concept of autonomy, *potential autonomy*,[1] associated with the maximization of a potential function, is related to graph structure, and uses this connection to derive results on the speed of convergence of a population to homogeneous behavior under log-linear dynamics when interactions are identical, symmetric coordination games. Here, it is shown that when interactions are non-identical and asymmetric, these ideas can be used to make statements about the long run behavior of players, in particular about the possibility of convergence to states in which different players play different strategies. Specifically, there is heterogeneity in players' raw preferences for one action over another, and the strength of any given player's preference is given by an individual-specific preference parameter. Conditions under which heterogeneous preferences lead to heterogeneous behavior are given. In particular, a class of graphs, *corpulent* graphs, is identified such that, for large enough graphs in this class, random diversity in ordinal preferences will nearly always lead to heterogeneity in behavior, regardless of the cardinal strength of the preferences.

When preferences are homogeneous, long run behavior under log-linear dynamics is independent of interaction structure (Blume, 1996). However, when preferences are heterogeneous, modifying the graph of interactions can affect long run behavior. Consequently, our results have design implications. For example, a planner may wish to design a network of interactions that leads to a particular pattern of behavior, such as the universal adoption of a new technology. Alternatively, a planner may be faced with a given interaction structure but have limited scope to influence the preferences of some of the individuals. We conclude Section 3 with a discussion of such issues.

The paper is organised as follows. Section 2 gives the basic model. Section

[1] Terminology introduced by Newton & Sercombe (2017) to distinguish potential autonomy from other forms of autonomy.

3 links heterogeneity and potential autonomy. Section 4 considers random preferences.

2. MODEL

The model is a standard one and we follow the notation of Newton & Sercombe (2017), which builds on that of Young (2011). Consider a simple, finite, connected graph $\Gamma = (V, E)$. The vertex set V represents a set of players. The edge set E, consisting of unordered pairs of elements of V, represents connections between players. If two vertices share an edge they are said to be *neighbors*. The number of neighbors of a vertex $i \in V$ is the *degree* of i. For $S \subseteq V$, $S \neq \emptyset$, denote by $d(S)$ the sum of the degrees of vertices in S. For $T, S \subseteq V$, denote by $d(T, S)$ the number of edges $\{i, j\} \in E$ such that $i \in T$ and $j \in S$. For notational convenience we write $d(\{i\})$ as $d(i)$ and $d(\{i\}, S)$ as $d(i, S)$. We write $V \setminus S$ for the complement of S in V.

Let $\{A, B\}$ be the (binary) set of strategies available to the players. A strategy profile σ is a function $\sigma : V \to \{A, B\}$ that associates each player with one of the two strategies. Let σ_S, σ_{-S} denote σ restricted to the domains S and $V \setminus S$ respectively. Let σ^A, σ^B be the strategy profiles such that for all $i \in V$, $\sigma^A(i) = A$, $\sigma^B(i) = B$. Denote by $V_A(\sigma) \subseteq V$ the set of players who play strategy A at profile σ and by $V_B(\sigma) \subseteq V$ the set of players who play strategy B at profile σ. Given the strategies played by i and j, an edge $\{i, j\} \in E$ generates payoffs for i and j as determined by the game in Figure 1. The payoff of player $i \in V$ at profile σ is then the sum of these payoffs over the edges he shares with each of his neighbors on the graph. Formally, player i's payoff at σ is

$$\pi_i(\sigma) = \begin{cases} \gamma_i \, d(i, V_A(\sigma)) & \text{if } \sigma(i) = A \\ (1 - \gamma_i) \, d(i, V_B(\sigma)) & \text{if } \sigma(i) = B \end{cases}. \tag{1}$$

This basic setup is identical to the model of Newton & Sercombe (2017) except for two differences. First, every pairwise interaction is restricted to have zero payoffs off-diagonal. Given that only individual agency is considered in the current paper (i.e. the unit of decision making is always a single player), this is without loss of generality (see the cited work for details). Second, payoffs are allowed to differ between players, whereas the cited work considers symmetric payoff matrices. We refer to γ_i as player i's *type*, which is the (threshold) fraction of i's neighbours required to play strategy B in order for i's payoff

	A	B
A	γ_i, γ_j	0
B	0	$1-\gamma_i, 1-\gamma_j$

Figure 1: For $\{i, j\} \in E$, for each combination of A and B, entries give payoffs for $i \in V$ and $j \in V$ respectively. Both $\gamma_i, \gamma_j \in (0,1)$.

from playing strategy B to be at least as high as his payoff from playing strategy A. Hence the appellation *threshold model* (Granovetter, 1978).

We note that this specification admits an exact potential function (Monderer & Shapley, 1996) given by

$$Potential(\sigma) = \sum_{\substack{\{i,j\}\in E:\\ \sigma(i)=A\\ \sigma(j)=A}} (\gamma_i+\gamma_j) + \sum_{\substack{\{i,j\}\in E:\\ \sigma(i)=A\\ \sigma(j)=B}} \gamma_i + \sum_{\substack{\{i,j\}\in E:\\ \sigma(i)=B\\ \sigma(j)=B}} 1 \tag{2}$$

The potential function aggregates information from the game in a way that retains information on the incentives of players under individual agency. Specifically, if we adjust the strategy of any single player, the change in his payoff equals the change in the potential function.

