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The group of automorphisms of the Lie algebra of derivations of a polynomial algebra

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Abstract

We prove that the group of automorphisms of the Lie algebra $\text{Der}_K(P_n)$ of derivations of a polynomial algebra $P_n = K[x_1, \ldots, x_n]$ over a field of characteristic zero is canonically isomorphic to the group of automorphisms of the polynomial algebra $P_n$.

Key Words: Group of automorphisms, monomorphism, Lie algebra, automorphism, locally nilpotent derivation.


1 Introduction

In this paper, module means a left module, $K$ is a field of characteristic zero and $K^*$ is its group of units, and the following notation is fixed:

- $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra over $K$ where $x^n := x_1^{a_1} \cdots x_n^{a_n}$,
- $G_n := \text{Aut}_K(P_n)$ is the group of automorphisms of the polynomial algebra $P_n$,
- $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives ($K$-linear derivations) of $P_n$,
- $D_n := \text{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ is the Lie algebra of $K$-derivations of $P_n$ where $[\partial, \delta] := \frac{\partial \delta - \delta \partial}{\partial}$,
- $\delta_1 := \text{ad}(\partial_1), \ldots, \delta_n := \text{ad}(\partial_n)$ are the inner derivations of the Lie algebra $D_n$ determined by the elements $\partial_1, \ldots, \partial_n$ (where $\text{ad}(a)(b) := [a, b]$),
- $\mathcal{G}_n := \text{Aut}_{\text{Lie}}(D_n)$ is the group of automorphisms of the Lie algebra $D_n$,
- $D := \bigoplus_{i=1}^n K \partial_i$,
- $\mathcal{H}_n := \bigoplus_{i=1}^n KH_i$ where $H_1 := x_1 \partial_1, \ldots, H_n := x_n \partial_n$,
- $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K x^n \partial^\beta$ is the $n$’th Weyl algebra,
- for each natural number $n \geq 2$, $u_n := K \partial_1 + P_1 \partial_2 + \cdots + P_{n-1} \partial_n$ is the Lie algebra of triangular polynomial derivations (it is a Lie subalgebra of the Lie algebra $D_n$) and $\text{Aut}_K(u_n)$ is its group of automorphisms.

The aim of the paper is to prove the following theorem.

**Theorem 1.1** $\mathcal{G}_n = G_n$.

**Structure of the proof.** (i) $G_n \subseteq \mathcal{G}_n$ via the group monomorphism (Lemma 2.3.3)

$G_n \to \mathcal{G}_n$, $\sigma \mapsto \sigma : \partial \mapsto \sigma(\partial) := \sigma \partial \sigma^{-1}$.

(ii) Let $\sigma \in \mathcal{G}_n$. Then $\partial'_1 := \sigma(\partial_1), \ldots, \partial'_n := \sigma(\partial_n)$ are commuting, locally nilpotent derivations of the polynomial algebra $P_n$ (Lemma 2.4.1(1)).
(iii) $\bigcap_{i=1}^n \ker P_n(\partial_i) = K$ (Lemma 2.6 (2)).

(iv) (crux) There exists a polynomial automorphism $\tau \in G_n$ such that $\tau \sigma \in \text{Fix}_{G_n}(\partial_1, \ldots, \partial_n)$ (Corollary 2.9).

(v) $\text{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \text{Sh}_n$ (Proposition 2.5 (3)) where

$$\text{Sh}_n := \{s_\lambda \in G_n \mid s_\lambda(x_1) = x_1 + \lambda_1, \ldots, s_\lambda(x_n) = x_n + \lambda_n\}$$

is the shift group of automorphisms of the polynomial algebra $P_n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in K^n$.

(vi) By (iv) and (v), $\sigma \in G_n$, i.e., $G_n = G_n$. $\square$

An analogue of the Jacobian Conjecture is true for $D_n$. The Jacobian Conjecture claims that certain monomorphisms of the polynomial algebra $P_n$ are isomorphisms: Every algebra endomorphism $\sigma$ of the polynomial algebra $P_n$ such that $J(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^*$ is an automorphism. The condition that $J(\sigma) \in K^*$ implies that the endomorphism $\sigma$ is a monomorphism.

**Conjecture.** Every homomorphism of the Lie algebra $D_n$ is an automorphism.

**Theorem 1.2** [7] Every monomorphism of the Lie algebra $u_n$ is an automorphism.

**Remark.** Not every epimorphism of the Lie algebra $u_n$ is an automorphism. Moreover, there are countably many distinct ideals $\{I_{\omega^n} \mid i \geq 0\}$ such that

$$I_0 = \{0\} \subset I_{\omega^n} \subset I_{2\omega^n} \subset \cdots \subset I_{i\omega^n} \subset \cdots$$

and the Lie algebras $u_n/I_{i\omega^n}$ and $u_n$ are isomorphic (Theorem 5.1 (1), [3]).

