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INFINITE-DIMENSIONAL INPUT-TO-STATE STABILITY AND ORLICZ SPACES*

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Abstract. In this work, the relation between input-to-state stability and integral input-to-state stability is studied for linear infinite-dimensional systems with an unbounded control operator. Although a special focus is laid on the case L^∞, general function spaces are considered for the inputs. We show that integral input-to-state stability can be characterized in terms of input-to-state stability with respect to Orlicz spaces. Since we consider linear systems, the results can also be formulated in terms of admissibility. For parabolic diagonal systems with scalar inputs, both stability notions with respect to L^∞ are equivalent.

Key words. Input-to-state stability, integral input-to-state stability, C_0-semigroup, admissibility, Orlicz spaces

AMS subject classifications. 93D20, 93C05, 93C20, 37C75

1. Introduction.

In systems and control theory, the question of stability is a fundamental issue. Let us consider the situation where the relation between the input (function) u and the state x is governed by the autonomous equation

$$\dot{x} = f(x,u), \quad x(0) = x_0. \tag{1.1}$$

One can then distinguish between *external stability*, that is, stability with respect to the input u, and *internal stability*, i.e. when $u = 0$. For the moment, f is assumed to map from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, and to be such that solutions x exist on $[0,\infty)$ for all inputs u in a function space Z. Already from this very general view-point, it seems clear that stability notions may strongly depend on the specific choice of Z (and its norm). The concept of *input-to-state stability* (ISS) combines both external and internal stability in one notion. If Z is chosen to be $L^\infty(0,\infty;U)$, $U = \mathbb{R}^m$, a system is called ISS (with respect to L^∞) if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \le \beta(\|x_0\|, t) + \gamma(\operatorname*{ess\,sup}_{s\in[0,t]} \|u(s)\|_U),$$

for all $t > 0$ and $u \in Z$. Here the sets $\mathcal{KL}$ and $\mathcal{K}$ refer to the classic comparison functions from nonlinear systems theory, see Section 2. Introduced by E. Sontag in 1989 [27], ISS has been intensively studied in the past decades; see [29] for a survey. A related stability notion is *integral input-to-state stability* (iISS) [28, 2], which means

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[0]The contents of this article emerged based on previous findings of the authors on input-to-state stability for parabolic systems that were published in the proceedings article [7]. However, this article provides a far more general and different approach using Orlicz spaces. This new approach also allowed to extend the theory essentially.

that for some $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$,

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \theta\left(\int_0^t \mu(\|u(s)\|)_U)\, ds\right), \tag{1.2}$$

for all $t > 0$ and $u \in Z = L^\infty(0, \infty; U)$. This property differs from ISS in the sense that it allows for unbounded inputs u that have "finite energy", see [28]. Many practically relevant systems are iISS whereas they are not ISS, see e.g. [19] for a detailed list. However, for linear systems, i.e., $f(x, u) = Ax + Bu$ with matrices A and B, iISS is equivalent to ISS. To some extent, this observation marks the starting point of this work.

In contrast to the well-established theory for finite-dimensions, a more intensive study of (integral) input-to-state stability for infinite-dimensional systems has only begun recently. We refer to [4, 5, 11, 12, 13, 16, 17, 18, 19, 20]. By nature, in the infinite-dimensional setting, the stability notions from finite-dimensions are more subtle. We refer to [21] for a listing of failures of equivalences around ISS known from finite-dimensional systems. In most of the mentioned infinite-dimensional references, systems of the form (1.1) with $f\colon X \times U \to X$ and Banach spaces X and U are considered. For linear equations, this setting corresponds to evolution equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1.3}$$

where B is a bounded control operator (note that for fixed t, $x(t) = x(t, \cdot)$ is a function and $\dot{x}$ denotes the time-derivative). Analogously to finite-dimensions, in this case, ISS and iISS are known to be equivalent, see e.g., [19, Cor. 2] and Proposition 2.14 below. However, concerning applications the requirement of bounded control operators B is rather restrictive. Typical examples for systems which only allow for a formulation with an unbounded B are boundary control systems. It is clear that such phenomena cannot occur for linear systems in finite-dimensions.

The main point of this paper is to relate and characterize (integral) input-to-state stability for linear, infinite-dimensional systems with unbounded control operators, i.e. systems of the form (1.3) with unbounded operators B. This is done by using the notion of *admissibility*, [25, 31], which also reveals the connection of the mentioned stability types with the boundedness of the linear mapping

$$Z \to X, \qquad u \mapsto x(t)$$

(for $x_0 = 0$). It is not surprising that the choice of topology for Z, the space of inputs u, is crucial here. However, looking at (1.2) for $x_0 = 0$, it is not clear how the right-hand side could define a norm for general functions μ and θ. The question of the right norm for Z motivates one to study ISS and iISS with respect to general spaces Z – not only $Z = L^\infty = L^\infty(0, \infty; U)$. For the precise definition of these notions, we refer to Section 2. We show that Z-ISS and Z-iISS are equivalent for $Z = L^p = L^p(0, \infty; U)$, $p \in [1, \infty)$. However, it turns out that this paves the way to characterize L^∞-iISS in terms of ISS. More precisely, we will show that L^∞-iISS is equivalent to ISS with respect to some *Orlicz space*. This is one of the main results of this work. Orlicz spaces (or Orlicz–Birnbaum spaces) appear naturally as generalizations of L^p-spaces and ISS with respect to such spaces can thus be seen as a generalization of classical stability notions. Other choices for general input functions have been made in the literature – like admissibility with respect to Lorentz spaces [6, 33] or Z-ISS with Z

	Eq. (1.3), B bounded	Eq. (1.3), B unbounded	Eq. (1.1), f nonlinear
$\dim X < \infty$	ISS $\Longleftrightarrow$ iISS	ISS $\Longleftrightarrow$ iISS	ISS $\underset{\not\Leftarrow}{\Longrightarrow}$ iISS
$\dim X = \infty$	ISS $\Longleftrightarrow$ iISS	ISS $\underset{(\overset{?}{\Longrightarrow})}{\Longleftarrow}$ iISS	not clear

Table 1.1
The relation between ISS and iISS (with respect to L^∞) in various settings.

being a Sobolev space [9, 18].

As we will see, it is plain that Z-iISS always implies Z-ISS for linear systems. The converse direction, for $Z = L^\infty$, remains open in general. It is known that ISS is equivalent to admissibility (together with exponential stability). We will show that L^∞-iISS in fact implies *zero-class admissibility* [8, 34], which is slightly stronger than admissibility, see Proposition 2.13. In Table 1.1, the relation of L^∞-ISS and L^∞-iISS in the various above-mentioned settings is depicted schematically.

In Section 2, we will discuss the setting and formally introduce the stability notions mentioned above. This includes a general abstract definition of ISS, iISS and admissibility with respect to some function space Z. Furthermore, we will give some basic facts about their relation.

Section 3 deals with the characterization of ISS and iISS in terms of Orlicz-space-admissibility. As a main result, we show that L^∞-iISS is equivalent to ISS with respect to some Orlicz space E_Φ, where Φ denotes a Young function, Theorem 3.16. Moreover, we show that ISS with respect to an Orlicz space is a natural generalization of classic L^p-ISS that "interpolates" the notions of L^1- and L^∞-ISS, Theorems 3.17 and 3.19.

In Section 4, we consider parabolic diagonal systems with scalar input. More precisely, we assume that A possesses a Riesz basis of eigenvectors with eigenvalues lying in a sector in the open left half-plane. For this class of systems we show that L^∞-ISS implies ISS with respect to some Orlicz space and thus, by the results of Section 3, the equivalence between iISS and ISS, known in finite dimensions, holds for this class of systems. Moreover, it turns out that any linear, bounded operator from U to the extrapolation space X_{-1} is L^∞-admissible, which yields a characterization of ISS. The results of this section partially generalize results that were already indicated in [7].

We illustrate the obtained results by examples in Section 5. In particular, we present a parabolic diagonal system which is L^∞-ISS, but not L^p-ISS for any $p \in [1,\infty)$. Finally, we conclude by drawing a connection between the question whether L^∞-ISS implies L^∞-iISS and a problem due to G. Weiss.

2. Stability notions for infinite-dimensional systems.

2.1. The setting and definitions. In this article we study systems $\Sigma(A,B)$ of the following form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \geq 0, \tag{2.4}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and B is a linear and bounded operator from a Banach space U to the extrapolation space X_{-1}. Note that B is possibly unbounded from U to X. Here X_{-1} is the completion of X with respect to the norm

$$\|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X,$$

for some $\beta \in \rho(A)$, the resolvent set of A. It can be shown that the semigroup $(T(t))_{t\geq 0}$ possesses a unique extension to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} with generator A_{-1}, which is an extension of A. Thus we may consider equation (2.4) on the Banach space X_{-1} and therefore for $u \in L^1_{loc}(0,\infty;U)$, the *(mild) solution of* (2.4) is given by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,ds, \qquad t \geq 0. \tag{2.5}$$

In this paper, we will consider the following types of function spaces.

Assumption 2.1. For a Banach space U, let $Z \subseteq L^1_{loc}(0,\infty;U)$ be such that for all $t > 0$

(a) $Z(0,t;U) := \{f \in Z \mid f|_{[t,\infty)} = 0\}$ becomes a Banach space of functions on the interval $(0,t)$ with values in U (in the sense of equivalence classes w.r.t. equality almost everywhere),

(b) $Z(0,t;U)$ is continuously embedded in $L^1(0,t;U)$, that is, there exists $\kappa(t) > 0$ such that for all $f \in Z(0,t;U)$ it holds that $f \in L^1(0,t;U)$ and

$$\|f\|_{L^1(0,t;U)} \leq \kappa(t)\|f\|_{Z(0,t;U)}.$$

(c) For $u \in Z(0,t;U)$ and $s > t$ we have $\|u\|_{Z(0,t;U)} = \|u\|_{Z(0,s;U)}$.