In the current context, the potential function is important in determining long run behaviour under a dynamic process of strategic adjustment. Specifically, we consider the individualistic asynchronous logit dynamic. Let σ^t denote a strategy profile at time $t \in \mathbb{N}$. At time t, a single vertex $i \in V$ is chosen uniformly at random and with probability

$$\frac{e^{\frac{1}{\eta}\pi_i(A,\sigma_{-i}^{t-1})}}{e^{\frac{1}{\eta}\pi_i(A,\sigma_{-i}^{t-1})} + e^{\frac{1}{\eta}\pi_i(B,\sigma_{-i}^{t-1})}}, \qquad \eta > 0,$$

we let $\sigma^t(i) = A$. Otherwise we let $\sigma^t(i) = B$. For $j \in V$, $j \neq i$, let $\sigma^t(j) = \sigma^{t-1}(j)$.

This process has a unique invariant probability measure μ_η on the state space $\{A,B\}^V$. Blume (1993) shows that as $\eta \to 0$, all mass under μ_η accumulates on the states σ that globally maximize $Potential(\sigma)$. That is, the global maximizers of $Potential(\cdot)$ are the *stochastically stable* (Young, 1993) states of the process.

3. FIXED PREFERENCES AND AUTONOMY

Adapting the terminology of Newton & Sercombe (2017), in turn inspired by Young (2011), a set of players S is *autonomous* if there is some reasonable expectation that players in the set will come to play a subprofile of strategies σ_S regardless of the behaviour of those outside of S. Young (2011) discusses autonomy in terms of potential maximization. Newton & Sercombe (2017) refer to this form of autonomy as *potential autonomy* to distinguish it from *agency autonomy* driven by collective agency. Here we only deal with potential autonomy. A set of players S is σ_S^*-autonomous if, for any strategies played by players outside of S, a higher potential is attained when players in S play σ_S^* than when they play any other strategies.

Definition 1. *$S \subseteq V$ is σ_S^*-autonomous if, for all σ such that $\sigma_S \neq \sigma_S^*$,*

$$Potential\,(\sigma_S^*, \sigma_{-S}) > Potential\,(\sigma).$$

Autonomy will be used to examine the possibility of heterogeneity in preferences leading to heterogeneity in behavior, specifically the possibility of multiple strategies being played at stochastically stable states. Let S^A be the set of players who, all else equal, have a preference for strategy A, and let S^B be the set of players who have a preference for strategy B. That is,

$$S^A := \{i \in V : \gamma_i > 1/2\} \qquad \text{and} \qquad S^B := \{i \in V : \gamma_i < 1/2\}.$$

Let σ^P be the state at which each player plays his preferred strategy. That is,

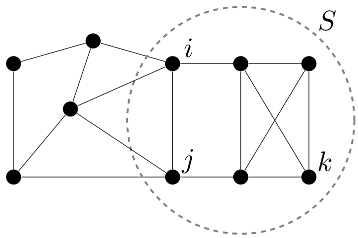
$$\sigma^P(i) = \begin{cases} A & \text{if } i \in S^A \\ B & \text{if } i \in S^B \end{cases}.$$

We now state our first result.

Lemma 1. *Fix a graph $\Gamma = (V, E)$ and a set of types $\{\gamma_i\}_{i \in V}$.*

[1a] *σ^P is the unique stochastically stable state if and only if S^A is $\sigma^A_{S^A}$-autonomous and S^B is $\sigma^B_{S^B}$-autonomous.*

[1b] *σ^A is a stochastically stable state if and only if there does not exist $S \subseteq V$ such that S is σ^B_S-autonomous.*



$$CK(S) = \min_{S' \subseteq S} \frac{d(S',S)}{d(S')} = \frac{3}{8}$$

$$Pl(S) = \frac{d(S,S)}{d(S)} = \frac{9}{22}$$

Figure 2: A graph $\Gamma = (V,E)$ and a subset of vertices $S \subset V$ that illustrate aspects of Lemmas 1, 2, 3. See text for discussion.

[1c] σ^B *is a stochastically stable state if and only if there does not exist* $S \subseteq V$ *such that* S *is* σ^A_S*-autonomous.*

The "if" part of [1a] and the "only if" parts of [1b],[1c] follow immediately from the definition of σ_S-autonomy. The "only if" part of [1a] follows from complementarity of the arguments in the potential function, specifically the fact that if S^A is not $\sigma^A_{S^A}$-autonomous, then $\sigma_{S^A} = \sigma^A_{S^A}$ cannot uniquely maximize potential given $\sigma_{S^B} = \sigma^B_{S^B}$. The "if" parts of [1b],[1c] are proved by showing that if, from σ^A or σ^B, potential can be increased by changing the strategies of a set of players, then some subset of this set must be autonomous for their new strategies.

If the conditions of neither [1b] nor [1c] are met, that is there exist $S, S' \subseteq V$ such that S is σ^B_S-autonomous and S' is $\sigma^A_{S'}$-autonomous, then any stochastically stable state will be heterogeneous. However, such a state will only involve each player playing his preferred strategy if the condition in [1a] holds. If neither [1a] nor [1b] nor [1c] holds, then any stochastically stable state will be both heterogeneous and involve some players playing their less preferred strategy.

