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The Attrition Dynamics of Multilateral War

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We extend classical force-on-force combat models to study the attrition dynamics of three-way and multilateral war. We introduce a new multilateral combat model—the multiduel—which generalizes the Lanchester models, and solve it under an objective function which values one's own surviving force minus that of one's enemies. The outcome is stark: either one side is strong enough to destroy all the others combined, or all sides are locked in a stalemate which results in collective mutual annihilation. The situation in Syria fits this paradigm.

Key words: Lanchester model; multiplayer nonzero-sum game

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1. Introduction

Conventional attrition models of armed conflicts (Ancker Jr 1995, Washburn and Kress 2009, Kress 2012) usually feature a duel between two sides (or coalitions thereof) out of which only one side eventually prevails as the victor. Such models have been extensively used to evaluate force structure, military operational concepts, and tactics (Bracken et al. 1995). The most common combat attrition models are Lanchester's aimed-fire (square law) and ancient (linear law) models (Washburn and Kress 2009). Another combat attrition model is the Salvo model (Hughes 1995, Armstrong 2004), which captures the attrition process due to an exchange of fire salvos typical mostly in modern missile warfare.

In this paper, we define and solve a new model of multilateral war, the Lanchester aimed-fire multiduel, in which each player's objective is to maximize the difference between its own numbers and the sum of its enemies' when the war ends. We show that unless there exists a player so strong that it can guarantee to win regardless of what the others do, the outcome is a gradual stalemate that culminates in mutual annihilation of all players. This remarkable conclusion is independent of the number of players and attrition data of a conflict. In the case of three players—known as *truel*—our conclusion stands in contrast to a range of results for sequential-engagement scenarios in which one player, often the weakest, can achieve a clear advantage (Kilgour and Brams 1997, Caplow 1956).

The war in Syria since 2011 serves as a motivating example for our model. This war presents a different paradigm than typical two-sided force-on-force engagements, with several players—the Assad regime and its Iranian and Hezboulla affiliates, Free Syrian Army, Kurdish militia, ISIS, and Jabhat al-Nusra—fighting each other for dominance over territory and people. This paradigm leads to an all-out war in which each player chooses how it divides its combat effort among its foes. Based on our multiduel attrition model, we cautiously speculate that absent an overall agreement among the various players, the war in Syria will prolong towards mutual annihilation, unless a significant and largely invulnerable external force such as Russia intervenes to make one player dominant.

The rest of this paper proceeds as follows. Section 2 introduces the Lanchester aimed-fire multiduel, and identifies situations where one player can guarantee itself a win. Section 3 shows that if no player is strong enough to guarantee a win, then the only possible outcome—assuming players are selfish—is mutual annihilation of all players. Section 4 offers conclusion.

2. The Lanchester Model

Consider a conflict situation comprising n players who fight each other, and delineate the *state* of the war by $\boldsymbol{x} = (x_1, \ldots, x_n)$, the players' current force sizes. The attrition (kill-rate) caused by player i to player j is θ_{ij}, for $i \neq j$; $i,j = 1,\dots,n$. A *fire allocation rule* is an $n \times n$ matrix $\boldsymbol{\alpha} = [\alpha_{ij}]$, with which player i allocates a fraction α_{ij} of its firepower at player j. A *policy* for player i is a state-dependent firepower allocation rule $\alpha_{ij}(\boldsymbol{x})$. For a given initial state $(x_1(0),\dots,x_n(0))$, a set of policies induces a force trajectory $\{(x_1(t),\dots,x_n(t)), t \geq 0\}$, according to the Lanchester model (Lanchester 1916):

$$\frac{dx_j(t)}{dt} = -\sum_{i \neq j, i=1}^{n} \alpha_{ij}(\boldsymbol{x}(t))\theta_{ij}x_i(t), \qquad j = 1,\dots,n, \tag{1}$$

while $\boldsymbol{x}(t) \geq 0$. These differential equations imply that the attrition continues unless all players die out possibly except for one. In other words, either a player will emerge as the only survivor while all other players die out, or all players head for mutual annihilation.

We say a player is *dominant* if it can defeat the alliance of all other players. In other words, a dominant player can guarantee a win regardless of what the other players do. A player is *pseudo-dominant* if it can guarantee a tie for itself—no other players can win—regardless of what the other players do.

If all players but one effectively act in alliance—concentrating all their fire on the remaining one player—then the multiduel model reduces to a two-player model with one side consisting of many heterogeneous force types. Lin and MacKay (2014) characterizes the optimal policy in this situation, which can be used to derive conditions for dominance in our multiduel model.