(d) $Z(0,t;U)$ is invariant under the left-shift and reflection, i.e., $S_\tau Z(0,t;U) \subset Z(0,t;U)$ and $R_t Z(0,t;U) \subset Z(0,t;U)$, where

$$S_\tau u = u(\cdot + \tau), \quad R_t u = u(t - \cdot),$$

and $\tau > 0$. Furthermore, $\|S_\tau\|_{\mathcal{L}(Z(0,t;U))} \leq 1$ and R_t is isometric.

(e) For all $u \in Z$ and $0 < t < s$ it holds that $u|_{(0,t)} \in Z(0,t;U)$ and

$$\|u|_{(0,t)}\|_{Z(0,t;U)} \leq \|u|_{(0,s)}\|_{Z(0,s;U)}.$$

If additionally we have in (b) that

$$\kappa(t) \to 0, \quad \text{as } t \searrow 0, \tag{B}$$

then we say that Z satisfies *condition (B)*.

For example, $Z = L^p$ refers to the spaces $L^p(0,t;U)$, $t > 0$, for fixed $1 \leq p \leq \infty$ and U. Other examples can be given by Sobolev spaces and the Orlicz spaces $L_\Phi(0,t;U)$ and $E_\Phi(0,t;U)$, see the appendix. If $p > 1$ (including $p = \infty$) and Φ is a Young function, then L^p, E_Φ and L_Φ satisfy Condition (B), thanks to Hölder's inequality. Clearly, L^1 does not satisfy condition (B).

In general, the state $x(t)$ given by (2.5) lies in X_{-1} for $u \in L^1_{loc}$ and $t > 0$. The notion of *admissibility* ensures that indeed $x(t) \in X$.

Definition 2.2. *We call the system* $\Sigma(A,B)$ admissible with respect to Z *(or* Z-admissible*), if*

$$\int_0^t T_{-1}(s)Bu(s)\,ds \in X \tag{2.6}$$

for all $t > 0$ and $u \in Z(0,t;U)$. If $\Sigma(A,B)$ is admissible with respect to Z, then all mild solutions (2.5) are in X and by the closed graph theorem there exists a constant $c(t)$ (take the infimum over all possible constants) such that

$$\left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| \le c(t)\|u\|_{Z(0,t;U)}. \tag{2.7}$$

Moreover, it is easy to see that $\Sigma(A,B)$ is admissible if (2.6) holds for one $t > 0$.

Definition 2.3. *We call the system* $\Sigma(A,B)$ infinite-time admissible with respect to Z *(or* Z-infinite-time admissible*), if the system is admissible with respect to* Z *and* $c_\infty := \sup_{t>0} c(t)$ *is finite. We call the system* $\Sigma(A,B)$ zero-class admissible with respect to Z *(or* Z-zero-class admissible*), if it is admissible with respect to* Z *and* $\lim_{t\to 0} c(t) = 0$.

Remark 2.4. Clearly, zero-class admissibility and infinite-time admissibility imply admissibility respectively.

Since $Z \subseteq L^1_{loc}(0,\infty;U)$, for any $u \in Z$ and any initial value x_0, the mild solution x of (2.4) is continuous as function from $[0,\infty)$ to X_{-1}. Next we show that zero-class admissibility guarantees that x even lies in $C(0,\infty;X)$.

Proposition 2.5. *If* $\Sigma(A,B)$ *is* Z*-zero-class admissible, then for every* $x_0 \in X$ *and every* $u \in Z$ *the mild solution of* (2.4)*, given by* (2.5)*, satisfies* $x \in C([0,\infty);X)$.

Proof. Since x is given by (2.5), it suffices to consider the case $x_0 = 0$. Let $u \in Z$. We have to show that $t \mapsto \Phi_t u := \int_0^t T_{-1}(s)Bu(s)\,ds$ is continuous. The proof is divided into two steps.
First, note that $t \mapsto \Phi_t u$ is right-continuous on $[0,\infty)$. In fact, by

$$\Phi_{t+h}u - \Phi_t u = T(t)\int_0^h T_{-1}(s)Bu(s+t)\,ds,$$

$h > 0$, and Z-zero-class admissibility, it follows that

$$\|\Phi_{t+h}u - \Phi_t u\| \le c(h)\|T(t)\|\|u(\cdot+t)\|_{Z(0,h;U)} \to 0$$

for $h \searrow 0$ (where we used properties (d), (e) of Z).
Second, we show that $t \mapsto \Phi_t$ is left-continuous on $(0,\infty)$. Since $(\Phi_t - \Phi_{t-h})u = (\Phi_t - \Phi_{t-h})u|_{(0,t)}$, we can assume that $u \in Z(0,t;U)$. Clearly,

$$(\Phi_t - \Phi_{t-h})u = T(t-h)\int_0^h T_{-1}(s)Bu(s+t-h)\,ds.$$

It follows that

$$\begin{aligned}\left\|\int_0^h T_{-1}(s)Bu(s+t-h)\,ds\right\| &\le c(h)\|u(\cdot+t-h)\|_{Z(0,h;U)}\\ &\le c(h)\|u(\cdot+t-h)\|_{Z(0,t;U)}\\ &\le c(h)\|u\|_{Z(0,t;U)} \xrightarrow{h\searrow 0} 0,\end{aligned}$$

where the last two inequalities hold by properties (e) and (d) of Z. Since $(T(t))_{t\ge 0}$ is uniformly bounded on compact intervals, we conclude that $\|\Phi_{t+h}u - \Phi_t u\| \to 0$ as $h \to 0$. ∎

Remark 2.6. If $\Sigma(A,B)$ is admissible with respect to L^p, $1 \le p < \infty$, then, by Hölder's inequality, $\Sigma(A,B)$ is L^q-zero-class admissible for any $q > p$. Thus, Proposition 2.5 implies that the mild solution of (2.4) lies in $C(0,\infty;X)$ for all $u \in L^q$. Moreover, this continuity even holds for $u \in L^p$, which was already shown by G. Weiss in his seminal paper [31, Prop. 2.3] on admissible control operators. However, there, a direct, but similar proof is used without using the notion of zero-class admissibility. As stated in [31, Problem 2.4], it is an interesting open problem whether the continuity of x is implied by L^∞-admissibility. By Proposition 2.5, the answer is 'yes' in the case of L^∞-zero-class admissibility. See also Section 6.

To introduce input-to-state stability, we will need the following well-known function classes from Lyapunov theory. Here, $\mathbb{R}_0^+$ denotes the set of nonnegative real numbers.

$$\begin{aligned}
\mathcal{K} &= \{\mu\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \mu(0)=0,\ \mu \text{ continuous, strictly increasing}\},\\
\mathcal{K}_\infty &= \{\theta \in \mathcal{K} \mid \lim_{x\to\infty} \theta(x) = \infty\},\\
\mathcal{L} &= \{\gamma\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid \gamma \text{ continuous, strictly decreasing}, \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &= \{\beta\colon (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot,t) \in \mathcal{K}\ \forall t \ge 0 \text{ and } \beta(s,\cdot) \in \mathcal{L}\ \forall s > 0\}.
\end{aligned}$$

DEFINITION 2.7. *The system* $\Sigma(A,B)$ *is called* input-to-state stable with respect to Z *(or* Z-ISS*), if there exist functions* $\beta \in \mathcal{KL}$ *and* $\mu \in \mathcal{K}_\infty$ *such that for every* $t \ge 0$*,* $x_0 \in X$ *and* $u \in Z(0,t;U)$

(i) $x(t)$ *lies in* X *and*

(ii) $\|x(t)\| \le \beta(\|x_0\|,t) + \mu(\|u\|_{Z(0,t;U)})$.

The system $\Sigma(A,B)$ *is called* integral input-to-state stable with respect to Z *(or* Z-iISS*), if there exist functions* $\beta \in \mathcal{KL}$*,* $\theta \in \mathcal{K}_\infty$ *and* $\mu \in \mathcal{K}$ *such that for every* $t \ge 0$*,* $x_0 \in X$ *and* $u \in Z(0,t;U)$

(i) $x(t)$ *lies in* X *and*

(ii) $\|x(t)\| \le \beta(\|x_0\|,t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\, ds\right)$.

The system $\Sigma(A,B)$ *is called* uniformly bounded energy bounded state with respect to Z *(or* Z-UBEBS*), if there exist functions* $\gamma,\theta \in \mathcal{K}_\infty$*,* $\mu \in \mathcal{K}$ *and a constant* $c > 0$ *such that for every* $t \ge 0$*,* $x_0 \in X$ *and* $u \in Z(0,t;U)$

(i) $x(t)$ *lies in* X *and*

(ii) $\|x(t)\| \le \gamma(\|x_0\|) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\, ds\right) + c$.

Remark 2.8.
1. By the inclusion of L^p spaces on bounded intervals we obtain that L^p-ISS (L^p-iISS, L^p-UBEBS) implies L^q-ISS (L^q-iISS, L^q-UBEBS) for all $1 \le p < q \le \infty$. Further the inclusions $L^\infty \subseteq E_\Phi \subseteq L_\Phi \subseteq L^1$ and $Z \subseteq L^1_{loc}$ yield a corresponding chain of implications of ISS, iISS and UBEBS.
2. Note that in general the integral $\int_0^t \mu(\|u(s)\|_U)\, ds$ in the inequalities defining Z-iISS and Z-UBEBS may be infinite. In that case, the inequalities hold trivially. This indicates that the major interest in iISS and UBEBS lies in the case $Z = L^\infty$, in which the integral is always finite.