Theorems 1.2 and Conjecture have bearing of the Jacobian Conjecture and the Conjecture of Dixmier [8] for the Weyl algebra $A_n$ over a field of characteristic zero that claims: every homomorphism of the Weyl algebra is an automorphism. The Weyl algebra $A_n$ is a simple algebra, so every algebra endomorphism of $A_n$ is a monomorphism. This conjecture is open since 1968 for all $n \geq 1$. It is stably equivalent to the Jacobian Conjecture for the polynomial algebras as was shown by Tsuchimoto [9], Belov-Kanel and Kontsevich [7] (see also [2] for a short proof which is based on the author’s new inversion formula for polynomial automorphisms [1]).

An analogue of the Conjecture of Dixmier is true for the algebra $\mathbb{I}_1 := K \langle x, \frac{d}{dx}, f \rangle$ of polynomial integro-differential operators.

**Theorem 1.3** (Theorem 1.1, [3]) Each algebra endomorphism of $\mathbb{I}_1$ is an automorphism.

In contrast to the Weyl algebra $A_1 = K \langle x, \frac{d}{dx} \rangle$, the algebra of polynomial differential operators, the algebra $\mathbb{I}_1$ is neither a left/right Noetherian algebra nor a simple algebra. The left localizations, $A_{1,\theta}$ and $\mathbb{I}_{1,\theta}$, of the algebras $A_1$ and $I_1$ at the powers of the element $\theta = \frac{d}{dx}$ are isomorphic. For the simple algebra $A_{1,\theta} \simeq \mathbb{I}_{1,\theta}$, there are algebra endomorphisms that are not automorphisms [3].

The group of automorphisms of the Lie algebra $u_n$. In [3], the group of automorphisms $\text{Aut}_K(u_n)$ of the Lie algebra $u_n$ of triangular polynomial derivations is found $(n \geq 2)$, it is isomorphic to an iterated semi-direct product (Theorem 5.3, [4]),

$$\mathbb{T}^n \rtimes (\text{UAut}_K(P_n)_n \rtimes (\mathbb{F}_{n}^n \times \mathbb{E}_n))$$

where $\mathbb{T}^n$ is an algebraic $n$-dimensional torus, $\text{UAut}_K(P_n)_n$ is an explicit factor group of the group $\text{UAut}_K(P_n)$ of unitriangular polynomial automorphisms, $\mathbb{F}_{n}^n$ and $\mathbb{E}_n$ are explicit groups that are isomorphic respectively to the groups $\mathbb{I}$ and $\mathbb{I}^{n-2}$ where $\mathbb{I} := (1 + t^2 K[[t]], \cdot) \simeq K^N$.
and \( J := (tK[[t]], +) \simeq K^N \). Comparing the groups \( G_n \) and \( \text{Aut}_K(u_n) \) we see that the group \( (\text{UAut}_K(P_n)_n \) of polynomial automorphisms is a tiny part of the group \( \text{Aut}_K(u_n) \) but in contrast \( G_n = \text{Aut}_K(P_n) \). It is shown that the adjoint group of automorphisms \( \mathcal{A}(u_n) \) of the Lie algebra \( u_n \) is equal to the group \( \text{UAut}_K(P_n)_n \) (Theorem 7.1, [6]). Recall that the adjoint group \( \mathcal{A}(G) \) of a Lie algebra \( G \) is generated by the elements \( e^{ad(g)} := \sum_{i \geq 0} \frac{ad(g)^i}{i!} \in \text{Aut}_K(G) \) where \( g \) runs through all the locally nilpotent elements of the Lie algebra \( G \) (an element \( g \) is a locally nilpotent element if the inner derivation \( ad(g) := [g, \cdot] \) of the Lie algebra \( G \) is a locally nilpotent derivation).

## 2 Proof of Theorem 1.1

This section can be seen as a proof of Theorem 1.1. The proof is split into several statements that reflect ‘Structure of the proof of Theorem 1.1’ given in the Introduction.

The Lie algebra \( D_n \) is \( Z^n \)-graded. The Lie algebra

\[
D_n = \bigoplus_{\alpha \in N^n} \bigoplus_{i=1}^n Kx^\alpha \partial_i
\]

is a \( Z^n \)-graded Lie algebra

\[
D_n = \bigoplus_{\beta \in Z^n} D_{n, \beta} \quad \text{where} \quad D_{n, \beta} = \bigoplus_{\alpha-e_i=\beta} Kx^\alpha \partial_i,
\]

i.e. \( [D_{n, \alpha}, D_{n, \beta}] \subseteq D_{n, \alpha+i} \) for all \( \alpha, \beta \in N^n \) where \( e_1 := (1, 0, \ldots, 0), \ldots, e_n := (0, \ldots, 0, 1) \) is the canonical free basis for the free abelian group \( Z^n \). This follows from the commutation relations

\[
[x^\alpha \partial_i, x^\beta \partial_j] = \beta_i x^{\alpha+\beta-e_i} \partial_j - \alpha_j x^{\alpha+\beta-e_j} \partial_i.
\]