Lemma 1. *Consider the standpoint of player 1, and without loss of generality rearrange players* $2,\dots,n$ *such that*

$$\theta_{12}\theta_{21} \leq \theta_{13}\theta_{31} \leq \cdots \leq \theta_{1n}\theta_{n1}.$$

For given initial forces $x_1,\dots,x_n$*, player 1 dominates if and only if*

$$x_1^2 > \sum_{i=2}^{n} \frac{\theta_{i1}}{\theta_{1i}} x_i^2 + 2 \sum_{i<j; i,j=2}^{n} \frac{\theta_{i1}}{\theta_{1j}} x_i x_j. \tag{2}$$

Player 1 is pseudo-dominant, if the preceding is changed to an equality.

Proof. Assuming that players $2,\dots,n$ form an alliance and allocate all their fire to player 1 at all time, we want to determine the minimal initial force required of player 1 in order to dominate.

According to Theorem 5 in Lin and MacKay (2014), player 1's optimal policy is to allocate all its fire at player n, then player $n-1$, and so on, in order to eliminate players $n, n-1, \ldots, 2$ in sequence. The result then follows by applying player 1's optimal policy in Theorem 1 in Lin and MacKay (2014). $\square$

The condition in Eq. (2), when applied to each of the n players, divides the state space $\Omega \equiv \{(x_1, \ldots, x_n) : x_i \geq 0, i = 1, \ldots, n\}$ into $n+1$ disjoint regions $D_1, ..., D_n, N$ such that player i is dominant in D_i, $i = 1, \ldots, n$, and $N \equiv \Omega \setminus \cup_{i=1}^n D_i$ is the *nondominant region* in which no player is dominant. The case $n = 3$ is illustrated in Figure 1. The surface OQR separates D_1 from N, and likewise ORP and OPQ separate D_2 and D_3 from N, respectively. The line OR defines the states where a duel between players 1 and 2 heads for mutual annihilation; that is, $x_3 = 0$ and $\theta_{12} x_1^2 = \theta_{21} x_2^2$; see Lanchester (1916). Thus, Ω has three dominant *triangular cones* D_1, D_2, D_3, which meet at three lines OP, OQ, OR, and surround the nondominant region N.

If a state belongs to a dominant region, then the corresponding dominant player will use its optimal sequential strategy described in Lemma 1 to guarantee a win. But if a state belongs to the nondominant region, what will happen? In order to study this question, we first prove that if the state belongs to the nondominant region N, then there exists a fire allocation rule $\boldsymbol{\alpha} = [\alpha_{ij}]$ that decreases each player's number at the same proportional rate, namely $x_i(t) = x_i(0)e^{-\lambda t}$ for some $\lambda > 0$.

Lemma 2. *For every fire allocation rule $\boldsymbol{\alpha} = [\alpha_{ij}]$, there exists a state $s_A(\boldsymbol{\alpha}) \in \Omega$, the* annihilating state*, from which the outcome is mutual annihilation for all players.*

Proof. The Perron-Frobenius theorem asserts for any positive matrix the existence of a largest real eigenvalue λ and a corresponding Perron-Frobenius (PF) eigenvector whose entries may be chosen

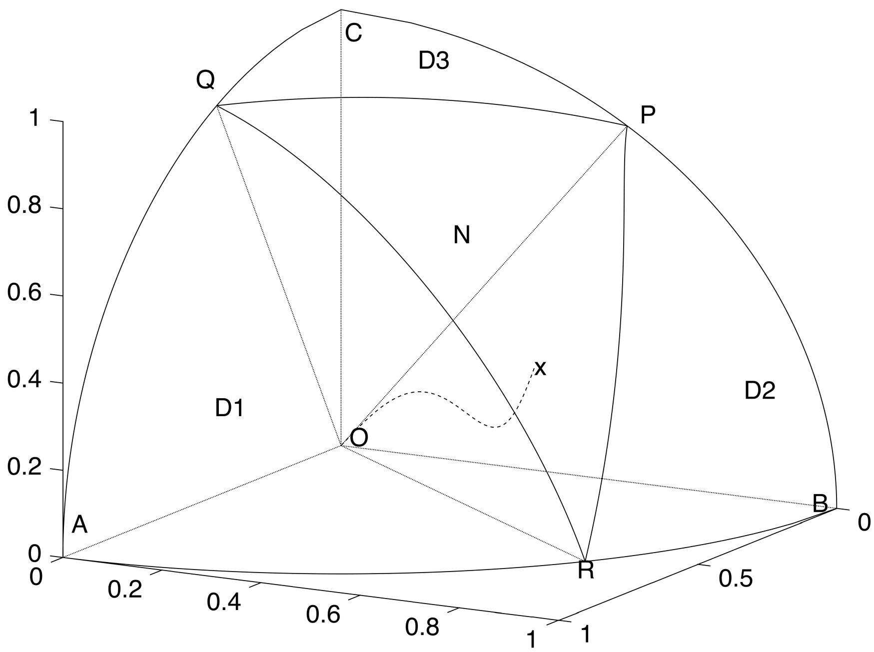

Figure 1 The case with $n = 3$ players, where x marks initial state and the dashed trajectory marks a path to mutual annihilation. The sphere octant is divided into four triangular cones D_1, D_2, D_3, N, separated by surfaces OPQ, OQR, and ORP.