2.2. Relations between the stability notions. Recall that the semigroup $(T(t))_{t\ge 0}$ is called exponentially stable, if there exist constants $M,\omega > 0$ such that

$$\|T(t)\| \le M e^{-\omega t}, \quad t \ge 0. \tag{2.8}$$

Lemma 2.9. *Let $(T(t))_{t\geq 0}$ be exponentially stable and $\Sigma(A,B)$ be Z-admissible. Then the following holds.*

(i) $\Sigma(A,B)$ is infinite-time Z-admissible.

(ii) $\Sigma(A,B)$ is Z-iISS if and only if there exist $\theta\in\mathcal{K}_\infty$ and $\mu\in\mathcal{K}$ such that for every $u\in Z(0,1;U)$,

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \leq \theta\left(\int_0^1 \mu(\|u(s)\|_U)\,ds\right). \tag{2.9}$$

Moreover, if (2.9) holds, then $\Sigma(A,B)$ is Z-iISS with the same choice of μ.

Proof. By the representation of the solution (2.5) for $x_0=0$, it follows that the condition in (ii) is necessary for Z-iISS. For the sufficiency it is enough to consider $x_0=0$ by exponential stability. Therefore, both (i) and (ii) hold if we can show that there exists $C>0$ such that for any $t>0$ and $u\in Z(0,t;U)$, there exists $\tilde{u}\in Z(0,1;U)$ such that the following three inequalities hold:

$$\begin{aligned}
\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| &\leq C\left\|\int_0^1 T_{-1}(s)B\tilde{u}(s)\,ds\right\|,\\
\|\tilde{u}\|_{Z(0,1;U)} &\leq \|u\|_{Z(0,t;U)},\\
\int_0^1 \mu(\|\tilde{u}(s)\|_U)\,ds &\leq \int_0^t \mu(\|u(s)\|_U)\,ds \quad \forall \mu\in\mathcal{K}.
\end{aligned}$$

Without loss of generality, we assume that $t\in\mathbb{N}$, otherwise extend u suitably by the zero-function. By splitting the integral, substitution and the fact that $\Sigma(A,B)$ is Z-admissible, we get for $u\in Z(0,t;U)$,

$$\begin{aligned}
\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| &= \left\|\sum_{k=0}^{t-1}\int_k^{k+1} T_{-1}(s)Bu(s)\,ds\right\|\\
&= \left\|\sum_{k=0}^{t-1} T(k)\int_0^1 T_{-1}(s)Bu(s+k)\,ds\right\|\\
&\leq \sum_{k=0}^{t-1}\|T(k)\| \max_{k=0,..,t-1}\left\|\int_0^1 T_{-1}(s)Bu(s+k)\,ds\right\|\\
&\leq C\cdot \max_{k=0,..,t-1}\left\|\int_0^1 T_{-1}(s)Bu(s+k)\,ds\right\|,
\end{aligned}$$

where $C<\infty$ only depends on the exponentially stable semigroup $(T(t))_{t\geq 0}$. Choose $\tilde{u}=u(\cdot+k_0)|_{(0,1)}$, where k_0 is the argument such that the above maximum is attained. Clearly, $\int_0^1\mu(\|\tilde{u}(s)\|_U)\,ds\leq\int_0^t\mu(\|u(s)\|_U)\,ds$. We now use the properties of Z described in Assumption 2.1. By (d), $u(\cdot+k_0)\in Z(0,t;U)$ and $\|u(\cdot+k_0)\|_{Z(0,t;U)}\leq \|u\|_{Z(0,t;U)}$. Therefore, Property (e) implies that $\tilde{u}\in Z(0,1;U)$ with $\|\tilde{u}\|_{Z(0,1;U)}\leq \|u(\cdot+k_0)\|_{Z(0,t;U)}\leq\|u\|_{Z(0,t;U)}$. ∎

Note that (i) in Lemma 2.9 for the case $Z=L^p$ is well-known and can e.g. be found in [30] for $p=2$.

Proposition 2.10. *Let $Z\subseteq L^1_{loc}(0,\infty;U)$ be a function space. Then we have:*

(i) The following statements are equivalent

(a) *$\Sigma(A,B)$ is Z-ISS,*

(b) $\Sigma(A,B)$ *is Z-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable,*
(c) $\Sigma(A,B)$ *is Z-infinite-time admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

(ii) If $\Sigma(A,B)$ *is Z-iISS, then the system is Z-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable,*

(iii) If $\Sigma(A,B)$ *is Z-UBEBS, then the system is Z-admissible and* $(T(t))_{t\geq 0}$ *is bounded, that is, (2.8) holds for* $\omega = 0$.

Proof. Clearly, Z-ISS, Z-iISS and Z-UBEBS imply Z-admissibility (consider $x_0 = 0$ in (2.5) and observe that $x(t) \in X$ for all $t > 0$). Further, Z-admissibility and exponential stability of $(T(t))_{t\geq 0}$ show Z-ISS, see Remark 2.4. If, $\Sigma(A,B)$ is Z-ISS or Z-iISS, by setting $u = 0$, it follows that $\|T(t)\| < 1$ for sufficiently large t, which shows that $(T(t))_{t\geq 0}$ is exponentially stable. It is easy to see that Z-UBEBS implies boundedness of $(T(t))_{t\geq 0}$. Finally, by Remark 2.4 items (b) and (c) in (i) are equivalent. ∎

Proposition 2.11. *If* $1 \leq p < \infty$*, then the following are equivalent*

(i) $\Sigma(A,B)$ *is L^p-ISS,*

(ii) $\Sigma(A,B)$ *is L^p-iISS,*

(iii) $\Sigma(A,B)$ *is L^p-UBEBS and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

Proof. Clearly, by the definition of iISS and UBEBS, (ii) $\Rightarrow$ (iii). By Proposition 2.10, (iii)$\Rightarrow$(i). Thus in view of Proposition 2.10 it remains to show that L^p-infinite-time admissibility and exponential stability imply L^p-iISS. Indeed, L^p-infinite-time admissibility and exponential stability show for $x_0 \in X$ and $u \in L^p(0,t;U)$ that

$$\begin{aligned}\|x(t)\| &\leq M\mathrm{e}^{-\omega t}\|x_0\| + c_\infty \|u\|_{L^p(0,t;U)} \\ &= M\mathrm{e}^{-\omega t}\|x_0\| + c_\infty \left(\int_0^t \|u(s)\|_U^p \, ds\right)^{1/p},\end{aligned}$$

which shows L^p-iISS. ∎

Remark 2.12. Let $1 \leq p < \infty$. If the system $\Sigma(A,B)$ is L^p-admissible and $(T(t))_{t\geq 0}$ is exponentially stable, then the system $\Sigma(A,B)$ is L^p-ISS with the following choices for the functions β and μ:

$$\beta(s,t) := M\mathrm{e}^{-\omega t}s \quad \text{and} \quad \mu(s) := c_\infty s.$$

Here the constants M and ω are given by (2.8) and $c_\infty = \sup_{t\geq 0} c(t)$.

Proposition 2.13. *If* $\Sigma(A,B)$ *is L^∞-iISS, then* $\Sigma(A,B)$ *is L^∞-zero-class admissible.*

Proof. If $\Sigma(A,B)$ is L^∞-iISS, then there exist $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for all $t > 0$, $u \in L^\infty(0,t;U)$, $u \neq 0$

$$\frac{1}{\|u\|_\infty}\left\|\int_0^t T_{-1}(s)Bu(s)\, ds\right\| \leq \theta\left(\int_0^t \mu\left(\tfrac{\|u(s)\|_U}{\|u\|_\infty}\right) ds\right). \tag{2.10}$$

Since the function μ is monotonically increasing and $\|u(s)\|_U \leq \|u\|_\infty$ a.e., the right-hand side of (2.10) is bounded above by $\theta(t\mu(1))$ which converges to zero as $t \searrow 0$. ∎

We illustrate the relations of the different stability notions with respect to L^∞ discussed above in the diagram depicted in Figure 2.1.