Clearly, for all \( i, j = 1, \ldots, n \) and \( \alpha \in N^n \),

\[
[H_j, x^\alpha \partial_i] = \begin{cases} 
\alpha_j x^\alpha \partial_i & \text{if } j \neq i, \\
(\alpha_i - 1)x^\alpha \partial_i & \text{if } j = i,
\end{cases}
\]

\[
[\partial_j, x^\alpha \partial_i] = \alpha_j x^{\alpha-e_j} \partial_i.
\]

The support \( \text{Supp}(D_n) := \{ \beta \in Z^n \mid D_{n, \beta} \neq 0 \} \) is a submonoid of \( Z^n \). Let us find the support \( \text{Supp}(D_n) \), the graded components \( D_{n, \beta} \) and their dimensions \( \dim_K D_{n, \beta} \). For each \( i = 1, \ldots, n \), let \( N^{n,i} := \{ \alpha \in N^n \mid \alpha_i = 0 \} \) and \( P^\beta_n := \ker P_n(\partial_i) \). It follows from the decompositions \( P_n = P^\beta_n \oplus P_n x_i \) for \( i = 1, \ldots, n \) that

\[
D_n = \bigoplus_{i=1}^n (P^\beta_n \oplus P_n x_i) \partial_i = \bigoplus_{i=1}^n P^\beta_n \partial_i \oplus \bigoplus_{i=1}^n P_n H_i,
\]

\[
D_n = \bigoplus_{i=1}^n P^\beta_n \partial_i \oplus \bigoplus_{\alpha \in N^n} x^\alpha H_n.
\]

Hence,

\[
\text{Supp}(D_n) = \prod_{i=1}^n \left( N^{n,i} - e_i \right) \bigcap N^n.
\]

\[
D_{n, \beta} = \begin{cases} 
\text{\( x^\alpha \partial_i \) if } \beta = \alpha - e_i \in N^{n,i} - e_i, \\
\text{\( x^\beta H_n \) if } \beta \in N^n.
\end{cases}
\]

\[
\dim_K D_{n, \beta} = \begin{cases} 
1 & \text{if } \beta = \alpha - e_i \in N^{n,i} - e_i, \\
n & \text{if } \beta \in N^n.
\end{cases}
\]
Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ be its Lie subalgebra. The centralizer $C_{\mathcal{G}}(\mathcal{H}) := \{ x \in \mathcal{G} \mid [x, \mathcal{H}] = 0 \}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$. In particular, $Z(\mathcal{G}) := C_{\mathcal{G}}(\mathcal{G})$ is the centre of the Lie algebra $\mathcal{G}$. The normalizer $N_{\mathcal{G}}(\mathcal{H}) := \{ x \in \mathcal{G} \mid [x, \mathcal{H}] \subseteq \mathcal{H} \}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, it is the largest Lie subalgebra of $\mathcal{G}$ that contains $\mathcal{H}$ as an ideal.

Let $V$ be a vector space over $K$. A $K$-linear map $\delta : V \to V$ is called a locally nilpotent map if $V = \bigcup_{i \geq 1} \ker(\delta^i)$ or, equivalently, for every $v \in V$, $\delta^i(v) = 0$ for all $i \gg 1$. When $\delta$ is a locally nilpotent map in $V$ we also say that $\delta$ acts locally nilpotently on $V$. Every nilpotent linear map $\delta$, that is $\delta^n = 0$ for some $n \geq 1$, is a locally nilpotent map but not vice versa, in general. Let $\mathcal{G}$ be a Lie algebra. Each element $a \in \mathcal{G}$ determines the derivation of the Lie algebra $\mathcal{G}$ by the rule $\text{ad}(a): \mathcal{G} \to \mathcal{G}$, $b \mapsto [a, b]$, which is called the inner derivation associated with $a$. The set $\text{Inn}(\mathcal{G})$ of all the inner derivations of the Lie algebra $\mathcal{G}$ is a Lie subalgebra of the Lie algebra $(\text{End}_K(\mathcal{G}), [\cdot, \cdot])$ where $[f, g] := fg - gf$. There is the short exact sequence of Lie algebras

$$0 \to Z(\mathcal{G}) \to \mathcal{G} \xrightarrow{\text{ad}} \text{Inn}(\mathcal{G}) \to 0,$$

that is $\text{Inn}(\mathcal{G}) \simeq \mathcal{G}/Z(\mathcal{G})$ where $Z(\mathcal{G})$ is the centre of the Lie algebra $\mathcal{G}$ and $\text{ad}([a, b]) = [\text{ad}(a), \text{ad}(b)]$ for all elements $a, b \in \mathcal{G}$. An element $a \in \mathcal{G}$ is called a locally nilpotent element (respectively, a nilpotent element) if so is the inner derivation $\text{ad}(a)$ of the Lie algebra $\mathcal{G}$.