Considering Figure 2 and ignoring terms that shall be defined later in the paper, we can, for example, state by [1b] that if the subset of vertices S is σ^B_S-autonomous, then σ^A is not a stochastically stable state. Conversely if there exists no such σ^B_S-autonomous set (over all subsets of vertices), then σ^A is stochastically stable.

Young (2011) shows that potential autonomy depends on the graph theoretic property of close-knittedness, which measures how well integrated each subset of a group of players is with the rest of the group. Our precise definition of close-knittedness follows Newton & Sercombe (2017).[2] The *close-knittedness* of a set $S \subseteq V$ is given by

$$CK(S) := \min_{S' \subseteq S} \frac{d(S', S)}{d(S')}.$$

Young (2011) links potential autonomy and close-knittedness to discuss the speed of convergence to homogeneous strategy profiles. Under heterogeneous preferences, these connections can be used to make statements about the stochastic stability of heterogeneous strategy profiles. Similarly to Proposition 2 of Young (2011), we obtain the following lemma.

Lemma 2. *Fix a graph $\Gamma = (V, E)$ and a set of types $\{\gamma_i\}_{i \in V}$. Then, for any nonempty $S \subseteq V$,*

[2a] *If $CK(S) > \max_{i \in S} 1 - \gamma_i$, then S is σ_S^A-autonomous.*

[2b] *If S is σ_S^A-autonomous, then $CK(S) > \min_{i \in S} 1 - \gamma_i$.*

[2c] *If $CK(S) > \max_{i \in S} \gamma_i$, then S is σ_S^B-autonomous.*

[2d] *If S is σ_S^B-autonomous, then $CK(S) > \min_{i \in S} \gamma_i$.*

That is, sets S with high $CK(S)$ are more likely to be potential autonomous and vice versa. The min and max operators enter because of the heterogeneity of the values of γ_i for $i \in S$. High values of γ_i make $i \in S$ more amenable to playing A, and low values do the opposite. As, by definition, $CK(S) \in [0, 1/2]$ and $\gamma_i \in (0,1)$, it can never be the case that both $CK(S) > 1 - \gamma_i$ and $CK(S) > \gamma_i$, so the conditions in [2a] and [2c] never hold simultaneously.

Returning to Figure 2, we see that $CK(S) = 3/8$. Consequently, by [2c], if every $l \in S$ has type $\gamma_l < 3/8$, then S is σ_S^B-autonomous. Conversely, [2d] tells us that if S is σ_S^B-autonomous, then at least one player $l \in S$ has $\gamma_l < 3/8$. The reason that the converse does not imply the inequality for all players in S is that the subset $S' \subseteq S$ that determines the value of $CK(S)$ need not include all of the players in S. In Figure 2, we see that the minimum over $d(S', S)/d(S')$ is attained when $S' = \{i, j\}$. The constraints on type that σ_S^B-autonomy of S places on vertices such as k which lie outside of this subset are less tight.

If we restrict the set of types, $\{\gamma_i\}_{i \in V}$, so that there are two types of player, those with $\gamma_i = \gamma_A > 1/2$ who prefer strategy A, and those with $\gamma_i = \gamma_B < 1/2$

[2] Young (2011) refers to a set S as 'r-close knit' if $CK(S) \geq r$.

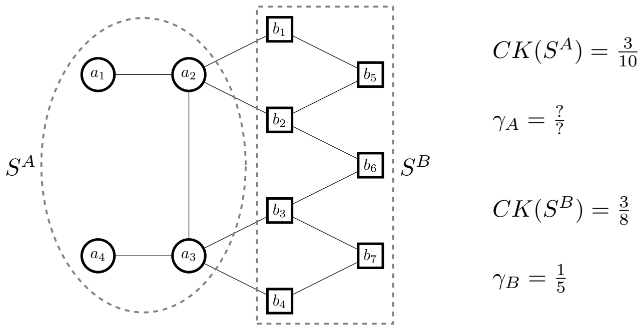


Figure 3: A graph $\Gamma = (V, E)$ and a subset of vertices $S \subset V$. As per the assumptions of Corollary 1, there are two types, γ_A and γ_B, with $\gamma_B < 1/2 < \gamma_A$. Consequently, for $i \in S^A = \{a_1, \ldots, a_4\}$, $\gamma_i = \gamma_A$ and for $i \in S^B = \{b_1, \ldots, b_7\}$, $\gamma_i = \gamma_B$. In the text we use this example to illustrate the use of Lemmas 1, 2 and Corollary 1 in the design of interaction structures.

who prefer strategy B, then we have a network version of the Language Game of Neary (2012). Under this restriction $\gamma_A = \max_{i \in S^A} \gamma_i = \min_{i \in S^A} \gamma_i$ and $\gamma_B = \max_{i \in S^B} \gamma_i = \min_{i \in S^B} \gamma_i$, so Lemma 2 and Lemma 1[a] can be used to give necessary and sufficient conditions for stochastic stability of σ^P in terms of close-knittedness of S^A and S^B. This is captured in the following corollary.