to be positive. The annihilating state $s_A(\boldsymbol{\alpha})$ is then the left PF eigenvector of the matrix

$$\begin{pmatrix} 0 & \theta_{12}\alpha_{12} & \theta_{13}\alpha_{13} & \dots & \theta_{1n}\alpha_{1n} \\ \theta_{21}\alpha_{21} & 0 & \theta_{23}\alpha_{23} & \dots & \theta_{2n}\alpha_{2n} \\ \theta_{31}\alpha_{31} & \theta_{32}\alpha_{32} & 0 & \dots & \theta_{3n}\alpha_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{n1}\alpha_{n1} & \theta_{n2}\alpha_{n2} & \theta_{n3}\alpha_{n3} & \dots & 0 \end{pmatrix}.$$

As the state evolves according to Eq. (1), the force numbers decline as $(x_1(t), \dots, x_n(t)) = (x_1(0), \dots, x_n(0))\, e^{-\lambda t}$, which approaches $(0, \dots, 0)$ as $t \to \infty$. $\square$

The set $\Gamma \equiv \{s_A(\boldsymbol{\alpha}) \,|\, \alpha_{ii} = 0; \alpha_{ij} \geq 0; \sum_{j \neq i} \alpha_{ij} = 1, i = 1, \dots, n\}$ contains the range of s_A for all possible fire allocation rules $\boldsymbol{\alpha}$. In order to prove that there exists a fire allocation rule $\boldsymbol{\alpha}$ for every state in N such that the n players head for mutual annihilation, we show next that $N \subseteq \Gamma$.

THEOREM 1. $N \subseteq \Gamma$.

Proof. We first consider the standpoint of player 1, and without loss of generality rearrange players $2, \dots, n$ such that

$$\theta_{12}\theta_{21} \le \theta_{13}\theta_{31} \le \cdots \le \theta_{1n}\theta_{n1}. \tag{3}$$

Consider a fire allocation rule, with player 1 distributing its fire over the other $n-1$ players, while each of the other $n-1$ players direct all their fire at player 1, as follows,

$$\begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{4}$$

The set

$$T_1 \equiv \{s_A(\boldsymbol{\alpha}) | \boldsymbol{\alpha} \text{ takes form (4)}\}$$

forms a boundary of Γ. Intuitively, if all other players form an alliance against player 1, and player 1 can ensure mutual annihilation with some fire allocation rule, then player 1 can defeat the alliance with its optimal policy. We next show that player 1 dominates at any point in T_1.

Any point $(x_1, x_2, \dots, x_n) \in T_1$ must satisfy

$$(x_1\, x_2\, \cdots\, x_n) \begin{pmatrix} 0 & \theta_{12}\alpha_{12} & \theta_{13}\alpha_{13} & \dots & \theta_{1n}\alpha_{1n} \\ \theta_{21} & 0 & 0 & \dots & 0 \\ \theta_{31} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{n1} & 0 & 0 & \dots & 0 \end{pmatrix} = \lambda(x_1\, x_2\, \cdots\, x_n),$$

which is equivalent to

$$\sum_{i=2}^{n} \theta_{i1} x_i = \lambda x_1, \tag{5}$$

$$\theta_{1i}\alpha_{1i}x_1 = \lambda x_i, \qquad i = 2, \dots, n. \tag{6}$$

From Eq. (6), we get

$$x_i = \frac{\theta_{1i}\alpha_{1i}x_1}{\lambda}, \qquad i = 2, \ldots, n, \tag{7}$$

and substitute it into Eq. (5) to get

$$\lambda^2 = \sum_{i=2}^{n} \alpha_{1i}\theta_{1i}\theta_{i1}. \tag{8}$$

We next show that if $(x_1, \ldots, x_n) \in T_1$, then it satisfies Eq. (2). Using Eq. (7), the right-hand side of Eq. (2) becomes