**The Cartan subalgebra $\mathcal{H}_n$ of $D_n$.** A nilpotent Lie subalgebra $C$ of a Lie algebra $\mathcal{G}$ is called a Cartan subalgebra of $\mathcal{G}$ if it coincides with its normalizer. We use often the following obvious observation: *An abelian Lie subalgebra that coincides with its centralizer is a maximal abelian Lie subalgebra.*

**Lemma 2.1**

1. $\mathcal{H}_n$ is a Cartan subalgebra of $D_n$.

2. $\mathcal{H}_n = C_{D_n}(\mathcal{H}_n)$ is a maximal abelian subalgebra of $D_n$.

**Proof.** Statements 1 and 2 follows from (6) and (7). □

$P_n$ is a $D_n$-module. The polynomial algebra $P_n$ is a (left) $D_n$-module: $D_n \times P_n \to P_n$, $(\partial, p) \mapsto \partial \ast p$. In more detail, if $\partial = \sum_{i=1}^{n} a_i \partial_i$ where $a_i \in P_n$ then

$$\partial \ast p = \sum_{i=1}^{n} a_i \frac{\partial p}{\partial x_i}.$$ 

The field $K$ is a $D_n$-submodule of $P_n$ and

$$\bigcap_{i=1}^{n} \ker_{P_n}(\partial_i) = K. \quad (8)$$

**Lemma 2.2** The $D_n$-module $P_n/K$ is simple with $\text{End}_{D_n}(P_n/K) = \text{id}$ where $\text{id}$ is the identity map.

**Proof.** Let $M$ be a nonzero submodule of $P_n/K$ and $0 \neq p \in M$. Using the actions of $\partial_1, \ldots, \partial_n$ on $p$ we obtain an element of $M$ of the form $\lambda x_i$ for some $\lambda \in K^*$. Hence, $x_i \in M$ and $x^\alpha = x^\alpha \partial_i \ast x_i \in M$ for all $0 \neq \alpha \in \mathbb{N}^n$. Therefore, $M = P_n/K$. Let $f \in \text{End}_{D_n}(P_n/K)$. Then applying $f$ to the equalities $\partial_i \ast (x_1 + K) = \delta_{i1}$ for $i = 1, \ldots, n$, we obtain the equalities

$$\partial_i \ast f(x_1 + K) = \delta_{i1} \text{ for } i = 1, \ldots, n.$$ 

Hence, $f(x_1 + K) \in \bigcap_{i=2}^{n} \ker_{P_n/K}(\partial_i) \cap \ker_{P_n/K}(\partial_1^2) = (K[x_1]/K) \cap \ker_{P_n/K}(\partial_1^2) = K(x_1 + K)$. So, $f(x_1 + K) = \lambda(x_1 + K)$ and so $f = \lambda \text{id}$, by the simplicity of the $D_n$-module $P_n/K$. □
The $G_n$-module $D_n$. The Lie algebra $D_n$ is a $G_n$-module,
\[ G_n \times D_n \to D_n, \ (\sigma, \partial) \mapsto \sigma(\partial) := \sigma \partial \sigma^{-1}. \]

Every automorphism $\sigma \in G_n$ is uniquely determined by the elements
\[ x'_1 := \sigma(x_1), \ldots, x'_n := \sigma(x_n). \]

Let $M_n(P_n)$ be the algebra of $n \times n$ matrices over $P_n$. The matrix $J(\sigma) := (J(\sigma)_{ij}) \in M_n(P_n)$, where $J(\sigma)_{ij} = \frac{\partial x'_i}{\partial x_j}$, is called the Jacobian matrix of the automorphism (endomorphism) $\sigma$ and its determinant $J(\sigma) := \det J(\sigma)$ is called the Jacobian of $\sigma$. So, the $j$'th column of $J(\sigma)$ is the gradient $\text{grad} x'_j := (\frac{\partial x'_1}{\partial x_j}, \ldots, \frac{\partial x'_n}{\partial x_j})^T$ of the polynomial $x'_j$. Then the derivations
\[ \partial'_1 := \sigma \partial_1 \sigma^{-1}, \ldots, \partial'_n := \sigma \partial_n \sigma^{-1} \]
are the partial derivatives of $P_n$ with respect to the variables $x'_1, \ldots, x'_n$.

\[ \partial'_i = \frac{\partial}{\partial x'_i}, \ldots, \partial'_n = \frac{\partial}{\partial x'_n}. \tag{9} \]

Every derivation $\partial \in D_n$ is a unique sum $\partial = \sum_{i=1}^n a_i \partial_i$ where $a_i = \partial * x_i \in P_n$. Let $\partial := (\partial_1, \ldots, \partial_n)^T$ and $\partial' := (\partial'_1, \ldots, \partial'_n)^T$ where $T$ stands for the transposition. Then
\[ \partial' = J(\sigma)^{-1} \partial, \ i.e. \ \partial'_i = \sum_{j=1}^n (J(\sigma)^{-1})_{ij} \partial_j \text{ for } i = 1, \ldots, n. \tag{10} \]

In more detail, if $\partial' = A \partial$ where $A = (a_{ij}) \in M_n(P_n)$, i.e. $\partial_i = \sum_{j=1}^n a_{ij} \partial_j$. Then for all $i, j = 1, \ldots, n$,
\[ \delta_{ij} := \partial'_i * x'_j = \sum_{k=1}^n a_{ik} \frac{\partial x'_j}{\partial x_k} \]
where $\delta_{ij}$ is the Kronecker delta function. The equalities above can be written in the matrix form as $AJ(\sigma) = 1$ where 1 is the identity matrix. Therefore, $A = J(\sigma)^{-1}$.