Corollary 1. *Fix a graph $\Gamma = (V, E)$ and a set of types $\{\gamma_i\}_{i \in V}$, such that for all $i \in V$, $\gamma_i \in \{\gamma_A, \gamma_B\}$, $\gamma_B < 1/2 < \gamma_A$. Profile σ^P is the unique stochastically stable state if and only if $CK(S^A) > 1 - \gamma_A$ and $CK(S^B) > \gamma_B$.*

Consider a social planner who wishes to use Lemmas 1, 2 and Corollary 1 to induce a particular pattern of behavior. Suppose the planner is faced with the interaction structure and type profile in Figure 3.

As $CK(S^B) = 3/8 > 1/5 = \gamma_B = \max_{i \in S^B} \gamma_i$, we have, by [2c] of Lemma 2, that S^B is $\sigma^B_{S^B}$-autonomous. Consequently, all players in S^B play B at any stochastically stable state.

It remains to determine the behavior of players in S^A. We find the values of γ_A such that S^A is $\sigma^A_{S_A}$-autonomous. Note that $\gamma_i = \gamma_A$ for all $i \in S^A$. Consequently, $\min_{i \in S^A} \gamma_i = \max_{i \in S^A} \gamma_i = \gamma_A$, so [2a] and [2b] of Lemma 2 imply that S^A is $\sigma^A_{S^A}$-autonomous if and only if $CK(S^A) > 1 - \gamma_A$. Computation yields that $CK(S^A) = 3/10$, so we have that S^A is $\sigma^A_{S^A}$-autonomous if and only if $\gamma_A > 7/10$.

Therefore, if $\gamma_A > 7/10$, then σ^P is the unique stochastically stable state, as predicted by Corollary 1. Further, it can be checked that if $\gamma_A \leq 7/10$, then not only is S^A not $\sigma^A_{S^A}$-autonomous, but no subset $S \subset S^A$ is σ^A_S-autonomous, so by [1c] of Lemma 1, σ^B is stochastically stable.

Now consider the case of a social planner who wishes to induce σ^P but is faced with $\gamma_A < 7/10$. To overcome this problem, she would like to increase the close-knittedness of S^A to obtain a lower threshold value of γ_A. Assume she has the resources to make one of three kinds of amendment: she can add an edge, delete an edge, or change the type of a player.

Adding an edge between players in S^B will increase $CK(S^B)$ while leaving $CK(S^A)$ unaffected. Adding an edge between a player in S^A and a player in S^B will decrease $CK(S^A)$ and $CK(S^B)$. Adding an edge between players in S^A, for example $\{a_1, a_3\}$, will increase $CK(S^A)$ to $1/3$, which in turn lowers the threshold on γ_A to $2/3$. If $\gamma_A > 2/3$, then σ^P will become uniquely stochastically stable. Conversely, if $\gamma_A < 2/3$, then no single additional edge can make σ^P stochastically stable.

Suppose instead that the planner deletes an edge. For this deletion to increase $CK(S^A)$, it must be an edge from a player in S^A to a player in S^B, for example $\{a_2, b_1\}$. This also increases $CK(S^A)$ to $1/3$ and so lowers the threshold on γ_A to $2/3$.

Finally, consider the planner changing the type of a single player. If she converts either a_1 or a_4 to type γ_B, then $CK(S^A)$ is reduced to $2/9$, whereas if she converts either a_2 or a_3, then $CK(S^A)$ is reduced further to $1/6$. Such a conversion might be useful if the planner were trying to encourage uniform adoption of strategy B.[3] In the other direction, if the planner were to convert b_4 to type γ_A, then $CK(S^A)$ would increase to $1/3$ and $CK(S^B)$ would decrease to $5/14$, which is small enough that S^B would still be $\sigma^B_{S^B}$-autonomous.

4. RANDOM PREFERENCES

In this section we consider random preferences and give conditions under which we can usually expect any stochastically stable state to exhibit heterogeneity in strategies. Specifically, we show that as long as there is some diversity in ordinal preferences, then there will usually be diversity of behavior on sufficiently large graphs within a specific class.

First, we shall give conditions under which homogeneous states cannot

[3] It should be remarked that in some contexts, the switch of a player from type γ_A to type γ_B could increase $CK(S^A)$.

be stochastically stable when preferences are fixed. This shall depend on the *plumpness* of sets $S \subseteq V$, which we define as

$$Pl(S) := \frac{d(S,S)}{d(S)}.$$

To use terminology from Young (2011), plumpness measures the *area* $d(S,S)$ of a set S relative to its *perimeter* $d(S, V \setminus S)$.[4]

It is immediate from their definitions that $CK(S) \leq Pl(S) \leq 1/2$. The difference in their definitions relates to potential as follows. From profile σ^B, if S is sufficiently plump relative to $\max_{i \in S} 1 - \gamma_i$, then potential *increases* if we switch S to play σ^A_S. In contrast, Lemma 2 shows that, if S is sufficiently close-knit, then σ^A_S *maximizes* potential given $\sigma^B_{V \setminus S}$. In both cases, σ^B is not stochastically stable.