$$\begin{aligned}
&\sum_{i=2}^{n} \frac{\theta_{i1}}{\theta_{1i}} \left(\frac{\theta_{1i}\alpha_{1i}x_1}{\lambda}\right)^2 + 2 \sum_{i<j;i,j=2}^{n} \frac{\theta_{i1}}{\theta_{1j}} \left(\frac{\theta_{1i}\alpha_{1i}x_1}{\lambda}\right)\left(\frac{\theta_{1j}\alpha_{1j}x_1}{\lambda}\right) \\
&= \frac{x_1^2}{\lambda^2}\left(\sum_{i=2}^{n} \alpha_{1i}^2\theta_{1i}\theta_{i1} + 2\sum_{i<j;i,j=2}^{n} \alpha_{1i}\alpha_{1j}\theta_{1i}\theta_{i1}\right) \\
&\le \frac{x_1^2}{\lambda^2}\left(\sum_{i=2}^{n} \alpha_{1i}^2\theta_{1i}\theta_{i1} + \sum_{i<j;i,j=2}^{n} \alpha_{1i}\alpha_{1j}(\theta_{1i}\theta_{i1} + \theta_{1j}\theta_{j1})\right),
\end{aligned} \tag{9}$$

where the inequality follows due to Eq. (3), since $i < j$ in the summation. The inequality is strict, unless $\theta_{1i}\theta_{i1}$ is the same for all $i = 2, \ldots, n$. Furthermore,

$$\begin{aligned}
&\sum_{i<j;i,j=2}^{n} \alpha_{1i}\alpha_{1j}(\theta_{1i}\theta_{i1} + \theta_{1j}\theta_{j1}) \\
&\quad = \sum_{i<j;i,j=2}^{n} \alpha_{1j}(\alpha_{1i}\theta_{1i}\theta_{i1}) + \sum_{i<j;i,j=2}^{n} \alpha_{1i}(\alpha_{1j}\theta_{1j}\theta_{j1}) \\
&\quad = \sum_{i=2}^{n}\left(\left(\sum_{j=1,j\ne i}^{n} \alpha_{1j}\right)\alpha_{1i}\theta_{1i}\theta_{i1}\right) \\
&\quad = \sum_{i=2}^{n}(1-\alpha_{1i})\alpha_{1i}\theta_{1i}\theta_{i1}.
\end{aligned}$$

Therefore, the right-hand side of Eq. (9) becomes

$$\frac{x_1^2}{\lambda^2}\left(\sum_{i=2}^{n} \alpha_{1i}^2\theta_{1i}\theta_{i1} + \sum_{i=2}^{n}(1-\alpha_{1i})\alpha_{1i}\theta_{1i}\theta_{i1}\right) = \frac{x_1^2}{\lambda^2}\left(\sum_{i=2}^{n}\alpha_{1i}\theta_{1i}\theta_{i1}\right) = x_1^2,$$

which proves Eq. (2), where the last equality is due to Eq. (8). In other words, player 1 dominates at any point in T_1.

From each player's standpoint, we can repeat the same argument to arrive at the same conclusion. That is, by defining $T_2, \ldots, T_n$ analogously, we can show that player i dominates at any point in T_i, for $i = 1, \ldots, n$. Because Γ is enclosed by the union of $T_1, \ldots, T_n$, it follows that $N \subseteq \Gamma$. $\square$

Theorem 1 shows that for any state $\boldsymbol{x} \in N$, there exists a fire allocation $\boldsymbol{\alpha}$ such that $x_i(t) = x_i e^{-\lambda t}$ for some $\lambda > 0$. In fact, there are almost always infinitely many such fire allocations, aside from singular cases. In other words, it is always possible—and fairly easy—for the n players to find a fire allocation that does not shift the current *balance of power*, by keeping the ratios $x_i(t)/x_j(t)$ fixed throughout. If all players collectively adopt such a fire allocation, then the resulting force trajectory is a straight line toward $(0,\dots,0)$. If some players are unhappy with the *status quo* and try to shake up the balance of power, then the *relative* force sizes $x_i(t)/x_j(t)$ may change over time, for some i and j. Is it possible for a player to outwit the others to become the sole winner? We next define an n-person nonzero-sum game and use Nash equilibria to show that the only possible outcome is mutual annihilation, as long as each player is selfish.

3. A Nonzero-Sum Game and Its Nash Equilibria

The Lanchester multiduel model in Section 2 has in general no analogue of the square law which characterizes the Lanchester duel. Instead, consider it as a nonzero-sum game, where player i freely chooses its policy $\alpha_{ij}(\boldsymbol{x})$ and seeks to maximize its payoff

$$V_i = x_i(\infty) - \sum_{j \neq i} x_j(\infty), \qquad i = 1,\dots,n.$$

In other words, each player wants to maximize its remaining force size at the end of the war, if it wins. If a player is annihilated, then its goal is to minimize the remaining force size of the eventual winner. If the players head for mutual annihilation, then each player's payoff is 0.