Suppose that a group $G$ acts on a set $S$. For a nonempty subset $T$ of $S$, $\text{St}_G(T) := \{ g \in G \mid gT = T \}$ is the stabilizer of the set $T$ in $G$ and $\text{Fix}_G(T) := \{ g \in G \mid gt = t \text{ for all } t \in T \}$ is the fixator of the set $T$ in $G$. Clearly, $\text{Fix}_G(T)$ is a normal subgroup of $\text{St}_G(T)$.

The maximal abelian Lie subalgebra $\mathcal{D}_n$ of $D_n$.

**Lemma 2.3**

1. $C_{D_n}(\mathcal{D}_n) = \mathcal{D}_n$ and so $\mathcal{D}_n$ is a maximal abelian Lie subalgebra of $D_n$.

2. $\text{Fix}_{G_n}(\mathcal{D}_n) = \text{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = Sh_n$.

3. $\mathcal{D}_n$ is a faithful $G_n$-module, i.e. the group homomorphism $G_n \to \mathbb{G}_n, \sigma \mapsto \sigma : \partial \mapsto \sigma \partial \sigma^{-1}$, is a monomorphism.

4. $\text{Fix}_{G_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{ e \}$.

**Proof.** 1. Statement 1 follows from (2).

2. Let $\sigma \in \text{Fix}_{G_n}(\mathcal{D}_n)$ and $J(\sigma) = (J_{ij})$. By (10), $\partial = J(\sigma) \partial$, and so, for all $i, j = 1, \ldots, n$,
\[ \delta_{ij} = \partial'_i * x'_j = \sum_{k=1}^n a_{ik} \frac{\partial x'_j}{\partial x_k} \]
for some scalars $\lambda_i \in K$, and so $\sigma \in S_{h_n}$.

3 and 4. Let $\sigma \in \text{Fix}_{G_n} = (\partial_1, \ldots, \partial_n, H_1, \ldots, H_n)$. Then $\sigma \in \text{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = Sh_n$, by statement 2. So, $\sigma(x_1) = x_1 + \lambda_1, \ldots, \sigma(x_n) = x_n + \lambda_n$ where $\lambda_i \in K$. Then $x_i \partial_i = \sigma(x_i \partial_i) = (x_i + \lambda_i) \partial_i$ for $i = 1, \ldots, n$, and so $\lambda_1 = \cdots = \lambda_n = 0$. This means that $\sigma = e$. So, $\text{Fix}_{G_n} = (\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{ e \}$ and $D_n$ is a faithful $G_n$-module. $\square$

By Lemma 2.3(3), we identify the group $G_n$ with its image in $\mathbb{G}_n$. 


Lemma 2.4 1. $D_n$ is a simple Lie algebra.

2. $Z(D_n) = \{0\}$.

3. $[D_n, D_n] = D_n$.

Proof. 1. Let $0 \neq a \in D_n$ and $a = (a)$ be the ideal of the Lie algebra $D_n$ generated by the element $a$. We have to show that $a = D_n$. Using the inner derivations $\delta_1, \ldots, \delta_n$ we see that $\delta_i \in a$ for some $i$. Then $a = D_n$ since

$$x^\alpha \partial_j = (\alpha_i + 1)^{-1}[\partial_i, x^{\alpha+\epsilon_j} \partial_j] \in a$$

for all $\alpha$ and $j$.

2 and 3. Statements 2 and 3 follow from statement 1. \(\square\)

Proposition 2.5 1. $\text{Fix}_{G_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{e\}$.

2. Let $\sigma, \tau \in G_n$. Then $\sigma = \tau$ iff $\sigma(\partial_i) = \tau(\partial_i)$ and $\sigma(H_i) = \tau(H_i)$ for $i = 1, \ldots, n$.

3. $\text{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \text{Sh}_n$.