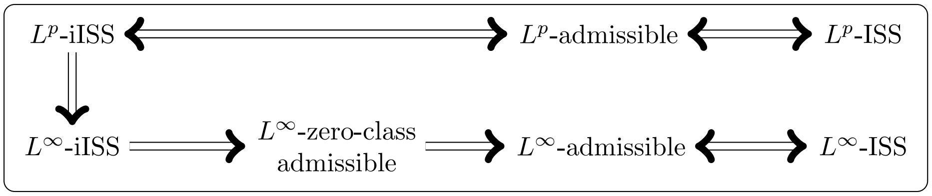

FIG. 2.1. *Relations between the different stability notions with respect to L^p, $p < \infty$, and L^∞ for a system $\Sigma(A, B)$, where it is assumed that the semigroup is exponentially stable.*

PROPOSITION 2.14. *Suppose that B is a bounded operator from U to X and $Z \subseteq L^1_{loc}(0, \infty; U)$ is a function space as in Section 2.1. Then the following statements are equivalent.*

(i) $(T(t))_{t\geq 0}$ *is exponentially stable,*
(ii) $\Sigma(A, B)$ *is Z-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable,*
(iii) $\Sigma(A, B)$ *is Z-infinite-time admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable,*
(iv) $\Sigma(A, B)$ *is Z-ISS,*
(v) $\Sigma(A, B)$ *is Z-iISS,*
(vi) $\Sigma(A, B)$ *is Z-UBEBS and* $(T(t))_{t\geq 0}$ *is exponentially stable,*
(vii) $\Sigma(A, B)$ *is L^1_{loc}-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

If Z satisfies Assumption (B), then the above assertions are equivalent to

(viii) $\Sigma(A, B)$ *is Z-zero-class admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

Proof. By Proposition 2.10 we have (v) $\Rightarrow$ (vi) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), and Proposition 2.11 and Remark 2.8 prove (vii) $\Rightarrow$ (v). The implication (i) $\Rightarrow$ (vii) follows from the fact that by the boundedness of B we have $x(t) \in X$ for all $t \geq 0$ and all $u \in L^1(0, t; U)$. Clearly, (viii) $\Rightarrow$ (ii). Thus it remains to show that if Z satisfies Assumption (B), then (i) $\Rightarrow$ (viii). Let $(T(t))_{t\geq 0}$ be exponentially stable, that is, there exist constants $M, \omega > 0$ such that (2.8) holds. Therefore, for any $u \in L^1(0, t; U)$,

$$\begin{aligned} \|x(t)\| &\leq M\mathrm{e}^{-\omega t}\|x_0\| + M\|B\| \int_0^t \mathrm{e}^{-\omega(t-s)}\|u(s)\|_U \, ds \\ &\leq M\mathrm{e}^{-\omega t}\|x_0\| + M\|B\| \int_0^t \|u(s)\|_U \, ds. \qquad (2.11) \end{aligned}$$

Using that $Z(0, t; U)$ is continuously embedded in $L^1(0, t; U)$, we conclude that

$$\|x(t)\| \leq M\mathrm{e}^{-\omega t}\|x_0\| + M\|B\|\kappa(t)\|u\|_{Z(0,t;U)} \qquad (2.12)$$

for all $t \geq 0$. If Assumption (B) holds, then the embedding constants $\kappa(t)$ tend to 0 as $t \searrow 0$. Hence, (2.12) shows that (i) implies (viii). ∎

For the special case $Z = L^p(0, \infty; U)$, parts of the equivalences in Proposition 2.14 can already be found in [19].

Remark 2.15. Note that in Proposition 2.14, the assertions are independent of Z as the assertions only rest on exponential stability. In particular, if one of the equivalent conditions hold, then the system $\Sigma(A, B)$ is L^p-ISS with the following choices for the functions β and μ

$$\beta(s, t) := M\mathrm{e}^{-\omega t}s \quad \text{and} \quad \mu(s) := \frac{M}{\omega q}\|B\|s,$$

where q is the Hölder conjugate of p, and L^p-iISS with

$$\beta(s,t) := M\mathrm{e}^{-\omega t}s, \qquad \mu(s) := s, \quad \text{and} \quad \theta(s) := sM\|B\|.$$

Here the constants M and ω are given by (2.8). Although, in this case a system is L^p-ISS or L^p-iISS for all p if this holds for some p, the choices for the functions μ, however, do depend on p. Note that if B is unbounded, then the question whether a system is L^p-ISS or L^p-iISS crucially depends on p.
Furthermore, note that in the trivial case $X = U = \mathbb{C}$ and $A = -1$, $B = 1$, we have that the system $\Sigma(A, B)$ is not L^1-zero-class admissible.

3. IISS from the viewpoint of Orlicz spaces. In this section we relate L^∞-ISS and L^1-ISS to ISS with respect to Orlicz spaces E_Φ corresponding to a Young function Φ. The use of Orlicz spaces is motivated by the idea of understanding the integral appearing in the definition of iISS, (1.2), as some type of norm. For the definition and fundamental properties of Orlicz spaces and Young functions, we refer to the Appendix. The main results of this section are summarized in the following three theorems.

THEOREM 3.16. *The following statements are equivalent.*
(i) There is a Young function Φ such that the system $\Sigma(A, B)$ is E_Φ-ISS,
(ii) $\Sigma(A, B)$ is L^∞-iISS,
(iii) $(T(t))_{t\geq 0}$ is exponentially stable and there is a Young function Φ such that the system $\Sigma(A, B)$ is E_Φ-UBEBS.

If Φ satisfies the Δ_2-condition (see Definition A.42) more can be said.

THEOREM 3.17. *If Φ is a Young function that satisfies the Δ_2-condition, then the following are equivalent.*
(i) $\Sigma(A, B)$ is E_Φ-ISS,
(ii) $\Sigma(A, B)$ is E_Φ-iISS,
(iii) $\Sigma(A, B)$ is E_Φ-UBEBS and $(T(t))_{t\geq 0}$ is exponentially stable.

Remark 3.18. Since L^p-spaces are examples of Orlicz spaces where the Δ_2-condition is satisfied, Theorem 3.17 can be seen as a generalization of Proposition 2.11.

THEOREM 3.19. *The following statements are equivalent.*
(i) $\Sigma(A, B)$ is L^1-ISS,
(ii) $\Sigma(A, B)$ is L^1-iISS,
(iii) $\Sigma(A, B)$ is E_Φ-ISS for every Young function Φ.

The proofs of Theorems 3.16, 3.17 and 3.19 are given at the end of this section.

$$\begin{tikzcd}
E_\Phi\text{-iISS} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & E_\Phi\text{-admissible} \arrow[r, Leftrightarrow] & E_\Phi\text{-ISS} \\
L^\infty\text{-iISS} \arrow[r, Leftrightarrow] & \begin{array}{c} E_\Psi\text{-admissible} \\ \text{for some } \Psi \end{array} \arrow[r, Leftrightarrow] & \begin{array}{c} E_\Psi\text{-ISS} \\ \text{for some } \Psi \end{array}
\end{tikzcd}$$

FIG. 3.2. *Relations between the different stability notions with respect to Orlicz spaces for a system $\Sigma(A, B)$, where it is assumed that the semigroup is exponentially stable and that Φ satisfies the Δ_2-condition.*

LEMMA 3.20. *Let $\Sigma(A,B)$ be L^∞-iISS. Then there exist $\tilde{\theta},\Phi\in\mathcal{K}_\infty$ such that Φ is a Young function which is continuously differentiable on $(0,\infty)$ and*

$$\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| \le \tilde{\theta}\left(\int_0^t \Phi(\|u(s)\|_U)\,ds\right) \tag{3.13}$$

for all $t>0$ and $u\in L^\infty(0,t;U)$.

Proof. By assumption, $(T(t))_{t\ge0}$ is exponentially stable and there exist $\theta\in\mathcal{K}_\infty$ and $\mu\in\mathcal{K}$ such that (2.9) holds for $Z=L^\infty$. Without loss of generality we can assume that μ belongs to $\mathcal{K}_\infty$. By Lemma 14 in [23] there exist a convex function $\mu_v\in\mathcal{K}_\infty$ and a concave function $\mu_c\in\mathcal{K}_\infty$ such that both are continuously differentiable on $(0,\infty)$ and $\mu\le\mu_c\circ\mu_v$ holds on $[0,\infty)$. Now for any Young function $\Psi\colon[0,\infty)\to[0,\infty)$ it is straightforward to check that $\mu_c\circ\Psi^{-1}$ is a concave function and hence we have by Jensen's inequality

$$\begin{aligned}\theta\left(\int_0^1 \mu(\|u(s)\|_U)\,ds\right) &\le \theta\left(\int_0^1 \mu_c\circ\mu_v(\|u(s)\|_U)\,ds\right)\\ &\le (\theta\circ\mu_c\circ\Psi^{-1})\left(\int_0^1 (\Psi\circ\mu_v)(\|u(s)\|_U)\,ds\right).\end{aligned}$$

Using Remark 3.2.7 in [15] it is easy to see that $\Phi:=\Psi\circ\mu_v$ is a Young function. Taking $\tilde{\theta}:=\theta\circ\mu_c\circ\Psi^{-1}$ we obtain the desired estimate for $t=1$. By Lemma 2.9, the assertion follows. ∎

Proof of Theorem 3.16. (i) ⇒ (ii): Since $\Lambda(s)=s^2$ defines a Young function with $\Lambda(1)=1$, it can be easily seen that

$$\Phi_1(s)=\begin{cases}\Phi(s), & s<1,\\ \Phi(\Lambda(s)), & s\ge1,\end{cases}$$

defines another Young function such that $\Phi\le\Phi_1$. Furthermore, Φ_1 increases essentially more rapidly than Φ (see Def. A.43), since the composition $\Phi\circ\Lambda$ of two Young functions Φ,Λ is known to be increasing essentially more rapidly than Φ (see page 114 of [14]). We define $\theta\colon[0,\infty)\to[0,\infty)$ by

$$\theta(\alpha)=\sup\left\{\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \;\middle|\; u\in L^\infty(0,1;U),\ \int_0^1\Phi_1(\|u(s)\|_U)\,ds\le\alpha\right\},$$

for $\alpha>0$ and $\theta(0)=0$. Clearly, θ is non-decreasing. Admissibility with respect to E_Φ and Remark A.40.4 yield that for $u\in L^\infty(0,1;U)$,

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le c(1)\|u\|_{E_\Phi(0,1;U)} \le c(1)\left(1+\int_0^1\Phi_1(\|u(s)\|_U)\,ds\right).$$