In a nonzero-sum game, a set of individual players' policies form a Nash equilibrium if no player can improve its payoff by switching to a different policy on its own (Nash 1950). Write S_1 for the set where the inequality in Eq. (2) holds with equality, and define $S_2,\dots,S_n$ analogously. In other words, player i is pseudo-dominant in S_i, which separates D_i from N, and $\{S_1,\dots,S_n\}$ form the boundaries of N. In Figure 1, S_1 is the surface OQR.

Theorem 2. *In the multiduel game, a set of policies $[\alpha_{ij}(\boldsymbol{x})]$ forms a Nash equilibrium, if and only if, for $\boldsymbol{x} \in D_i \cup S_i$, $i = 1,\dots,n$, we have that*

$$\alpha_{ji}(\boldsymbol{x}) = 1, \tag{10}$$

for all $j \neq i$, and

$$\alpha_{ik}(\boldsymbol{x}) = 1, \tag{11}$$

where $k = \arg\max_{j \neq i, x_j \geq 0}\{\theta_{ij}\theta_{ji}\}$. The values of $[\alpha_{ij}(\boldsymbol{x})]$ for $\boldsymbol{x} \in N \setminus \cup_{i=1}^{n} S_i$ are irrelevant.

Proof. First, we prove Eqs. (10) and (11) are sufficient conditions. Suppose $[\alpha_{ij}(\boldsymbol{x})]$ meets Eqs. (10) and (11).

1. If $\boldsymbol{x} \in D_i \cup S_i$, then due to Eq. (11), it follows from Lin and MacKay (2014) that player i's policy maximizes $x_i(\infty)$; in other words, V_i decreases if player i does not use this policy. In addition, if player $j \neq i$ does not allocate all its fire at player i as instructed in Eq. (10), then $x_i(\infty)$ will increase, so V_j decreases.
2. If $\boldsymbol{x} \in N$, then $\alpha_{ij}(\boldsymbol{x})$ ensures a force trajectory that converges to $(x_1(\infty), \ldots, x_n(\infty)) = (0, \ldots, 0)$, which gives $(V_1, \ldots, V_n) = (0, \ldots, 0)$. No matter what player i does, the state $\boldsymbol{x}(t)$, $t \geq 0$, will never cross over S_i to D_i. Hence, $V_i \leq 0$, for $i = 1, \ldots, n$. In other words, player i cannot improve V_i by deviating from its policy.

Since no player can improve its payoff by switching to a different policy, $[\alpha_{ij}(\boldsymbol{x})]$ forms a Nash equilibrium.

Next, we prove Eqs. (10) and (11) are necessary conditions. Suppose $[\alpha_{ij}(\boldsymbol{x})]$ does not meet Eqs. (10) and (11).

1. There exists some state $\boldsymbol{x}$, such that $\boldsymbol{x} \in D_i \cup S_i$, but $\alpha_{ij}(\boldsymbol{x})$ does not satisfy Eq. (11). Player i can switch to Eq. (11) to increase $x_i(\infty)$ due to Lin and MacKay (2014), thus increasing V_i.
2. There exists some state $\boldsymbol{x}$, such that $\boldsymbol{x} \in D_i \cup S_i$, but $\alpha_{ji}(\boldsymbol{x})$ does not satisfy Eq. (10); that is, $\alpha_{ji}(\boldsymbol{x}) < 1$ for some $j \neq i$. Player j can switch to $\alpha_{ji}(\boldsymbol{x}) = 1$ to bring down $x_i(\infty)$, thus increasing V_j.

Hence, $[\alpha_{ij}(\boldsymbol{x})]$ does not form a Nash equilibrium, if it violates Eqs. (10) or (11). □

In other words, if $x \in D_i \cup S_i$, then the dominant (or pseudo-dominant) player will kill off the other players according to the optimal order, while all the other players will fire at the dominant (or pseudo-dominant) player. In state $\boldsymbol{x} \in N \setminus \cup_{i=1}^{n} S_i$, surprisingly, it does not matter what a player does. The final outcome is the same—mutual annihilation—for any starting point in N, and no player can do anything to evade it. It is always possible for the players to adopt a fire allocation that keeps $x_i(t)/x_j(t)$ unchanged for all pairs i and j. It is also possible that some player—by itself or through cooperation with the others—tries to tilt $x_i(t)/x_j(t)$ in its favor. The crucial observation is that $\boldsymbol{x}(t)$ may take a variety of paths toward the point of mutual annihilation, but it will never step outside of N. The case $n = 3$ can be visualized in Figure 1, which depicts a trajectory of $\boldsymbol{x}(t)$ approaching the origin, while staying in N at all time. If $\boldsymbol{x}(t)$ gets close to S_1 (surface OQR in Figure 1), then the other players are motivated to allocate more fire at player 1 to pull the state away from it. If $\boldsymbol{x}(t)$ ever reaches S_1, where player 1 is pseudo-dominant, then the Nash equilibrium policies described in Theorem 2 will ensure mutual annihilation to yield $V_i = 0$ for all i. In other words, no player can develop a strategy to outwit the others to score a win; instead, all players are doomed to mutual annihilation.