Proof. 1. Let $\sigma \in F := \text{Fix}_{G_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n)$. We have to show that $\sigma = e$. Since $\sigma \in \text{Fix}_{G_n}(H_1, \ldots, H_n)$, the automorphism $\sigma$ respects the weight decomposition of $D_n$. By \(\Box\), $\sigma(x^\alpha \partial_i) = \lambda_{\alpha,i} x^\alpha \partial_i$ for all $\alpha \in \mathbb{N}^{n \times n}$ and $i = 1, \ldots, n$ where $\lambda_{\alpha,i} \in K$. Clearly, $\lambda_{0,i} = 1$ for $i = 1, \ldots, n$. Since $\sigma \in \text{Fix}_{G_n}(\partial_1, \ldots, \partial_n)$, by applying $\sigma$ to the relations $\alpha_j x^{\alpha - \epsilon_j} \partial_i = [\partial_j, x^\alpha \partial_i]$, we get the relations

$$\alpha_j \lambda_{\alpha - \epsilon_j,i} x^{\alpha - \epsilon_j} \partial_i = [\partial_j, \lambda_{\alpha,i} x^\alpha \partial_i] = \alpha_j \lambda_{\alpha,i} x^{\alpha - \epsilon_j} \partial_i.$$

Hence $\lambda_{\alpha,i} = \lambda_{\alpha - \epsilon_j,i}$ provided $\alpha_j \neq 0$. We conclude that all the coefficients $\lambda_{\alpha,i}$ are equal to one of the coefficients $\lambda_{\epsilon_j,i}$ where $i, j = 1, \ldots, n$ and $i \neq j$. The relations $\partial_j = [\partial_i, x_i \partial_j] = \lambda_{\epsilon_j,i} \partial_j$, hence all the coefficients $\lambda_{\epsilon_j,i}$ are equal to 1. So, $\sum_{i=1}^n P_i \partial_i \subseteq F := \text{Fix}_{D_n}(\sigma) := \{\partial \in D_n | \sigma(\partial) = \partial\}$. To finish the proof of statement 1 it suffices to show that $x^\alpha H_i \in F$ for all $\alpha \in \mathbb{N}^{n \times n}$ and $i = 1, \ldots, n$, see \([\Box]\) and \([\Box]\). We use induction on $|\alpha| := a_1 + \cdots + a_n$. If $|\alpha| = 0$ the statement is obvious as $\sigma \in F$. Suppose that $|\alpha| > 0$. Using the commutation relations

$$[\partial_j, x^\alpha H_i] = \begin{cases} \alpha_j x^{\alpha - \epsilon_j} H_i & \text{if } j \neq i, \\ (\alpha_i + 1) x^\alpha \partial_i & \text{if } j = i, \end{cases} \quad (11)$$

the induction and the previous case, we see that

$$[\partial_j, \sigma(x^\alpha H_i) - x^\alpha H_i] = 0 \quad \text{for } i = 1, \ldots, n.$$

Therefore, $\sigma(x^\alpha H_i) - x^\alpha H_i \in C_{D_n}(D_n) = D_n$. Since the automorphism $\sigma$ respects the weight decomposition of $D_n$, we must have $\sigma(x^\alpha H_i) - x^\alpha H_i \in x^\alpha H_n \cap D_n = \{0\}$. Hence, $x^\alpha H_i \in F$, as required.

2. Statement 2 follows from statement 1.

3. Clearly, $\text{Sh}_n \subseteq F = \text{Fix}_{G_n}(\partial_1, \ldots, \partial_n)$. Let $\sigma \in F$ and $H'_1 := \sigma(H_1), \ldots, H'_n := \sigma(H_n)$. Applying the automorphism $\sigma$ to the commutation relations $[\partial_j, H'_i] = \delta_{ij} \partial_i$ gives the relations $[\partial_j, H'_i] = \delta_{ij} \partial_i$. By taking the difference, we see that $[\partial_i, H'_i - H_i] = 0$ for all $i$ and $j$. Therefore, $H'_i = H_i + \delta_i$ for some elements $d_i \in C_{D_n}(D_n) = D_n$ (Lemma \([\Box]\) (3)), and so $d_i = \sum_{j=1}^n \lambda_{ij} \partial_j$ for some elements $\lambda_{ij} \in K$. The elements $H'_1, \ldots, H'_n$ commute, hence

$$[H'_j, \partial_i] = [H_i, \partial_j] \quad \text{for all } i, j,$$
or equivalently,
\[ \lambda_{ij} \partial_j = \lambda_{ji} \partial_i \text{ for all } i, j. \]
This means that \( \lambda_{ij} = 0 \) for all \( i \neq j \), i.e.
\[ H'_i = H_i + \lambda_{ii} \partial_i = (x_i + \lambda_{ii}) \partial_i = s_{\lambda}(H_i) \]
where \( s_{\lambda} \in \text{Sh}_n \), \( s_{\lambda}(x_i) = x_i + \lambda_{ii} \) for all \( i \). Then \( s_{\lambda}^{-1} \sigma \in \text{Fix}_{\text{Sh}_n}(\partial_1, \ldots, \partial_n, H_1, \ldots, H_n) = \{ e \} \) (statement 2), and so \( \sigma = s_{\lambda} \in \text{Sh}_n. \)

\[ \text{Lemma 2.6 Let } \sigma \in \mathbb{G}_n \text{ and } \partial' := \sigma(\partial_1), \ldots, \partial'_n := \sigma(\partial_n). \text{ Then} \]
1. \( \partial'_1, \ldots, \partial'_n \) are commuting, locally nilpotent derivations of \( P_n \).
2. \( \bigcap_{i=1}^n \ker_{P_n}(\partial'_i) = K. \)

