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Comparing the direct normal form method with harmonic balance and the method of multiple scales

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Abstract

Approximate analytical methods have been used extensively for finding approximate solutions to nonlinear ordinary differential equations. In this paper we compare the recently developed direct normal form transformation with two other very well known and long standing methods, harmonic balance and the method of multiple scales. We will show that the direct normal form method combines some of the key advantages of harmonic balance and multiple scales whilst reducing some of the limitations.

Keywords: nonlinear;, vibration;, normal form;, harmonic balance;, multiple scales

1. Introduction

Approximate analytical methods have been used extensively for finding approximate solutions to nonlinear ordinary differential equations. For example, the method of harmonic balance offers a very straightforward approach, that can be applied to a wide variety of systems. Unlike perturbation methods, it has no limitation on the size of the nonlinear terms, or the size of the amplitude of response. However, it has some significant limitations, most notably the terms in the solution that are "unbalanced".

Perturbation methods are typically applied under the constraint that the nonlinear terms are small in comparison to the linear terms, a situation often referred to as *weak* nonlinearity. The method of multiple scales is widely used to provide approximate solutions for systems with weakly nonlinear functions [10]. Approximate solutions can also be obtained using normal form transformations, see for example [5]. In recent work Neild & Wagg [12] showed that systems of weakly coupled second-order nonlinear differential equations have a direct normal form transformation. Comprehensive discussions of normal form theory can be found, for example, in [1,6–9]. The direct method proposed in [12] is applied directly to the governing second-order equations of motion without transforming into a system of first-order equations. One of the main benefits of this type of direct normal form transformation, and the motivation for

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this paper, is that it can give better approximations to backbone curves [2–4]. In the literature, the standard example by which these methods are demonstrated is the undamped, unforced Duffing equation, which we introduce next.

2. Example : the undamped, unforced Duffing oscillator

The undamped, unforced Duffing oscillator can be written as

$$\ddot{x} + \omega_n^2 x + \alpha x^3 = 0, \tag{1}$$

where x is displacement, ω_n is the linear natural frequency, α is a nonlinear coefficient and overdot represents differentiation with respect to time. We assume that the initial conditions for this system are time $t_0 = 0$, displacement $x(0) = A$ and velocity $\dot{x}(0) = 0$.

2.1. Direct normal form

The idea behind this normal form approach is to apply a coordinate transform to the equation of motion to give a resonant equation of motion, in terms of a new coordinate u. The resonant equation can be solved exactly using

$$u = u_p + u_m = \frac{A}{2}\mathrm{e}^{\mathrm{i}(\omega_r t-\phi_0)} + \frac{A}{2}\mathrm{e}^{-\mathrm{i}(\omega_r t-\phi_0)}, \tag{2}$$

where A is displacement amplitude, ω_r is the response frequency, ϕ_0 is phase lag, and subscripts p and m denote plus and minus exponential terms respectively. Approximations come in the transform which is not exact. To proceed we rewrite (1) as

$$\ddot{x} + \omega_n^2 x + \varepsilon n(x_p, x_m) = 0, \tag{3}$$

where ε has been introduced to indicate that the nonlinear term is assumed small. The nonlinear term x^3 is replaced by $(x_p + x_m)^3$ to give

$$n(x_p, x_m) = \alpha(x_p^3 + 3x_p^2 x_m + 3x_p x_m^2 + x_m^3) = [\alpha \;\; 3\alpha \;\; 3\alpha \;\; \alpha]\mathbf{x}^*, \tag{4}$$

where the vector $\mathbf{x}^*$ is defined by the form of the nonlinear term, and in this example is given as $\mathbf{x}^* = [x_p^3, x_p^2 x_m, x_p x_m^2, x_m^3]^T$. Next a near-identity nonlinear transform from x to u is applied, giving

$$x = u + \varepsilon\mathbf{b}\mathbf{u}^* \quad \text{where} \quad \mathbf{b} = [b_1 \;\; b_2 \;\; b_3 \;\; b_4], \quad \text{and} \quad \mathbf{u}^* = [u_p^3, u_p^2 u_m, u_p u_m^2, u_m^3]^T, \tag{5}$$

where the structure of $\mathbf{u}^*$ is defined to be exactly the same as $\mathbf{x}^*$, and the base solutions are given in (2). The near-identity transformation will lead to a new governing equation for the Duffing system in the form

$$\ddot{u} + \omega_n^2 u + \varepsilon n_u(u_p, u_m) = 0, \tag{6}$$

where the transformed nonlinear terms, $n_u(u_p, u_m)$, is reduced to only the essential nonlinear terms: the normal form.