Thus the outcome of the game is either a win by a dominant player who is capable of defeating the alliance formed by all other players, or attritional stalemate leading to mutual annihilation without a winner. This conclusion does not rely on any specific mathematical assumptions in the model; it is much more general. For instance, any objective function which is monotonically increasing in a player's own surviving number and decreasing in its opponents' would give the same results. The same conclusion also holds if there are small perturbations, such as a small misstep by a player, a small force-recruitment effort by a player, a small change in the attrition rate θ_{ij}, a small random event, or an introduction of additional nondominant players. As long as the small perturbation moves the state to another point still in the nondominant region, the result of mutual annihilation will remain true. In practice, each player is motivated to move the state toward the center part of the nondominant region—away from its boundaries $\cup_{i=1}^{n} S_i$—so as to keep the mutual annihilation result as robust as possible against potential unpredictable events.

Is it possible for some players to form a temporary alliance to eliminate some other players first, before fighting it out among themselves? In theory, Nash equilibrium does not preclude alliances, but it is practically impossible when only 3 players remain. If players 1 and 2 form a temporary alliance with the goal to eliminate player 3, then player 3's best response is to allocate its fire carefully so that at its own demise, players 1 and 2 head for mutual annihilation—tracking the straight line OR in Figure 1. However, pushing the state toward the straight line OR in Figure 1 *with precision* is practically impossible, if we allow small perturbations in the system. When the state gets very close to the line OR, or to any boundary of N, a small perturbation will tip the state outside of N, which is undesirable for all but one player. What is more likely to happen in the real world is, again, for all players to keep the state in the center part of the nondominant region in order to avoid the emergence of a dominant player.

4. Conclusion

Models are abstractions of realities (Epstein 2008), and our model is no exception. However, with all the caveats associated with Lanchester modeling, we believe that the model realistically captures the essential dynamics of multilateral war. The results are general and robust: they also apply to Lanchester's linear model where attrition is fixed; see Appendix A. The insight is crystal clear: either a single player is strong enough to beat all other opponents combined, or all players share an ineluctable fate of a prolonged attritional stalemate that will culminate in mutual annihilation.

These two possible outcomes are in contrast to the conventional Lanchester aimed-fire duel and stochastic duels (Kikuta 1986, Lin 2014), which always (except for singular cases) uniquely determine a winner. They also contrast those in the Salvo duel model (Armstrong 2004), which allows the possibility of no attrition at all, if each player has a strong defensive mechanism—such as using surface-to-air missiles to intercept the enemy's incoming missiles. The two possible outcomes of our multiduel model are also strikingly different from those in three-way fights—known as *truels* (Kilgour and Brams 1997)—where a common feature, with survival as everyone's goal, is that the apparently weakest player has a surprisingly high chance of being the last man standing. Similar situations have long been known in sociology, where "the triadic situation often favors the weak over the strong" (Caplow 1956). This result can be recovered in our model with a different, more tactical and defensive objective, in which each player has the instantaneous goal of maximizing only the decline of its own casualty rate; see Appendix B. The central insight to emerge from our model is that, in a war of attrition, as long as each player values its opponents' destruction as well as its own survival, the players are destined for mutual annihilation.

The situation in Syria since 2011 appears to conform more to our model than to truels or more nuanced coalitional models (Mesterton-Gibbons et al. 2011). The various players in this conflict—Assad regime, Free Syrian Army, ISIS, and others—have been entangled in a violent conflict, killing each other, with no end in sight. In terms of our model, all the Syrian players have been in the nondominant region. Based on the insights gleaned from our model, and absent any external "shock" such as an effective introduction of non-conventional weapons by one of the players, we speculate that there are only two ways to stop the attritional process before all players become dysfunctional: (a) a political agreement that stops the violence; or (b) an intervention by a significant external force that supports one of the players and pushes the state to a dominant region of our model. The intervention of Russia in supporting the Assad regime suggests that the latter solution may prevail.

Appendix A: The Lanchester Ancient Model

The article mainly concerns the Lanchester aimed-fire (square law) model, but an analogous result holds for the Lanchester ancient (linear law) model (Lanchester 1916). In the Lanchester ancient model, a player has constant fire power as long as it is still *alive*. We use the same notation as in the article, but in the ancient warfare model with n players, the state evolution is governed by

$$\frac{dx_j(t)}{dt} = -\sum_{i\neq j, x_i>0} \alpha_{ij}(\boldsymbol{x}(t))\theta_{ij}, \qquad j=1,\dots,n. \tag{12}$$

The counterpart to Lemma 1 is presented below.