Hence, $\theta(\alpha)<\infty$ for all $\alpha\ge0$.
If we can show that $\lim_{t\searrow0}\theta(t)=0$, then, by Lemma 2.5 in [3], there exists $\tilde{\theta}\in\mathcal{K}_\infty$ such that $\theta\le\tilde{\theta}$ pointwise. Therefore, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. By the definition of θ, for any $n\in\mathbb{N}$ there exists $u_n\in L^\infty(0,1;U)$ such that

$$\int_0^1\Phi_1(\|u_n(s)\|_U)\,ds\le\alpha_n$$

and

$$\left|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\|\right| < \frac{1}{n}. \tag{3.14}$$

Hence the sequence $(\|u_n(\cdot)\|_U)_{n\in\mathbb{N}}$ is Φ_1-mean convergent to zero (see Def. A.41). By Theorem A.44, the sequence even converges to zero with respect to the norm of the space $L_\Phi(0,1)$, and thus also in $E_\Phi(0,1)$. Hence

$$\lim_{n\to\infty} \|u_n\|_{E_\Phi(0,1;U)} = \lim_{n\to\infty} \|\|u_n(\cdot)\|_U\|_{E_\Phi(0,1)} = 0,$$

where we used Remark A.40.2. Hence, by admissibility,

$$\left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\| \le c(1)\|u_n\|_{E_\Phi(0,1;U)} \to 0,$$

as $n\to\infty$. Altogether we obtain that

$$\begin{aligned}\theta(\alpha_n) &\le \left|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\|\right| + \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\| \\ &\le \frac{1}{n} + c(1)\|u_n\|_{E_\Phi(0,1;U)},\end{aligned}$$

and thus, $\lim_{n\to\infty}\theta(\alpha_n) = 0$.

Therefore, there exists $\tilde\theta \in \mathcal{K}_\infty$ such that $\theta \le \tilde\theta$ pointwise. Furthermore, Φ_1 is a Young function, in particular we have $\Phi_1 \in \mathcal{K}_\infty$. The definition of θ yields that

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le \theta\left(\int_0^1 \Phi_1(\|u(s)\|_U)\,ds\right) \le \tilde\theta\left(\int_0^1 \Phi_1(\|u(s)\|_U)\,ds\right)$$

for all $u \in L^\infty(0,1;U)$. By Lemma 2.9, we conclude that $\Sigma(A,B)$ is L^∞-iISS.

(ii) $\Rightarrow$ (i): Now assume that $\Sigma(A,B)$ is L^∞-iISS. We need to show that for some Young function Φ the system $\Sigma(A,B)$ is E_Φ-ISS. By Proposition 2.10(i) it suffices to show that there is a Young function Φ such that $\int_0^t T_{-1}(s)Bu(s)\,ds \in X$ for all $u \in E_\Phi(0,t)$. Note that since $E_\Phi(0,t;U) \subset L^1(0,t;U)$ for any Young function Φ, the integral always exists in X_{-1}. By assumption, $\int_0^t T_{-1}(s)Bu(s)\;ds \in X$ for all $u \in L^\infty(0,t)$. By Lemma 3.20, there exist $\tilde\theta \in \mathcal{K}_\infty$ and a Young function Φ such that (3.13) holds. Let $u \in E_\Phi$. By definition, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset L^\infty(0,t;U)$ such that $\lim_{n\to\infty}\|u_n - u\|_{E_\Phi(0,t;U)} = 0$. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $E_\Phi(0,t;U)$, we can assume without loss of generality that $\|u_n - u_m\|_{E_\Phi(0,t;U)} \le 1$ for all $m,n \in \mathbb{N}$. By [15, Lemma 3.8.4 (i)] this implies that for all $n,m\in\mathbb{N}$,

$$\int_0^t \Phi(\|u_n(s) - u_m(s)\|_U)\,ds \le \|u_n - u_m\|_{E_\Phi(0,t;U)}.$$

Together with (3.13) and the monotonicity of $\tilde\theta$, this yields

$$\begin{aligned}\left\|\int_0^t T_{-1}(s)B(u_n(s) - u_m(s))\,ds\right\| &\le \tilde\theta\left(\int_0^t \Phi(\|u_n(s) - u_m(s)\|_U)\,ds\right) \\ &\le \tilde\theta\left(\|u_n - u_m\|_{E_\Phi(0,t;U)}\right).\end{aligned}$$

Hence $(\int_0^t T_{-1}(s)Bu_n(s)\,ds)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and thus converges. Let y denote its limit. Since $E_\Phi(0,t;U)$ is continuously embedded in $L^1(0,t;U)$, see Remark A.40.3, it follows that

$$\lim_{n\to\infty}\int_0^t T_{-1}(s)Bu_n(s)\,ds = \int_0^t T_{-1}(s)Bu(s)\,ds$$

in X_{-1}. Since X is continuously embedded in X_{-1}, we conclude that

$$y = \int_0^t T_{-1}(s)Bu(s)\,ds.$$

Thus, we have shown that $\int_0^t T_{-1}(s)Bu(s)\,ds \in X$ for all $u \in E_\Phi$ and hence $\Sigma(A,B)$ is admissible with respect to E_Φ.

(i) $\Rightarrow$ (iii): This follows since for all $u \in E_\Phi(0,t;U)$ it holds that $u \in \tilde{L}_\Phi(0,t;U)$ and

$$\|u\|_{E_\Phi} \le 1 + \int_0^t \Phi(\|u(s)\|_U)\,ds,$$

see Remark A.40.4.

(iii) $\Rightarrow$ (i): This follows by (iii) and (i) of Proposition 2.10. ☐

Proof of Theorem 3.17. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) follow, analogously as for the L^p-case, by Proposition 2.10.

(i) $\Rightarrow$ (ii): Similarly to the proof of Theorem 3.16, we can define a non-decreasing function θ by

$$\theta(\alpha) = \sup\left\{\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \;\middle|\; u \in E_\Phi(0,1;U), \int_0^1 \Phi(\|u(s)\|_U)\,ds \le \alpha\right\},$$

for $\alpha > 0$ and $\theta(0) := 0$. By E_Φ-admissibility and Remark A.40.4, we have that

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le c(1)\|u\|_{E_\Phi(0,1;U)} \le c(1)\left(1 + \int_0^1 \Phi(\|u(s)\|_U)\,ds\right),$$

for $u \in E_\Phi(0,1;U) \subset \tilde{L}_\Phi(0,t;U)$. Hence, θ is well-defined. In analogy to the proof of Theorem 3.16, it remains to show that θ is right-continuous at 0. This follows because Φ satisfies the Δ_2-condition. In fact, if the latter is true, it is known that a sequence $(u_n)_{n\in\mathbb{N}}$ in E_Φ converges to 0 if and only if the sequence is Φ-mean convergent to zero (see Def. A.41). Therefore, $\alpha_n \searrow 0$ implies that there exists a sequence $u_n \in E_\Phi(0,1;U)$ that converges to 0 in E_Φ and such that

$$\left|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}Bu_n(s)\,ds\right\|\right| \le \frac{1}{n}, \quad n \in \mathbb{N}.$$

By E_Φ-admissibility, we conclude that $\theta(\alpha_n) \to 0$ as $n \to \infty$.
Hence, by Lemma 2.4 in [3], we find $\tilde{\theta} \in \mathcal{K}_\infty$ such that $\theta \le \tilde{\theta}$ pointwise. By definition of θ, this implies

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le \tilde{\theta}\left(\int_0^1 \Phi(\|u(s)\|_U)\,ds\right)$$

for all $u \in E_\Phi(0,1;U)$. Finally, Lemma 2.9 yields that $\Sigma(A,B)$ is E_Φ-iISS. ☐

Proof of Theorem 3.19. By Propositions 2.10 and 2.11, we only need to show the equivalence of (i) and (iii). That (i) implies (iii) follows immediately since E_Φ is continuously embedded in L^1.
Conversely, let $\Sigma(A,B)$ be E_Φ-admissible for every Young function Φ. According to Proposition 2.10 (a), we have to show that $\Sigma(A,B)$ is L^1-admissible. Let $t>0$ and $u\in L^1(0,t;U)$. It remains to prove that $\int_0^t T_{-1}(s)Bu(s)\,ds \in X$. By [14, p. 61], there exists a Young function Φ satisfying the Δ_2-condition such that $\|u(\cdot)\|_U \in L_\Phi$[1]. The Δ_2-condition implies that $E_\Phi = L_\Phi$ and $E_\Phi(0,t;U) = L_\Phi(0,t;U)$, see [24, p. 303] or [26, Thm. 5.2]. Thus $\int_0^t T_{-1}(s)Bu(s)\,ds \in X$ by assumption. ∎

PROPOSITION 3.21. *Let $\Sigma(A,B)$ be L^∞-ISS. If there exist a nonnegative function $f\in L^1(0,1)$, $\theta\in\mathcal{K}$, a constant $c>0$ and a Young function μ such that for every $u\in L^1(0,1;U)$ with $\int_0^1 f(s)\mu(\|u(s)\|_U)\,ds<\infty$ one has*

$$\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \le c+\theta\left(\int_0^1 f(s)\mu(\|u(s)\|_U)\,ds\right),$$

then $\Sigma(A,B)$ is L^∞-iISS.