**Proof.** 1. The derivations \( \partial'_1, \ldots, \partial'_n \) commute since \( \partial_1, \ldots, \partial_n \) are commute. The inner derivations \( \delta_1, \ldots, \delta_n \) of the Lie algebra \( D_n \) are commuting and locally nilpotent. Hence, inner derivations
\[ \delta'_1 := \text{ad}(\partial'_1), \ldots, \delta'_n := \text{ad}(\partial'_n) \]
of the Lie algebra \( D_n \) are commuting and locally nilpotent. The vector space \( P_n \partial'_i \) is closed under the derivations \( \delta'_j \) since
\[ \delta'_j(P_n \partial'_i) = [\partial'_j, P_n \partial'_i] = (\partial'_j * P_n) \cdot \partial'_i \subseteq P_n \partial'_i. \]
Therefore, \( \partial'_1, \ldots, \partial'_n \) are locally nilpotent derivations of the polynomial algebra \( P_n \).
2. Let \( \lambda \in \bigcap_{i=1}^n \ker_{P_n}(\partial'_i). \) Then
\[ \lambda \partial'_i \in C_{D_n}(\partial'_1, \ldots, \partial'_n) = \sigma(C_{D_n}(\partial_1, \ldots, \partial_n)) = \sigma(C_{D_n}(D_n)) = \sigma(D_n) = \sigma(\bigoplus_{i=1}^n K \partial_i) = \bigoplus_{i=1}^n K \partial'_i, \]
since \( C_{D_n}(D_n) = D_n \), Lemma 2.3(1). Then \( \lambda \in K \) since otherwise the infinite dimensional space \( \bigoplus_{i \geq 0} K \lambda^i \partial'_i \) would be a subspace of a finite dimensional space \( \sigma(D_n). \) □

The following lemma is well-known and it is easy to prove.

**Lemma 2.7** Let \( \partial \) be a locally nilpotent derivation of a commutative \( K \)-algebra \( A \) such that \( \partial(x) = 1 \) for some element \( x \in A \). Then \( A = A^{\partial}[x] \) is a polynomial algebra over the ring \( A^{\partial} := \ker(\partial) \) of constants of the derivation \( \partial \) in the variable \( x \).

The next theorem is the most important point in the proof of Theorem 1.1 and, roughly speaking, the main reason why Theorem 1.1 holds.

**Theorem 2.8** Let \( \partial'_1, \ldots, \partial'_n \) be commuting, locally nilpotent derivations of the polynomial algebra \( P_n \) such that \( \bigcap_{i=1}^n \ker_{P_n}(\partial'_i) = K. \) Then there exist polynomials \( x'_1, \ldots, x'_n \in P_n \) such that
\[ \partial'_i * x'_j = \delta_{ij}. \] (12)

Moreover, the algebra homomorphism
\[ \sigma : P_n \rightarrow P_n, \ x_1 \mapsto x'_1, \ldots, x_n \mapsto x'_n \]
is an automorphism such that \( \partial'_i = \sigma \partial_i \sigma^{-1} = \frac{\partial}{\partial x'_i} \) for \( i = 1, \ldots, n. \)
Proof. Case $n = 1$: By Lemma 2.7, the derivation $\partial'_1$ of the polynomial algebra $P_1$ is a locally nilpotent derivation with $K'_1 := \ker_{P_1}(\partial'_1) = K$. Hence, $\partial'_1 + x'_1 = 1$ for some polynomial $x'_1 \in P_1$. By Lemma 2.7, $K[x_1] = K'_1[x'_1] = K[x'_1]$, and so $\sigma : K[x_1] \rightarrow K[x_1], x \mapsto x'_1$, is an automorphism such that $\sigma_{\partial'_1} = \frac{d}{dx'_1} = \sigma_{\partial'_1} \sigma^{-1}$.

Case $n \geq 2$. Let $K'_i := \ker_{P_n}(\partial'_i)$ for $i = 1, \ldots, n$. Clearly, $K \subseteq K'_i$.

(i) $K'_i \neq K$ for $i = 1, \ldots, n$: If $K'_i = K$ for some $i$ then by the same argument as in the case $n = 1$ there exists a polynomial $x'_i \in P_1$ such that $\partial'_i + x'_i = 1$, and so $P_n = K'_i[x'_i] = K[x'_i]$, a contradiction.

(ii)(a) Suppose that $m < n$, i.e. $\partial'_1 + x'_m = \delta_m$ for all $i = 1, \ldots, n$. By Lemma 2.7, $P_n = K'_n[x'_n]$. The algebra $K'_n$ admits the set of commuting, locally nilpotent derivations

$$\partial''_n := \partial'_1|_{K'_n}, \ldots, \partial''_{n-1} := \partial'_{n-1}|_{K'_n}$$

with $\bigcap_{i=1}^{n-1} \ker_{K'_n}(\partial''_i) = K'_n \cap \bigcap_{i=1}^{n-1} K'_i = K$.