LEMMA 3. *For given initial forces $x_1,\dots,x_n$, consider the standpoint of player 1, and without loss of generality rearrange players $2,\dots,n$ such that*

$$\frac{\theta_{12}\theta_{21}}{x_2} \le \frac{\theta_{13}\theta_{31}}{x_3} \le \cdots \le \frac{\theta_{1n}\theta_{n1}}{x_n}.$$

Player 1 dominates if and only if

$$x_1 > \sum_{j=2}^{n} \left(\sum_{k=j}^{n} \frac{x_k}{\theta_{1k}} \theta_{j1} \right). \tag{13}$$

In addition, the optimal policy is for player 1 to allocate all its fire at player n, then player $n-1$, and so on, in order to eliminate players $n, n-1, \ldots, 2$ in sequence.

Proof. Assuming that players $2, \ldots, n$ form an alliance and allocate all their fire to player 1 at all time, we want to determine the minimal initial force required for player 1 to win. Among all policies that end up killing off the other players in an arbitrary order $i_1, i_2, \ldots, i_{n-1}$, the policy that minimizes player 1's loss is the one that allocates all its fire at player i_1, then player i_2, and so on, in order to eliminate players $i_1, \ldots, i_{n-1}$ in sequence. Therefore, it remains to find which of the $(n-1)!$ sequences is optimal. By comparing the two sequences $i, j, \ldots$ and $j, i, \ldots$, it is straightforward to show the optimal sequence as stated.

Using the optimal policy, the time it takes to eliminate player j is

$$\sum_{k=j}^{n} \frac{x_k}{\theta_{1k}},$$

during which time player 1's loss due to player j's fire is

$$\left(\sum_{k=j}^{n} \frac{x_k}{\theta_{1k}} \right) \theta_{j1}.$$

Hence, the right-hand side in Eq. (13) is the total loss of player 1 until it eliminates all other players, which concludes the proof. □

The counterpart to Lemma 2 is presented below.

Lemma 4. *For every fire allocation rule $\boldsymbol{\alpha} = [\alpha_{ij}]$, there exists a state $s_A(\boldsymbol{\alpha}) \in \Omega$, the* annihilating state*, from which the outcome is annihilation for all players.*

Proof. According to Eq. (12), for a fire allocation rule $\boldsymbol{\alpha}$, the n players head for mutual annihilation if

$$\frac{x_j}{\sum_{i \neq j} \alpha_{ij} \theta_{ij}} = \lambda, \qquad j = 1, \ldots, n, \tag{14}$$

for some $\lambda > 0$. Therefore, for an arbitrary $\lambda > 0$, let $x_j = \lambda \sum_{i \neq j} \alpha_{ij} \theta_{ij}$, and the state $(x_1, \ldots, x_n)$ is an annihilating state of $\boldsymbol{\alpha}$. □

The set $\Gamma \equiv \{s_A(\boldsymbol{\alpha}) \,|\, \alpha_{ii} = 0; \alpha_{ij} \geq 0; \sum_{j \neq i} \alpha_{ij} = 1, i = 1, \ldots, n\}$ contains the range of s_A for all possible fire allocation rules $\boldsymbol{\alpha}$. The counterpart to Theorem 1 is presented below.

Theorem 3. $N \subseteq \Gamma$.

Proof. We first consider the standpoint of player 1, and without loss of generality rearrange players $2,\dots,n$ such that

$$\frac{\theta_{12}\theta_{21}}{x_2} \le \frac{\theta_{13}\theta_{31}}{x_3} \le \cdots \le \frac{\theta_{1n}\theta_{n1}}{x_n}.$$

Consider a policy, with player 1 distributing α_{1i} of its fire at player $i \neq 1$, while each of the other $n-1$ players direct all its fire at player 1. According to Eq. (14), for the n players to head for mutual annihilation, we require that

$$\frac{x_1}{\sum_{i\neq 1}\theta_{i1}} = \frac{x_2}{\alpha_{12}\theta_{12}} = \frac{x_3}{\alpha_{13}\theta_{13}} = \cdots = \frac{x_n}{\alpha_{1n}\theta_{1n}} = \lambda > 0. \tag{15}$$

The set

$$T_1 \equiv \{s_A(\boldsymbol{\alpha})|\boldsymbol{\alpha} \text{ satisfies (15)}\}$$

forms a boundary of Γ. We next show that player 1 dominates at any point in T_1.