Proof. By Theorem 3.16 and Proposition 2.10 it is sufficient to show that there is a Young function Φ such that the system $\Sigma(A,B)$ is E_Φ-admissible. Theorem A.33 implies that there exists a Young function Ψ such that $f\in \tilde{L}_\Psi(0,1)$. Let $\tilde{\Phi}$ be the complementary Young function to Ψ. We define the Young function Φ by $\Phi := \tilde{\Phi}\circ\mu$. Using Remark A.36 for $u\in E_\Phi(0,1;U)$ we obtain

$$\begin{aligned}\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| &\le c+\theta\left(\int_0^1 f(s)\mu(\|u(s)\|_U)\,ds\right)\\ &\le c+\theta\left(\int_0^1 \Psi(f(s))\,ds+\int_0^1 \tilde{\Phi}(\mu(\|u(s)\|_U)\,ds\right).\end{aligned}$$

This shows that for all $u\in E_\Phi(0,1;U)$ we have

$$\int_0^1 T_{-1}(s)Bu(s)\,ds \in X,$$

that is, $\Sigma(A,B)$ is E_Φ-admissible. ∎

4. Stability of parabolic diagonal systems.

In the previous section we have proved that for infinite-dimensional systems L^∞-iISS implies L^∞-ISS. It is an open question whether the converse implication holds. Here, we give a positive answer for parabolic diagonal systems, which are a well-studied class of systems in the literature, see e.g. [30].
Throughout this section we assume that $U=\mathbb{C}$, $1\le q<\infty$ and that the operator A

[1] In [14, p. 61] it is actually shown that for given $f\in L^1(0,t)$, there exists a Young function Q such that $f\in L_{Q\circ Q}(0,t)$ and such that Q satisfies the Δ'-condition, i.e.,

$$\exists c, u_0>0\ \forall u,v\ge u_0: \quad Q(uv)\le cQ(u)Q(v).$$

In fact, it is easy to see that this property implies that $Q\circ Q$ satisfies

$$\forall u\ge u_0: \quad (Q\circ Q)(\ell u)\le k(\ell)(Q\circ Q)(u),$$

for some $\ell>1$ and $k(\ell)>0$, which is known to be equivalent to $Q\circ Q$ satisfying the Δ_2-condition, see [14, p. 23].

possesses a q-Riesz basis of eigenvectors $(e_n)_{n\in\mathbb{N}}$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ lying in a sector in the open left half-plane $\mathbb{C}_-$. More precisely, $(e_n)_{n\in\mathbb{N}}$ is a q-Riesz basis of X, if $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis and for some constants $c_1, c_2 > 0$ we have

$$c_1 \sum_k |a_k|^q \leq \left\| \sum_k a_k e_k \right\|^q \leq c_2 \sum_k |a_k|^q$$

for all sequences $(a_k)_{k\in\mathbb{N}}$ in $\ell^q = \ell^q(\mathbb{N})$. Thus without loss of generality we can assume that $X = \ell^q$ and that $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of ℓ^q. We further assume that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ lies in $\mathbb{C}$ with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and that there exists a constant $k > 0$ such that $|\operatorname{Im}\lambda_n| \leq k|\operatorname{Re}\lambda_n|$, $n \in \mathbb{N}$, i.e., $(\lambda_n)_n \subset \mathbb{C} \setminus S_{\pi/2+\theta}$ for some $\theta \in (0, \pi/2)$, where

$$S_{\pi/2+\theta} = \{z \in \mathbb{C} \mid |z| > 0, |\arg z| < \pi/2 + \theta\}.$$

Then the linear operator $A\colon D(A) \subset \ell^q \to \ell^q$, given by

$$Ae_n = \lambda_n e_n, \qquad n \in \mathbb{N},$$

and $D(A) = \{(x_n) \in \ell^q \mid \sum_n |x_n\lambda_n|^q < \infty\}$, generates an analytic exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on ℓ^q, which is given by $T(t)e_n = \mathrm{e}^{t\lambda_n} e_n$. An easy calculation shows that the extrapolation space $(\ell^q)_{-1}$ is given by

$$(\ell^q)_{-1} = \left\{ x = (x_n)_{n\in\mathbb{N}} \mid \sum_n \frac{|x_n|^q}{|\lambda_n|^q} < \infty \right\},$$

$$\|x\|_{X_{-1}} = \|A^{-1}x\|_{\ell^q}.$$

Thus the linear bounded operator B from $\mathbb{C}$ to $(\ell^q)_{-1}$ can be identified with a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ satisfying

$$\sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\lambda_n|^q} < \infty.$$

Thanks to the sectoriality condition for $(\lambda_n)_{n\in\mathbb{N}}$ this equivalent to

$$\sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} < \infty.$$

The following result shows that, under the above assumptions, the system $\Sigma(A, B)$ is L^∞-iISS. Thus for this class of systems L^∞-iISS is equivalent to L^∞-ISS, and both notions are implied by $B \in (\ell^q)_{-1}$, that is, $\sum_n \frac{|b_n|^q}{|\lambda_n|^q} < \infty$. The following theorem generalizes the main result in [7], where the case $q = 2$ is studied.

THEOREM 4.22. *Let $U = \mathbb{C}$, and suppose that the operator A possesses a q-Riesz basis of X that consists of eigenvectors $(e_n)_{n\in\mathbb{N}}$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ lying in a sector in the open left half-plane $\mathbb{C}_-$ with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and $B \in \mathcal{L}(\mathbb{C}, X_{-1})$. Then the system $\Sigma(A, B)$ is L^∞-iISS, and hence also L^∞-ISS and L^∞-zero-class admissible.*

Remark 4.23. In the situation of Theorem 4.22, $\Sigma(A, B)$ is L^∞-iISS if and only if $\Sigma(A, B)$ is L^∞-ISS.

Proof of Theorem 4.22. Without loss of generality we may assume $X = \ell^q$ and that $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of ℓ^q. Let $f\colon (0,\infty) \to [0,\infty)$ be defined by

$$f(s) = \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-1}} \mathrm{e}^{\operatorname{Re}\lambda_n s}.$$

Then it is easy to see that f belongs to $L^1(0,\infty)$. Now for $u \in L^1(0,1)$ with $\int_0^1 f(s)|u(s)|^q\,ds < \infty$ we obtain (denoting by q' the Hölder conjugate of q)

$$\begin{aligned}
\left\| \int_0^1 T_{-1}(s)Bu(s)\,ds \right\|_{\ell^q}^q &= \sum_{n\in\mathbb{N}} |b_n|^q \left| \int_0^1 \mathrm{e}^{\lambda_n s} u(s)\,ds \right|^q \\
&\le \sum_{n\in\mathbb{N}} |b_n|^q \left(\int_0^1 \mathrm{e}^{\operatorname{Re}\lambda_n s} |u(s)|\,ds \right)^q \\
&= \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{(\operatorname{Re}\lambda_n)^q} \left(\int_0^1 |\operatorname{Re}\lambda_n| \mathrm{e}^{\operatorname{Re}\lambda_n s} |u(s)|\,ds \right)^q \\
&\le \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{(\operatorname{Re}\lambda_n)^q} \left(\int_0^1 |\operatorname{Re}\lambda_n| \mathrm{e}^{\operatorname{Re}\lambda_n s} |u(s)|^q\,ds \right) \left(\int_0^1 |\operatorname{Re}\lambda_n| \mathrm{e}^{\operatorname{Re}\lambda_n s}\,ds \right)^{q/q'} \\
&\le \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} \left(\int_0^1 |\operatorname{Re}\lambda_n| \mathrm{e}^{\operatorname{Re}\lambda_n s} |u(s)|^q\,ds \right) \\
&= \int_0^1 \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-1}} \mathrm{e}^{\operatorname{Re}\lambda_n s} |u(s)|^q\,ds \\
&= \int_0^1 f(s)|u(s)|^q\,ds \\
&< \infty.
\end{aligned}$$

This shows that the system $\Sigma(A,B)$ is L^∞-ISS and the claim now follows from Proposition 3.21. ∎

Remark 4.24. Theorem 4.22 states that L^∞-admissibility implies E_Φ-admissibility for some Young function Φ in the case of parabolic diagonal systems. A natural question is whether Φ can always be chosen such that the Δ_2-condition is satisfied. Looking at the proof and having in mind that L^1 equals the union of all spaces E_Ψ where Ψ satisfies the Δ_2-condition, this could be expected. However, the answer is negative, which can be seen as follows. For a Young function Φ satisfying the Δ_2-condition there exist constants $x_0 > 0$ and $p \in \mathbb{N} \setminus \{1\}$ such that

$$\Phi(x) \le x^p, \qquad x > x_0,$$

see [14, p. 25]. This implies that $E_\Phi \supset L^p$, see e.g. [15, Sec. 3.17]. However, there exists Young functions that do not satisfy the latter estimate, e.g., $\Phi(x) = \mathrm{e}^{x-1} - x - \mathrm{e}^{-1}$. In Example 5.29, $\Sigma(A,B)$ is not L^p-admissible for any $p < \infty$, which, with the above reasoning, implies that the system cannot be E_Φ-admissible for any Φ satisfying the Δ_2-condition.

LEMMA 4.25. *Let μ be a positive regular Borel measure supported on a sector S_ϕ with $\phi \in (0, \frac{\pi}{2})$, and let $1 \le q < \infty$. Then the following are equivalent:*

(i) The Laplace transform $\mathcal{L}\colon L^\infty(0,\infty) \to L^q(\mathbb{C}_+,\mu)$ is bounded,
(ii) The function $s \mapsto 1/s$ lies in $L^q(\mathbb{C}_+,\mu)$.