Lemma 3. *Fix a graph* $\Gamma = (V, E)$ *and a set of types* $\{\gamma_i\}_{i \in V}$. *Then, for any nonempty* $S \subseteq V$,

[3a] *If* $Pl(S) > \max_{i \in S} \gamma_i$, *then* σ^A *is not stochastically stable and there exists* $S' \subseteq S$ *such that* S' *is* $\sigma^B_{S'}$*-autonomous.*

[3b] *If* $Pl(S) > \max_{i \in S} 1 - \gamma_i$, *then* σ^B *is not stochastically stable and there exists* $S' \subseteq S$ *such that* S' *is* $\sigma^A_{S'}$*-autonomous.*

Returning to Figure 2, we see that $Pl(S) = 9/22 > 3/8 = CK(S)$. Thus, for example, if $\max_{i \in S} \gamma_i = 2/5$, we have that $Pl(S) > \max_{i \in S} \gamma_i > CK(S)$. Consequently, we cannot use [2c] to state whether S is σ^B_S-autonomous. That is, we do not know whether potential is always maximized when players in S play B. However, we can use [3a] to infer that potential is higher when all players in S play B than when all players in S play A. Furthermore, we know that S contains a subset $S' \subseteq S$ such that S' is $\sigma^B_{S'}$-autonomous. It can be checked by calculation that the subset $S' \subset S$ comprising the rightmost 4 vertices of S is indeed $\sigma^B_{S'}$-autonomous when $\max_{i \in S} \gamma_i = 2/5$.

Consider a sequence of graphs $\{\Gamma_k\}_{k \in \mathbb{N}_+}$, $\Gamma_k = (V_k, E_k)$. We say that such a sequence is *corpulent* if, for any target level of plumpness, there is a growing

[4] To see this, note that

$$Pl(S) = \frac{d(S,S)}{d(S)} = \frac{d(S,S)}{d(S, V \setminus S) + 2d(S,S)}$$

which is increasing in the ratio of area to perimeter.

number of non-intersecting sets of bounded size which are at least as plump as the target level.

Definition 2. *A sequence of graphs* $\{\Gamma_k\}_{k\in\mathbb{N}_+}$ *is corpulent if, for all* $\phi \in (0, 1/2)$*, there exists* $l \in \mathbb{N}_+$*, such that for all* $n \in \mathbb{N}_+$*, there exists* $\bar{k}$*, such that for all* $k \geq \bar{k}$*,* $\Gamma_k = (V_k, E_k)$ *contains subsets* $\{S^m\}_{m=1}^{m=n}$*,* $S^m \subset V_k$*, such that* $|S^m| \leq l$*,* $S^m \cap S^{m'} = \emptyset$ *for* $m \neq m'$*, and* $Pl(S^m) \geq \phi$ *for all m.*

It follows from the definition that any corpulent sequence will be increasing in size so that $\lim_{k\to\infty} |V_k| = \infty$. Some examples of corpulent families of graphs are square lattices with von-Neumann neighborhood or Moore neighborhood, the Kagome lattice, and the ring (see Figure 4).

The idea of a corpulent sequence of graphs is that as the graphs in such a sequence increase in size, they include an arbitrarily large number of arbitrarily plump subsets. To illustrate this, consider the case when Γ_k is the k by k square lattice with von-Neumann neighborhood. Assume some target level of plumpness, $\phi \in (0, 1/2)$. Consider a subset S of such a Γ_k, such that S is composed of a $\sqrt{l}$ by $\sqrt{l}$ block of vertices so that $|S| = l$, and $d(i) = 4$ for all $i \in S$ (see Figure 5). Then $d(S,S) = 2\sqrt{l}(\sqrt{l}-1)$ and $d(s) = 4l$, so $Pl(S) = (\sqrt{l}-1)/2\sqrt{l}$. This implies that if we choose l large enough, then $Pl(S) > \phi$. For any positive integer n, we can then choose k large enough that Γ_k includes n such sets $S^1, S^2, \ldots, S^n$ that do not intersect one another, thus satisfying the definition of corpulence.

Now, let each γ_i, $i \in V$, be independently drawn according to a probability measure $\mathbb{P}$ on the Borel sets of $(0,1)$. We say that preferences are *ordinally diverse* if there is nonzero probability of a given player $i \in V$ having an ordinal preference for A over B and vice versa.

Definition 3. *Preferences are ordinally diverse if* $\mathbb{P}[\gamma_i \in (0, 1/2)] > 0$ *and* $\mathbb{P}[\gamma_i \in (1/2, 1)] > 0$.

For a given graph $\Gamma = (V, E)$, abuse notation to let $\mathbb{P}[\mathscr{H} \,|\, \Gamma]$ be the probability that one of the two homogeneous states, σ^A or σ^B, is stochastically stable when the types $\{\gamma_i\}_{i\in V}$ are determined according to $\mathbb{P}$.

We can now state the main result of this section. When preferences are ordinally diverse, large corpulent graphs will nearly always have heterogeneity in strategies at stochastically stable states. That is, ordinal diversity in preferences implies diversity in behavior for large graphs within these families.