Substituting (5) into (3) leads to

$$\ddot{u} + \varepsilon\mathbf{b}\ddot{\mathbf{u}}^* + \omega_n^2(u + \varepsilon\mathbf{b}\mathbf{u}^*) + \varepsilon n(u_p + u_m + O(\varepsilon^1)) = 0. \tag{7}$$

where we have substituted $u = u_p + u_m$ in the nonlinear term n with base solutions from (2). Note that ε^2 terms and higher are neglected (which is why the transformation is not exact). A detuning of the form $\omega_n^2 = \omega_r^2 - \varepsilon(\omega_r^2 - \omega_n^2)$ is now introduced, as described in [13]. Then eliminating $\ddot{u}$ using (6) in combination with the detuning gives an equation for the ε^1 terms as

$$\varepsilon^1 : \quad -\mathbf{b}\ddot{\mathbf{u}}^* - \omega_r^2\mathbf{b}\mathbf{u}^* = n - n_u. \tag{8}$$

Both the nonlinear terms on the right-hand side are now functions of u_p and u_m, and so can be written as functions of the vector $\mathbf{u}^*$. Therefore we define $n = [n]\mathbf{u}^*$ and $n_u = [n_u]\mathbf{u}^*$ where $[n] = [\alpha \;\; 3\alpha \;\; 3\alpha \;\; \alpha]$ and $[n_u] = [n_{u1} \;\; n_{u2} \;\; n_{u3} \;\; n_{u4}]$, following the structure of (4), with $x_i \to u_i$. The derivative of $\mathbf{u}^*$ can be obtained by first noting that

$$\mathbf{u}^* = \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix} = \frac{A^3}{8}\begin{bmatrix} \mathrm{e}^{\mathrm{i}3(\omega_r t-\phi_0} \\ \mathrm{e}^{\mathrm{i}(\omega_r t-\phi_0)} \\ \mathrm{e}^{-\mathrm{i}(\omega_r t-\phi_0)} \\ \mathrm{e}^{-\mathrm{i}3(\omega_r t-\phi_0)} \end{bmatrix}, \quad \text{and so} \quad \ddot{\mathbf{u}}^* = -\mathbf{dd} \circ \mathbf{u}^* \qquad \text{where} \qquad \mathbf{dd} = \omega_r^2 \begin{bmatrix} 9 & 1 & 1 & 9 \end{bmatrix}^T, \tag{9}$$

and where $\circ$ is the Hadamard product (element-wise matrix multiplication). Substituting this into (8) gives

$$\mathbf{b}\left[(\mathbf{dd} - \omega_r^2\mathbf{1}) \circ \mathbf{u}^*\right] = [n]\mathbf{u}^* - [n_u]\mathbf{u}^*, \tag{10}$$

where $\mathbf{1}$ is a vector the same size as $\mathbf{dd}$ with every element being one. For multi-degree-of-freedom systems the same approach can be taken which results in a slightly more complex formula as presented in, for example, [12]. From (10) we now obtain

$$\omega_r^2[8b_1 \;\; 0 \;\; 0 \;\; 8b_4]\mathbf{u}^* = \left([\alpha \;\; 3\alpha \;\; 3\alpha \;\; \alpha]^T - [n_{u1} \;\; n_{u2} \;\; n_{u3} \;\; n_{u4}]\right)\mathbf{u}^*. \tag{11}$$

The left hand side gives two zero terms which means that two of the n_{ui} coefficients on the right-hand side have to be non-zero for the equation to be satisfied. This gives

$$\mathbf{b} = \begin{bmatrix} \frac{\alpha}{8\omega_r^2} & 0 & 0 & \frac{\alpha}{8\omega_r^2} \end{bmatrix}, \quad [n_u] = \begin{bmatrix} 0 & 3\alpha & 3\alpha & 0 \end{bmatrix}. \tag{12}$$

It is important to note that there is some freedom of choice between the $\mathbf{b}$ and $[n_u]$ coefficients in (11). However one of the advantages of this method is that the non-resonant and only the non-resonant terms in $\mathbf{u}^*$ are removed from the transformed equation of motion using the straightforward approach demonstrated with this example.