Proof. (i) $\Rightarrow$ (ii): Taking $f(t) = 1$ for $t \geq 0$ we have that $\mathcal{L}f(s) = 1/s$ and the result follows.
(ii) $\Rightarrow$ (i): For $f \in L^\infty(0,\infty)$ and $s \in \mathbb{C}_+$ we have

$$\left|\int_0^\infty f(t)\mathrm{e}^{-st}\,dt\right| \leq \|f\|_\infty \int_0^\infty |\mathrm{e}^{-st}|\,dt \leq \|f\|_\infty/(\operatorname{Re} s) \leq M\|f\|_\infty/|s|,$$

where M is a constant depending only on ϕ. Now Condition (ii) implies that $\mathcal{L}$ is bounded. ∎

THEOREM 4.26. *Suppose that A possesses a q-Riesz basis of X consisting of eigenvectors $(e_n)_{n\in\mathbb{N}}$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ lying in a sector in the open left half-plane $\mathbb{C}_-$ and $B \in X_{-1}$. Then the following assertions are equivalent.*
(i) $\Sigma(A,B)$ is infinite-time L^∞-admissible,
(ii) $\sup_{\lambda\in\mathbb{C}_+} \|(\lambda - A)^{-1}B\| < \infty$,
(iii) The function $s \mapsto 1/s$ lies in $L^q(\mathbb{C}_+,\mu)$, where μ is the measure $\sum |b_k|^2\delta_{-\lambda_k}$.

Proof. By [9, Thm 2.1], admissibility is equivalent to the boundedness of the Laplace transform $\mathcal{L}\colon L^\infty(0,\infty) \to L^q(\mathbb{C}_+,\mu)$, and hence (i) and (iii) are equivalent by Lemma 4.25. Note that

$$\|(\lambda - A)^{-1}B\|^q = \sum_k \frac{|b_k|^q}{|\lambda - \lambda_k|^q}.$$

Now if (ii) holds, then (iii) also holds, letting $\lambda \to 0$. Conversely, if (iii) holds, then by sectoriality we have that

$$\sum_k \frac{|b_k|^q}{|\operatorname{Re}\lambda_k|^q} < \infty,$$

and hence $\sum_k |b_k|^q/|\lambda - \lambda_k|^q$ is bounded independently of $\lambda \in \mathbb{C}_+$, that is, (ii) holds. ∎

Remark 4.27. Let $\mathfrak{b}_p(X)$ denote the set of L^p-admissible control operators from $\mathbb{C}$ to X for a given A. By Theorem 4.22, we have that $\mathfrak{b}_\infty(X) = X_{-1}$ for exponentially stable, parabolic diagonal systems. Using [32, Theorem 6.9], and the inclusion of the L^p-spaces, we obtain the following chain of inclusions for $X = \ell^q$ with $q > 1$[2]

$$X = \mathfrak{b}_1(X) \subset \mathfrak{b}_p(X) \subset \mathfrak{b}_\infty(X) = X_{-1}. \tag{4.15}$$

It is not so hard to show that the equality $\mathfrak{b}_\infty(X) = X_{-1}$ does not hold in general if the exponential stability is dropped. In fact, a counterexample on $X = \ell^2$ with the standard basis is given by $\lambda_n = 2^n$, $n \in \mathbb{Z}$, $b_n = 2^n/n$ for $n > 0$ and $b_n = 2^n$ for $n < 0$.

The relations of the different stability notions with respect to L^∞ for parabolic diagonal systems are summarized in the diagram shown in Figure 4.3.

5. Some Examples.

Example 5.28. Let us consider the following boundary control system given by the one-dimensional heat equation on the spatial domain $[0,1]$ with Dirichlet boundary

[2]here, $q = 1$ is also allowed if $(T^*(t))_{t\geq 0}$ is strongly continuous.

$$
\begin{tikzcd}
L^\infty\text{-iISS} \arrow[r, Leftrightarrow] & \begin{array}{c} L^\infty\text{-zero-class} \\ \text{admissible} \end{array} \arrow[r, Leftrightarrow] & L^\infty\text{-admissible} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & L^\infty\text{-ISS} \\
 & & B \in X_{-1} &
\end{tikzcd}
$$

FIG. 4.3. *Relations between the different stability notions for parabolic diagonal system (assuming that the semigroup is exponentially stable).*

control at the point 1,

$$
\begin{aligned}
x_t(\xi,t) &= a x_{\xi\xi}(\xi,t), \quad \xi \in (0,1),\ t>0,\\
x(0,t) &= 0, \quad x(1,t) = u(t), \quad t>0,\\
x(\xi,0) &= x_0(\xi),
\end{aligned}
$$

where $a>0$. It can be shown that this system can be written in the form $\Sigma(A,B)$ in (2.4). Here $X = L^2(0,1)$ and

$$
\begin{aligned}
Af &= f'', \quad f \in D(A),\\
D(A) &= \left\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \right\}.
\end{aligned}
$$

Moreover, with $\lambda_n = -a\pi^2 n^2$,

$$
Ae_n = \lambda_n e_n, \quad n \in \mathbb{N},
$$

where the functions $e_n = \sqrt{2}\sin(n\pi\cdot)$, $n \geq 1$, form an orthonormal basis of X. With respect to this basis, the operator $B = a\delta_1'$ can be identified with $(b_n)_{n\in\mathbb{N}}$ for $b_n = (-1)^n \sqrt{2} a n\pi$, $n \in \mathbb{N}$. Therefore,

$$
\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\lambda_n|^2} = \frac{1}{3} < \infty,
$$

which shows that $B \in X_{-1}$. By Theorem 4.22, we conclude that the system is L^∞-iISS. Moreover, we obtain the following L^∞-ISS and L^∞-iISS estimates:

$$
\begin{aligned}
\|x(t)\|_{L^2(0,1)} &\leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + \frac{1}{\sqrt{3}}\|u\|_{L^\infty(0,t)},\\
\|x(t)\|_{L^2(0,1)} &\leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + c\left(\int_0^t |u(s)|^p ds\right)^{1/p},
\end{aligned}
$$

for $p>2$ and some constant $c = c(p) > 0$. For the second inequality, we used the fact that $\Sigma(A,B)$ is even L^p-admissible for $p>2$, as it can be shown by applying Theorem 3.5 in [9]. We note that a slightly weaker L^∞-ISS estimate for this system can also be found in [12].

Example 5.29. As remarked, Example 5.28 provides a system $\Sigma(A,B)$ which is even L^p-admissible for $p>2$. In the following we present a system which is L^∞-admissible, but not L^p-admissible for any $p<\infty$. In order to find such an example, we use the characterization of L^p-admissibility from [9, Thm. 3.5].
Let $X = \ell^2$ and let $(\lambda_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ define a parabolic diagonal system $\Sigma(A,B)$ as in

Section 4. Furthermore, let $p \in (2,\infty)$. Then $\Sigma(A,B)$ is infinite-time L^p-admissible if and only if

$$\left(2^{-\frac{2n(p-1)}{p}}\mu(Q_n)\right)_{n\in\mathbb{Z}} \in \ell^{\frac{p}{p-2}}(\mathbb{Z}),$$

where $\mu = \sum_{n\in\mathbb{Z}} |b_n|^q \delta_{\lambda_n}$ and $Q_n = \{z \in \mathbb{C} \mid \operatorname{Re} z \in (2^{n-1}, 2^n]\}$, $n \in \mathbb{Z}$.
We choose $\lambda_n = -2^n$ and $b_n = \frac{2^n}{n}$ for $n \in \mathbb{N}$. Clearly, $B = (b_n) \in X_{-1}$. Then we have that

$$2^{-\frac{2n(p-1)}{p}}\mu(Q_n) = 2^{-\frac{2n(p-1)}{p}}\frac{2^{2n}}{n^2} = \frac{2^{\frac{2n}{p}}}{n^2},$$

and thus for $p > 2$,

$$\left(\left(2^{-\frac{2n(p-1)}{p}}\mu(Q_n)\right)^{\frac{p}{p-2}}\right)_{n\in\mathbb{Z}} = \left(\frac{2^{\frac{2n}{p-2}}}{n^{\frac{2p}{p-2}}}\right)_{n\in\mathbb{Z}} \notin \ell^1.$$

Hence, $\Sigma(A,B)$ is not L^p-admissible for any $p > 2$, and therefore also not for any $p \geq 1$. However, since $\sum_{n\in\mathbb{N}} |b_n|^2/|\operatorname{Re}\lambda_n|^2 = \sum_{n\in\mathbb{N}} 1/n^2 < \infty$, Theorem 4.22 shows that $\Sigma(A,B)$ is L^∞-iISS and, in particular infinite-time L^∞-admissible.
We observe that by Theorem 3.16, there exists a Young function Φ such that $\Sigma(A,B)$ is E_Φ-admissible. However, as the system is not L^p-admissible, such Φ cannot satisfy the Δ_2-condition, see Remark 4.24.

6. Conclusions and Outlook. In this paper, we have studied the relation between input-to-state stability and integral input-to-state stability for linear infinite-dimensional systems with a (possibly) unbounded control operator and inputs in general function spaces. In this situation, ISS is equivalent to admissibility together with exponential stability of the semigroup. We have related the notions of iISS with respect to L^1 and L^∞ to ISS with respect to Orlicz spaces. The known result that ISS and iISS are equivalent for L^p-inputs with $p < \infty$, was generalized to Orlicz spaces that satisfy the Δ_2-condition. Moreover, we have shown that for parabolic diagonal systems and scalar input, the notions of L^∞-iISS and L^∞-ISS coincide.

Among possible directions for future research are the investigation of the non-analytic diagonal case, general analytic systems and the relation of zero-class admissibility and input-to-state stability. Recently, the results on parabolic diagonal systems have been adapted to more general situations of analytic semigroups – the crucial tool being the holomorphic functional calculus for such semigroups [10]. Furthermore, versions ISS and iISS for strongly stable semigroups rather than exponentially stable can be studied, see [22].