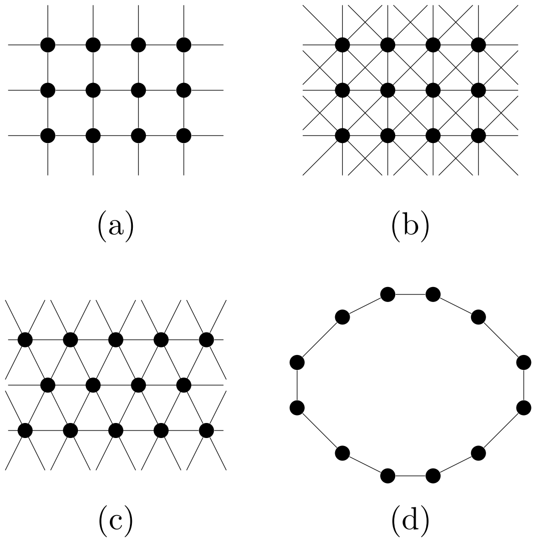


Figure 4: Examples of corpulent families of graphs include (a) square lattice with von-Neumann neighborhood; (b) square lattice with Moore neighborhood; (c) Kagome lattice; (d) the ring.

Theorem 1. *If* $\{\Gamma_k\}_{k\in\mathbb{N}_+}$ *is corpulent and preferences are ordinally diverse then* $\lim_{k\to\infty}\mathbb{P}[\mathscr{H}\,|\,\Gamma_k]=0$.

The intuition behind the theorem is that large graphs in corpulent sequences have large numbers of very plump sets of vertices. Indeed, for an arbitrary target level of plumpness it is possible to choose a graph large enough that it has an arbitrary number of non-intersecting sets that are at least as plump as the target level. Ordinal diversity ensures that, usually, at least some of these sets will be composed solely of players with an ordinal preference in favour of strategy A and some will be composed solely of players with an ordinal preference for strategy B. These preferences may be cardinally very weak, but this does not matter as the target level of plumpness can be adjusted to take account of this. Consequently, large corpulent graphs under random preferences will usually contain sets of players with homogeneous ordinal preferences that are plump enough, per Lemma 3, to destabilize homogeneous behavior in the population.

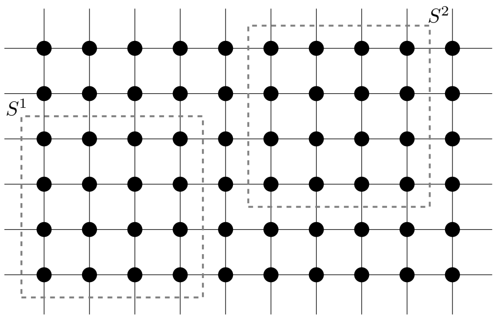


Figure 5: Non-intersecting sets S^1, S^2 such that $l = |S^1| = |S^2| = 16$ and $Pl(S^1) = Pl(S^2) = 3/8$. Arbitrarily large square lattices with von-Neumann neighborhood can include arbitrarily large numbers of such sets.

A. APPENDIX

Proof of Lemma 1[a].

($\Leftarrow$) Assume that S^A is $\sigma^A_{S^A}$-autonomous and S^B is $\sigma^B_{S^B}$-autonomous. As S^A is $\sigma^A_{S^A}$-autonomous, any state σ^* that maximizes potential and is thus stochastically stable must, by Definition 1, have $\sigma^*_{S^A} = \sigma^A_{S^A}$. Similarly, $\sigma^*_{S^B} = \sigma^B_{S^B}$. Therefore, $\sigma^* = (\sigma^A_{S^A}, \sigma^B_{S^B}) = \sigma^P$.

($\Rightarrow$) Assume that σ^P is uniquely stochastically stable and thus uniquely maximizes potential. If S^A is not $\sigma^A_{S^A}$-autonomous, then, by Definition 1, for some σ, $\sigma_{S^A} \neq \sigma^A_{S^A}$,

$$\text{Potential}\,(\sigma^A_{S^A}, \sigma_{-S^A}) \leq \text{Potential}\,(\sigma). \tag{3}$$

Note that, by (2), edges between vertices playing different strategies give lower potential than edges between vertices playing the same strategy. Therefore, (3) implies

$$\text{Potential}\,(\sigma^A_{S^A}, \sigma^B_{-S^A}) \leq \text{Potential}\,(\sigma_{S^A}, \sigma^B_{-S^A}). \tag{4}$$

but as $\sigma^P = (\sigma^A_{S^A}, \sigma^B_{-S^A})$, inequality (4) implies that σ^P does not uniquely maximize potential, so σ^P is not uniquely stochastically stable. Contradiction. Therefore, S^A must be $\sigma^A_{S^A}$-autonomous. Similarly, S^B must be $\sigma^B_{S^B}$-autonomous. $\square$

Proof of Lemma 1[b] *(and by analogy,* [c]*).*

($\Rightarrow$) Assume that σ^A is stochastically stable and thus maximizes potential. If there exists $S \subseteq V$ such that S is σ^B_S-autonomous, then by Definition 1,

$$\text{Potential}\,(\sigma^B_S, \sigma^A_{-S}) > \text{Potential}\,(\sigma^A), \tag{5}$$

contradicting σ^A being a potential maximizer. Therefore, there does not exist $S \subseteq V$ such that S is σ^B_S-autonomous.