The near-identity transform, to order ε^1 may now be written as

$$x = u + \varepsilon\mathbf{b}\mathbf{u}^* = \frac{A}{2}(\mathrm{e}^{\mathrm{i}(\omega_r t-\phi_0)} + \mathrm{e}^{-\mathrm{i}(\omega_r t-\phi_0)}) + \varepsilon\begin{bmatrix} \frac{\alpha}{8\omega_r^2} & 0 & 0 & \frac{\alpha}{8\omega_r^2} \end{bmatrix}\mathbf{u}^* = A\cos(\omega_r t - \phi_0) + \varepsilon\frac{\alpha A^3}{32\omega_r^2}\cos(3(\omega_r t - \phi_0)). \tag{13}$$

Note that the harmonics are orthogonal in this solution, which is in contrast to the classical normal form which is based on state-space equations [11].

Then from (6), along with (12), the transformed equation of motion may be written as

$$\ddot{u} + \omega_n^2 u + \varepsilon 3\alpha(u_p^2 u_m + u_p u_m^2) = 0. \tag{14}$$

To obtain the frequency amplitude relationship for the backbone curve, we substitute the base solutions for u_p and u_m into (14) and then exactly balance either the $\mathrm{e}^{\mathrm{i}(\omega_r t-\phi_0)}$ or $\mathrm{e}^{-\mathrm{i}(\omega_r t-\phi_0)}$ terms (there are no other terms as the non-resonant terms have been removed) to give

$$\omega_r = \sqrt{\omega_n^2 + \varepsilon\frac{3\alpha}{4}A^2}. \tag{15}$$

2.2. Harmonic balance

The method of harmonic balance assumes a trial solution for x. The simplest assumption is a single term harmonic solution, such as $x \approx A\cos(\omega_r t)$, where ω_r represents the nonlinear response frequency (which is amplitude dependent), and A is the amplitude of response which is equal to the initial displacement. This is because at time $t_0 = 0$ we have $x = A\cos(0) = A$ and $\dot{x}(0) = -\omega_r A\sin(0) = 0$. Substituting the trial solution into (1) gives

$$(\omega_n^2 - \omega_r^2)A\cos(\omega_r t) + \frac{\alpha}{4}A^3[3\cos(\omega_r t - \phi_0) + \cos(3(\omega_r t - \phi_0))] \approx 0. \tag{16}$$

Now the idea is to "balance" the terms in the equation. So comparing coefficients of the $\cos(\omega_r t - \phi_0)$ terms gives the amplitude frequency relationship

$$\omega_r^2 \approx \omega_n^2 + \frac{3\alpha}{4}A^2 \rightsquigarrow \omega_r \approx \sqrt{\omega_n^2 + \frac{3\alpha}{4}A^2}. \tag{17}$$

This is a first approximation to the backbone curve relating nonlinear response frequency ω_r to the amplitude of oscillation A. In this first approximation to the solution, (16), there is a $\cos(3(\omega_r t - \phi_0))$ term that is not balanced with anything else, so in order to balance this term we need to add an additional term to the trial solution, $x = A(\cos(\omega_r t - \phi_0) + B\cos(3(\omega_r t - \phi_0)))$. This will lead to a solution with $3\omega_r$ frequency terms balanced but new higher frequency terms occur which are now unbalanced. The logical conclusion is to include all possible harmonic terms.

An alternative approach is to use perturbation methods which allow the accuracy of the solution to be controlled by monitoring the relative size of the terms. This is based on the assumption that the nonlinear terms are small, and a parameter ε is used to identify these small terms. We now apply this approach to the Duffing example using the method of multiple time scales.