Using Eq. (15), the right-hand side of Eq. (13) is

$$\sum_{j=2}^{n}\left(\sum_{k=j}^{n}\frac{x_k}{\theta_{1k}}\theta_{j1}\right) = \lambda\sum_{j=2}^{n}\left(\sum_{k=j}^{n}\alpha_{1k}\theta_{j1}\right) \le \lambda\sum_{j=2}^{n}\theta_{j1} = x_1,$$

with equality if and only if $\alpha_{1n}=1$. Thus, according to Eq. (13), player 1 dominates at any point in T_1.

From each player's standpoint, we can repeat the same argument to arrive at the same conclusion. That is, by defining $T_2,\dots,T_n$ analogously, we can show that player i dominates at any point in T_i, for $i=1,\dots,n$. Because Γ is the enclosure of $T_1,\dots,T_n$, it follows that $N \subseteq \Gamma$. $\square$

Appendix B: A Defensive Tactical Objective

In various multiduel models with three players—known as *truels*—in which survival is the common goal, the apparently-weakest player has a surprisingly high chance of winning, since it presents the least threat to the others. Similar ideas and results can be recovered in our aimed-fired model via a different objective function, where player i maximizes the rate of reduction of its casualty rate, namely $\ddot{x}_i$. We make a simplification in which player i has kill rate θ_i against either opponent, $i=1,2,3$, and let $\theta_1 \ge \theta_2 \ge \theta_3$ without loss of generality. Player 1 is the strongest player, while player 3 is the weakest.

Taking the second derivative of x_1 with respect to t yields

$$\ddot{x}_1 = \theta_1(\alpha_{12}\theta_2\alpha_{21} + \alpha_{13}\theta_3\alpha_{31})x_1 + \alpha_{12}\text{-independent terms},$$

which player 1 wants to maximize, and similarly for x_2 and x_3. Hence, the players' best responses are

- Player 1: choose $\alpha_{12}=1$ (resp. $\alpha_{12}=0$) according as $\theta_2\alpha_{21}-\theta_3\alpha_{31}>0$ (resp. <0).
- Player 2: choose $\alpha_{23}=1$ (resp. $\alpha_{23}=0$) according as $\theta_3\alpha_{32}-\theta_1\alpha_{12}>0$ (resp. <0).
- Player 3: choose $\alpha_{31}=1$ (resp. $\alpha_{31}=0$) according as $\theta_1\alpha_{13}-\theta_2\alpha_{23}>0$ (resp. <0).

In terms of $(\alpha_{12},\alpha_{23},\alpha_{31})$, the line from $(1,0,0)$ to $(1,0,1)$, in which players 1 and 2 fight each other while player 3 is neutral between them, is stable, since $\theta_1\geq\theta_2\geq\theta_3$. The only other semi-stable point is $(0,1,1)$, at which players 1 and 2 attack player 3, but player 2 is then neutral as between attacking player 1 and attacking player 3.

One could go further and make the determination of policies a differential game, in which the players all rapidly adjust their policies, on a timescale much smaller than that of the attrition, in proportion to the advantage gained by doing so. With $\tau=t^2_{\text{attrition}}/t_{\text{adaptive}}$, we have

$$\frac{1}{\tau}\frac{d\alpha_{12}}{dt}=\theta_1(\theta_2\alpha_{21}-\theta_3\alpha_{31}),$$

$$\frac{1}{\tau}\frac{d\alpha_{23}}{dt}=\theta_2(\theta_3\alpha_{32}-\theta_1\alpha_{12}),$$

$$\frac{1}{\tau}\frac{d\alpha_{31}}{dt}=\theta_3(\theta_1\alpha_{13}-\theta_2\alpha_{23}).$$

This is a dynamical system within the unit cube of admissible policies. An analysis of the system's stable regions and basins of attraction again shows an advantage for player 3. The line from $(1,0,0)$ to $(1,0,1)$, in which players 1 and 2 fight each other while player 3 is neutral between them, is again stable. There is a stable line segment from $(0,1-\theta_3/\theta_2,1)$ to $(0,1,1)$, with a smaller basin, along which players 1 and 3 fight while player 2 divides its fire. If $\theta_2+\theta_3>\theta_1$, there is also a short stable segment from $(1-\theta_2/\theta_1,1,0)$ to $(\theta_3/\theta_1,1,0)$ in which players 2 and 3 fight, while player 1's fire is finely balanced between players 2 and 3 in such a way that neither player 2 nor player 3 is suffering so badly from that fire as to wish to attack player 1 instead. If $\theta_2+\theta_3<\theta_1$, no such region exists, and player 1 is never left unattacked.

This scenario most closely matches the typical insights associated with classic truels (Kilgour and Brams 1997): The (in some sense) weakest player gains advantage from being the least immediate threat to the others. But note that this only applies because of our choice of objective function, in which each player cares only for its own hurt, in contrast to the objective in the main article, which was not merely to win but, if winning was impossible, to cause as many casualties as possible to opponents.

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