($\Leftarrow$) Assume that there does not exist $S \subseteq V$ such that S is σ^B_S-autonomous. If σ^A is not stochastically stable then it does not maximize potential. Amongst all maximizers of potential, choose one, denoted σ^*, such that, denoting $S = \{i \in V : \sigma^*_i = B\}$, for any $S' \subset S$, $\sigma' = (\sigma^B_{S'}, \sigma^A_{-S'})$ does not maximize potential. Then we have that for all $\sigma_S \neq \sigma^B_S$,

$$\text{Potential}\,(\sigma^*) = \text{Potential}\,(\sigma^B_S, \sigma^A_{-S}) > \text{Potential}\,(\sigma_S, \sigma^A_{-S}). \tag{6}$$

Note that by (2), edges between vertices playing different strategies give lower potential than edges between vertices playing the same strategy. Therefore, (6) implies that for any σ, $\sigma_S \neq \sigma^B_S$,

$$\text{Potential}\,(\sigma^B_S, \sigma) > \text{Potential}\,(\sigma), \tag{7}$$

which is the definition of S being σ^B-autonomous. Contradiction. Therefore, σ^A is stochastically stable. □

Proof of Lemma 2[a] *(and by analogy,* [c]*).*

Assume that $CK(S) > \max_{i \in S} 1 - \gamma_i$. Note that as, by (2), edges between vertices playing different strategies give lower potential than edges between vertices playing the same strategy, S is σ^A_S-autonomous if and only if for all $S' \subseteq S$,

$$\text{Potential}\,(\sigma^A_S, \sigma^B_{V \setminus S}) > \text{Potential}\,(\sigma^A_{S \setminus S'}, \sigma^B_{S'}, \sigma^B_{V \setminus S}). \tag{8}$$

Substituting from (2),

$$\begin{aligned}&\text{Potential}\,(\sigma_S^A,\sigma_{V\setminus S}^B)-\text{Potential}\,(\sigma_{S\setminus S'}^A,\sigma_{S'}^B,\sigma_{V\setminus S}^B) \\ &=\sum_{i\in S'} d(i,V\setminus S)(\gamma_i-1)+\sum_{i\in S'} d(i,S\setminus S')(\gamma_i)+\sum_{i,j\in S'}(\gamma_i+\gamma_j-1). \\ &> d(S',V\setminus S)\left(\min_{i\in S'}\gamma_i-1\right)+d(S',S\setminus S')\left(\min_{i\in S'}\gamma_i\right) \\ &\qquad +d(S',S')\left(2\min_{i\in S'}\gamma_i-1\right).\end{aligned} \tag{9}$$

Now,

$$CK(S)>\max_{i\in S}1-\gamma_i=1-\min_{i\in S}\gamma_i, \tag{10}$$

so by definition of $CK(S)$, we have, for all $S'\subseteq S$,

$$\begin{aligned}&\frac{d(S',S)}{d(S')}>1-\min_{i\in S}\gamma_i \\ &\implies d(S',S)-d(S')\left(1-\min_{i\in S}\gamma_i\right)>0 \\ &\implies d(S',S)-d(S')\left(1-\min_{i\in S'}\gamma_i\right)>0 \\ &\implies \underbrace{d(S',S\setminus S')+d(S',S')}_{=d(S',S)} \\ &\qquad -\underbrace{\left(d(S',V\setminus S)+d(S',S\setminus S')+2d(S',S')\right)}_{=d(S')}\left(1-\min_{i\in S'}\gamma_i\right)>0 \\ &\implies d(S',V\setminus S)\left(\min_{i\in S'}\gamma_i-1\right)+d(S',S\setminus S')\left(\min_{i\in S'}\gamma_i\right) \\ &\qquad +d(S',S')\left(2\min_{i\in S'}\gamma_i-2\right)>0.\end{aligned} \tag{11}$$

So (9) and (11) together imply that for all $S'\subseteq S$,

$$\text{Potential}\,(\sigma_S^A,\sigma_{V\setminus S}^B)-\text{Potential}\,(\sigma_{S\setminus S'}^A,\sigma_{S'}^B,\sigma_{V\setminus S}^B)>0, \tag{12}$$

which implies (8), so S is σ_S^A-autonomous. □

Proof of Lemma 2[b] *(and by analogy,* [d]*).*
Assume that S is σ^A_S-autonomous. Then for all $S' \subseteq S$, (8) holds, so

$$\begin{aligned}
&\text{Potential}\,(\sigma^A_S, \sigma^B_{V\setminus S}) - \text{Potential}\,(\sigma^A_{S\setminus S'}, \sigma^B_{S'}, \sigma^B_{V\setminus S}) > 0 \qquad (13)\\
&\implies \sum_{i\in S'} d(i, V\setminus S)(\gamma_i - 1) + \sum_{i\in S'} d(i, S\setminus S')(\gamma_i) + \sum_{i,j\in S'} (\gamma_i + \gamma_j - 1) > 0\\
&\implies d(S', V\setminus S)\left(\max_{i\in S'} \gamma_i - 1\right) + d(S', S\setminus S')\left(\max_{i\in S'} \gamma_i\right)\\
&\qquad + d(S', S')\left(2\max_{i\in S'} \gamma_i - 1\right) > 0\\
&\implies d(S', S) - d(S')\left(1 - \max_{i\in S'} \gamma_i\right) > 0 \quad \text{[by similar algebra to (11)]}\\
&\implies d(S', S) - d(S')\left(1 - \max_{i\in S} \gamma_i\right) > 0\\
&\implies \frac{d(S', S)}{d(S')} > 1 - \max_{i\in S} \gamma_i,
\end{aligned}$$

which implies that

$$CK(S) > 1 - \max_{i\in S} \gamma_i = \min_{i\in S} 1 - \gamma_i. \tag{14}$$

□

Proof of Corollary 1.
Note that

$$\min_{i\in S^A} \gamma_i = \gamma_A = \max_{i\in S^A} \gamma_i, \qquad \min_{i\in S^B} \gamma_i = \gamma_B = \max_{i\in S^B} \gamma_i. \tag{15}$$