2.3. Method of multiple time scales

In this method we assume a solution of the form

$$x(t) = X_0(\tau,\ T) + \varepsilon X_1(\tau,\ T) + O(\varepsilon^2). \tag{18}$$

where it is assumed that the ε^2 and higher terms are insignificant. Each of the X_i terms is assumed to be a function of two time-scales: fast-time over which oscillations occur $\tau = \omega t$; and slow-time over which the amplitudes evolve $T = \varepsilon t$. These times τ and T are treated as independent variables, such that derivatives with respect to t can be expressed, to order ε^1, as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\partial x}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} = \omega\frac{\partial x}{\partial \tau} + \varepsilon\frac{\partial x}{\partial T}, \qquad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \omega^2\frac{\partial^2 x}{\partial \tau^2} + 2\omega\varepsilon\frac{\partial^2 x}{\partial T\partial \tau} + O(\varepsilon^2).$$

Here there is a choice, the fast time frequency, ω, can be set to the linear natural frequency ω_n or to the response frequency ω_r giving results that are slightly different. We first consider using $\tau = \omega t = \omega_n t$ and then $\tau = \omega t = \omega_r t$.

2.3.1. Fast time: $\omega = \omega_n$

If we take $\omega = \omega_n$, the frequency of the response ω_r is captured through the combination of the two timescales. Substituting the expressions for the time derivatives into (1), and including ε, gives

$$\left(\omega_n^2\frac{\partial^2 x}{\partial \tau^2} + 2\omega_n\varepsilon\frac{\partial^2 x}{\partial T\partial \tau}\right) + \omega_n^2 x + \varepsilon\alpha x^3 = 0,$$

where terms of order ε^2 and above have been neglected. Now substituting (18) into this and balancing in terms of the order of ε gives

$$\varepsilon^0 : \frac{\partial^2 X_0}{\partial \tau^2} + X_0 = 0, \qquad \varepsilon^1 : \frac{\partial^2 X_1}{\partial \tau^2} + X_1 = -\frac{1}{\omega_n^2}\left(2\omega_n\frac{\partial^2 X_0}{\partial T\partial \tau} + \alpha X_0^3\right). \tag{19}$$

Both these equations are linear in terms of X_0 and X_1 respectively, with the X_0 terms in the second expression acting as an *excitation* of the X_1 system. The general solution to Eq. (19) is taken to be

$$X_0 = A(T)\cos(\tau + \phi(T)), \tag{20}$$

where $A(T)$ and $\phi(T)$ are slowly time varying amplitude and phase respectively. This solution can be substituted into the second equation of (19) to give

$$\frac{\partial^2 X_1}{\partial \tau^2} + X_1 = \frac{2}{\omega_n}\left(\frac{\partial A(T)}{\partial T}\sin(\tau + \phi(T)) + A(T)\frac{\partial \phi(T)}{\partial T}\cos(\tau + \phi(T))\right) - \alpha\frac{A(T)^3}{4\omega_n^2}\left[3\cos(\tau + \phi(T)] + \cos(3[\tau + \phi(T)]).\right. \tag{21}$$

The $\cos(\tau + \phi)$ and $\sin(\tau + \phi)$ terms on the right-hand side will create secular terms in X_1. To avoid this, we require that

$$\frac{\partial A(T)}{\partial T} = 0, \quad \text{and} \quad \frac{2}{\omega_n}A(T)\frac{\partial \phi(T)}{\partial T} - \alpha\frac{3A(T)^3}{4\omega_n^2} = 0 \quad \text{giving} \quad A(T) = \text{constant} = A \quad \text{and,} \quad \phi(T) = \frac{3\alpha}{8}\frac{A^2}{\omega_n}T + \phi_0,$$

where ϕ_0 is an integration constant representing a phase offset at time zero. Hence, using (20), we can write

$$X_0 = A\cos(\omega_r t + \phi_0), \quad \text{with:} \quad \omega_r = \omega_n + \varepsilon\frac{3\alpha}{8}\frac{A^2}{\omega_n} + O(\varepsilon^2), \tag{22}$$