Finally, we mention that the existence of a counterexample for one of the unknown (converse) implications in Figure 2.1 can be related to the following open question posed by G. Weiss in [31, Problem 2.4].

Question A: *Does the mild solution x belong to $C([0,\infty),X)$ for any $x_0 \in X$ and $u \in Z = L^\infty(0,\infty;U)$ provided that $\Sigma(A,B)$ is L^∞-admissible?*

Although we do not provide an answer to this question, we relate it to

PROPOSITION 6.30. *At least one of the following assertions is true.*
1. *The answer to Question A is positive for every system $\Sigma(A,B)$.*
2. *There exists a system $\Sigma(A_0,B_0)$, with A_0 generating an exponentially stable semigroup and $\Sigma(A_0,B_0)$ is L^∞-admissible, but not L^∞-zero-class admissible.*

Proof. This follows directly from Proposition 2.5. ∎

Appendix A. Orlicz Spaces. In this section we recall some basic definitions and facts about Orlicz spaces. More details can be found in [14, 15, 1, 35]. For the generalization to vector-valued functions see [24, VII, Sec. 7.5]. In the following $I \subset \mathbb{R}$ is an open bounded interval, U is a Banach space and $\Phi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a function.

Definition A.31. *The* Orlicz class $\tilde{L}_\Phi(I;U)$ *is the set of all equivalence classes (w.r.t. equality almost everywhere) of Bochner-measurable functions* $u\colon I \to U$ *such that*

$$\rho(u;\Phi) := \int_I \Phi(\|u(x)\|_U)\, dx < \infty.$$

In general, $\tilde{L}_\Phi(I;U)$ is not a vector space. Of particular interest are Orlicz classes generated by Young functions.

Definition A.32. *A function* $\Phi : [0,\infty) \to \mathbb{R}$ *is called a* Young function (or Young function generated by φ) *if*

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \qquad t \geq 0,$$

where the function $\varphi\colon [0,\infty) \to \mathbb{R}$ *has the following properties:* φ *is right-continuous and nondecreasing,* $\varphi(0) = 0$*,* $\varphi(s) > 0$ *for* $s > 0$ *and* $\lim_{s\to\infty} \varphi(s) = \infty$*.*

Theorem A.33 ([15, Thm. 3.2.3 and Thm. 3.2.5]). *Let* Φ *be a Young function. Then* $\tilde{L}_\Phi(I;U)$ *is a convex set and* $\tilde{L}_\Phi(I;U) \subset L^1(I;U)$*. Conversely, for* $u \in L^1(I;U)$ *there is a Young function* Φ *such that* $u \in \tilde{L}_\Phi(I;U)$*.*

Definition A.34. *Let* Φ *be the Young function generated by* φ*. Then* Ψ *defined by*

$$\Psi(t) = \int_0^t \psi(s)\, ds \quad \textit{with} \quad \psi(t) = \sup_{\varphi(s)\leq t} s, \quad t \geq 0,$$

is called the complementary function *to* Φ*.*

The complementary function of a Young function is again a Young function. If φ is continuous and strictly increasing in $[0,\infty)$, i.e. belongs to $\mathcal{K}_\infty$, then ψ is the inverse function φ^{-1} and vice versa. We call Φ and Ψ a *pair of complementary Young functions.*

Theorem A.35 (Young's inequality, [35, Thm. I, p. 77]). *Let* Φ*,* Ψ *be a pair of complementary Young functions and* φ*,* ψ *their generating functions. Then*

$$uv \leq \Phi(u) + \Psi(v), \qquad \forall u, v \in [0,\infty).$$

Equality holds if and only if $v = \varphi(u)$ *or* $u = \psi(v)$*.*

Remark A.36. Let Φ, Ψ be a pair of complementary Young functions, $u \in \tilde{L}_\Phi(I)$ and $v \in \tilde{L}_\Psi(I)$. By integrating Young's inequality we get

$$\int_I |u(x)v(x)|\, dx \leq \rho(u;\Phi) + \rho(v;\Psi)$$

We are now in the position to define the Orlicz spaces for which several equivalent definitions exist. Here we use the so-called *Luxemburg norm.*

Definition A.37. *For a Young function Φ, the set $L_\Phi(I;U)$ of all equivalence classes (w.r.t. equality almost everywhere) of Bochner-measurable functions $u\colon I \to U$ for which there is a $k > 0$ such that*

$$\int_I \Phi(k^{-1}\|u(x)\|_U)\, dx < \infty$$

is called the Orlicz space*. The* Luxemburg norm *of $u \in L_\Phi(I;U)$ is defined as*

$$\|u\|_\Phi := \|u\|_{L_\Phi(I;U)} := \inf\left\{ k > 0 \;\middle|\; \int_I \Phi(k^{-1}\|u(x)\|)\, dx \le 1 \right\}.$$

For the choice $\Phi(t) := t^p$, $1 < p < \infty$, the Orlicz space $L_\Phi(I;U)$ equals the vector-valued L^p-spaces with equivalent norms.

Theorem A.38 ([15, Thm. 3.9.1]). *$(L_\Phi(I;U), \|\cdot\|_\Phi)$ is a Banach space.*

Clearly, $L^\infty(I,U)$ is a linear subspace of $L_\Phi(I,U)$.

Definition A.39. *The space $E_\Phi(I,U)$ is defined as*

$$E_\Phi(I,U) = \overline{L^\infty(I,U)}^{\|\cdot\|_{L_\Phi(I;U)}}.$$

The norm $\|\cdot\|_{E_\Phi(I;U)}$ refers to $\|\cdot\|_{L_\Phi(I;U)}$.

If $U = \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then we write $L_\Phi(I) := L_\Phi(I;\mathbb{K})$ and $E_\Phi(I) := E_\Phi(I;\mathbb{K})$ for short.

Remark A.40. The Banach spaces $E_\Phi(I;U)$ and $L_\Phi(I;U)$ have the following properties:

1. $E_\Phi(I;U)$ is separable, see e.g. [26, Thm. 6.3].
2. For a measurable $u\colon I \to U$, $u \in L_\Phi(I;U)$ if and only if $f = \|u(\cdot)\|_U \in L_\Phi(I)$. This follows from the fact that $\|u\|_\Phi = \|f\|_\Phi$. Thus, $(u_n)_{n\in\mathbb{N}} \subset L_\Phi(I;U)$ converges to 0 if and only if $(\|u_n(\cdot)\|_U)_{n\in\mathbb{N}}$ converges to 0 in $L_\Phi(I)$.
3. Let Φ, Ψ be a pair of complementary Young functions. The extension of Hölder's inequality to Orlicz spaces reads: for any $u \in L_\Phi(I)$ and $v \in L_\Psi(I)$, it holds that $uv \in L^1(I)$ and

$$\int_I |u(s)v(s)|\, ds \le 2\|u\|_{L_\Phi(I)}\|v\|_{L_\Psi(I)},$$

see [15, Thm. 3.7.5 and Rem. 3.8.6]. This implies that for $u \in L_\Phi(I;U)$,

$$\|u\|_{L^1(0,t;U)} = \int_0^t \|u(s)\|_U\, ds \le 2\|\chi_{(0,t)}\|_\Psi \|u\|_\Phi,$$

i.e., $L_\Phi(I;U)$ is continuously embedded in $L^1(I;U)$. Moreover, $\|\chi_{(0,t)}\|_\Psi \to 0$ as $t \searrow 0$, where $\chi_{(0,t)}$ denotes the characteristic function of the interval $(0,t)$.
4. $E_\Phi(I;U) \subset \tilde{L}_\Phi(I;U) \subset L_\Phi(I;U)$, see e.g. [26, Thm. 5.1]. For $u \in \tilde{L}_\Phi(I;U)$,

$$\|u\|_\Phi \le \rho(\|u(\cdot)\|_U;\Phi) + 1 < \infty.$$

Definition A.41 (Φ-mean convergence). *A sequence $(u_n)_{n\in\mathbb{N}}$ in $L_\Phi(I)$ is said to converge in Φ-mean to $u \in L_\Phi(I)$ if*

$$\lim_{n\to\infty} \rho(u_n - u;\Phi) = \lim_{n\to\infty} \int_I \Phi(|u_n(x) - u(x)|)\, dx = 0.$$

Definition A.42. *We say that a Young function* Φ *satisfies the* Δ_2*-condition if*

$$\exists k > 0, u_0 \geq 0 \ \forall u \geq u_0 : \quad \Phi(2u) \leq k\Phi(u).$$

It holds that $E_\Phi(I;U) = \tilde{L}_\Phi(I;U) = L_\Phi(I;U)$ if Φ satisfies the Δ_2-condition.

Definition A.43. *Let* Φ *and* Φ_1 *be two Young functions. We say that the function* Φ_1 increases essentially more rapidly *than the function* Φ *if, for arbitrary* $s > 0$*,*

$$\lim_{t\to\infty} \frac{\Phi(st)}{\Phi_1(t)} = 0.$$

Theorem A.44 ([14, Thm. 13.4]). *Let* Φ, Φ_1 *be Young functions such that* Φ_1 *increases essentially more rapidly than* Φ*. If* $(u_n)_{n\in\mathbb{N}} \subset \tilde{L}_{\Phi_1}(I)$ *converges to* 0 *in* Φ_1*-mean, then it also converges in the norm* $\|\cdot\|_\Phi$*.*

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