Therefore, by Lemma 2[a,b], S^A is $\sigma^A_{S^A}$-autonomous if and only if $CK(S^A) > 1 - \gamma_A$.
Similarly, by Lemma 2[c,d], S^B is $\sigma^B_{S^B}$-autonomous if and only if $CK(S^B) > \gamma_B$.
So S^A is $\sigma^A_{S^A}$-autonomous and S^B is $\sigma^B_{S^B}$-autonomous if and only if $CK(S^A) > 1 - \gamma_A$ and $CK(S^B) > \gamma_B$.
By Lemma 1[a], S^A is $\sigma^A_{S^A}$-autonomous and S^B is $\sigma^B_{S^B}$-autonomous if and only if σ^P is the unique stochastically stable state. □

Proof of Lemma 3[a] *(and by analogy,* [b]*).*
We have

$$Pl(S) = \frac{d(S,S)}{d(S)} > \max_{i \in S} \gamma_i \tag{16}$$
$$\implies d(S) \max_{i \in S} \gamma_i - d(S,S) < 0.$$

Now the potential difference between σ^A and $(\sigma^B_S, \sigma^A_{V \setminus S})$ equals

$$\begin{aligned}
&\text{Potential}\,(\sigma^A) - \text{Potential}\,(\sigma^B_S, \sigma^A_{V \setminus S}) \\
&= \sum_{i \in S} d(i, V \setminus S)\, \gamma_i + \sum_{i,j \in S} \gamma_i + \gamma_j - 1 \\
&\leq d(S, V \setminus S) \max_{i \in S} \gamma_i + d(S,S) \left(2 \max_{i \in S} \gamma_i - 1 \right) \\
&= \underbrace{(d(S) - 2d(S,S))}_{= d(S, V \setminus S)} \max_{i \in S} \gamma_i + d(S,S) \left(2 \max_{i \in S} \gamma_i - 1 \right) \\
&= d(S) \max_{i \in S} \gamma_i - d(S,S) \\
&\underbrace{<}_{\text{by (16)}} 0.
\end{aligned} \tag{17}$$

Therefore, σ^A does not maximize potential and is thus not stochastically stable. Consider σ_S that maximize potential given $\sigma_{V \setminus S} = \sigma^A_{V \setminus S}$. Consider such a σ_S, denoted σ^*_S, such that, denoting $S^* = \{i \in S : \sigma^*_i = B\} \subseteq S$, for any $S' \subset S^*$, $\sigma' = (\sigma^B_{S'}, \sigma^A_{-S'})$ does not maximize potential. Then we have that, for all $\sigma_{S^*} \neq \sigma^B_{S^*}$,

$$\text{Potential}\,(\sigma^B_{S^*}, \sigma^A_{-S^*}) > \text{Potential}\,(\sigma_S, \sigma^A_{-S^*}). \tag{18}$$

Note that by (2), edges between vertices playing different strategies give lower potential than edges between vertices playing the same strategy. Therefore, (6) implies that for any σ, $\sigma_{S^*} \neq \sigma^B_{S^*}$,

$$\text{Potential}\,(\sigma^B_{S^*}, \sigma) > \text{Potential}\,(\sigma), \tag{19}$$

which is the definition of S^* being σ^B-autonomous. $\square$

Proof of Theorem 1.

As preferences are assumed to be ordinally diverse, by Definition 3, there exists $\phi < 1/2$ such that $\mathbb{P}[(0,\phi)] =: \rho > 0$.

As $\{\Gamma_k\}_{k\in\mathbb{N}_+}$ is corpulent, by Definition 2 there exists l such that for all $n \in \mathbb{N}_+$, there exists $\bar{k}(n)$ such that for $k \geq \bar{k}(n)$ we can choose non-intersecting sets $\{S^m\}_{1\leq m\leq n}$, $|S^m| \leq l$ such that $Pl(S^m) \geq \phi$.

For given Γ_k, S^m, as $|S^m| \leq l$, the probability that all $i \in S^m$ have $\gamma_i < \phi$, and hence $\phi > \max_{i\in S^m}$, is bounded below by $\rho^l > 0$. The probability that this holds for at least one such S^m is thus bounded below by $1-(1-\rho^l)^n$ for $k \geq \bar{k}$. This probability approaches one as $n \to \infty$.

So, with probability approaching one as $k \to \infty$, there exists $S^m \subseteq V_k$ such that $Pl(S^m) \geq \phi > \max_{i\in S^m} \gamma_i$, and by Lemma 3, σ^A is not stochastically stable. A similar argument holds for σ^B. $\square$

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