where we have recalled that $\tau = \omega_n t$ and $T = \varepsilon t$ such that $\tau + \frac{3\alpha}{8}\frac{A^2}{\omega_n}T = \omega_r t$. In addition (21) simplifies to

$$\frac{\partial^2 X_1}{\partial \tau^2} + X_1 = -\frac{\alpha A^3}{4\omega_n^2}\cos\left(3\left(\tau + \frac{3\alpha}{8}\frac{A^2}{\omega_n}T + \phi_0\right)\right), \quad \text{which gives} \quad X_1 = \frac{\alpha A^3}{32\omega_n^2}\cos(3(\omega_r t + \phi_0)). \tag{23}$$

As a result the order ε^1 solution, $x = X_0 + \varepsilon X_1$, is given by

$$x = A\cos(\omega_r t + \phi_0) + \frac{\alpha A^3}{32\omega_n^2}\cos(3(\omega_r t + \phi_0)) + O(\varepsilon^2) \quad \text{with:} \quad \omega_r = \omega_n + \varepsilon\frac{3\alpha}{8\omega_n}A^2 + O(\varepsilon^2). \tag{24}$$

2.3.2. Fast time: $\omega = \omega_r$

Alternatively, rather than setting the fast time to be $\omega t = \omega_n t$, we could set it to $\omega t = \omega_r t$, where ω_r is the, as yet unknown, response frequency. Substituting the expressions for the time derivatives into (1) and adding ε now gives, to order ε^1,

$$\left(\omega_r^2 \frac{\partial^2 x}{\partial \tau^2} + 2\omega_r \varepsilon \frac{\partial^2 x}{\partial T \partial \tau}\right) + \omega_n^2 x + \varepsilon \alpha x^3 = 0. \tag{25}$$

Now, before balancing for ε^i, a small detuning parameter is introduced to relate ω_r and ω_n. There are (at least) two ways of doing this, depending on whether the aim is to eliminate ω_n or ω_r from (25). Here we will derive a solution using $\omega_n = \omega_r(1 + \varepsilon\gamma)$ and then eliminate ω_n. In this expression γ is the detuning parameter and we assume that the response frequency is close to the natural frequency, hence it is of order ε. The alternative is to write $\omega_r = \omega_n(1 + \varepsilon\mu)$ (where μ is a detuning parameter) and eliminate ω_n. This approach is identical to using $\omega = \omega_n$ i.e. (24).

Substituting $\omega_n = \omega_r(1 + \varepsilon\gamma)$ into (25) and balancing gives

$$\varepsilon^0 : \frac{\partial^2 X_0}{\partial \tau^2} + X_0 = 0, \qquad \varepsilon^1 : \frac{\partial^2 X_1}{\partial \tau^2} + X_1 = -\frac{1}{\omega_r^2}\left(2\gamma\omega_r^2 X_0 + 2\omega_r \frac{\partial^2 X_0}{\partial T \partial \tau} + \alpha X_0^3\right), \tag{26}$$

As before, (20) the solution to the first equation in (26) may be written as $X_0 = A(T)\cos(\tau + \phi(T))$, although note that now $\tau = \omega_r t$ whereas before $\tau = \omega_n t$. Substituting this solution into the right-hand side of (26) and removing secular terms gives

$$\frac{\partial A(T)}{\partial T} = 0 \quad \text{and} \quad 2\gamma\omega_r^2 A(T) + 2\omega_r A(T)\frac{\partial \phi(T)}{\partial T} + \frac{3\alpha A(T)^3}{4} = 0 \tag{27}$$