(ii)(b) Suppose that $m < n - 1$. By (12),

$$K \cdot x'_{m+1} + K = K \cdot x'_{m+2} + K = \cdots = K \cdot x'_n + K,$$

and so $\lambda_j := \partial''_j \cdot x'_n \in K$ for $j = m + 1, \ldots, n - 1$. Hence, $(\partial''_j - \lambda_j \partial''_j) \cdot x'_n = 0$ for $j = m + 1, \ldots, n - 1$. A linear combination of commuting, locally nilpotent derivations is a locally nilpotent derivation (the proof boils down to the case $\partial + \delta$ of two commuting, locally nilpotent derivations, then the result follows from $(\partial + \delta)^m = \sum_{i=0}^{m} \binom{m}{i} \partial^i \delta^{m-i}$ and $\partial \delta^{m-i} = \delta^{m-i} \partial$).

Using the set of commuting, locally nilpotent derivations $\partial''_1, \ldots, \partial''_n$ that satisfy (12), we obtain the set of commuting, locally nilpotent derivations

$$\delta_{i} := \partial''_1, \ldots, \delta_{m} := \partial''_m, \delta_{m+1} := \partial''_{m+1} - \lambda_{m+1} \partial''_m, \ldots, \delta_{n-1} := \partial''_{n-2} - \lambda_{n-2} \partial''_{n-1}, \delta_{n} := \partial''_{n-1}$$

that satisfy (12) with

$$\delta_{i} \cdot x'_n = \delta_{i} \qquad \text{for } i = 1, \ldots, n.$$

Then repeating the arguments of (ii)(a), we see that $P_n = K'_n[x'_n]$. The algebra $K'_n$ admits the set of commuting, locally nilpotent derivations

$$\partial''_1 := \delta''_1|_{K'_n}, \ldots, \partial''_{n-1} := \delta''_{n-1}|_{K'_n}$$

with

$$\bigcap_{i=1}^{n-1} \ker_{K'_n}(\partial''_i) = K'_n \cap \bigcap_{i=1}^{n-1} \ker_{P_n}(\delta'_i) = K'_n \cap \bigcap_{i=1}^{n-1} \ker_{P_n}(\partial''_i) = \bigcap_{i=1}^{n-1} K'_i = K.$$

(iii) Using the cases (ii)(a) and (ii)(b) $n - 1$ more times we find polynomials $x'_1, \ldots, x'_n$ and commuting set of locally nilpotent derivations of $P_n$, say, $\Delta_1, \ldots, \Delta_n$ that satisfy (12) and such that

$$(\alpha) \Delta_i \cdot x'_j = \delta_{ij} \quad \text{for all } i, j = 1, \ldots, n;$$

and the derivations $\partial'_j$ acts locally nilpotently on the algebra $A^{\partial'_j}$. Therefore, for each index $j = m + 1, \ldots, n$, there exists an element $x'_j \in A$ such that $\partial'_j \cdot x'_j = 1$, and so (Lemma 2.7)

$$A = A^{\partial'_j}[x'_j] = K[x'_j] \quad \text{for } j = m + 1, \ldots, n.$$
Corollary 2.9
Let $\varphi$ ism (see (\(\Delta = \Lambda\partial\)))

Theorem 2.8. By Theorem 2.8, \(\partial\) is invertible.

Indeed, by (\(\alpha\), $\Lambda \cdot (\partial_i \ast x_j) = 1$, the identity $n \times n$ matrix. Hence, (\(\partial_i \ast x_j\) · $\Lambda = 1$, as required.

(v) Let $x'_1, \ldots, x'_n$ be the set of polynomials as in the theorem. Then $\sigma$ is an algebra automorphism (see (\(\gamma\)) and (iv)) such that $\partial'_i = \sigma \partial_i \sigma^{-1} = \frac{\partial'}{\partial x'_i}$ for $i = 1, \ldots, n$. $\square$

Corollary 2.9 Let $\sigma \in \mathbb{G}_n$. Then $\tau \sigma \in \text{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$ for some $\tau \in G_n$.

Proof. By Lemma 2.8, the elements $\partial'_1 := \sigma(\partial_1), \ldots, \partial'_n := \sigma(\partial_n)$ satisfy the assumptions of Theorem 2.8. By Theorem 2.8, $\partial'_1 := \tau^{-1}(\partial_1), \ldots, \partial'_n := \tau^{-1}(\partial_n)$ for some $\tau \in G_n$. Therefore, $\tau \sigma \in \text{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n)$. $\square$

Proof of Theorem 1.1 Let $\sigma \in \mathbb{G}_n$. By Corollary 2.9, $\tau \sigma \in \text{Fix}_{\mathbb{G}_n}(\partial_1, \ldots, \partial_n) = \text{Sh}_n$ (Proposition 2.5(3)). Therefore, $\sigma \in G_n$, i.e. $\mathbb{G}_n = G_n$. $\square$

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References

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