These can be solved to give $A(T) = \text{constant} = A$, $\phi(T) = \phi_0$ and $\gamma = -\frac{3\alpha A^2}{8\omega_r^2}$, such that

$$X_0 = A\cos(\omega_r t + \phi_0) \quad \text{with:} \quad \omega_r = \omega_n + \varepsilon\frac{3\alpha A^2}{8\omega_r} + O(\varepsilon^2) \tag{28}$$

where we have recalled that $\omega_n = \omega_r(1 + \varepsilon\gamma)$. With (28), (26) simplifies to

$$\frac{\partial^2 X_1}{\partial \tau^2} + X_1 = -\frac{\alpha A^3}{4\omega_r^2}\cos\left(3(\tau + \phi_0)\right) \quad \text{and so } X_1 \text{ may be written as} \quad X_1 = \frac{\alpha A^3}{32\omega_r^2}\cos\left(3(\tau + \phi_0)\right). \tag{29}$$

The ε^1 solution for the response, $x = X_0 + \varepsilon X_1$ is therefore

$$x = A\cos(\omega_r t + \phi_0) + \frac{\alpha A^3}{32\omega_r^2}\cos\left(3(\tau + \phi_0)\right) + O(\varepsilon^2) \quad \text{with:} \quad \omega_r = \omega_n + \varepsilon\frac{3\alpha A^2}{8\omega_r} + O(\varepsilon^2). \tag{30}$$

Comparing direct normal form to this version of the multiple scales method, it can be seen that the expression for x, (13) and (30), is the same. However the response frequency, (15), differs slightly – expanding (15) using a Taylor series expansion confirms that they agree to order ε^1 but this solution contains additional higher order ε terms giving the additional accuracy. This is also obtained from the harmonic balance expression for the backbone curve, (17). As a numerical example, backbone curves for the Duffing oscillator are shown in Figure 1. In this figure, the backbone curves, which give the amplitude-frequency behaviour, are shown for the undamped, unforced Duffing system. It is clear that at higher response amplitudes, the multiple scales method is less accurate than the other methods. This difference in apparent accuracy can be explained by comparing the ω_r expressions in (15), (17), (24) and (30). Note that the direct normal form and harmonic balance have exactly the same expressions when $\varepsilon = 1$. Multiple scales has an expression that is consistent with the others if the square root function is approximated for small amplitudes.

3. Conclusions

In this paper we have made a comparison between the direct normal form method, harmonic balance and the method of multiple scales. Using the example of an undamped, unforced Duffing oscillator, it has been shown that for approximating backbone curves, all methods give good accuracy when the response amplitude is low. However, the

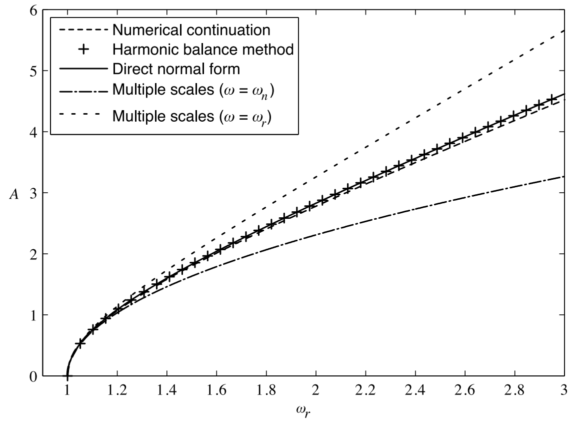

Fig. 1: Backbone curves for the undamped Duffing oscillator (defined in Section 2). Parameter values: $\omega_n = 1$, $\varepsilon = 1$ and $\alpha = 0.5$. The dashed line is the curve determined by numerical continuation (so is taken to be the most accurate). The solid line is from the direct normal form transformation, and the crosses represent the harmonic balance method. The multiple scales solutions are represented by short dashes and dash-dotted lines for the $\omega = \omega_r$ and $\omega = \omega_n$ cases respectively.

direct normal form and harmonic balance both give good accuracy as the response amplitude increases. Furthermore, the direct normal form technique gives a response frequency, ω_r, estimate that is the same as harmonic balance; however the harmonic balance solution only describes the fundamental component. Additionally, the direct normal form and harmonic balance methods appear more robust to detuning of the response frequency from the linear natural frequency. It has been shown in previous literature that the prediction of the fundamental component by the direct normal form technique is robust to the detuning that is chosen, [13]. In contrast, the multiple scales technique is influenced by the choice of the fast time scale.

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